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Kerri Spooner

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Using mathematical modelling to provide students with a contextual learning experience of differential equations

Kerri Spooner

Department of Mathematical Sciences, Auckland University of Technology, Auckland, New Zealand

ABSTRACT

Gaining useful insight into real-world problems through mathematical modelling is a valued activity across several disciplines including mathematics, biology, computer science and engineering. Differential equations are a valuable tool used in modelling. Modelling provides a way for students to engage with differential equations within a contextual environment. Teaching mathematics in context has the potential of giving students something to anchor the mathematics to and hence act as cognitive roots With this in mind, in what ways can lecturers use mathematical modelling to provide students with contextual learning experiences of differential equations? A New Zealand study was carried out involving three case studies. Each case study comprised of a mathematical modelling course, lecturer participant and student participants. Modelling examples and activities that involved differential equations were part of all course content. In this paper, I will present information on each course, the main modelling activities each course used, and examples of students' use of differential equations for these activities. Insights discovered into the common practices of the three lecturers regarding using mathematical modelling to teach differential equations will also be presented.

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KEYWORDS
Mathematical modelling; Differential equations; Teaching approaches

1. Introduction

A New Zealand study was carried out involving three case studies. Each case study compromised a mathematical modelling course, lecturer participant and student participants. Informed consent was obtained from participants through participant information sheets and signed consent forms.

All lecturers had a different approach to teaching modelling. Modelling examples and activities that involved differential equations were part of all course content. In this paper, I will present information on each course, the main modelling activities each course used and examples of students' use of differential equations for these activities. Modelling techniques, including differential equations, were part of course contents though different modelling activities were used for each course. Each lecturer placed a different emphasis on the modelling process. All modelling activities took place within student groups.

CONTACT Kerri Spooner kerri.spooner@aut.ac.nz Department of Mathematical Sciences, Auckland University of Technology, Private Bag 92006, Auckland 1142, New Zealand

Table 1. Summary of modelling activities for each course.

Course	Modelling activity used	Focus on the modelling cycle	Students' responsibility for decisions
Course one	Open-ended	Explicit	All decisions, including the foundation model to use
Course two	Case Studies	Implicit	Modification of model only
Course three	Popular Culture	Explicit	All decisions except for the foundation model

A summary of the type of modelling activity used, focus on the modelling cycle and student responsibility for decisions during the activity is given in Table 1.

2. Course one – open-ended modelling activities

For the first case study, the lecturer taught modelling techniques and the modelling processes during lectures and tutorials, followed by the students participating in a group open-ended modelling activity worth 5% of their final grade.

Preparation from the lecturer to enable student participation in the modelling activity included: developing techniques for modelling; acquiring knowledge of the process of mathematical modelling; becoming familiar with skills for effective group/teamwork and understanding the criteria for the assessment of the modelling project.

Developing techniques for modelling were taught during lectures and included mathematical techniques, methods and tools. Differentiation techniques including analytical, numerical and integration techniques were taught. In addition, probability, vectors and matrices were covered in the course material. In interviews with the lecturer, specific tools for modelling covered in lecture material included the application of proportionality relationships, dimensional analysis, optimisation, rates of change, physical laws including Newton's second law and curve fitting.

Acquiring knowledge of the process of mathematical modelling occurred through explicit teaching of a generic modelling process and contextual exposure to this process throughout the course. The course was 'embedded within the context of the modelling process' (Spooner, 2020, p. 484) with the lecturer illustrating how the modelling techniques feed into the modelling process. Exposure to the modelling process occurred at the onset of the course with the first teaching module dedicated to the modelling process. Additional exposure to the modelling process occurred with the lecturer role modelling the use of the modelling process in lectures in applications throughout the course, referencing the modelling process in worked assignment solutions and the modelling process detailed in assessment instructions. The generic modelling process used was to identify the problem, including clarifying the problem, listing and classifying the factors, making assumptions and formulating a precise problem statement; formulating a mathematical model involving representing factors by mathematical symbols, making assumptions about how the factors are related and translating the assumptions into mathematical equations and inequalities; solving the model, involving using analytic and numerical methods to obtain a solution; interpreting the mathematical solution and comparing the solution with reality. 'The lecturer utilised the generic modelling cycle, along with a 'keep it simple' principle when role modelling the modelling problems and in assessment solutions' (Spooner, 2021, p. 586).

