Wijsman Convergence: Topological Properties and Embedding

Jiling Cao

School of Computing and Mathematical Sciences
Auckland University of Technology
Private Bag 92006
Auckland 1142
New Zealand

Email: jiling.cao@aut.ac.nz

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Outline

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1. What is Wijsman convergence?
   - The motivation
   - Wijsman's work
2. Topological properties: a brief review
   - Wijsman topologies
   - Two classical results
   - Three problems
3. Recent progress and open questions
   - Amsterdam and other properties
   - The Baire property
   - Normality and embedding

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The motivation

R. Wijsman and D. Burkholder, *Optimum properties and admissibility of sequential tests*, Ann. Math. Statist. **34** (1963), 1-17.

Problem

A_n, A closed convex in $\mathbb{R}^2$; x is in none of the A_n nor in A. Consider the supporting lines through x of A_n and of A. If $A_n \xrightarrow{?} A$, is the same true for the supporting lines?

The solution was published in:
R. Wijsman, *Convergence of sequences of convex sets, cones and functions II*, Trans. Amer. Math. Soc. **123** (1966), 32-45.

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Definition

Call $A_n \overset{Wijsman}{\longrightarrow} A$, if $d(A_n, x) \to d(A, x)$ for every $x \in \mathbb{R}^2$, where

$$d(A, x) = \inf\{d(a, x) : a \in A\}.$$

This can be extended to the setting of a metric space.

Legendre-Fenchel transformation

X is real normed linear, $f : X \to \mathbb{R} \cup \{+\infty\}$ is nontrivial. Then $f^* : X^* \to \mathbb{R} \cup \{+\infty\}$ defined by

$$\forall p \in X^*, \quad f^*(p) = \sup_{x \in X}[\langle p, x\rangle - f(x)]$$

is called the convex conjugate of f, and $\mathcal{F} : f \mapsto f^*$ is called the *Legendre-Frenchel* transformation.

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Epigraph

The *epigraph* of $f : X \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$\mathrm{epi}(f) = \{(x, \alpha) : x \in X, \alpha \in \mathbb{R} \text{ and } \alpha \geq f(x)\}.$$

1. f is l.s.c. if and only if $\mathrm{epi}(f)$ is closed in $X \times \mathbb{R}$.
2. f is convex if and only if $\mathrm{epi}(f)$ is convex in $X \times \mathbb{R}$.

Wijsman's Theorem; 1966

On $\mathbb{R}^n$, for nontrivial l.s.c. convex functions, $epi(f_n) \xrightarrow{\text{Wijsman}} epi(f)$ if, and only if $epi(f_n^*) \xrightarrow{\text{Wijsman}} epi(f^*)$.

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Wijsman topologies

Definition

$CL(X)$ = the set of nonempty closed sets of (X, d). Under

$$A \leftrightarrow d(A, \cdot) : X \to \mathbb{R},$$

$CL(X) \hookrightarrow \mathbb{R}^X$. The *Wijsman topology* $\tau_{w(d)}$ on $CL(X)$ is just the subspace topology.

Facts

1. The Legedere-Fenchel transformation is continuous w.r.t. Wijsman topologies.

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1. The Legedere-Fenchel transformation is continuous w.r.t. Wijsman topologies.
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A brief summary

1. Investigating properties of the Wijsman topology.
2. Extending Wijsman's theorem to infinite dimensions?
3. Determine relations of Wijsman convergence with others.
4. Using graphical approach to study function spaces.

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Two classical results

Lechicki-Levi Theorem; 1987

Let (X, d) be a metric space. Then (X, d) is separable if, and only if $(CL(X), \tau_{w(d)})$ is metrizable.

For any dense subset $\{x_n : n \in \mathbb{N}\} \subseteq X$,

$$\varrho_d(A, B) = \sum_{n=1}^{\infty} \frac{|d(x_n, A) - d(x_n, B)| \wedge 1}{2^n}$$

defines a compatible metric.

Beer-Costantini Theorem; 1990s

X is a Polish space if, and only if $(CL(X), \tau_{w(d)})$ is Polish for any compatible metric d on X.

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Three problems we concerned

Motivated by two classical results just mentioned, we are considering the following three problems.

Three problems

1. Is complete metrizability of (X, d) equivalent to some completeness properties of $(CL(X), \tau_{w(d)})$?
2. When is $(CL(X), \tau_{w(d)})$ a Baire space?
3. When is $(CL(X), \tau_{w(d)})$ a normal space?

Problem 1 was an oral question by Beer. Zsilinszky started to study Problem 2 in 1996. Problem 3 is closely related to an open question posed by Di Maio and Meccariello in 1998.

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Problem 1: Čech completeness

Čech completeness is a natural candidate for Problem 1.

