

Pricing Variance Swaps under Stochastic Volatility and Stochastic Interest Rate

Teh Raihana Nazirah Binti Roslan

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Primary Supervisor: Professor Jiling Cao

Declaration

I hereby declare that this submission is my own work and that, to the best of my knowledge and belief, it contains no material previously published or written by another person (except where explicitly defined in the acknowledgements), nor material which to a substantial extent has been submitted for the award of any other degree or diploma of a university or other institution of higher learning.

Auckland, May 2016



Teh Raihana Nazirah Binti Roslan

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Abstract

In this thesis, we study the issue of pricing discretely-sampled variance swaps under stochastic volatility and stochastic interest rate. In particular, our modeling framework consists of the equity which follows the dynamics of the Heston stochastic volatility model, whereas the stochastic interest rate is driven by the Cox-Ingersoll-Ross (CIR) model. We first extend the framework of [119] by incorporating the CIR interest rate into their Heston model for pricing discretely-sampled variance swaps. We impose partial correlation between the asset price and the volatility, and derive a semi-closed form pricing formula for the fair delivery price of a variance swap. Several numerical examples and comparisons are provided to validate our pricing formula, as well as to show the effect of stochastic interest rate on the pricing of variance swaps.

In addition, the pricing of discretely-sampled variance swaps with full correlation among the asset price, interest rate as well as the volatility is investigated. This offers a more realistic model with practical importance for pricing and hedging. Since this full correlation model is incompliant with the analytical tractability property, we determine the approximations for the non-affine terms by following the approach in [55] and present a semi-closed form approximation formula for the fair delivery price of a variance swap. Our results confirm that the impact of the correlation between the stock price and the interest rate on variance swaps prices is very crucial. Besides that, the impact of correlation coefficients becomes less apparent as the number of sampling frequencies increases for all cases.

Finally, the issue of pricing discretely-sampled variance swaps under stochastic volatility and stochastic interest rate with regime switching is also discussed. This model is an extension of the corresponding one in [34] and is capable of capturing several macroeconomic issues such as alternating business cycles. Our semi-closed form pricing formula is proven to

achieve almost the same accuracy in far less time compared with the Monte Carlo simulation. Through numerical examples, we discover that prices of variance swaps obtained from the regime switching Heston-CIR model are significantly lower than those of its non-regime switching counterparts. Furthermore, when allowing the Heston-CIR model to switch across three regimes, it is observable that the price of a variance swap is cheapest in the best economy, and most expensive in the worst economy among all.

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Chapter 1

Introduction

1.1 Background

This section is devoted to provide some insights regarding the background of important aspects in this thesis. Subsection 1.1.1 describes some information on volatility derivatives. Following this section is Subsection 1.1.2 which introduces stochastic interest rate.

1.1.1 Volatility Derivatives

Basically, volatility derivatives are financial derivatives whose values depend on the future levels of volatility. A major difference between volatility derivatives and other standard derivatives lies in the volatility term which not only determines the final calculation points, but also exists in the payoff formulation. According to Demeterfi et al. [29], volatility derivatives are traded for decision-making between long or short positions, trading spreads between realized and implied volatility, and hedging against volatility risks. Without doubt, these are due to the captivating traits of volatility itself such as mean-reversion, sensitivity to risks and non-positive relationship with the stock or indices. The utmost advantage of volatility derivatives is their capability in providing direct exposure towards the asset's volatility without being burdened with the hassles of continuous delta-hedging. This is due to the fact that constant buying and selling activities in delta-hedging would result in high transaction costs and liquidity issues. The tremendous spike in the trading volume of volatility derivatives recently can be related to their importance in providing volatility exposures to market practitioners. The Chicago Board Options Exchange (CBOE) reported that the average daily trading volume of futures on its VIX volatility index showed an increasing trend, and climbed

26 percent from 159,498 in 2013 to 200,521 in 2014. An increment of 11 percent was also exhibited by the VIX options in the same period, with the same overall rising pattern from the year 2006 up to 2014.

Figure 1.1: Average daily trading volume of VIX futures.