Becoming familiar with skills for effective group/teamwork occurred within a pre-modelling project tutorial. The pre-modelling tutorial involved activities for students to get to know each other and tips for working effectively within groups.

Understanding the criteria for the assessment of the modelling project, including a recap of the modelling process, was also part of the pre-modelling project tutorial. In addition, the project instructions included resources required and allowed, time management, referencing material and report writing, as well as a project template for the group's final report.

Course one's modelling activity consisted of an ill-defined and open-ended problem to be modelled with relevant physics and mathematics. The ill-defined and open nature of the problem made the problem open for interpretation. This allowed students 'to engage in, and independently develop, a group-directed modelling process' (Spooner, 2021, p. 585) to develop a mathematical model for the problem. Questions used respectively in different semesters include: *How high can you jump from a building without causing injury? If a CubeSat (a square-shaped miniature satellite) hit the moon, how big would the resulting crater be?* Students worked on the question given in their semester for 7 h in pre-determined groups of 4. The groups were selected by the lecturer to ensure a spread of abilities and genders within the groups.

While *If a CubeSat hit the moon, how big would the resulting crater be?* did not generate student models using differential equations, some students used differential equation techniques while modelling *How high can you jump from a building without causing injury?* For example, based on Newton's laws including Newton's second law and laws for kinetic and potential energy, Steph's group initially generated a model to represent the downward and upward forces generated on the bones from the jump as being equivalent in order for the bone to break. Though, they abandoned this model when they realised time (t) would be needed in order for them to find the height of the fall (x). The following is the group's initial model that was later discarded, as presented in their written report:

$$mg - cv^2 = -m\frac{d^2x}{dt^2},$$

$$\frac{mg - cv^2}{-m} = \frac{d^2x}{dt^2},$$

$$\frac{dx}{dt} = \int \frac{mg - cv^2}{-m},$$

$$x = \int (\frac{mg - cv^2}{-m})dt.$$

The possibilities of an open-ended modelling activity are that they allow a student to creatively adapt a differential equation to the situation by applying a modelling process. This allows students to experience first-hand connecting calculus to real-world applications. The creative aspect and open-ended nature of the modelling allow students to be fully engaged in the process. Full engagement in the process generated student benefits of retention and adaptivity of the skills gained through their experience. This leaves students with a first-hand application of differential equations to model a real-world situation. This, in turn, should help develop students' confidence, or self-belief, in themselves as users of mathematics.

3. Course two – case studies

For the second case study, modelling techniques, including mathematical tools, were taught during lectures and tutorials. The students experienced the modelling process by studying modelling case studies. Presenting the case studies to the class contributed to 10% of their final grade.

In preparation for the case studies the topics taught included: discrete equations, ordinary differential equations in general, models of population dynamics using ordinary differential equations, recap on solving differential equations and the difference between difference equations and differential equations.

Students were presented with three case studies. The three cases were: *Case Study: Prickly pear meets its moth* (Barnes & Fulford, 2015); *Case Study: Possums threaten New Zealand cows* (Barnes & Fulford, 2015) and *Case Study: It's hot and study in the attic* (Barnes & Fulford, 2015). Students were asked to read the case studies and then rank the case studies from 1 to 3 according to which case study they preferred to work on. Using the students' ranking, the lecturer then matched students with case studies ensuring that all student groups were doing a different case study. Three student groups were generated with 3, 2 and 2 students.

The modelling activity consisted of groups preparing and giving 3 separate presentations, based on their case study, to the class. The first presentation involved explaining the background of the model, presenting the model and defining the variables of the model. The second presentation involved presenting the analysis of the model, including equilibrium points, phase portraits and graphs. The third presentation involved identifying the limitations of the model and possible modifications to the model, followed by working through a modification to the model, including presenting the analysis of the modification and its implications. At the end of each presentation, the audience was expected to ask questions, with assessment marks given for quality questions asked.