Example (Costantini, 1998)

Take X to be the real line $\mathbb{R}$ with the discrete topology. In 1998, Costantini constructed a three-valued metric d on X such that $(CL(X), \tau_{w(d)})$ is not Čech complete.

This example tells that Čech completeness is unfortunately not a good candidate for Problem 1.

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Problem 2: Amsterdam properties

Base compact spaces

A space X is (resp. *countably*) *base compact* with respect to an open base $\mathfrak{B}$ if X is regular such that $\bigcap_{F \in \mathcal{F}} \overline{F} \neq \varnothing$ for each (resp. countable) centered family $\mathcal{F} \subseteq \mathfrak{B}$.

Subcompact spaces

A space X is (resp. *countably*) *subcompact* with respect to an open base $\mathfrak{B}$ if X is regular such that $\bigcap_{F \in \mathcal{F}} \overline{F} \neq \varnothing$ for each (resp. countable) regular filterbase $\mathcal{F} \subseteq \mathfrak{B}$.

If "regular" is replaced by "quasi-regular" and "base" is replaced by "π-base", the resulting spaces are called almost base compact, almost countably base compact, almost subcompact, almost countably subcompact, respectively.

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Amsterdam properties: non-separable case

For metrizable spaces, (countable) subcompactness is equivalent to complete metrizability [de Groot; 1963].

Example (Cao & Junnila; 2010)

There is a metric space (X, d) of the first category such that $(CL(X), \tau_{w(d)})$ is countably base compact.

Sketch of the example

Give a cardinal $\kappa \geq \omega_1$, consider the Baire metric d_κ on κ^ω

$$d_\kappa(x, y) = \begin{cases} 0, & \text{if } x = y; \\ 2^{-n}, & \text{if } x \neq y \text{ and } n \text{ is the least with } x(n) \neq y(n). \end{cases}$$

Let $X := (\kappa^\omega)_0$ be the set of $(\kappa^\omega, d_\kappa)$ consisting of elements which are eventually zero, and d be the relative metric on X.

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For each $n < \omega$, put

$$F_n = \{x \in X : x(i) = 0 \text{ when } i \geq n\}.$$

Then, F_n is a nowhere dense subspace of (X, d) and $X = \bigcup_{n<\omega} F_n$. Thus, (X, d) is of the first category.

Let $\mathcal{E}$ the family of all open balls, and $\mathfrak{B}$ be the collection consisting of all sets $\mathcal{G}_{\mathcal{F},\mathcal{A}} \subseteq CL(X)$ of the form

$$\mathcal{G}_{\mathcal{F},\mathcal{A}} = \left(X \setminus \bigcup \mathcal{F}\right)^{+} \cap \bigcap_{E \in \mathcal{A}} E^{-},$$

where $\mathcal{F}$ and $\mathcal{A}$ are finite subfamilies of $\mathcal{E}$. It can be verified that $(CL(X), \tau_{W(d)})$ is a countably base-compact space with respect to $\mathfrak{B}$. $\square$

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Amsterdam properties: separable case

Example (Cao & Junnila; 2010)

There is a separable metric space (Y, ρ) of the first category such that $(CL(Y), \tau_{W(\rho)})$ is almost countably subcompact.

$Y = (\omega^\omega)_0 \times \omega^\omega$ with the box metric ρ. Then, Y is separable, of the first category. It can be shown that $(CL(Y), \tau_{W(\rho)})$ is almost countably subcompact with some π-base $\mathfrak{B}$. $\square$.

Remarks

The construction of the π-base $\mathfrak{B}$ in the previous example is somehow tedious. Recently, Piatkiewicz and Zsilinszky observed that $(CL(Y), \tau_{W(\rho)})$ is in fact almost countably base compact via an indirect approach.

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Problem 1: properties defined by games

In 2010, Piatkiewicz and Zsilinszky showed the following

(Strong) α-favorability

$\left(CL(Y), \tau_{W(\rho)}\right)$ is α-favorable; $\left(CL(X), \tau_{W(d)}\right)$ is strongly α-favorable.

Two examples discussed in this subsection tell us that Amsterdam properties as well as (strong) α-favorability may not be good candidates for Problem 1.

Open questions

1. If (X, d) is base compact, must $\left(CL(X), \tau_{W(d)}\right)$ be base compact?
2. If (X, d) is subcompact, must $\left(CL(X), \tau_{W(d)}\right)$ be subcompact?

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Two examples discussed in this subsection tell us that Amsterdam properties as well as (strong) α-favorability may not be good candidates for Problem 1.

Open questions

1. If (X, d) is base compact, must $(CL(X), \tau_{W(d)})$ be base compact?
2. If (X, d) is subcompact, must $(CL(X), \tau_{W(d)})$ be subcompact?

