ALTERNATING α-SERIES PROCESS

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The alternating α-series process (AAS process) can be used to study ageing repairable systems whose lifetime can be modelled as an alternating sequence of operational times and repair times. In this paper the AAS process is defined, and two key counting processes associated with the AAS process are analysed. The AAS process and results in this paper can be applied in fields such as warranty, maintenance and reliability cost analysis.

1. Introduction

Consider a repairable system, for which the lifetime can be modelled as an alternating sequence of operational times and repair times. A variety of models have been used to study this type of alternating system. For example, [7] used an alternating renewal process to evaluate the warranty costs over a finite time horizon for this type of system. For systems, which are impacted by ageing, the operational times tend to decrease and repair times tend to increase over time. Extending the geometric process (GP) [9] to an alternating process, [2] introduced the alternating geometric (AG) process, which models operational times using a decreasing geometric process and repair times using an increasing geometric process, and demonstrate its application to warranty cost analysis. [1] studied two counting processes associated with the AG process. In addition to the geometric and AG processes, there are a large number of other stochastic processes that can be used to model trends in inter-event times; see [3] for a review of these processes. One example of such a process is the α-series (AS) process, which was introduced by [5]. The main advantage of the AS process over the GP is that, under certain conditions, the number of events observed within a finite time interval, under an decreasing AS process has a finite expected value, unlike the decreasing GP. For more on the AS process see [6].

The main goal of this work is to introduce and study the alternating α-series (AAS) process. Extensions of the results of [1,2] relating to the two counting processes associated with the alternating α-series process are also discussed.

The structure of the paper is as follows. In Section 2, the α-series process is defined and the alternating α-series process is introduced, along with two key counting processes associated with it. In Section 3, two approaches for computing the mean and variance of the two counting processes are provided. The accuracy of these approaches is demonstrated using numerical examples in Section 4. Section 5 concludes the paper.

2. AS process and AAS process

2.1. *Definitions*

Definition 1. **α-series Process:** Let $\{X_n, n = 1, 2, \ldots\}$ be a sequence of independent, non-negative random variables. If the distribution function of X_n is given by $F_{X_n}(t) = F(n^\alpha x)$ for $n = 1, 2, \ldots$, where $\alpha \in \mathbb{R}$, then $\{X_n, n = 1, 2, \ldots\}$ is called an α-series process with parameters $\{\alpha, F_{X_1}(t)\}$.

The AS process can be used to model inter-event times in which a trend is observed. An AS process is stochastically increasing if $\alpha < 0$ and stochastically decreasing if $\alpha > 0$. If $\alpha = 0$, then the AS process is a renewal process.

Next, using the AS process the alternating α-series process (AAS process) is defined. The AAS can be used to model ageing systems with stochastically decreasing operational times and stochastically increasing repair times. Consider a repairable item, which initially operates for a length of time X_1 and then fails and is repaired for a length of time Y_1. After the repair, the item is again operational for a time X_2, which is followed by a repair for a time Y_2, and so on. The process is thus defined by a sequence of alternating operational and repair times, so it is called an *alternating* process.

Definition 2. Alternating α - series Process: Let $\{X_n\}_1^\infty$ and $\{Y_n\}_1^\infty$ be independent sequences of random variables. If the sequence of the operational times $\{X_i\}_1^\infty$ is a stochastically decreasing AS process with parameters $\{\alpha, F_{X_1}(t)\}$, $\alpha > 0$ and the sequence of repair times $\{Y_i\}_1^\infty$ is a stochastically increasing AS process with parameters $\{\beta, F_{Y_1}(t)\}$, $\beta < 0$, then the corresponding alternating process is referred to as an alternating α - series (AAS) process with parameters $\{\alpha, F_{X_1}(t); \beta, F_{Y_1}(t)\}$.

