

ASSESSING CORE STABLE COALITIONS BASED ON SOCIAL NETWORK STRUCTURES

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Declaration

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Signature of candidate

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Abstract

Game theoretic techniques have become deliberate with social network analysis. Studies show that contemporary approach on social network analysis is unable to collectively evaluate the rationality of individuals and synergies that occur between them. Thus, game theory has been selected as an alternate approach for social network analysis to overcome such shortcomings (Narahari, 2011). A field of social network analysis is to examine the strength of ties within a social group and this is referred to as *social cohesion*.

The study of social groups and their tendency to stay in unity is highly correlated to interpersonal relationships and the benefits one can gain to remain in a group — whether it be monetary, popularity, social influence or social needs of an individual (Liu & Wei, 2016). Building upon this foundation, we design a type of coalitional game where the social influence rating of members is affected based on the affiliated type of network structure. We first define group cohesion and then assess cohesion on special classes of graphs via the core stability of a coalition. We then study the core stability of a special class of weighted graph followed by the implementation of weighted graphs as a regular expression which can be read by a finite automaton.

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Chapter 1

Introduction

1.1 Introduction

Social networks have become crucial in many aspects of human life. The social surrounding of a person can influence a person's choice of purchase, their judgment towards another person and even the decision for joining a social group. Humans feel the needs to create social ties with others as due to their desire to feel a sense of belonging — being in social groups or commercial organizations. The question on how social groups are bound together has been an ongoing research for sociologists since Durkheim (Moody & White, 2003). The analysis on the basis of group unity enables sociologists to clarify important aspects of social unity. These aspects include conformity of groups, the emergent norms, and social classes (Semin & Fiedler, 1996) (Friedkin & Johnsen, 2011) (Hogg, 1992). A study adapted from Jones (Jones, Livingstone & Manstead, 2011) showed that a good social group identification and strength of bonds within a group can affect a child's reaction towards bullying as well as their group emotion affects the propensity to react towards the actions of bullying.

Formation of a social group occur when ties are developed between members of a group. The *cohesion* of a social group refers to the strength of bonds between members

of a group and how they tend to “stick together” as a whole. In a study by Berman (Berman & Phillips, 2004), social cohesion is referred to as one of the four important elements of social quality — the other three being socio-economic status, social inclusion and empowerment. These elements are said to have an inherent correlation between self-realization and the composition of collective identities. Discussions on the subject matter of social cohesion can be seen as a complex task. Social cohesion, as discussed in (Beck, 2001), has significance in the process of formation, preserving formation or disrupting social groups and the underlying social structures that hold a social network.

One aspect of social cohesion as defined by sociologist Durkheim (Berman & Phillips, 2004) is that the key component of social unity is the constant allocation of diverse tasks. This holds true since groups strive to complete tasks that are given at hand due to the benefit gained by cooperating since collective expertise and resource findings usually result in a better outcome. Following this motivation, we want to evaluate how a person’s goal can be satisfied via the distribution of the accumulated outcomes. Game theoretic techniques have been selected as alternative approaches for conventional social network analysis to study such a problem assuming that every person is rational. A branch of game theory is extended to *coalitional games* which study stable coalitions between a group of players such that the collective gain of the group is divided in a way that satisfies individual needs.

Motivated by those factors mentioned previously, we combine the idea of coalitional games and social networks, and then define a coalitional game on a set of rational players that are in the form of social networks where every vertex in the social network represents a player and an edge represents a type of *relationship* between two players. The outcome of a game is dependent on the structure of the social network that should portray the social needs of an individual (Liu & Wei, 2016). While coalitional game theory is applicable to such a study, there still are certain aspects that need to be taken

into consideration when building our model. For instance, coalitional game theory does not examine cohesion of a group since notions of network structures are not thought-through even though cohesion directly relates to stability.

We now consider the notion of social influence and relate this to network formations. Naturally, humans form networks based on a simple decision: whether to become friends or to create a family. A study from Christakis and Fowler (Fowler & Christakis, 2008) shows that the decision to form networks is much more complicated than it is, that is, multiple decisions on forming networks lead to much more complex variations of networks. In their study, they have found that a person can influence another to study for a test, gain weight or even start to abuse alcohol. Thus a person would form a friendship or join a social group if they are able to “impact” — regardless an impact is positive or negative, and regardless an impact is on a social group or an individual.

Subsequently, we define payoffs following the notions of social influence: People will only stay in groups where they feel of importance and that they are able to impact an individual or a group. That is, the payoff of any player in a subnetwork should provide information of an individual’s *positional advantage*. Here we only consider those concepts of positional properties that contribute as factors that affect a person’s positional advantage. We view *social influence rating* as an index of measurement for inter-relational bonds and likings amongst people. Consequently, the higher the rating is, the more important and influential the person is in a group. We measure the level of influence by adopting the idea of degree centrality as proposed in (Liu & Wei, 2016), that is, payoffs are measured in terms of the degree of a vertex.

We now provide an overview of our main contributions in this thesis. Our first result proposes an *influence game* on a social network and describes cohesion in terms of the solution concepts of a coalitional game. In particular, we associate cohesion with the core stable concept which is one of the main solution concepts in the study of coalitional game theory. Building upon this, our second result is to implement the notion of core

stability of our game and its consistency on special classes of social network structures. We then apply theoretic concepts to a class of graphs known as the *weighted graphs*. In this study we only consider weighted paths. Lastly we show that we are able to implement such paths as an automaton.

The motivation of this thesis is a result of the developing research in the area of social network based games. In (Galeotti, Goyal, Jackson & Vega-Redondo, 2009) and (Jackson & Zenou, 2014), Galeotti and Jackson's aim was to study how a person's behavior and payoff are affected by the type of social network structure. Their focus was on the behavior of the surrounding neighbors of a player and the affects they have on the payoff gained, but our focus here is that players are not affected by surrounding neighbors and are considerate of themselves being better off. Furthermore, Saad et al. (Saad, Han, Debbah, Hjørungnes & Basar, 2009) discuss three types of coalitional games namely: canonical coalitional games, coalitional formation games and coalitional graph games. In each of these games, Saad et al. differentiate each game by their respective payoff functions and latter show the applications on network analysis tailored for network engineers. We adopt the coalitional formation games in our study. This thesis extends to the study in (Liu & Wei, 2016), where the game is defined under popularity and we review a few special classes of networks that have been assessed in the paper then investigate classes that have not yet been studied. Also, we extend the study to the class of weighted path graphs, following the idea in (Brown & Housman, 1988), and show how it provides a much more complex situation when strength of ties is taken into consideration for a social network structure.

Chapter 2

Preliminaries

In this chapter, Section 2.1 consists of preliminaries on both Cooperative Game Theory and Graph Theory. Here we address important fundamentals that will be utilized throughout the study. Section 2.2 then assesses the core stability of several classes of (unweighted) graphs each of which represents a form of social network structure.

2.1 Fundamental Concepts

2.1.1 Cooperative Game Theory

Game theory provides a formal language and analytical structure for interdependent decision making amongst rational players in competitive environments. There have been many applications of game theory to a variety of situations where players' choices of interactions influence the outcome. One branch of game theory studies the behavior of rational players when they cooperate with one another under the condition that each player is guaranteed an advantage. This branch of game theory is known as the Cooperative Game Theory.

Under the branch of cooperative games, there are two main categories, one in which

players are able to compare and transfer utility (*transferable utility games* or TU games), and one in which it is not possible for players to compare utility (*non-transferable utility games* or NTU games). In the context of TU games, often the total worth of the coalition is associated with a real quantifiable value and this value is distributed between the coalition members with no restrictions imposed. NTU games however define the worth of the coalition as a “consequence” in a way that the outcome (payoff) of a player in the coalition is dependent on the collective actions of members in the coalition.

One crucial part of cooperative game theory is to determine the payoff distributions for the players within different collaboration scenarios. TU games were introduced by Neuman and Morgenstern (Neumann & Morgensterni, 1972) to model payoff distributions whereby utilities are freely transferable. As for the latter where utility is non-transferable, Aumann and Peleg (Aumann & Peleg, 1960) formalized this scenario and specified that the games are, instead of having one worth, described by classifying all potential payoffs for each member of the coalition.

In this thesis we strictly put our focus on games with transferable utilities. A formal definition of cooperative TU games is given as follows:

Definition 2.1.1 *A cooperative game with transferable utility (TU game) is a pair (N, v) where N is a (finite) set of players, and $v(\cdot) : 2^N \rightarrow \mathbb{R}$ is a characteristic function such that $v(\emptyset) = 0$.*

A *coalition* is a non-empty subset S of N and the *grand coalition* is the set N as a whole. A *coalition structure* $\mathcal{C} = \{C_1, \dots, C_l\}$, is defined as a *partition* of N , that is, $\forall i \neq j \Rightarrow C_i \cap C_j = \emptyset$, and $\bigcup_{i=1}^l C_i = N$. The value of a coalition $S \in \mathcal{C}$ is given by $v_{\mathcal{C}}(S)$.

Definition 2.1.2 *The characteristic function is said to be **superadditive** if for all*

$C_1, C_2 \subseteq N$ and $C_1 \cap C_2 = \emptyset$, we have

$$v(C_1 \cup C_2) \geq v(C_1) + v(C_2).$$

The concept of *superadditivity* argues that by cooperating (forming large coalitions out of smaller disjoint coalitions), the value is at least that of the value obtained from forming smaller disjoint coalitions. In a superadditive game, coalition guarantees a higher (if not equal) worth. For further readings on the types of characteristic functions of cooperative games, the reader is referred to (Airiau, 2013) and (Branzei, Dimitrov & Tijs, 2008).

The characteristic function v quantifies the worth of a coalition in a game such that for every coalition $S \subseteq N$, $v(S)$ is the generated worth to be shared amongst members of S . For any coalition S , the *cardinality* $|S|$ denotes the number of players in the coalition. The total utility is distributed to players with a rule that constitutes fairness (a rule of equal proportions). The vector $\mathbf{x} \in \mathbb{R}^{|S|}$ with each element x_i representing the utility gained (payoff) for player $i \in S$ describes how the worth is shared between members of coalition S and this vector \mathbf{x} is the *payoff distribution*. We also use the notation $\mathbf{x}(S) = \sum_{i \in S} x_i$ to represent the total payoff of coalition S .

A solution concept for cooperative games is to assign a set of outcomes (payoff distributions) to each game. Each solution concept represents the consequences of the players for the coalition formation and this should provide a set of agreed terms that are stable in some sense. The stability requirement is that there should be no incentive for players to form coalitions on their own. The most relevant set of solution concepts that can be found in Osborne and Rubinstein (Osborne & Rubinstein, 2007) are the *core*, the *Shapley value*, the *kernel*, the *nucleolus* and the *bargaining set*. For simplicity, we restrict our study to only the core.

The concept of *core* was introduced by Gillies (Gillies, 1953). This solution concept corresponds to the idea of Nash equilibrium of noncooperative games, that is, if there is no incentive for players to breach the current coalition to obtain an outcome that is

better for all the members of the coalition, then the outcome is stable. Extending this concept to TU games, a payoff is stable with the condition that no coalition can obtain a payoff that is greater than the all members' payoffs combined.

Before formally introducing the definition of the core, we first define an *outcome* for a TU game as follows.

Definition 2.1.3 *An outcome of a game is a payoff distribution that is both efficient and individually rational for all players whereby*

***Efficiency:** The payoff distribution for each player is the division of the worth of the grand coalition $\mathbf{x}(N) = v(N)$, i.e., there is no loss of utility with the vastness of the population.*

***Individual rationality:** A player i only decides to join a coalition if $x_i \geq v(\{i\})$, i.e., a player is better off by joining a coalition than being on its own, (Airiau, 2013).*

For any coalition $S \subseteq N$, a payoff vector $\mathbf{x} \in \mathbb{R}^{|S|}$ is *S-feasible* if $\mathbf{x}(S) \leq v(S)$. We extend this notion to define an *objection*. Let (N, v) be a cooperative TU game and take a coalition structure \mathcal{C} .