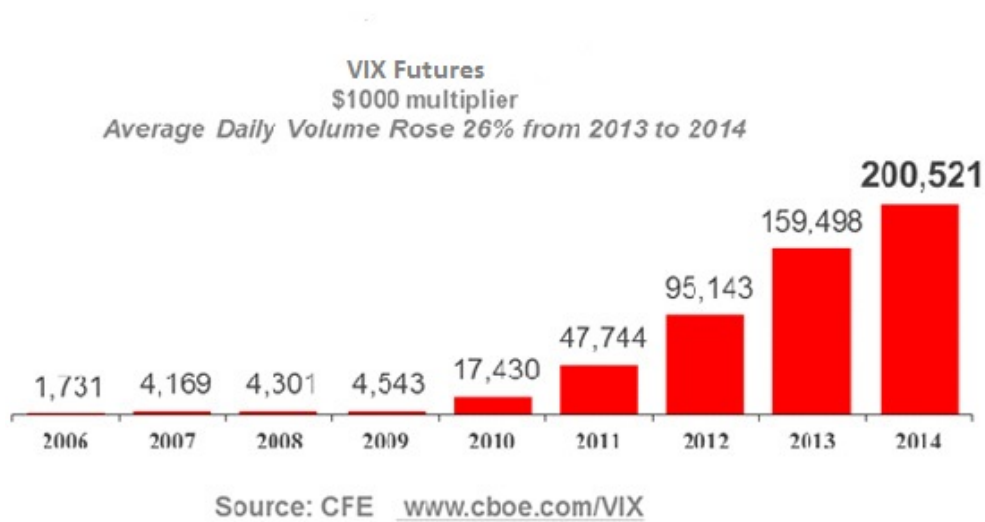


Figure 1.2: Average daily trading volume of VIX options.



Historically, M. Brenner and D. Galai were the pioneers of trading activities for volatility derivatives back in 1993 where implied volatilities of at-the-money (ATM) options were used to develop a volatility index. In the same year, a volatility index known as VIX was launched by CBOE. VIX offered an alternative for Brenner and Galai's approach by focusing on the one-month implied volatilities of S&P 100 index options. Starting from this point, other countries such as Germany and Austria also announced their volatility indexes respectively. By 1996, the trend of trading a class of volatility derivatives known as the volatility swaps was observed. This was followed by the trading of variance swaps back in 1998 due to the impact brought by the crash of the Long Term Capital Management. By 2005, the third generation of variance derivatives such as options on realized variance, conditional variance swaps and corridor variance swaps have started to trade actively. Further details regarding the evolution of volatility derivatives can be found in [22].

Generally, volatility can be measured in three main ways, namely historical volatility, implied volatility and model-based volatility. Historical volatility is mainly related to previous standard deviation of financial returns involving a specified time period. Examples of volatility derivatives written on this historical volatility measure include variance swaps, volatility swaps and futures on realized variance launched by CBOE. For the implied volatility, it ascertains the volatility by matching volatilities from the market with some specific pricing model. The VIX of CBOE estimates this type of volatility measure of the S&P 500 index. Finally, the model-based volatility is defined in the class of stochastic volatility models as done in [66, 101] and others.

Volatility derivatives' transactions involve two main methods known as static replication and delta hedging. For the static replication, the essential ingredients are continuum of strikes from market prices, deals occurring only at initial and maturity times, and existence of futures market. This method is suitable for options insensitive to price changes by mixing uniform weighting of returns from high and low option strikes. It is more favourable than the ATM implied volatilities because it does not assume constant volatility or continuous underlying price process. Moreover, it retains the volatility responsiveness along the time interval even throughout cases of extreme price changes. In order to overcome this, the method of delta hedging is proposed. Here, the assumption of continuous semi-martingale is imposed on the underlying futures price process, whereas the assumption for the volatility does not change from previous. The replication conducted using the Black model with constant volatility (the hedging volatility) will produce errors based on the nature of volatility itself which is stochastic. Some rules are followed in order to determine whether the errors induce profit or loss. A loss

is observed if the hedging volatility is always less than the realized volatility and vice versa. The ultimate profit and loss for an option can be incurred from daily summation throughout the option's lifetime, which reflects the evaluation formula for a variance swap. The only difference is that delta hedging holds the path-dependence property since its weights has subordinations with the option's gamma. Little and Pant [80] suggested using stochastic volatility models or specifying the payment function to relax the path-dependency property.

In Chapter 2 of this thesis, we will introduce a special type of volatility derivatives, namely variance swaps. Variance swaps were first launched in 1998 due to the breakthrough of volatility derivatives in the market. An extensive review of variance swaps can be found in [11].