2.2. *AAS Process – Counting Process 1: $N(\mathrm{t})$ number of cycles completed by time* t

Consider an AAS process with parameters $\{\alpha, F_{X_1}(t); \beta, F_{Y_1}(t)\}$, with $\alpha > 0$ and $\beta < 0$. Let a cycle be defined as a period of time consisting of an operational time followed by the corresponding repair time. Denote by $Z_n = X_n + Y_n$, the length of the n^{th} cycle with cumulative distribution function (CDF) $H_n(t)$, where

$$H_n(z) = F_{X_n} * F_{Y_n}(z), \tag{1}$$

and "*" denotes a convolution. Let $T_n = \sum_{i=1}^{n}(X_i + Y_i)$. Then, the CDF of T_n is given by

$$G_n(t) = P(T_n \leq t) = H_1 * H_2 * \cdots * H_n(t). \tag{2}$$

The number of AAS process cycles completed by time t is given by

$$N(t) = \sup\{n : T_n \leq t\}, \tag{3}$$

and it is well-known, see [10], that

$$\{N(t) \geq n\} \iff \{T_n \leq t\}. \tag{4}$$

2.3. *AAS Process – Counting Process 2: $M(\mathrm{t})$ number of failures up to time* t

Now, consider another counting process, $M(t)$, which represents the number of failures occurring before time t. Denote by $Z'_n = Y_n + X_{n+1}$ the length of the n^{th} shifted cycle, i.e., the sum of the n^{th} repair and $(n+1)^{\text{th}}$ operational times, $n = 1, 2, 3 \ldots$. with CDF $H'_n(z) = F_{Y_n} * F_{X_{n+1}}(z)$. The time until the completion of the $(n-1)^{\text{th}}$ shifted cycle is $T'_n = X_1 + \sum_{i=1}^{n-1} Z'_i$, for

$n = 1, 2, \ldots$, with the empty sum for $n = 1$ equal to 0. Denote by $G'_n(t)$ the CDF of T'_n, then $G'_n(t) = F_1 * H'_1 * H'_2 * \cdots * H'_{n-1}(t)$. Results similar to (3) and (4) can be derived for $M(t)$.

3. Mean and variance of the counting processes $\mathbf{N}(t)$ and $\mathbf{M}(t)$

In this section two approaches for the computation of the mean and variance of the counting processes $N(t)$ and $M(t)$ are outlined. These approaches extend the method used to compute the mean and variance for the GP [4] and the AG process [1]. Refer to [1] for further details of the approaches outlined below.

3.1. *Computing E[N(t)] and Var[N(t)]*

Let the mean and variance of the number of cycles, $N(t)$, be denoted by $\mathrm{E}[N(t)]$ and $\mathrm{Var}[N(t)]$ respectively. Using (4) and the standard approach for deriving results for the renewal (geometric) function [10] ([8]), the following formulae for the mean $\mathrm{E}[N(t)]$ and the variance $\mathrm{Var}[N(t)]$ functions are obtained:

$$\mathrm{E}[N(t)] = \sum_{k=1}^{\infty} G_k(t), \quad t \geq 0 \quad \text{and} \tag{5}$$

$$\mathrm{Var}[N(t)] = 2\sum_{k=1}^{\infty} k\, G_k(t) - \mathrm{E}[N(t)](1 + \mathrm{E}[N(t)]), \quad t \geq 0. \tag{6}$$

As in [1], we consider two approaches for the computation of $\mathrm{E}[N(t)]$ and $\mathrm{Var}[N(t)]$.

Approach A

In order to approximate $\mathrm{E}[N(t)]$ and $\mathrm{Var}[N(t)]$, we adapt the approach outlined in [1,4]. Consider a uniform partition of $[0, t]$ into m equal subintervals, then for $n \geq 2$ and $i = 1, 2, \ldots, m$, $G_n(t_i)$ in (2) can be approximated as follows

$$\tilde{G}_n(t_i) = \sum_{j=1}^{i} \frac{\tilde{G}_{n-1}(t_{i-j+1}) + \tilde{G}_{n-1}(t_{i-j})}{2} \left(H_n(t_j) - H_n(t_{j-1})\right), \tag{7}$$

where $t_i = \frac{it}{m}$ and $\tilde{G}_1(t_i) = H_1(t_i)$ for $i = 1, 2, \ldots, m$. Using (5), (6) and (7), approximations of $\mathrm{E}[N(t)]$ and $\mathrm{Var}[N(t)]$ can be obtained. For more details see [1].