Definition 2.1.4 *Let $\mathbf{y} \in \mathbb{R}^{|N|}$ be an outcome, and $S \subseteq N$ be a coalition. We say that S is an **objection** to \mathbf{y} if there is an S -feasible payoff vector $\mathbf{x} \in \mathbb{R}^{|S|}$ such that $\mathbf{x}_k > \mathbf{y}_k$ for any $k \in S$. Alternatively, we say that S **blocks** \mathbf{y} through \mathbf{x} .*

In other words, a player k is better off joining a coalition S .

Definition 2.1.5 *The core of a TU game is the set of all outcomes \mathbf{y} such that there are no objections by means of any set $S \subseteq N$.*

We associate the notation of *core* with “stability” since no members of a coalition will receive a better payoff by leaving the former coalition. Though the core may seem to be a much more enticing and preferred way of studying stability, the core of a TU game may be non-existent in some cases, that is, the core can be empty. However, for

the purpose of this study we assume that the TU game is superadditive and the core is non-empty.

2.1.2 Graph Theory

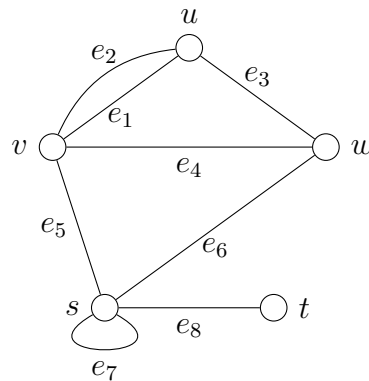
In 1736, Swiss mathematician Leonhard Paul Euler solved the Königsberg Bridge Problem using graph theory and thenceforth known as the inventor of graph theory (Biggs, Loyds & Wilson, 2006). The very first book on graph theory was written in 1936 (after a good 200 years) by Dénes König (König, 1950) and since then graph theory has expanded into a prominent branch of mathematics.

Graph theory is now considered one of the primary mathematical research tools due to its wide range of applications in a variety of fields including computer science, electrical engineering, sociology, marketing, business studies and so on (Bondy & Murty, 2002). Networks, being social, technological, biological or informatics, form one of the major areas that relates to graph theory. Typically, connected networks can be expressed in the form of graphs (Acemoglu & Ozdaglar, 2009),(Zafarani, Abbasi & Liu, 2014) where a set of people (or objects) are represented by a set of *nodes*, also commonly known as *vertices*, while the connections between them are known as the set of *edges*.

Following this notion, we formally introduce some basics and common notations used in graphs.

Definition 2.1.6 A **graph** G consists of a (finite) vertex set V and a (finite) edge set E which consists of 2-element subsets of V . The elements in the set V are called vertices (or nodes) of G and the elements in the set E are called the edges of G .

We write $G = (V, E)$ where G is a graph with vertex set V and edge set E . $V(G)$ and $E(G)$ are often used to denote the vertex and edge sets of G respectively rather than writing V and E . If there is an edge joining vertices u and v , we write $e = \{u, v\}$.

Figure 2.1: A drawing of a graph G

Definition 2.1.7 A **loop** is an edge that joins a vertex to itself. Two or more edges that are connected to the same pair of vertices are known as **parallel edges**. A graph containing no loops or parallel edges is called a **simple graph**.

Note that if there is a loop at a vertex u , then we write $e = \{u, u\}$ instead of $e = \{u\}$.

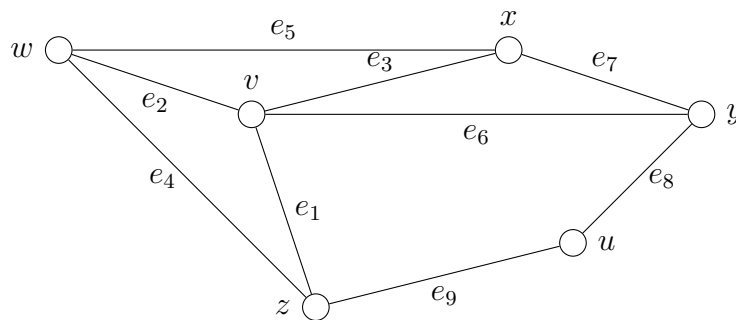
Figure 2.2: A drawing of a simple connected graph H

Figure 2.2 shows an example of a simple graph. Simple graphs are undoubtedly more commonly used to model networks (Kleinberg & Easley, 2010) under the study of network analysis based on the belief that there is at most *one link* between any two people or objects.

Definition 2.1.8 A vertex u is said to be **adjacent** to vertex v in a graph G if $e = \{u, v\}$ is an edge of G . In this case, e is called **incident** with u and v .

Example 2.1 In Fig. 2.1, edge e_7 is a loop, while edges e_1 and e_2 are parallel. The edge e_3 is incident with vertex u and vertex w , thus u and w are adjacent.

Definition 2.1.9 The **degree** (or valency) of a vertex v , denoted $d(v)$, is defined as the number of edges incident with v .

A loop that is incident with vertex v adds 2 to the degree of v . A vertex v of degree 0 is known as an *isolated vertex*. In any graph G , an edge $\{u, v\}$ represents a certain relation between the two vertices u and v .

Definition 2.1.10 A **path** from a vertex u to vertex v is a finite alternating sequence of vertices and edges such that no vertex (and thus no edge) appears more than once.

Example 2.2 Fig. 2.2 shows an example of a simple connected graph. There exists a path from u to v such that

$$ue_9ze_4we_5xe_7ye_6v$$

and no vertices or edges appear twice.

Definition 2.1.11 In a graph G , vertices u and v are connected if there exists a path from u to v . A graph G is said to be **connected** if every pair of vertices in G are connected.

Example 2.3 Fig. 2.2 shows a connected graph H since for every pair of vertices in graph H there exists a path.

For any vertex v of graph G , we let the set of vertices that are connected to v be $\zeta(v)$. An *induced subgraph* of G by the set $\zeta(v)$ is defined to be the graph having the vertex set $\zeta(v)$ and those edges that are connected to v in G . For further readings on graph theory readers are referred to (Clark & Holton, 2005).

2.2 Influence Games and Core Stable Coalitions

In this section we review some of the definitions and results from (Liu & Wei, 2016), expanding some of the details for more elaborate results. In Section 2.2.1 we introduce a coalitional game that merges both literatures and then present some new notations that will be used in this study. The latter presents and analyses the effects of various social network formations on the solution of the coalitional game, the *core*. Our focus here is to study which special classes of graphs belong in the *core*, i.e. contain no *objections*.

2.2.1 Groundwork and Notations

A *social network* can be regarded as an undirected graph $G = (V, E)$ where V denotes the set of vertices and E the set of (undirected) edges. We define a coalitional game on G whereby the set of vertices V denotes a set of rational players. The edges in the set E represents a form of social relation between two players u and v , being friends, colleagues, acquaintances or a consanguinity.

Definition 2.2.1 A coalitional game (TU game) is a pair (V, φ) , where V is the set of rational players and $\varphi(\cdot) : 2^V \rightarrow \mathbb{R}$ is a characteristic function.

A coalition structure \mathcal{C} is a partition of the set of players V , i.e., a collection of coalitions, $\mathcal{C} = \{C_1, \dots, C_k\}$ such that $\bigcup_{1 \leq i \leq k} C_i = V$ and for $\forall i \neq j, C_i \cap C_j = \emptyset$. The value of a coalition $S \in \mathcal{C}$ is given by $\varphi_{\mathcal{C}}(S)$. The *grand coalition structure* is the set of V itself, such that $\mathcal{C}_G = \{V\}$.

Let (V, φ) be a TU game. For any coalition $S \subseteq V$, we adopt the same notation as for a *payoff distribution* $\mathbf{x} \in \mathbb{R}^{|S|}$ such that each component x_i is the *utility* of player $i \in S$. A coalition structure is *stable* provided that no players will benefit from joining a new coalition. Let $\mathbf{z} \in \mathbb{R}^{|V|}$ be a payoff distribution. Recall that a set of players $S \subseteq V$

is an objection to z if there exist a S -feasible payoff distribution \mathbf{x} such that $\forall u \in S$, $\mathbf{x}_u > z_u$.

Definition 2.2.2 A coalition structure \mathcal{C} corresponding to a coalitional game $\Gamma = (V, \varphi)$ is **core stable** if it contains no objections.

Definition 2.2.3 Let \mathcal{C} be a coalition structure corresponding to a coalitional game $\Gamma = (V, \varphi)$. We say that a member S of \mathcal{C} **blocks** \mathcal{C} if S is an objection to some outcome $\mathbf{y} \in \mathbb{R}^{|N|}$.

It is clear that if a coalition structure is blocked by any of its members then it is not core stable.

A person's decision to affiliate oneself to a social group partially depends on whether one could have an impact (or influence) on members of the group, being negative (e.g. smoking, alcohol abuse) or positive (e.g. exercise, eating healthy etc.) social impacts. Following this intuition, we propose a type of coalitional game that reveals a person's level of influence, also referred to as the social influence rating. Studies from preceding years in the area of social networks have been highly associated with *centrality* which *measures* the level of "importance" of a person (Zafarani et al., 2014).

A centrality measure, whether it is *KATZ*, *degree*, *betweenness*, or *eigenvector* etc., provides information on a person's strategical advantage (Liu & Wei, 2016). In the study of social networking, we often think of people with the highest number of connections to be important. The idea of *degree centrality* uses this underlying concept and puts it into measure. In other words, the degree centrality of a vertex $u \in V$ is the *degree* of vertex u which we denote as $d(u)$. In this thesis, we denote the degree of a player u in a coalition S with $d_S(u)$.

We apply the notion of degree centrality and relate it to a person's influential status which is vital in terms of a person's social need or commercial affairs (Freeman, 1978),

(Gruman, Schneider & Coutts, 2007). Psychological studies indicate that a person's level of influence highly correlates to social group formations, decision making in business engagements, group tasks handling etc. (Guimond, 2006), (Christakis & Fowler, 2009).

Definition 2.2.4 *The social influence rating of a vertex u in a subset $S \subseteq V$ is measured by $\chi_S(u) = \frac{d_S(u)}{|S|}$.*

Consequently, an isolated vertex (singleton) has social influence rating $\chi_{\{u\}}(u) = 0$ for every vertex u . For simplicity, we will reinstate the term *social influence rating* with the term *influence level*.

Let U be a nonempty subset of the vertex set V of G . A subgraph H of G , induced by U , is a graph such that the vertex set of H is U and the edges of H are those edges of G whose ends are contained in U . If $t \in H$ is connected to every vertex in H , then t has the highest level of influence in H where $\chi_H(t) = \frac{|H|-1}{|H|}$. Thus the range of the influence level of any player is $0 \leq \chi < 1$.

Definition 2.2.5 *The influence game on $G = (V, E)$ is a coalition game*

$\Gamma(G) = (V, \varphi)$ *such that $\varphi : V \times 2^V \rightarrow [0, 1)$ is described by $\varphi(u, S) = \chi_S(u)$.*

The set of *outcomes* of the influence game $\Gamma(G)$ represents a form of settlement that bonds the set of players in such a way that no player receives an incentive by disrupting the formation.

Lemma 2.2.6 (Euler's Handshaking Lemma) (Goodaire & Parmenter, 2002) *For any graph $G = (V, E)$, the sum of the degrees of the vertices is twice the number of edges; that is*

$$\sum_{v \in V} d(v) = 2|E|.$$

Let $S \subseteq V$ be a coalition. We use $E(S)$ to denote the set of edges whose ends in are in S . The total utility of S is described by taking the *average degree* in S ; that is

$$\frac{\sum_{u \in S} d_S(u)}{|S|} = \sum_{u \in S} \chi_S(u) = \sum_{u \in S} \varphi(u, S).$$

Applying Euler's lemma, we obtain

$$\frac{\sum_{u \in S} d_S(u)}{|S|} = \frac{2|E(S)|}{|S|}.$$

The total utility is regarded as the synergy between players within the set S .