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Probem 2: the Baire property

Concerning Problem 2, we have the following partial results:

Partial results on Problem 2

Given a metric space (X, d), $(CL(X), \tau_{W(d)})$ is a Baire space, if

1. (X, d) *is separable and Baire* (Zsilinszky; 1996);
2. (X, d) *is complete* (Zsilinszky; 1998);
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Open question 3

If (X, d) is a metric Baire space, must $(CL(X), \tau_{W(d)})$ be a Baire space?

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Open question 3: barely Baire spaces

A Baire space X is called *barely Baire* if there is a Baire space Y such that $X \times Y$ is not Baire. Several examples of barely Baire spaces were constructed by Fleissner and Kunen in 1978.

1. There is a metric Baire space Y such that Y^2 is not Baire.
2. For every cardinal κ, there is a family of $\{X_\alpha : \alpha < \kappa\}$ of metric space such that $\prod\{X_\alpha : \alpha < \kappa, \alpha \neq \beta\}$ is Baire for every $\beta < \alpha$, but $\prod\{X_\alpha : \alpha < \kappa\}$ is not Baire.

Theorem (Cao & Junnila; 2010)

Let (X, d) be an ultrametric space such that no d-ball is covered by countably many smaller d-balls and no set $\{d(y, x) : y \in X\}$, where $x \in X$, has a non-zero accumulation point in $\mathbb{R}^+$. Then $(CL(X), \tau_{w(d)})$ is countably base-compact.

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Problem 2: hereditarily Baire?

Example (Zsilinszky; 1998)

In 1973, Aarts and Lutzer constructed a separable, hereditarily Baire metric space (X, d) such that $X \times X$ is not hereditarily Baire; In 1998, Zsilinzsky observed that $(CL(X), \tau_{w(d)})$ is not hereditarily Baire.

Example (Chaber & Pol; 2002)

In 2002, Chaber and Pol showed that if the set of points in (X, d) without any compact nbhd has weight $2^{\aleph_0}$, then $\mathbb{Q} \hookrightarrow (CL(X), \tau_{w(d)})$ as a closed subspace. Take $X := C(\beta\mathbb{N})$ with the metric d generated by the sup-norm. Then $(CL(X), \tau_{w(d)})$ is not hereditarily Baire.

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Problem 3: a question of Di Maio and Meccardiello

In 1998, Di Maio and Meccardiello posed the following open question, which is related to the normality problem.

Open question 4 (Di Maio & Meccardiello; 1998)

It is known that if (X,d) is a separable metric space, then $\left(CL(X),\tau_{w(d)}\right)$ is metrizable and so paracompact and normal. Is the opposite true? Is $\left(CL(X),\tau_{w(d)}\right)$ normal if, and only if $\left(CL(X),\tau_{w(d)}\right)$ is metrizable?

Next, we provide a partial answer to this question.

Theorem (Cao & Junnila)

For a metric space (X,d), $\left(CL(X),\tau_{w(d)}\right)$ is metrizable if, and only if $\left(CL(X),\tau_{w(d)}\right)$ is hereditarily normal.

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Hereditarily normal

Sketch of proof.

According to the Lechicki-Levi theorem, we only need to prove that if $(CL(X), \tau_{W(d)})$ is hereditarily normal, then (X, d) must be separable. If (X, d) is not separable, then we can find an $\varepsilon > 0$ and an ε-discrete closed subset $D \subseteq X$ with $|D| = \aleph_1$.

Next, we construct a homeomorphic embedding

$$\varphi : (\omega_1 + 1) \times (\omega_1 + 1) \to (CL(X), \tau_{W(d)})$$

such that $\varphi((\omega_1 + 1) \times (\omega_1 + 1))$ is a closed subspace of $(CL(X), \tau_{W(d)})$. Therefore, $(\omega_1 + 1) \times (\omega_1 + 1)$ is hereditarily normal. This is impossible, as showed by a classical result of Dieudonné. □

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Embedding into Wijsman hyperspaces

Let $\mathfrak{C}$ be the class of spaces embeddable as a closed subspace into the Wijsman hyperspace of a metric space.

Theorem (Cao & Junnila)

$\mathfrak{C}$ is closed hereditary and multiplicative, and contains all Dieudonné complete spaces.

We know that $\mathfrak{C}$ contains $\omega_1 + 1$, $\mathbb{Q}$, etc. A topological space is called *$\mathbb{N}$-compact* if it is homeomorphic to a closed subspace of a Cartesian product of copies of $\mathbb{N}$.

An observation

Each $\mathbb{N}$-compact space is embeddable into the Wijsman hyperspace of some discrete metric space as a closed subspace.

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