Each time students presented was a teaching and learning opportunity. The content presented in the presentation was used as teaching points by the lecturer for the class. For example:

The Ngaire and Natalia group used a method not covered in lectures for finding the general behaviour and stability of one of the equilibrium points in the presentation of the analysis of the *Possums threaten New Zealand cows* model. Ngaire and Natalia first presented the model including defining the variables and parameters, followed by how to find the equilibrium points and then presented how to find the eigenvalues of the Jacobian matrix. Presented below is a summary of the mathematics without a detailed student explanation:

As in the case study (Barnes & Fulford, 2015), the model is given as

$$\frac{dS}{dt} = b(S+I) - aS - \beta SI,$$
$$\frac{dI}{dt} = \beta SI - (\alpha_d + \alpha)I$$

where

$S =$ possums susceptible

$I =$ possums infected (assumed to be also contagious)

$b =$ natural birth rate
$a =$ natural death rate
$\alpha_d =$ infection transmission coefficient (density dependent)
$\beta =$ death rate from the disease.
The equilibrium points for the model can be established as follows:

$$(S, I) = (0, 0),$$
$$(S, I) = \left(\frac{\alpha_d + a}{\beta}, \frac{(b-a)(\alpha_d + a)}{\beta(\alpha_d + a - b)}\right).$$

To determine the stability and behaviour of the equilibrium points, the eigenvalues of the Jacobian matrix for the first equilibrium point can be found as $\lambda_1 = (b - a)$ and $\lambda_2 = -(\alpha_d + a)$.

For the second equilibrium, the algebra is not so straightforward. This is where Ngaire and Natalia, presented how, by examining the trace and determinant of the Jacobian matrix, the nature, stability and general information regarding the behaviour of the equilibrium points could be found without actually doing numerical calculations.

By examining the nature of the eigenvalues and the sign of the trace of the Jacobian matrix, the stability of the equilibrium point can be established. Ngaire and Natalia presented the following rationale to the class:
Case 1:

$$\sqrt{\text{Trace}^2(\text{J}) - 4\det(\text{J})} < 0$$

then eigenvalues will be complex numbers.
If Trace(J) < 0, then Re(λ) < 0. It is stable.
If Trace(J) > 0, then Re(λ) > 0. It is unstable.
Case 2:

$$\sqrt{\text{Trace}^2(\text{J}) - 4\det(\text{J})} > 0,$$

then the eigenvalues will be real numbers.
If det(J) > 0, then we have

$$\sqrt{\text{Trace}^2(\text{J}) - 4\det(\text{J})} < Trace(J)$$

Because $\text{Trace}^2(\text{J}) - 4\det(\text{J}) < \text{Trace}^2(\text{J})$ then
det(J) > 0.

- Trace(J) > 0, then $\lambda_1 > 0, \lambda_2 > 0$, so it is unstable.
- Trace(J) < 0, $\sqrt{\text{Trace}^2(\text{J}) - 4\det(\text{J})} < 0$, then it is a stable complex number.

If det (J) < 0, then we have

$$\sqrt{\text{Trace}^2(\text{J}) - 4\det(\text{J})} > Trace(J).$$

Because $\text{Trace}^2(\text{J}) - 4\det(\text{J}) > \text{Trace}^2(\text{J})$
$det(J) < 0$.
$\lambda_1 > 0, \lambda_2 < 0,$

So, it is a saddle point, then it is unstable.

By examining the Jacobian matrix $\left(\frac{\alpha_d+a}{\beta}, \frac{(b-a)(\alpha_d+a)}{\beta(\alpha_d+a-b)}\right)$ the trace was determined to be Trace (J) $= \frac{(b-a)(\alpha_d+a)}{\beta(\alpha_d+a-b)}$.

Trace $>$ 0 when $\alpha_d + a <$ b and equilibrium point is unstable.

Trace $<$ 0 when $\alpha_d + a >$ b and equilibrium point is stable.