Social cohesion has been defined as the tendency of a group to remain in unity to satisfy members in terms of social needs (Liu & Wei, 2016). We adopt the notation of social cohesion and use *core stability* of the influence game $\Gamma(G)$ to express cohesion of a group. In particular, if a coalition structure \mathcal{C} is not core stable, there must be a set S of players whose coalition will allow every agent to receive a gain more than their former coalition \mathcal{C} .

Definition 2.2.7 *If the grand coalition structure \mathcal{C}_G of a social network $G = (V, E)$ corresponding to the influence game $\Gamma(G)$ is core stable, then G is cohesive.*

Based on the definition above, all members in the grand coalition structure \mathcal{C}_G will find no incentive to leave the formation and join another coalition.

Theorem 2.2.8 (Connectivity). *If a coalition structure \mathcal{C} of G is core stable then any induced subgraph in $S \in \mathcal{C}$ is connected. Thus a network G is cohesive only if it is connected.*

Proof: Suppose T and W are two non-empty sets and there are no edges between any pair of vertices between T and W . Take $S = T \cup W$. Thus we have for any $v \in W$:

$$\chi_W(v) = \frac{d_W(v)}{|W|} > \frac{d_S(v)}{|S|} = \chi_S(v).$$

Thus W blocks the coalition structure \mathcal{C} . □

If there exists no path in G then no two players will benefit by forming a coalition by Theorem 2.2.7.

Definition 2.2.9 *A set $S \subseteq V$ is known as a **social party** of G if the subnetwork induced on S is connected. Then a **party structure** is a coalition structure that only contains social parties.*

In (Dunbar, 1993), evolutionary anthropologist Robin Dunbar claims that every person has a limited capacity in his/her social circle. According to Dunbar, the number of people whom a person can maintain a meaningful relationship is approximately 150, known as *Dunbar Number* nowadays. Based on this idea, we provide the next theorem that signifies cohesion to be a *property of small networks*.

Theorem 2.2.10 *Suppose the maximum degree of a graph $G = (V, E)$ is denoted as $\delta(G)$. Then G is cohesive only when $|V| \leq 2\delta(G)$ and $|V| \neq 1$.*

Proof: It is trivial that G is not cohesive when $|V| = 1$. We assume cohesion for $|V| > 2\delta(G)$. By Theorem 2.2.7, G is not cohesive if G is disconnected. So G must be connected and we pick two vertices $u, v \in V$ and let an edge $e \in E$ be incident to u and v . Thus

$$\sup \{d(u), d(v)\} \leq \delta(G) < \frac{|V|}{2}$$

and so we have

$$\sup \{\chi_V(u), \chi_V(v)\} < \frac{1}{2}.$$

Therefore the edge e blocks G , making G not cohesive. □

2.2.2 Core Stable Graph Classes

In this section, we assess core stability of some special classes of graphs corresponding to the influence game $\Gamma(G)$. A few classes have been investigated in (Liu & Wei, 2016) namely the complete graph, the star graph and the complete bipartite graph. In this study, we review these three classes of graphs and also present some classes that have not yet been assessed in (Liu & Wei, 2016).

Complete Graphs. A *complete graph* $G = (V, E)$ is a simple graph where every distinct pair of vertices is adjacent. A complete graph with n vertices is denoted by K_n . This type of graphs represent a very compact and sturdy network since every association is mutual.

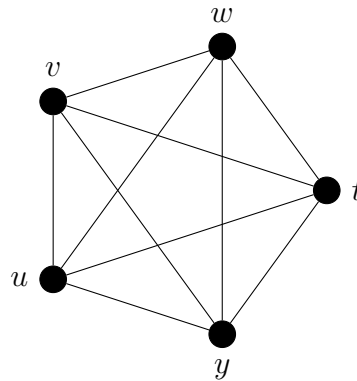


Figure 2.3: A Complete Graph K_5

Theorem 2.2.11 *Let K_n be a complete network on $G = (V, E)$. Then G is cohesive since only the grand coalition is core stable.*

Proof: Suppose we take a subnetwork S of a complete network K_n and let $u \in S$. Since we know that S is also a complete network, it follows that

$$\varphi_S(u) = \frac{|S|-1}{|S|} < \frac{|V|-1}{|V|} = \varphi_V(u)$$

since $|S| < |V|$. Thus in the grand coalition the players level of influence is maximal. \square

Star Graphs. A *star graph* \mathfrak{S}_n with n vertices v_1, v_2, \dots, v_n has single edges joining one (center) vertex v_1 to v_i (branches) for $2 \leq i \leq n$. Thus v_1 is said to have $b = n - 1$ branches. This formation represents a typical social network where one is associated to many others where no mutual interaction exists. In a star graph, the central person v_1 would like to be connected to as many people as possible.

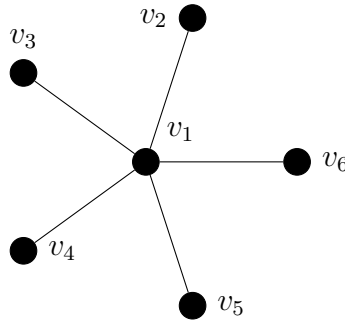


Figure 2.4: A Star Graph \mathfrak{S}_6

Theorem 2.2.12 *Let \mathfrak{S}_n be a star network. Suppose there is a party structure \mathcal{C} of \mathfrak{S}_n . Then \mathcal{C} is core stable if and only if the central vertex is adjacent to at least half the number of vertices. Hence any star network \mathfrak{S}_n is cohesive.*

Proof: Suppose we take any party structure \mathcal{C} . Let a social party be S with the center player v_1 that is adjacent to m other players, i.e. v_1 has m branches. For any player that does not belong in S has influence level of 0 since there are no interactions between other players apart from v_1 . Furthermore, $\chi_S(v_1) = \frac{m}{m+1}$ and $\chi_S(v_i) = \frac{1}{m+1}$ for any $v_i \in S$.

- We first look at the case where $m \geq \frac{b}{2}$. Let T be a social party in \mathcal{C} such that $T \neq S$ and T contains the center player v_1 . If $|T| \leq |S|$, then $\chi_T(v_1) \leq \chi_S(v_1)$. If

$|T| > |S|$, then there consists of some branch $t \in T \cap S$ where $\chi_T(t) < \chi_S(t)$. Therefore T will not block \mathcal{C} in any way. So \mathcal{C} is core stable.

- Now for case $m < \frac{b}{2}$. Suppose the set of branches not in S is denoted by S' . Since $|S'| > m$, then in this case for the center player v_1 , $\chi_S(v_1) < \chi_{S' \cup \{v_1\}}$ if center player v_1 joins S' instead. Hence $S' \cup \{v_1\}$ blocks \mathcal{C} .

It shows that \mathcal{C} is core stable iff $m \geq \frac{b}{2}$. \square

Complete Bipartite Graph. A *bipartite graph* is a simple graph whose vertex set can be partitioned into two non-empty sets X and Y with cardinality m and n respectively such that each edge of the graph is incident with a vertex in X and with a vertex in Y . A *complete bipartite graph* $K_{m,n}$ is where each vertex in X is adjacent to each vertex in Y . Suppose $m \geq n$ and let \mathcal{C} be a party structure of G . For a social party $S \in \mathcal{C}$, let $S = S_1 \cup S_2$ such that $S_1 \subseteq X$ and $S_2 \subseteq Y$. We denote the cardinalities $|S_1|$ and $|S_2|$ as L_S and R_S respectively.

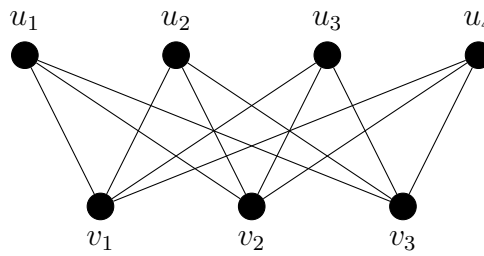


Figure 2.5: A Complete Bipartite Graph $K_{4,3}$

Lemma 2.2.13 *The party structure \mathcal{C} is core stable only if $L_S \geq R_S$ for any $S \in \mathcal{C}$.*

Proof: Let there be a social party $S \in \mathcal{C}$ with $L_S < R_S$. As we have $m \geq n$, then suppose there is a $T \in \mathcal{C}$ where $T = T_1 \cup T_2$ with $T_1 \subseteq X$ and $T_2 \subseteq Y$. Also $L_T = |T_1|$ and $R_T = |T_2|$ and let $L_T > R_T$. Pick two players $u \in S_2$ and $v \in T_1$. Thus

$$\chi_S(u) = \frac{R_S}{L_S + R_S} < \frac{1}{2}, \text{ and } \chi_T(v) = \frac{L_T}{L_T + R_T} < \frac{1}{2}$$

Hence if u and v form a party H , then

$$\chi_H(u) = \chi_H(v) = \frac{1}{2}.$$

Thus H blocks \mathcal{C} . □

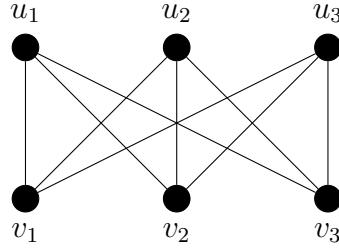


Figure 2.6: A Complete Bipartite Graph $K_{3,3}$

We now look into *complete bipartite graphs* $K_{n,n}$ where $n > 0$. This particular structure is core stable only if each of the vertices in X is linked to distinct vertices in Y . This situation is known as a *perfect matching* of $K_{n,n}$.

Theorem 2.2.14 *For any social party $S \in \mathcal{C}$, a party structure \mathcal{C} of $K_{n,n}$ is core stable if and only if $L_S = R_S$.*

Proof: Lemma 2.2.12 implies that for any party structure \mathcal{C} with social party $S \in \mathcal{C}$, if $L_S = R_S$, then for any player $u, v \in S$,

$$\chi_S(u) = \chi_S(v) = \frac{1}{2}.$$

Therefore \mathcal{C} is core stable since for any set $T \subseteq V$ there contain some players that have influence level of at most $\frac{1}{2}$, thus giving players no incentive to break the party structure.

□

Following the review on the core stability of complete networks, star networks and complete bipartite networks as shown in (Liu & Wei, 2016), we now present the results on core stability of path, cycles and wheel networks.

Path Graphs. A *path graph* P_n of n vertices is a graph where the vertices can be arranged in order v_1, v_2, \dots, v_n such that the edges are $\{v_i, v_{i+1}\}$ and $i = 1, 2, \dots, n - 1$. Subsequently, a path with n vertices will have two terminal vertices of degree 1 and the other non-terminal $n - 2$ vertices of degree 2. We let a terminal vertex correspond to a *minor player*. A path with a single vertex v is called a *singleton* (isolated vertex) such that the $\chi_{P_n}(u) = 0$. Naturally, any player u would not want to be on its own and would be connected to at least one other player, so that $\chi_{P_n}(u) = \frac{1}{2}$.

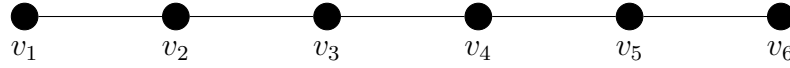


Figure 2.7: A Path Graph P_6

Theorem 2.2.15 Let P_n be a path network on $G = (V, E)$. A party structure \mathcal{C} of P_n is core stable only if $2 \leq n \leq 4$.