Approach B

Using the definition of the expected value and variance of a discrete random variable, and the distribution $P(N(t) = k)$, $k = 0, 1, 2, \ldots$, $t \geq 0$, then

$$\mathrm{E}[N(t)] = \sum_{k=0}^{\infty} k\, P(N(t) = k) \quad \text{and} \tag{8}$$

$$\mathrm{Var}[N(t)] = \sum_{k=0}^{\infty} k^2\, P(N(t) = k) - (E[N(t)])^2. \tag{9}$$

Using (4), the distribution of $N(t)$ can be obtained as follows

$$\begin{aligned} P(N(t) = k) &= P(N(t) \geq k) - P(N(t) \geq k+1) \\ &= P(T_k \leq t) - P(T_{k+1} \leq t) \\ &= G_k(t) - G_{k+1}(t). \end{aligned} \tag{10}$$

Using an appropriate method to approximate $G_k(t)$ (e.g., (7)), and by truncating the infinite series in (8) and (9), the mean $\mathrm{E}[N(t)]$ and variance $\mathrm{Var}[N(t)]$ can be computed.

3.2. *Computing E[M(t)] and Var[M(t)]*

Let the mean and variance of the number of failures, $M(t)$, be denoted by $\mathrm{E}[M(t)]$ and $\mathrm{Var}[M(t)]$ respectively. Approximations for $\mathrm{E}[M(t)]$ and $\mathrm{Var}[M(t)]$ can be obtained by appropriately adjusting (7) (Approach A) and (8), (9) and (10) (Approach B).

4. Numerical Results

In this section the computation of $\mathrm{E}[N(t)]$ and $\mathrm{Var}[N(t)]$, associated with the AAS process, are compared using simulation and the two numerical approaches outlined above. Similar results were obtained and analysed for $\mathrm{E}[M(t)]$ and $\mathrm{Var}[M(t)]$ but are not reported here.

Table 1 contains the values of $\mathrm{E}[N(t)]$ and $\mathrm{Var}[N(t)]$ computed via the two numerical approaches and simulation, for an AAS process with $\mathrm{E}[X_1] = 3$, $\mathrm{Var}[X_1] = 9$, $\alpha = 1$, $\mathrm{E}[Y_1] = 0.01$, $\mathrm{Var}[Y_1] = 0.0001$, $\beta = -1$, and where F_{X_1} and F_{Y_1} are the exponential CDF with the corresponding parameters. The simulation values are the average values across 10,000,000 simulation runs.

As shown in Table 1, the numerical approaches produce values that are close to those obtained using simulation. We found that the mean and variance computed using Approach B begin to deviate as t increases, compared with the values computed using Approach A and simulation. We expect that this is due to computing error accumulation, however this is an area for further investigation. To demonstrate the variety of monotonic trends that can be modelled using an AAS process, we have computed the expected number of cycles $\mathrm{E}[N(t)]$ and the expected number of failures $\mathrm{E}[M(t)]$ (not included in this paper) for a range of parameter values. The results presented below were computed using both numerical approaches and simulation, however due to the similarity of the results, only those for Approach A are shown in the plots.

Table 1. Comparison of numerical approaches and simulation for an AASP with $\mathrm{E}[X_1] = 3$, $\mathrm{Var}[X_1] = 9$, $\alpha = 1$, $\mathrm{E}[Y_1] = 0.01$, $\mathrm{Var}[Y_1] = 0.0001$, β = -1, F_{X_1}, F_{Y_1} = exponential cdf. Settings for numerical approaches: ϵ=10^{-3}, t/m = 0.01, $\eta = 10^{-16}$. Settings for simulation: n = 10000000

t	$\mathrm{E}[N(t)]$			$\mathrm{Var}[N(t)]$		
Time	Approach A	Approach B	Simulation	Approach A	Approach B	Simulation
0.6	0.213682	0.213682	0.213738	0.250136	0.250136	0.250140
1.0	0.382684	0.382684	0.382808	0.504318	0.504318	0.504426
2.0	0.907106	0.907106	0.907360	1.582393	1.582393	1.583126
5.0	3.670939	3.670939	3.672390	12.160205	12.160205	12.167529
10.0	13.030337	13.030337	13.031078	59.528072	59.528072	59.546670
12.0	17.890105	17.890105	17.890020	76.948257	76.948257	76.967922
15.0	25.393798	25.393794	25.394476	89.810023	89.809996	89.826206
18.0	32.578429	32.575922	32.578501	89.169183	89.187123	89.171773
20.0	37.054373	37.004514	37.056137	84.715775	85.516449	84.763496