By students needing to present the case study and the analysis of the model, it forced students to engage with mathematics and explore ways of producing the analysis. Students giving presentations enabled the whole cohort to be exposed to content that had not been covered in class. Presenting the mathematics of the case studies allowed students to see the relevance of the mathematics and techniques, including the mathematics of, and techniques for, differential equations. Nicola illustrates how students found the presentation valuable commenting 'and I was like, that's cool that they covered it like. But yeah, it gives you, like, you never thought that it could be done that way sort of thing like it gave you an idea though. You could look at it this way'.

The lecturer used the student presentations to identify key teaching points relevant to the student's mathematical development for differential equations. In the case of teaching from Ngaire and Natalia's presentation, the lecturer focused the following lecture on the techniques the students had introduced to the class. This meant students could see the applicability and relevance of the mathematics as opposed to just being taught abstract skills. An abstract from the lesson observation is given below:

> For the 3rd fixed point [lecturer 2] introduced using a solution to the general Jacobian to show you could tell a lot about the behaviour of a solution (phase plane) without actually having to numerically calculate it. You could look at the different signs of the terms in the solution to determine the behaviour of the solution.

$$\text{General solution for eigenvalues of Jacobian}\lambda_{1,2} = \frac{Tr(J) \pm \sqrt{Tr^2(J) - 4det(J)}}{2}.$$

By Ngaire and Natalia presenting their analysis to the class, they exposed their peers to a new technique for modelling, enabled the lecturer to pull out the new mathematics and use it as a teaching point as part of the feedback feedforward session of the students' presentation. The students' presentation prompted the lecturer to follow this up by dedicating the next lesson to teaching this new technique in further detail. This created a situation for teaching as needed, showing students the relevance and applicability of mathematics.

Another example of using students' work as teaching points occurred when Nicola and Rhys's group presented a phase portrait to the class. After the presentation, the lecturer focused on the detail of phase portrait and used it as a teaching tool for behaviours of equilibrium points. In particular, the lecturer showed how the stability of the system is seen in a phase portrait. The teaching that occurred was described by Nicola as further developing our understanding of the 'stability and the different definitions of stability for that really oddly behaving fixed point' in their model. Again, the lecturer gave a follow-up lesson, this time on phase portraits.

As can be seen from these examples, using student presentations was a powerful way of teaching and enabled the lecturer to add to the students' knowledge by providing new mathematics and techniques relevant to what they were currently working on.

A benefit of the presentations was that the lecturer can use them as teaching points for the whole class. It was seen from the student interviews that in order for this to be successful, it is important to create an environment where students feel safe to make mistakes and have their mistakes and work in general, used as teaching points. A clear message communicated to students by lecturer 2 was not to be worried if the mathematics was correct. We learn more from our own and each other's mistakes than getting things right all the time.

Different from Course 1, the modelling cycle was only briefly taught at the beginning of the course. It was the lecturer's intention that students would implicitly experience the modelling cycle by getting to know the case study models and draw on this in the modification stage of their work. This was successful for some students, but not all.

4. Course three – popular culture

For the final case study, modelling techniques were taught during lectures and tutorials. Modelling was experienced by creating models for different fictional scenarios.

To prepare students for the mathematical modelling activity, the students were first introduced to the modelling cycle with the lecturer developing simple models using known mathematics. This exposed students to what mathematical models were and how they could be simply developed. In two lessons before the modelling activity, the lecturer delivered two lectures, primarily focused on the infectious disease S-I-R model, which students would use as the base for model development during the activity. The lectures were for students to get to know and be familiar with the S-I-R model and possible applications of the model. The lecturer role-modelled how to manipulate the model to fit different situations.

The mathematical modelling activity occurred in the lecture following the S-I-R lectures. This was a non-assessed modelling activity using fictional scenarios based around zombie movies and books. Students worked collaboratively to create a model to represent the movie or book situation assigned to their groups. The final fifteen minutes of the lecture were dedicated to student groups presenting what they had created. Student groups were selected by students receiving a make of sweets as they entered the lecturer theatre. All students who received a brand A sweet formed one group, brand B sweet formed another group, etc. Large groups were split as needed to achieve groups of 5 or 6 students each. As Leo explained in his interview '[lecturer 3] has pictures of all the wrappers and you go sit at the table with your wrapper.'