Proof: Assume core stability on a party structure \mathcal{C} with $n > 4$. Let S be a social party of \mathcal{C} with $n > 4$ which contains a player u . Suppose u is a minor player, then $\chi_S(u) = \frac{1}{n}$ but if u is not a minor then $\chi_S(u) = \frac{2}{n}$. Take any set $S' \neq S$ that contains u . Suppose $|S'| \leq |S|$. If u is a minor we have

$$\chi_{S'}(u) = \frac{1}{|S'|} \geq \frac{1}{|S|} = \chi_S(u).$$

Similarly, if u is not a minor then

$$\chi_{S'}(u) = \frac{2}{|S'|} \geq \frac{2}{|S|} = \chi_S(u).$$

Then suppose $|S'| > |S|$, then there is a $v \in S' \cap S$ such that if v is a minor we have

$$\chi_{S'}(v) = \frac{1}{|S'|} < \frac{1}{|S|} = \chi_S(v),$$

and also

$$\chi_{S'}(v) = \frac{2}{|S'|} < \frac{2}{|S|} = \chi_S(v)$$

if v is not a minor. Either way, S' blocks \mathcal{C} . Hence \mathcal{C} is not core stable when $n > 4$. \square

Recall that a coalition structure \mathcal{C} is a collection of coalitions such $\mathcal{C} = \{C_1, \dots, C_k\}$ such that $\forall i \neq j, C_i \cap C_j = \emptyset$ and $\bigcup_{1 \leq i \leq k} C_i = V$. We now turn our attention to coalition structures of path graphs P_n . From Theorem 2.2.14, we have established that a path with $n \leq 4$ is core stable. Here we define *forbidden structures* as coalition structures that are not core stable: e.g. if the sets S_1 and S_2 are in a coalition structure \mathcal{C} , members from either set receive an incentive by forming a new coalition. We study the forbidden structures in terms of the length of paths with maximum length 4 in a coalition and let $\phi(|S_1|, |S_2|)$ denote a structural pattern.

Theorem 2.2.16 *The following structural patterns are forbidden structures:*

$$\phi(1, 1), \phi(1, 3), \phi(1, 4), \phi(3, 3), \phi(3, 4) \text{ and } \phi(4, 4).$$

Proof: Let \mathcal{C} be a coalition structure and let S_1 and S_2 be represent two different coalitions in \mathcal{C} . We will prove each case individually.

- (i) Suppose we have $\phi(1, 1)$. This structure represents a set of two singletons where $u \in S_1$ and $v \in S_2$. Recall that for a singleton $u \in \mathcal{C}$, $\chi_{\mathcal{C}}(u) = 0$. Then if $S_1 \cup S_2 = T$ where $|T| = 2$, and for $u \in T$,

$$\chi_T(u) = \frac{1}{2} > 0 = \chi_{\mathcal{C}}(u).$$

Hence T blocks \mathcal{C} .

- (ii) Suppose we have $\phi(1, 3)$. Here we have a singleton $u \in S_1$ and a path consisting of 3 vertices. Pick a terminal vertex $v \in S_2$. Then $\chi_{\mathcal{C}}(v) = \frac{1}{3}$. If $S_1 \cup \{v\} = T$, such that $|T| = 2$ and

$$\chi_T(v) = \frac{1}{2} > \frac{1}{3} = \chi_{\mathcal{C}}(v).$$

Again, T blocks \mathcal{C} .

- (iii) Suppose we have $\phi(1, 4)$. Now we have a singleton $u \in S_1$ and a path consisting of 4 vertices. Pick a terminal vertex $v \in S_2$. Then $\chi_{\mathcal{C}}(v) = \frac{1}{4}$. If $S_1 \cup \{v\} = T$, such that $|T| = 2$ and

$$\chi_T(v) = \frac{1}{2} > \frac{1}{4} = \chi_{\mathcal{C}}(v).$$

Hence T blocks \mathcal{C} .

- (iv) Suppose we have $\phi(3, 3)$. For this structure, we have two paths consisting of 3 vertices each. Pick any terminal vertex from both coalitions such that $u \in S_1$ and $v \in S_2$ with

$$\chi_{\mathcal{C}}(v) = \chi_{\mathcal{C}}(u) = \frac{1}{3}.$$

Suppose $\{u\} \cup \{v\} = T$ so that $|T| = 2$, then

$$\chi_T(v) = \chi_T(u) = \frac{1}{2} > \frac{1}{3} = \chi_{\mathcal{C}}(u) = \chi_{\mathcal{C}}(v)$$

For obvious reasons, both terminal vertices will form a coalition T and so T blocks \mathcal{C} .

- (v) Suppose we have $\phi(3, 4)$. Now we look at a path of 3 vertices with a path of 4 vertices. Again we pick any terminal vertex from both coalitions $u \in S_1$ and $v \in S_2$ with $\chi_{\mathcal{C}}(v) = \frac{1}{3}$ and $\chi_{\mathcal{C}}(u) = \frac{1}{4}$. If $\{u\} \cup \{v\} = T$ so that $|T| = 2$. Then now

$$\chi_T(u) = \frac{1}{2} > \frac{1}{3} = \chi_{\mathcal{C}}(v)$$

and

$$\chi_T(v) = \frac{1}{2} > \frac{1}{4} = \chi_{\mathcal{C}}(v).$$

Both terminal vertices have an incentive to disrupt their former coalition, hence T blocks \mathcal{C} .

(vi) Suppose we have $\phi(4, 4)$. Here we have two paths of length 4. Let $u \in S_1$ and $v \in S_2$ be the terminal vertices and

$$\chi_{\mathcal{C}}(v) = \chi_{\mathcal{C}}(u) = \frac{1}{4}.$$

If $\{u\} \cup \{v\} = T$ such that $|T| = 2$, then

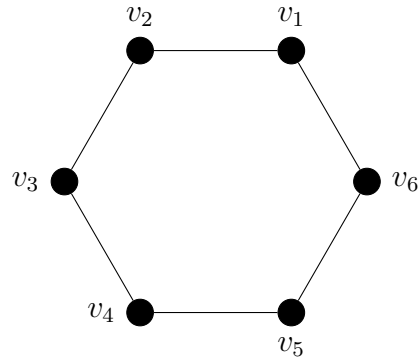
$$\chi_T(v) = \chi_T(u) = \frac{1}{2} > \frac{1}{4} = \chi_{\mathcal{C}}(u) = \chi_{\mathcal{C}}(v)$$

Again we show that both terminal vertices receives a higher gain when joining forces, hence T blocks \mathcal{C} .

Thus any coalitional structure \mathcal{C} containing such forbidden structures are not core stable.

□

Circuit Graphs. The *circuit graph* C_n has n vertices (v_1, v_2, \dots, v_n) where single edges join v_i to v_{i+1} for $1 \leq i \leq n$ and each subscript i is congruent modulo n , i.e. vertex v_n is then joined to v_1 to complete the circuit. This graph is of a form of a circle such that every vertex is in C_n is of degree 2.

Figure 2.8: A Circuit Graph C_6

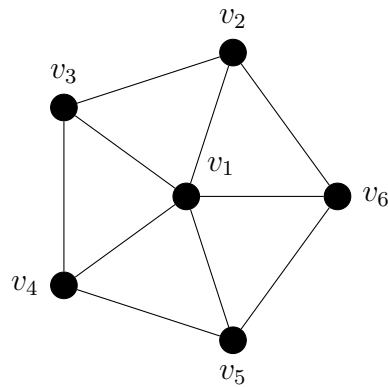
Theorem 2.2.17 *Let C_n be a circuit network on $G = (V, E)$. A party structure \mathcal{C} of C_n with at most $n = 4$ vertices is core stable.*

Proof: Let us assume core stability for $n > 4$. Take a party structure \mathcal{C} . Since the degree for every player $u \in \mathcal{C}$ is 2, we have $\chi_{\mathcal{C}}(u) = \frac{2}{n}$. Any subnetwork S of a circuit graph C_n is a path network and Theorem 2.2.14 shows that for any path network with $2 \leq n \leq 4$ players is core stable. Consequently, suppose we have a social party $S \in \mathcal{C}$ with at least 2 players, then

$$\chi_S(u) = \frac{1}{2} > \frac{2}{n} = \chi_{\mathcal{C}}(u)$$

for $n > 4$ in \mathcal{C} . Thus S blocks \mathcal{C} and so \mathcal{C} is core stable only if n is at most 4. \square

Wheel Graphs. A *wheel graph* W_n with $n \geq 4$ vertices (v_1, v_2, \dots, v_n) has only single edges joining v_1 (positioned in the inner center of the graph) to v_i and only single edges joining v_i to v_{i+1} for $2 \leq i \leq n$ while v_n joins back to v_2 , forming an outer circuit. It follows that the vertices placed “outside” the center vertex has degree 3 and the center vertex is always of degree $n - 1$. This type of graph is combination of a star graph S_n and a circuit graph C_n .

Figure 2.9: A Wheel Graph W_6

Theorem 2.2.18 *A party structure \mathcal{C} of a wheel network W_n is core stable only if $n \leq 6$.*

Proof: Assume core stability for $n > 6$. Take any party structure \mathcal{C} and suppose we take non-center players u and v in \mathcal{C} . If players u and v form a party S , then

$$\chi_{\mathcal{C}}(u) = \chi_{\mathcal{C}}(v) = \frac{3}{n} < \frac{1}{2} = \chi_S(v) = \chi_S(u)$$

for any $|\mathcal{C}| > 6$. Hence S blocks \mathcal{C} . Thus \mathcal{C} is not core stable for $n > 6$. \square

Chapter 3

Weighted Paths and Automata

In this chapter, we turn our attention to a *path graph* P_n with *weighted* edges and we denote this type of graphs as *weighted paths*. We will assess core stability on this type of graphs and subsequently implement our weighted edges as a *regular expression* that can be read by an *automaton*. Section 3.1 gives a brief introduction to Automata Theory, and Section 3.2 presents results on core stability of weighted paths together with automaton implementation.

3.1 Background

An automaton can be seen as an abstract computing device or “machine”. In 1930’s, before computers were invented, Alan Turing studied an abstract machine that contained properties and computational capabilities (at least to a certain extent) of computers being used today. Turing introduced an abstract machine known as *Turing machines* where his aim was to illustrate the limit of a computing machine which also applies to modern day machines.

Simpler types of machines, also called “finite automata,” were then studied by various researchers in the 1940’s and 1950’s. These automata turned out to have many

other applications instead of the original purpose of just modeling brain functions. The study of “grammars” began in the late 1950’s by linguist Noam Chomsky where his study then proposed that grammars have close relationships to abstract automata. Now, grammars provide a foundation for certain important software components (Hopcroft, Motwani & Ullman, 2001).

In computer science, finite automata and formal grammars provide some type of concepts that are used in the production of important kinds of software. Turing machines on the other hand provide other concepts that allows users to understand the limitations of the software. We first provide fundamentals of automata theory and then we will look into ideas that permeate automata theory: alphabets, strings, languages and regular expressions.

3.1.1 Automata Theory

An automaton has a system to scan and interpret input, where an input is a string over a set of symbols known as the alphabet. This input is inscribed as an “input file”, where an automaton is able to scan and interpret the input but is not able to change the input. An input file is divided into cells where each cell holds one symbol. The automaton contains a temporary “storage unit” that contains infinite number of cells where it is able to modify the contents of the cells. The automaton contains a finite set of *internal states* where states can be changed under certain rules.

Example 3.1 *We follow the simple example of a finite automaton from (Hopcroft et al., 2001).*

Figure 3.1 shows a simple finite automaton that models an on/off switch such that state q_0 represents “off” state and q_1 represents the “on” state. This machine allows users to press a button and obtain a different effect which depends on the state of the switch at the moment of time, that is, the machine is able to remember whether it is in the on

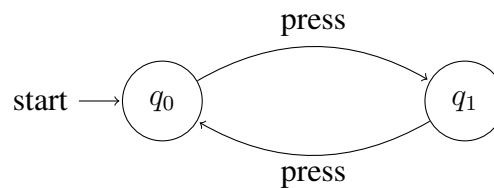


Figure 3.1: A simple finite automaton model

state or off state. For instance, if the switch is in the off state, then it changes to off state when the user presses the button and vice versa.