In Figure 1, the columns depict two different values of $\mathrm{E}[X_1]$ and the rows depict two different values of β. In the plot in the top left of Figure 1, the expected time until the first failure, $\mathrm{E}[X_1] = 0.03$, is similar to the first expected repair time, $\mathrm{E}[Y_1] = 0.01$. The AS process parameter for the repair times is $\beta = -5$, which means that the repair times increase rapidly. Consequently, the first two repairs occur very quickly as both $\mathrm{E}[X_1]$ and $\mathrm{E}[Y_1]$ are small. However, $\mathrm{E}[Y_n]$ soon begins to increase rapidly and becomes significantly larger than $\mathrm{E}[X_n]$. Since the system cannot fail while being repaired, the number of cycles increases at a slower rate for $n > 2$. The AS process parameter for the operational times α has little impact on the expected number of cycles as $\mathrm{E}[Y_n]$ is larger than $\mathrm{E}[X_n]$ for $n > 2$. A similar trend in $\mathrm{E}[N(t)]$ (an initial steep increase, followed by a more gradual increase) is also shown in the plot in the top right of Figure 1. However, in this case, due to the longer expected time until the first failure,

$E[X_1] = 3$, the impact is less severe. Notice that α has an impact here, with higher values of α leading to a more rapid decrease in the operational times, and thus a higher number of cycles. Similar trends were observed for $E[M(t)]$.

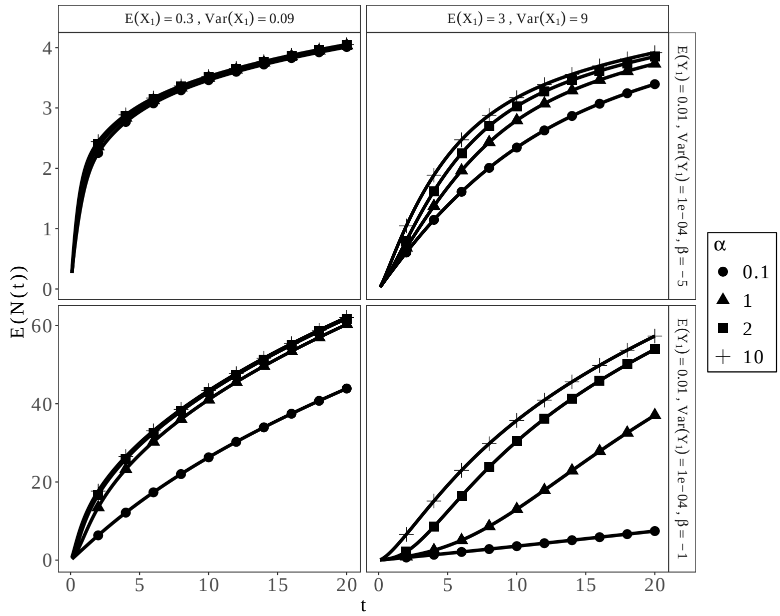

Figure 1. Expected number of completed AAS process cycles, $E[N(t)]$, for $\alpha \in \{0.1, 1, 2, 10\}$ for F_{X_1}, F_{Y_1} = exponential CDF. $E[N(t)]$ was computed using numerical approach A, with $\epsilon = 10^{-3}$, $t/m = 0.01$, and $\eta = 10^{-16}$.

In the two plots in the bottom row of Figure 1, $\beta = -1$, i.e., the increase in the repair times is slower than in the top row. Once again, when $E[X_1]$ is small, α does not have much impact as $E[X_n]$ are small compared with $E[Y_n]$. However, since $E[Y_n]$ increases slower than in the plots in top row, $E[N(t)]$ is much larger. When $E[X_1] = 3$, with $\alpha = 0.1$, the operational time decreases very slowly, leading to a low value of $E[N(t)]$. Similar trends were observed for $E[M(t)]$.

5. Conclusion

In this paper we define the AAS process and discuss two counting processes associated with the AAS process: (1) $N(t)$ - the number of cycles up to time t and (2) $M(t)$ - the number of failures up to time t, for $t > 0$. We discuss two numerical approaches for the approximation of their mean and variance functions.

For a system that can be successfully modelled by an AAS process with parameters $\{\alpha, F_{X_1}(t);\ \beta, F_{Y_1}(t)\}$ with ratios $\alpha > 0$ and $\beta < 0$, these results can be used to compute the mean value and the variance of the warranty cost under different warranty strategies. These results could be particularly useful for designing better warranty strategies as well as assisting the producers in allocating appropriate funds to the warranty reserves related to their products.

Acknowledgments

This research was supported by JSPS KAKENHI Grant-in-Aid for Scientific Research (C), Grant Number: 18K04621

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