Groups were then assigned one fictional zombie activity to model. The activities designed by the lecturer were:

> Night of the Living Dead: A radioactive leak at a nearby military base leads to dead bodies becoming reanimated as zombies. The zombies attack and kill humans and can only be stopped by decapitation or by burning.
>
> 28 Days Later: A bacteria developed to cure cancer is released from a scientific lab after animal protestors try to rescue lab chimpanzees. The bacteria called 'rage', is spread by contact with blood. Those infected by 'rage' are turned into super-violent monsters in under a minute and try to attack humans. Those infected by 'rage survive until they die of hunger.

> I am a Legend: A bacterial disease has the ability to reanimate dead bodies, turning them into 'feral' vampires-like creatures which cannot tolerate sunlight, garlic, etc. However, it is discovered that if a live person gets infected, they do not become feral. Instead, they become a sentient 'vampire' race which has the potential to replace humans.
>
> Walking Dead: All humans carry a pathogen which activates when the host dies, turning them into a zombie which shambles after humans, attacking them when they can. The pathogen can only be destroyed by damaging the host's brain or by fully destroying the body (e.g. by cremation).
>
> Brain Dead: The bite of a Sumatran Rat Monkey infects the main character of a movie. The main character declines and degrades, before eventually dying and being reanimated as a violet zombie. Other humans become zombies after they are killed. The only way to stop them is by fire and lawnmowers.

The students were given one lecture to collaboratively create a model to represent their given situation. The final ten minutes of the lecture were dedicated to student groups presenting what they had created.

Data were not collected on students' written work. However, students discussed in interviews how they went about developing their models. This is illustrated as follows:

Common for all student groups was to 'take the S-I-R model and change it' (Leo).

Hazel said 'we had to determine what were the important factors for our situation and incorporate them into the model'. Similarly, Leo and Luke both talked about 'identifying the variables and then assigning them to susceptible, infected or recovered' (Luke). Both Leo and Luke discussed how they then used their modified models to determine 'how the numbers of susceptible people and infected people and dead, and removed people changed over time' (Leo). Luke went on to say 'once we assigned the variables to the S-I-R model, we had to come up with another variable for the people that had been bitten but weren't infected or didn't die but they came back as a zombie' illustrating that modifying the S-I-R model for the situation was not straightforward and involved students actively having to think and apply themselves. Students enjoyed the modelling experience and got a lot out of the experience as Hannah illustrates when she said 'we got to apply modelling to, I wouldn't say a real situation, a situation and actually have a go at trying to model something. It made me feel smart. Ours was insanely complicated because we tried to incorporate all the variables, but we did alright. We ended up writing all the differential equations out and everything'. Hannah summarised the student experience of the activity by saying 'we learned about the SIR model and how to manipulate it to fit together.' (Hannah).

Interestingly to note, students commented on how participating by watching the presentations allowed them to 'listen and see how others would approach it' (Luke) commenting on how they 'could learn that way as well' (Hope). This is similar to the course 2 students' experience who found watching other students present their work a valuable learning experience.

4.1. Insights into common lecturer practices across all three courses

All lecturers designed and delivered their instruction such that it included opportunities for lecturers to *role-model mathematical modelling behaviours, provide resources and promote independence* for students. All lecturers created opportunities for students to work independently, while also promoting independence as a key modelling behaviour for students to develop.

4.2. Conclusion

By providing opportunities for students to use differential equations while mathematically modelling enables students to appreciate the usefulness of mathematics and the diversity of its uses and applications. It is hoped that using differential equations in contexts students' understanding of mathematics and its applicability to situations is realised. Students experience applying mathematics in non-standard ways and creatively adjust and modify the mathematics for the specific situation under investigation. There is value in students' developing models, even if the development does not produce a final useful model. The use of modelling activities provides a rich contextual learning environment for differential equations and mathematics in general. Learning is realised for students through independent engagement in the activities and by presenting and seeing each other's work.

Ethics approval

Approved by the Auckland University of Technology Ethics Committee. Reference number 16/387.

Disclosure statement

No potential conflict of interest was reported by the author.

ORCID

Kerri Spooner http://orcid.org/0000-0002-1539-8248

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