For any finite automata, each state is represented by circles and the arcs between states are designated a set of “inputs” which represent external impacts on the automata system. In the set of states, there is one state that is the *start* state q_0 where this is placed at the beginning of the system. Often there are one or more states that are *final* or *accepting* states $\{q_1, \dots, q_n\}$ where $\forall n \in \mathbb{N}$.

There are two types of automata; *deterministic finite automata* (DFA) and *non-deterministic finite automata* (NFA). Before providing formal definitions for both DFA and NFA, we first look at the central concepts of automata theory. These concepts include a set of symbols called *alphabets*, a sequence of symbols from an alphabet called *strings* and a set of strings from the same alphabet called a *language*.

Definition 3.1.1 An *alphabet* is a finite set Σ consisting of elements called *symbols*.

Example 3.2 Common alphabets include

1. $\Sigma_1 = \{a, b, c, \dots, y, z\}$ — set of all lower-case letters of the English language.
2. $\Sigma_2 = \{0, 1\}$ — the binary numbers.
3. $\Sigma_3 = \{0, 1, 2, \dots, 9\}$ — the set of decimal digits.

We can describe any set as an alphabet as long as it is a finite set.

Definition 3.1.2 A *string* (or *word*) w is a finite sequence of symbols constructed from an alphabet Σ .

A string is a permutation of symbols chosen from an alphabet and a string w is of the form $w = a_1a_2\cdots a_n$ with each $a_i \in \Sigma$. For the alphabet Σ_2 from the example above, we have 01001 and 110 that are strings from the alphabet Σ_2 . There is a string that is considered for any set of alphabets, called the *empty string*. This string is denoted by ε and contains no symbols whatsoever. Subsequently the *reverse string* w^R is written as $w^R = a_n a_{n-1} \cdots a_1$. The *length* of a string w is a number of symbols that are in the string and this is denoted as $|w|$. For instance, 01001 has length 5 and ε has length 0.

Definition 3.1.3 Given an alphabet Σ , the set of all strings of a certain length k is denoted by Σ^k .

Example 3.3 With any given alphabet Σ , note that $\Sigma^0 = \{\varepsilon\}$. Suppose we take an alphabet $\Sigma_1 = \{a, b\}$, thus, $\Sigma_1^1 = \{a, b\}$, $\Sigma_1^2 = \{aa, ab, ba, bb\}$,

$$\Sigma_1^3 = \{aaa, aab, aba, abb, baa, bab, bba, bbb\}$$

and so on.

Some may be confused with both Σ_1 and Σ_1^1 as they contain the same set of elements. The first Σ_1 is an alphabet, where a and b are symbols. However Σ_1^1 is a set of strings of length 1 such that a and b are strings from the alphabet. The set of all strings over an alphabet Σ is denoted Σ^* . In particular, if $\Sigma_1 = \{a, b\}$ thus $\Sigma_1^* = \{\varepsilon, a, b, aa, ab, ba, bb, \dots\}$. Simply put, Σ^* can be rewritten as

$$\Sigma^* = \Sigma^0 \cup \Sigma^1 \cup \Sigma^2 \cup \Sigma^3 \dots \cup \Sigma^k$$

Also note that the set of all strings of an alphabet Σ excluding the empty string is denoted as Σ^+ , that is, $\Sigma^+ = \Sigma^1 \cup \Sigma^2 \cup \Sigma^3 \cup \dots$.

Definition 3.1.4 Let w and y be strings from an alphabet Σ . The **concatenation** of w and y is the string formed by joining the strings end-to-head and we denote this wy .

Example 3.4 Note that for any string w and an empty string ε , we have $\varepsilon w = w\varepsilon = w$. That is, ε is the **concatenation identity**. Now suppose $w = abaa$ and $y = baba$ then we have $wy = abaababa$ and $yw = babaabaa$.

For any string $w = a_1a_2\cdots a_i$ of length i and a string $y = b_1b_2\cdots b_j$ of length j , then by concatenating both strings we obtain a string of length $i + j$: $wy = a_1a_2\cdots a_ib_1b_2\cdots b_j$.

Definition 3.1.5 A **language** L over an alphabet Σ is a set of strings over Σ , that is, $L \subseteq \Sigma^*$.

Any language L over Σ does not necessarily include strings that contains every symbol in Σ . For instance, English is a collection of a set of strings over the alphabet that contains all the letters.

Example 3.5 Some simple examples of languages over an alphabet $\Sigma = \{0, 1\}$ are as follows:

1. $L_1 = \{w \in \Sigma^* : w \text{ has odd lengths of } 0\text{'s and equal lengths of } 1\text{'s}\}$
2. $L_2 = \{w \in \Sigma^* : |w| = 3\}$
3. $L_3 = \{w \in \Sigma^* : \text{the sum of the symbols is a prime}\}$

Note that the only restriction for any language L , the string over any alphabet must be finite. Since languages are merely set of strings, new languages could possibly be generated by applying standard operations on sets or string operations like *concatenation*. For instance, if L_1 and L_2 are languages over Σ then L_1L_2 , $L_1 \cap L_2$, $L_1 \cup L_2$ etc. are also languages over Σ . The *complement* L' of L is just $L' = \Sigma^* - L$.

Following the intuition that new languages can be constructed from existing languages by performing set or string operations, here we will look at how new languages are constructed by using the three operations on languages — kleene star ($*$), union ($+$) and concatenation (\cdot). Any combination of the three operations on languages, is called a *regular language*. The term *regular expressions* is then used to describe regular languages using specific formulas consisting of the three operators mentioned previously.

Definition 3.1.6 *A regular expression over the alphabet Σ is defined as follows:*

1. *If $\{a\} \in \Sigma^*$, a is a regular expression corresponding to the language $\{a\} \in \Sigma^*$.*
2. *ε is a regular expression corresponding to $\{\varepsilon\}$*
3. *For an empty language \emptyset , the corresponding regular expression is \emptyset .*
4. *If r and s are regular expressions over Σ corresponding to languages L_r and L_s respectively, then*
 - (a) *(rs) is also a regular expression corresponding to $L_r L_s$*
 - (b) *$(r + s)$ is also a regular expression corresponding to $L_r \cup L_s$*
 - (c) *s^* is also a regular expression corresponding to L_s^**
5. *Regular expressions can only be built up by finitely many applications of the operations mentioned in rule 1-4.*

Thus, a *regular language* over an alphabet Σ is a language that has some regular expression over Σ corresponding to it (Chakraborty, 2003). Note that there is an operator precedence where $(*)$ binds tighter than (\cdot) which binds tighter than $(+)$, thus $(*) > (\cdot) > (+)$.

Example 3.6 *A few examples of simple regular expressions are as follows:*

1. $L(a \cdot b) = \{ab\}$.
2. $L(b^*) = \{\varepsilon, b, bb, bbb, \dots\}$.
3. $L((a + b \cdot a) * (\varepsilon + b)) =$ all strings of a's and b's where no two b's are next to each other.

Following the central concepts of a finite automaton, we are now able to discuss how a *deterministic finite automaton* (DFA) is able to *recognize* a given language. For instance, given a language L and an input string w , we are able to design a machine M_L that will either *accept* the string if $w \in L$ or *reject* it otherwise.

Definition 3.1.7 A *Deterministic Finite Automaton* (DFA) is a 5-tuple $M = (Q, \Sigma_M, q_0, \delta, F)$ where

Q	=	Finite set of states
Σ_M	=	Finite set of input symbols
$q_0 \in Q$	=	Initial state
$\delta : Q \times \Sigma_M \rightarrow Q$	=	Transition function
$F \subseteq Q$	=	Set of accepting states

The term *deterministic* indicates that for each input, the automaton can transit it from its current state to only one other state. On the other hand, a *non-deterministic finite automaton* is able to transit it to several states at once.

Definition 3.1.8 A *Non-Deterministic Finite Automaton* (NFA) is a 5-tuple

$M = (Q, \Sigma_M, q_0, \delta, F)$ where

Q	= Finite set of states
Σ_M	= Finite set of input symbols
$q_0 \in Q$	= Initial state
$\delta : Q \times \Sigma_M \rightarrow 2^Q$	= Transition function
$F \subseteq Q$	= Set of accepting states

The notation 2^Q denotes the powerset of Q , that is the set of all possible subsets of Q .

Both definitions above show that NFA and DFA are exactly the same except for their respective transition functions δ . In the case of DFA, we are able to further expand the definition of δ to δ^* such that it accepts a string of symbols. In particular, if we have a string $w \in \Sigma^*$, a symbol $a \in \Sigma$ and a state $q_i \in Q$, $\delta(q_i, wa) = \delta(\delta^*(q_i, w), a)$.

NFA's can be generalized further by introducing a transition ε , that is, a transition between states only require the empty string ε as input. We next define an NFA with transition *varepsilon*, written as ε -NFA, the same as a regular NFA but has a different transition function $\delta^* : Q \times (\Sigma^* \cup \{\varepsilon\}) \rightarrow 2^Q$.

Definition 3.1.9 An ε -NFA is defined as a 5-tuple $M = (Q, \Sigma_M, q_0, \delta, F)$. The ε -NFA M is an NFA except for the transition function that is given by $\delta : Q \times (\Sigma \cup \{\varepsilon\}) \rightarrow 2^Q$. For a subset T of Q , the ε -closure of T denoted $\varepsilon(T)$ is a subset of Q such that

1. Every element of T is an element of $\varepsilon(T)$.
2. For $p \in \varepsilon(T)$, every element of $\delta(p, \varepsilon)$ is also an element of $\varepsilon(T)$.
3. Only elements of Q that are results of rules 1 and 2 above that can be elements in $\varepsilon(T)$.

Thus the ε -closure of a set T is the set of states that the elements are able to reach by using only the transition ε . If two mechanisms are able to accept the same group of languages then these two mechanisms are said to be *equivalent* (Gopalakrishnan, 2006). In particular, one is able to turn a regular expression into an equivalent DFA with two steps: (i) convert a regular expression to an NFA via Thompson's algorithm or McNaughton and Yamada's method (Chang & Paige, 1997), (ii) latter convert an NFA to a DFA using subset construction of Rabin and Scott (Singh, 2014). Note that a DFA is considered to be a unique class of an NFA and an NFA is also considered to be a unique class of ε -NFA. Therefore, any language that is recognized by a DFA can be recognized by an NFA which also can be recognized by a ε -NFA and the converses also hold (Ginzburg, 1968).

3.2 Automaton Formations

In this chapter, we first define a category of graphs having weights or numbers associated with each edge namely *weighted graphs*. We then assess the core stability of a *path graph* with weighted edges and implement the set of "weights" as a *regular expression* which can be read by an automaton.

3.2.1 Weighted Graphs

The edges (or links) in many networks are not solely in binaries — either there exists an edge or there is no edge, but have some number k known as weight that is associated with each edge which displays the strength of ties between one another.

Definition 3.2.1 A *weighted graph* is defined as a triple $G = (V, E, \varpi)$ where ϖ denotes the *weight function* $\varpi : E \rightarrow \mathbb{R}$.

Weighted graphs appear to have been used to model numerous problems where

objects or places (cities, people in groups, computers, etc.) are connected with one another with links of different weights (Newman, 2004). These weights represent various informations, being distance between two places, strength of ties between two people, cost of reaching an objection, the capacity of water flow in a pipe, etc. Note that we now need to consider the edge weights when examining the payoff of a player. For instance, suppose $v \in G$ is incident with three edges of weights m , l and k , thus $d(v) = \frac{m+l+k}{n}$ where n is the total number of vertices in G and $d(v) \in \mathbb{R}$. We describe the coalitional game on weighted graphs as follows:

Definition 3.2.2 *The influence game on $G = (V, E, \varpi)$ is a coalition game*

$\Gamma_{\varpi}(G) = (V, \varphi)$ such that $\varphi_{\varpi} : V \times 2^V \rightarrow \mathbb{R}$ is described by $\varphi_{\varpi}(u, S) = \chi_S(u)$.

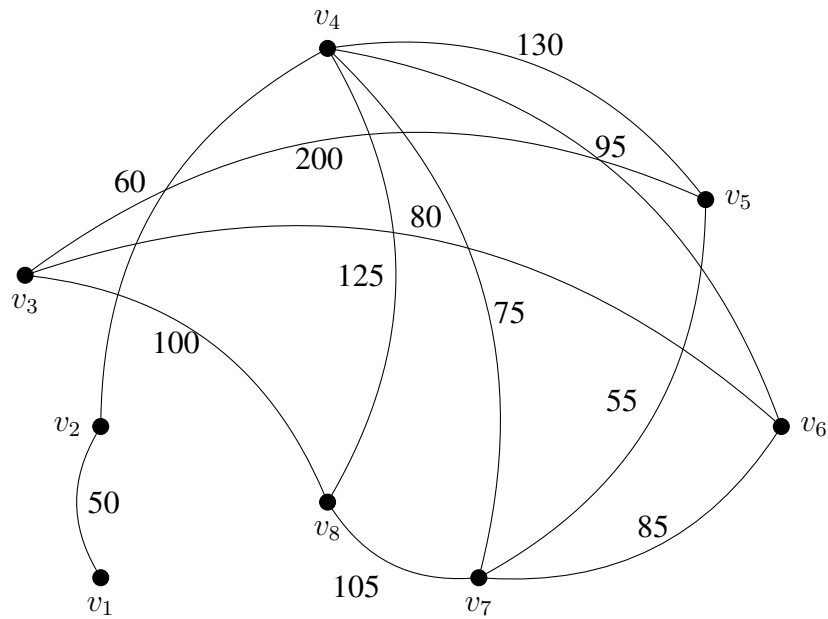


Figure 3.2: A drawing of a Weighted Graph G .

We draw our attention to a category of weighted graph called *weighted paths*. Recall that a *path graph* P_n is a sequence of vertices $v_i \in V$ such that the edges are $\{v_i, v_{i+1}\}$ and $i = 1, 2, \dots, n-1$. We now denote a *weighted path* as $\rho_n = (V, E, \varpi)$ where the weight $\varpi(e)$ represents a form of social tie between two players $v_i, v_{i+1} \in V$ that are

incident to e . We maintain the concept of degree centrality to study the influence level of a player v_i in a weighted path network. Let $\varpi_\ell(v_i)$ denote the weight of the edge incident to v_{i-1} and v_i and let $\varpi_r(v_i)$ denote the weight of the edge incident to v_i and v_{i+1} . Thus the influence level of a player v_i is now

$$\chi_{\rho_n}(u) = \frac{\varpi_\ell(v_i) + \varpi_r(v_i)}{n}$$

for weighted path graph ρ_n on n vertices. Following the proof on core stability of an unweighted path network P_n , we have established that a party structure of a path network is stable if $2 \leq n \leq 4$. This differs to the case of weighted graphs. We assess core stability of weighted path graphs as follows:

Theorem 3.2.3 *The core of a party structure \mathcal{C} on a weighted path network ρ_n with maximum edge weight $\varpi = m$ is non-empty only if $2 \leq n \leq 4$.*

Proof: Assume that the core is non-empty for $n > 4$. Take any party structure \mathcal{C} and let the max weight for an edge be m . The maximum possible influence level of a player is $\frac{m}{2}$ and the least possible influence level for any player is $\frac{1}{n}$. Assume also that any two players will only form a new coalition if both strictly benefit (gain a higher influence level). Since the party structure is core stable, then the influence level for any player $v_i \in \mathcal{C}$ has to be either $\chi_{\mathcal{C}}(v_i) \geq \frac{\varpi_\ell(v_i)}{2}$, or $\chi_{\mathcal{C}}(v_i) \geq \frac{\varpi_r(v_i)}{2}$ so that no player will form a party with either adjacent players. Suppose we pick a player v_4 that is incident to an

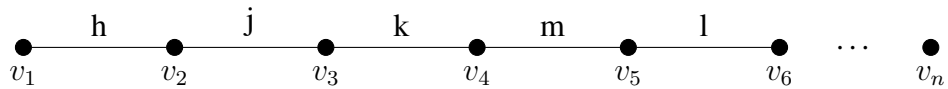


Figure 3.3: Party structure C on $n > 6$ vertices and maximum edge weight m .

edge with maximum weight m . We look into two cases for player v_4 , where player v_4 chooses to form a new coalition with either $\{v_4\} \cup \{v_3\} = S_1$ or $\{v_4\} \cup \{v_5\} = S_2$ forms. Note that since $n > 4$, we consider the lower bound on n such that $\lfloor n \rfloor = 5$.

Case 1 for S_1 :

$$\begin{aligned} \chi_{\mathcal{C}}(v_4) = \frac{k+m}{n} \geq \frac{k}{2} = \chi_{S_1}(v_4) \quad \text{and} \quad \chi_{\mathcal{C}}(v_3) = \frac{k+j}{n} \geq \frac{k}{2} = \chi_{S_1}(v_3) \\ 2k+2m \geq kn \qquad \qquad \qquad 2k+2j \geq kn \\ 2m \geq k(n-2) \qquad \qquad \qquad 2j \geq k(n-2) \\ m \geq \frac{k(n-2)}{2} \qquad \qquad \qquad j \geq \frac{k(n-2)}{2} \end{aligned}$$

Applying lower bound on n gives

$$m \geq \lfloor \frac{3k}{2} \rfloor \qquad \qquad \qquad j \geq \lfloor \frac{3k}{2} \rfloor$$

or

Case 2 for S_2 :

$$\begin{aligned} \chi_{\mathcal{C}}(v_4) = \frac{k+m}{n} \geq \frac{m}{2} = \chi_{S_2}(v_4) \quad \text{and} \quad \chi_{\mathcal{C}}(v_5) = \frac{l+m}{n} \geq \frac{m}{2} = \chi_{S_2}(v_5) \\ 2k+2m \geq mn \qquad \qquad \qquad 2l+2m \geq mn \\ 2k \geq m(n-2) \qquad \qquad \qquad 2l \geq m(n-2) \\ k \geq \frac{m(n-2)}{2} \qquad \qquad \qquad l \geq \frac{m(n-2)}{2} \end{aligned}$$

Applying lower bound on n gives

$$k \geq \lfloor \frac{3m}{2} \rfloor \qquad \qquad \qquad l \geq \lfloor \frac{3m}{2} \rfloor$$

As can be seen from case 2 for the coalition S_2 that $k, l \geq \lfloor \frac{3m}{2} \rfloor$ and this is not possible based on the fact that m is the maximum weight of the path ρ_n . Since S_2 blocks \mathcal{C} , thus this contradicts our assumption that the party structure \mathcal{C} with $n > 3$ is core stable.

Hence \mathcal{C} is not core stable if $n > 4$. □

Following Theorem 3.2.3, we specifically look at the case where $n = 4$ and assess which party structure \mathcal{C} is core stable.

Theorem 3.2.4 *Let \mathcal{C} be a party structure on a weighted path network ρ_n , with $n \geq 2$. Assume any one of the following conditions holds:*

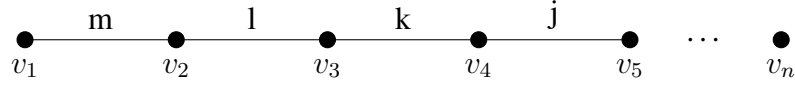
1. *The weights on the edges are strictly of ascending or descending order.*
2. *The weights for $n - 2$ consecutive edges are equal and strictly greater than the other weights.*
3. *All consecutive edges are of equal weights.*

Then \mathcal{C} is core stable only if $n \leq 4$.

Proof: Assume core stability for $n > 4$ on a party structure \mathcal{C} and suppose the maximum weight is m . We need to consider three cases such that: (i) \mathcal{C} has $n > 4$ vertices where the weights are strictly of ascending or descending order; (ii) \mathcal{C} has $n > 4$ vertices with $n - 2$ consecutive edges are of equal weights; (iii) \mathcal{C} has $n > 4$ vertices with all consecutive weights being equal. Note that the maximum influence level of a player is $\frac{m}{2}$. Take any non-minor player $v_i \in \mathcal{C}$. We shall assess each case as follows:

- (i.) **Case 1:** \mathcal{C} has $n > 4$ with weights of strictly in ascending or descending order.

Since the sequence is in order and a path is symmetrical, we need only prove one way. Here we will only look at a sequence of weights strictly in descending order. Suppose we pick a player v_2 and let the sequence of edge weights be $m > l > k > j > \dots > 1$. Suppose we take $\{v_2\} \cup \{v_1\} = S_1$. Then

Figure 3.4: Party structure \mathcal{C} with edge weights strictly in descending order.

$$\begin{aligned} \chi_{\mathcal{C}}(v_2) &= \frac{l+m}{n} \geq \frac{m}{2} = \chi_{S_1}(v_2) & \text{and} & \quad \chi_{\mathcal{C}}(v_1) = \frac{m}{n} \geq \frac{m}{2} = \chi_{S_1}(v_1) \\ 2l+2m &\geq mn & & \quad 2m \geq mn \\ \frac{2l+2m}{m} &\geq n & & \quad 2 \geq n \end{aligned}$$

Solving both equations to obtain

$$\begin{aligned} 2l+2m &\geq 2m & \text{when} & \quad 2 \geq n \\ 2l &\geq 0 \\ l &\geq 0 \end{aligned}$$

Suppose now we take $\{v_2\} \cup \{v_3\} = S_2$. Then

$$\begin{aligned} \chi_{\mathcal{C}}(v_2) &= \frac{l+m}{n} \geq \frac{l}{2} = \chi_{S_2}(v_2) & \text{and} & \quad \chi_{\mathcal{C}}(v_3) = \frac{l+k}{n} \geq \frac{l}{2} = \chi_{S_2}(v_3) \\ 2l+2m &\geq ln & & \quad 2l+2k \geq ln \\ \frac{2l+2m}{n} &\geq l & & \quad \frac{2l+2k}{n} \geq l \end{aligned}$$

Taking the lower bound on $n > 4$ where $\lfloor n \rfloor = 5$ and solving both equations yield

$$\begin{aligned} 2l+2m &\geq 5l & \text{and} & \quad 2l+2k \geq 5l \\ 2m &\geq 3l & & \quad 2k \geq 3l \\ m &\geq \frac{3l}{2} & & \quad k \geq \frac{3l}{2} \end{aligned}$$

Since the sequence is in descending order, the result where $k \geq \frac{3l}{2}$ can not hold.

Thus, S_1 and S_2 blocks \mathcal{C} .

- (ii.) **Case 2:** \mathcal{C} has $n > 4$ with $n - 2$ consecutive edges that are of equal weights strictly greater than the other weights. Suppose v_2 is adjacent to a minor player v_1 of weight $l < m$ and a non-minor player v_3 of weight m that is also adjacent to another player v_4 of edge weight m . Now suppose $\{v_2\} \cup \{v_3\} = S_2$. Then

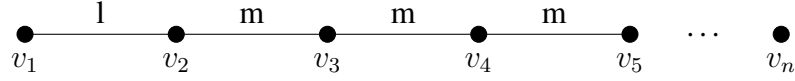


Figure 3.5: Party structure \mathcal{C} on $n > 4$ players $n - 2$ consecutive edge weights m .

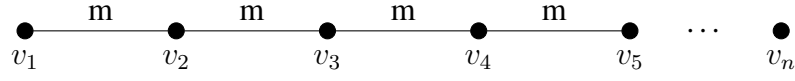
$$\begin{aligned} \chi_{\mathcal{C}}(v_2) &= \frac{l+m}{n} \geq \frac{m}{2} = \chi_{S_2}(v_2) & \text{and} & \quad \chi_{\mathcal{C}}(v_3) = \frac{2m}{n} \geq \frac{m}{2} = \chi_{S_2}(v_3) \\ 2l + 2m &\geq mn & & \quad 4m \geq mn \\ \frac{2l + 2m}{m} &\geq n & & \quad 4 \geq n \end{aligned}$$

Solving both equations gives

$$\begin{aligned} 2l + 2m &\geq 4m & \text{when} & \quad 4 \geq n \\ 2l &\geq 2m \\ l &\geq m \end{aligned}$$

This case is not valid as m is the maximum weight and l can not be greater than m and simultaneously it contradicts with our assumption that $n > 4$.

- (iii.) **Case 3:** \mathcal{C} has $n > 4$ with all edges are of equal weight m . Suppose v_2 is adjacent to a minor player v_1 and a non-minor player v_3 .

Figure 3.6: Party structure \mathcal{C} on $n > 4$ players with equal edge weights m .

(a) Suppose $\{v_2\} \cup \{v_3\} = S_2$. Then

$$\begin{aligned}\chi_{\mathcal{C}}(v_2) = \chi_{\mathcal{C}}(v_3) &= \frac{2m}{n} \geq \frac{m}{2} = \chi_{S_2}(v_3) = \chi_{S_2}(v_2) \\ 4m &\geq mn \\ 4 &\geq n\end{aligned}$$

(b) Now suppose $\{v_2\} \cup \{v_1\} = S_1$. Then

$$\begin{aligned}\chi_{\mathcal{C}}(v_1) = \frac{m}{n} \geq \frac{m}{2} = \chi_{S_1}(v_1) \quad \text{and} \quad \chi_{\mathcal{C}}(v_2) = \frac{2m}{n} \geq \frac{m}{2} = \chi_{S_1}(v_2) \\ 2m \geq mn \qquad \qquad \qquad 4m \geq mn \\ 2 \geq n \qquad \qquad \qquad 4 \geq n\end{aligned}$$

Thus, both inequalities can not hold simultaneously, and this again contradicts with our assumption for $n > 4$

All three cases above clearly show that either S_1 or S_2 blocks \mathcal{C} . Therefore, \mathcal{C} with $n = 4$ is only core stable if there is at least one of the three rules above hold. \square

Now we look at the core stable party structures \mathcal{C} for the case where $n = 3$.

Theorem 3.2.5 *Let ρ_n be a weighted path network with maximum edge weight $\varpi = m$. A party structure \mathcal{C} with $n = 3$ is core stable only if either of the following holds:*

1. *All consecutive edges are of equal weights.*
2. *Any edge with weight $\varpi > 2$ must not be adjacent to any edge with weight 1.*

Proof: Let \mathcal{C} be a party structure and let $\varpi = m$ be the maximum edge weight of ρ_n . We shall assess both cases separately.

- (i.) **Case 1.** The first case is trivial following the proof from Theorem 3.2.3 for equal consecutive weights for $n = 4$. The proof holds for any $n \leq 4$.
- (ii.) **Case 2.** Let \mathcal{C} be a party structure. Suppose we pick a non-minor player v_2 that is adjacent to a minor player v_1 of edge weight 1 and another minor player v_3 of edge weight k such that $2 \leq k \leq m$. Assume this structure is core stable. Thus,

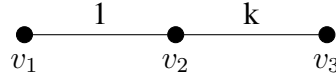


Figure 3.7: A party structure \mathcal{C} with $n = 3$.

$$\begin{aligned} \chi_{\mathcal{C}}(v_2) &= \frac{1+k}{3} \geq \frac{k}{2} = \chi_{\{v_2\} \cup \{v_3\}}(v_2) & \text{and} & & \chi_{\mathcal{C}}(v_3) &= \frac{k}{3} \geq \frac{k}{2} = \chi_{\{v_2\} \cup \{v_3\}}(v_3) \\ 2 + 2k &\geq 3k & & & 2k &\geq 3k \\ 2 &\geq k & & & 2 &\geq 3 \end{aligned}$$

The last two inequalities can not hold since $2 \not\geq 3$ is impossible and our assumption was that $k > 2$. It shows that if $\{v_2\} \cup \{v_3\}$, then the union will block \mathcal{C} .

Thus \mathcal{C} is only core stable if either one of the two cases hold. □

3.2.2 Regular Expressions and Automaton Implementation

From Theorem 3.2.3, we have established that a party structure \mathcal{C} of a weighted path network ρ_n of at most 4 players contains a non-empty core. The core stable weight structures for $n = 4$ and $n = 3$ have been identified through Theorem 3.2.4 and Theorem 3.2.5. As for the case of $n = 2$, this is trivial since no players want to be on their own — there contains no *singletons*.

Following the results above, we now examine how we are able to implement each core stable weight structure as a *regular expression* that is recognizable by a *finite automaton*. To do this we will first need to identify the alphabet, set of strings and define the languages constructed over the alphabet.

Alphabet, Strings and Language. Suppose we denote the weights ϖ of any weighted path ρ_n as an alphabet Σ such that

$$\Sigma = \{1, 2, 3, \dots, m, \hat{1}, \hat{2}, \hat{3}, \dots, \hat{m}\} \text{ where } m \in \mathbb{Z}^+.$$

Note that each $\hat{a} \in \Sigma$ represents a “break” of an edge: i.e. a player leaves the coalition to join another, causing the “removal” of the link between the player and another member of the coalition. We denote \hat{i} as a *hat symbol* and observe that

$$1 = \hat{1} < 2 = \hat{2} < \dots < m = \hat{m}$$

where m denotes the maximum weight. A *string* over Σ represents a sequence of weights along the path ρ_n . The strings that we will be assessing here are of maximum length 3 over Σ^+ where $\Sigma^+ = \Sigma^* - \{\varepsilon\}$, since the path of maximum 4 vertices and 3 edges has a non-empty core. We represent the possible strings of lengths 1,2 and 3 with a , ab and abc respectively, where $a, b, c \in \Sigma$. Note that $a = \hat{a}$, $b = \hat{b}$ and $c = \hat{c}$ as defined above. Now we are able to present the languages over Σ as follows:

1. $L_1 = \{\mathbf{p} | \mathbf{p} \in \Sigma^+ : |\mathbf{p}| = 1, \mathbf{p} \text{ consists of non-hat symbols } a\}$.
2. $L_2 = \{\mathbf{q} | \mathbf{q} \in \Sigma^+ : |\mathbf{q}| = 2, \mathbf{q} \text{ has consecutive non-hat symbols } ab \text{ and } a, b > 1\}$.
3. $L_3 = \{\mathbf{r} | \mathbf{r} \in \Sigma^+ : |\mathbf{r}| = 2, \mathbf{r} \text{ is the string that starts or ends with } \hat{1} \text{ and } a > 2\}$.
4. $L_4 = \{\mathbf{s} | \mathbf{s} \in \Sigma^+ : |\mathbf{s}| = 3, \mathbf{s} \text{ is the string } abc \text{ or } cba \text{ and } c > b > a\}$.
5. $L_5 = \{\mathbf{w} | \mathbf{w} \in \Sigma^+ : |\mathbf{w}| = 3, \mathbf{w} \text{ is the string } abb \text{ or } bba \text{ and } b > a\}$.

6. $L_6 = \{\mathbf{x} | \mathbf{x} \in \Sigma^+ : |\mathbf{x}| \leq 3, \mathbf{x} \text{ is the string consisting of the same symbols}\}.$

7. $L_7 = \{\mathbf{y} | \mathbf{y} \in \Sigma^+ : |\mathbf{y}| = 3, \mathbf{y} \text{ is the string } \hat{a}\hat{b}\hat{c} \text{ and } b < a, c\}.$

8. $L_8 = \{\mathbf{z} | \mathbf{z} \in \Sigma^+ : |\mathbf{z}| = 3, \mathbf{z} \text{ is the string } \hat{a}bc \text{ or } ab\hat{c} \text{ or } \hat{a}b\hat{c} \text{ and } b > a, c\}.$

Recall that operations on languages over the alphabet Σ create a new language over the alphabet Σ .

Regular Expressions. We now describe the regular expression representing each language as defined previously. Subsequently, we will apply simple operations (i.e. concatenation, union, and kleene star) on the regular expressions defined to obtain another regular expression.

Language	Corresponding Regular Expression
$\{a\}$	a
$\{ab\}$	ab
$\{a\hat{1}, \hat{1}a\}$	$a\hat{1} + \hat{1}a$
$\{abc, cba\}$	$abc + cba$
$\{abb, bba\}$	$abb + bba$
$\{a, aa, aaa\}$	$a + aa + aaa$
$\{\hat{a}\hat{b}\hat{c}\}$	$\hat{a}\hat{b}\hat{c}$
$\{\hat{a}bc, ab\hat{c}, \hat{a}b\hat{c}\}$	$\hat{a}bc + ab\hat{c} + \hat{a}b\hat{c}$

We are only interested in sequences of symbols that are not permitted, that is, strings that are not recognizable by an automaton, namely the *forbidden strings*. We associate these strings with certain structures that “ruin” coalition or social groups in such a way that these “disruptions” can not happen. We shall examine forbidden strings as follows:

Theorem 3.2.6 *A forbidden string is a string that contains any of the following sequence:*

1. \hat{a} or $\hat{a}\hat{b}$ or $\hat{a}\hat{b}\hat{c}$,
2. $a\hat{b}$ where $a < 2b$,
3. $\hat{a}b$ where $a > \frac{b}{2}$,
4. $\hat{a}bc$ where $b > a, c$,
5. $\hat{a}bc, ab\hat{c}$ or $\hat{a}b\hat{c}$ where $b < a, c$,
6. $\hat{a}\hat{b}\hat{c}$ or $a\hat{b}\hat{c}$.

Proof: We shall prove for each case as follows.

- (i.) The case where the string is \hat{a} or $\hat{a}\hat{b}$ or $\hat{a}\hat{b}\hat{c}$ is trivial since no player wants to be alone, i.e., no singletons.

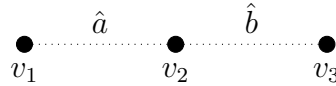


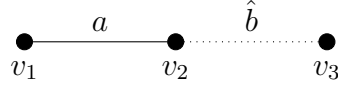
Figure 3.8: The string sequence that “breaks” the structure into singletons.

Any two singletons are better off by forming a coalition since

$$\chi_{\{v_1\}}(v_1) = 0 < \frac{a}{2} = \chi_{\{v_1\} \cup \{v_2\}}(v_1).$$

Thus it shows that a string containing a consecutive sequence of hat symbols $\hat{a} \in \Sigma$ is a forbidden string.

- (ii.) Case for $a\hat{b}$ where $a < 2b$. Let \mathcal{C} be a party structure. Suppose we assume that such a string is acceptable and let v_2 denote the non-minor player in the structure that is connected to two minor players v_1 and v_3 .

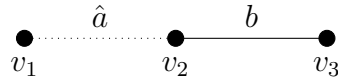
Figure 3.9: The string sequence $a\hat{b}$.

Therefore it implies that the disruption of the formation is agreeable as players are strictly better off such that

$$\begin{aligned} \chi_{\mathcal{C}}(v_2) &= \frac{a+b}{3} < \frac{a}{2} = \chi_{\{v_1\} \cup \{v_2\}}(v_2) \quad \text{and} \quad \chi_{\mathcal{C}}(v_1) = \frac{a}{3} < \frac{a}{2} = \chi_{\{v_1\} \cup \{v_2\}}(v_1) \\ 2a + 2b &< 3a & 2a < 3a \\ 2b &< a & 2 < 3 \end{aligned}$$

The last inequality contradicts with our assumption that $a < 2b$. Thus $a\hat{b}$ with $a < 2b$ is a forbidden string.

(iii.) Case for $\hat{a}b$ where $a > \frac{b}{2}$. Let \mathcal{C} be a party structure. Again we assume that the string is acceptable and pick player v_2 that is adjacent to v_1 and v_3 .

Figure 3.10: The string sequence $\hat{a}b$.

Thus we have

$$\begin{aligned} \chi_{\mathcal{C}}(v_2) &= \frac{a+b}{3} < \frac{b}{2} = \chi_{\{v_3\} \cup \{v_2\}}(v_2) \quad \text{and} \quad \chi_{\mathcal{C}}(v_3) = \frac{b}{3} < \frac{b}{2} = \chi_{\{v_3\} \cup \{v_2\}}(v_3) \\ 2a + 2b &< 3b & 2b < 3b \\ 2a &< b & 2 < 3 \end{aligned}$$

Again, the last inequality contradicts with our assumption that $2a > b$. Hence, $\hat{a}b$ with $a > \frac{b}{2}$ is a forbidden string.

(iv.) Case for $a\hat{b}c$ where $b > a, c$. Let \mathcal{C} be a party structure. Suppose we assume that the string is accepted on $b > a, c$ and pick non minor players v_2 and v_3 as shown below.

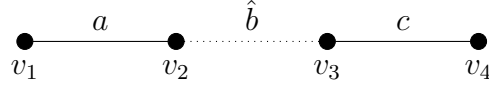


Figure 3.11: The string sequence $a\hat{b}c$.

Take $\{v_1\} \cup \{v_2\} = S_1$ and $\{v_3\} \cup \{v_4\} = S_2$. Thus

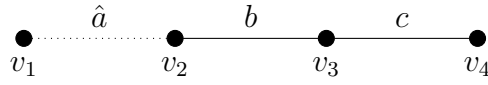
$$\begin{aligned} \chi_{\mathcal{C}}(v_1) &= \frac{a}{4} < \frac{a}{2} = \chi_{S_1}(v_1) & \text{and} & & \chi_{\mathcal{C}}(v_2) &= \frac{a+b}{4} < \frac{a}{2} = \chi_{S_1}(v_2) \\ 2a < 4a & & & & 2a + 2b < 4a \\ 2 < 4 & & & & 2b < 2a \end{aligned}$$

also

$$\begin{aligned} \chi_{\mathcal{C}}(v_3) &= \frac{b+c}{4} < \frac{c}{2} = \chi_{S_2}(v_3) & \text{and} & & \chi_{\mathcal{C}}(v_4) &= \frac{c}{4} < \frac{c}{2} = \chi_{S_2}(v_4) \\ 2b + 2c < 4c & & & & 2c < 4c \\ 2b < 2c & & & & 2 < 4 \end{aligned}$$

Either of inequalities $b < a$ or $b < c$ does not hold as this contradicts with our assumption that $b > a, c$. Therefore, the string $a\hat{b}c$ with $b > a, c$ is a forbidden string.

(v.) Case for $\hat{a}bc$, $ab\hat{c}$ or $\hat{a}b\hat{c}$ where $b < a, c$. Let \mathcal{C} be a party structure. Assume that the string is accepted on $b < a, c$ and we let v_2 be a non minor player. We will prove each cases separately.

Figure 3.12: The string sequence $\hat{a}bc$.

Case $\hat{a}bc$. Take $\{v_2\} \cup \{v_3\} \cup \{v_4\} = S_1$.

$$\chi_{\mathcal{E}}(v_2) = \frac{a+b}{4} < \frac{b}{3} = \chi_{S_1}(v_2) \quad \text{and} \quad \chi_{\mathcal{E}}(v_3) = \frac{b+c}{4} < \frac{b+c}{3} = \chi_{S_1}(v_3)$$

$$3a + 3b < 4b$$

$$3b + 3c < 4b + 4c$$

$$3a < b$$

$$-c < b$$

also

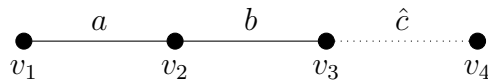
$$\chi_{\mathcal{E}}(v_4) = \frac{c}{4} < \frac{c}{3} = \chi_{S_1}(v_4)$$

$$3c < 4c$$

$$3 < 4$$

The inequalities $3a < b$ and $-c < b$ contradict with our assumption that $b < a, c$.

Thus, the string is a forbidden.

Figure 3.13: The string sequence $ab\hat{c}$.

Case $ab\hat{c}$. Take $\{v_1\} \cup \{v_2\} \cup \{v_3\} = S_2$.

$$\chi_{\mathcal{E}}(v_1) = \frac{a}{4} < \frac{a}{3} = \chi_{S_2}(v_1) \quad \text{and} \quad \chi_{\mathcal{E}}(v_2) = \frac{a+b}{4} < \frac{a+b}{3} = \chi_{S_2}(v_2)$$

$$3a < 4a$$

$$3a + 3b < 4a + 4b$$

$$3 < 4$$

$$-a < b$$

also

$$\chi_{\mathcal{E}}(v_3) = \frac{b+c}{4} < \frac{b}{3} = \chi_{S_2}(v_4)$$

$$3b + 3c < 4b$$

$$3c < b$$

Again, the inequality $-a < b$ and $3c < b$ contradicts our assumption that $b < a, c$ and thus this string is forbidden.

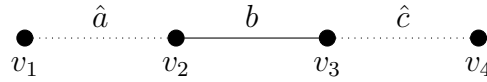


Figure 3.14: The string sequence $\hat{a}\hat{b}\hat{c}$.

Case $\hat{a}\hat{b}\hat{c}$. Take $\{v_2\} \cup \{v_3\} = S_3$.

$$\chi_{\mathcal{E}}(v_2) = \frac{a+b}{4} < \frac{b}{2} = \chi_{S_3}(v_2) \quad \text{and} \quad \chi_{\mathcal{E}}(v_3) = \frac{b+c}{4} < \frac{b}{2} = \chi_{S_3}(v_3)$$

$$2a + 2b < 4b$$

$$2b + 2c < 4b$$

$$2a < 2b$$

$$2c < 2b$$

$$a < b$$

$$c < b$$

Clearly, the two inequalities contradict our assumption that $b < a, c$. Therefore $\hat{a}\hat{b}\hat{c}$ with $b < a, c$ is also a forbidden string.

(vi.) Case for $\hat{a}\hat{b}\hat{c}$ or $\hat{a}\hat{b}\hat{c}$. Similar to the first case, this is also trivial since the structure partitions players into singletons. Thus the singletons v_1 and v_2 benefit if they

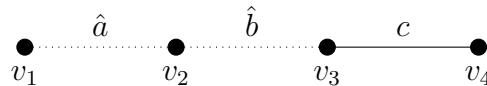


Figure 3.15: String Sequence

form a party such that

$$\chi_{\{v_1\}}(v_1) = 0 < \frac{a}{2} = \chi_{\{v_1\} \cup \{v_2\}}(v_1).$$

This also proves for the case $\hat{a}\hat{b}\hat{c}$. Thus, any structure that creates more than one singleton is not permitted and we conclude that $\hat{a}\hat{b}\hat{c}$ and $\hat{a}\hat{b}\hat{c}$ are forbidden strings.

This completes the proof for Theorem 3.2.6. \square

The proof has shown that a string with any of the above structure is not recognizable by any automaton.

Example 3.7 We demonstrate how we are able to convert the regular expression $abc + cba$ as presented above to an NFA using the Thompson-McNaughton-Yamada NFA.

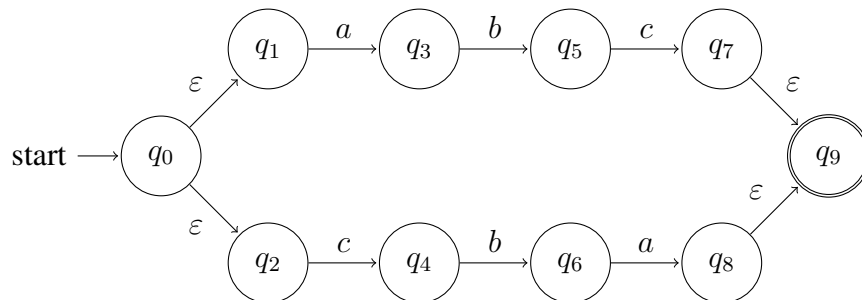


Figure 3.16: Visualization for the regular expression $abc + cba$ as an NFA.

Recall that every automaton has a single initial state represented as q_0 in Figure 3.16 and an NFA will have only one accepting state represented as q_9 . Since we are creating an NFA, we are able to begin with the empty expression ε that can be matched by an automaton which has an initial state q_0 , by using a transition ε to the accepting state rather than a symbol from the alphabet Σ .

Here we have a choice between the expressions abc and cba . We can construct the NFA for $abc + cba$ by creating an initial state with transition ε to the respective initial states of abc and cba automata, and followed by an accepting state also with transition ε from their respective accepting states. Figure 3.16 shows how the expression $abc + cba$ is read by an NFA. For the expression abc , we start with an empty expression ε and transit to state q_1 . If the automaton reads the symbol a , it will then transit to the next state q_3 , followed by the symbols b and c until it reaches the accepting state q_9 .

The above construction can be applied to other regular expressions as described previously. We are able construct a DFA from the NFA in Example 3.7 by using the Rabin and Scott's subset construction. Recall that any language that is recognized by a DFA is also recognized by an NFA. For further details on converting regular expressions to NFAs or DFAs, readers are referred to (Xavier, 2005),(Sipser, 2006),(Chang & Paige, 1997).

Chapter 4

Conclusions and Future Work

The study of social networks is considerably complex. Thus, it is not surprised that there are immense studies on this topic in the literature. Recent empirical work suggests that there is a link between social network structures, human behavior and economic outcomes. The principal motivation of this thesis is to describe the connections between social network structures with coalitional games and its effects on human behavior. In particular, we focus on the simple facets of networks, the number of interpersonal ties (degree of connections) and presume that each player in the network has complete information in regards to the number of ties of other players (special classes of graphs). This enables the formulation of a general framework for the type of games played on social networks. With this framework, we are able to capture the characteristics displayed by real world networks such as the cohesiveness of networks formed correlating to the degree of neighbors.

In this thesis, we have described a type of coalitional games that reflect a player's influence level in terms of a payoff function using degree centrality. Through game theoretic approaches, we are also able to discuss the notion of cohesion of groups based on payoff functions associated with each network structure as discussed in Section 2.1.2. Following the results on special classes of graphs, we can see that all types of (fixed)

social network structures have a core stable formation. Through this literature, players are able to improve their strategies to increase their influence in a group setting. We also show how the strength of ties (represented as the weight of an edge) between two players impacts on the core stability of the weighted path networks. Furthermore, we have implemented the set of weights as a regular expression which is recognizable by a finite automaton.

Due to the nature and time limit of this study, only a selection of graph classes has been studied and the only weighted graphs that have been assessed are the weighted path graphs. There have been a few future research directions that have emerged from this study. It would be of great interests to examine the other issues that have been identified through this study — can core stability of tree graphs and random graphs be examined using the same approach? Is an incomplete bipartite graph $K_{n,n}$ core stable? What happens when each vertex is weighted? What about the other solution concepts of coalitional games (e.g. Shapley value, kernel, nucleolus or stable set)?

Since payoff functions can be associated with various centrality measures, future work could include looking into applying other centrality measures (e.g. KATZ, closeness, betweenness or eigenvector centralities), considering different forms of graphs (e.g. directed, weighted vertices) and even study the core stability on much more complex graphs such as trees and random graphs. Furthermore, one could also explore generating automata (DFA or NFA) for different networks and study the computational complexity of each scenario.

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