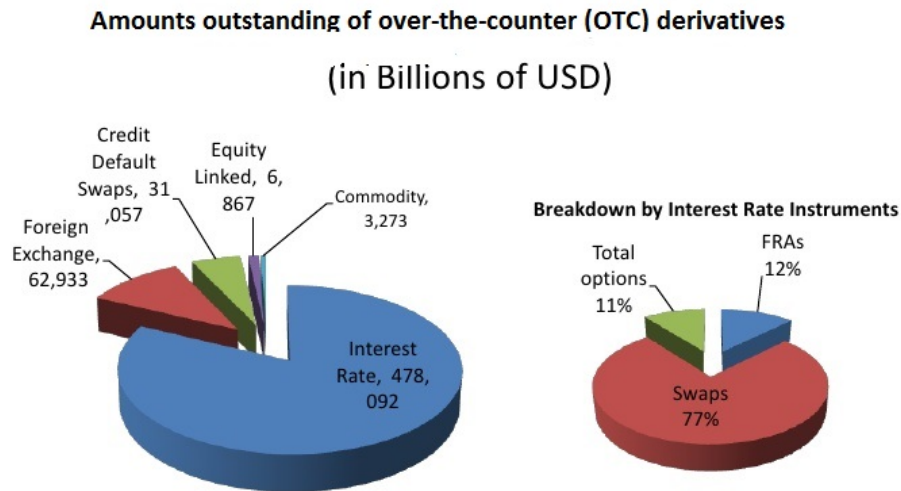
1.1.2 Stochastic Interest Rate

In today's modern financial world, interest rate and its ever-changing feature is one of the most debated issues among economists, investors and researchers. This is due to the fact that it possesses strong influence towards all types of derivative securities. Dynamics of the interest rate is determined by many factors, and this largely affects all other financial derivatives which are very responsive towards it. These financial derivatives are defined in [108] as interest-rate derivatives, which range from fixed-income contracts such as bond options, caps, floors, and swaptions, to more complicated and path-dependent contracts such as index amortizing rate swap. Basically, the financial derivatives values are indirectly derived from values of other traded equities, and their future prices will also be influenced by the future prices of those traded equities. Bank for International Settlements (BIS) reported ¹ that interest rate derivatives dominated over 82 percent of the total outstanding amount of over-the-counter (OTC) derivatives in 2010. This value was constructed by 77 percent of swaps, followed by 12 percent of forward rate agreement and 11 percent of total options respectively.

Factors that influence levels of market interest rate are recognized as expected levels of inflation, general economic activities, current status of surplus or deficit, foreign exchange market and political stability [51]. Without doubt, the interest rate concept has long been incorporated into our daily lives. One simple example is an expectation of money growth with certain rates after depositing money in bank accounts for some specific period. As for creditors, they would also expect some increase in the

¹<http://www.slideshare.net/francoischoquet/build-curve-cpt>

Figure 1.3: Trading volume of interest rate derivatives and the respective instruments breakdown as in June 2010.



Source: BIS June 2010 S/A Survey

amount received later from the debtors, with certain rates incurred on the borrowed amount. The greater the term to maturity involved, the greater the uncertainty would be. Practitioners in the financial and economics world have long realized how crucial the role played by interest rate is. First, pricing of actuarial commodities or any other assets in the market is figured out through discounted cash flows calculated until maturity. These discounts would rely on default-free dynamic interest rate, also named as spot rates. Secondly, interest rate also greatly influences actions taken in businesses or organizations. The rate of return that could be gained during choosing investment opportunities offering equal risks would concern interest rate. Moreover, if there are optional chances that involve different rates but looks promising enough, then the ones that come out with less interest rate will likely be considered. In addition, the government will also be assisted in dealing with pricing issues and choosing debt opportunities with the least cost, along with other financial policies.

Cox et al. [27] criticized the four prominent theories of the Term Structure of Interest Rates (TSIR) which comprised of the Expectations Hypothesis, Liquidity Preference Theory, Market Segmentation Theory and Preferred Habitat Theory. They stressed on

the importance of encompassing the uncertainties which later led to investigation of stochastic interest rate models. Yet, it is undeniable that this goes inextricably with the real scenarios occurring in the financial and economics world. Perturbations of liquid zero-coupon bond prices, along with insufficiency in their amounts are among the realities that have to be faced in today's market. In addition, since the financial world is full of uncertainties, analysis of the spot rates and forward rates becomes more and more complicated and hard. There are also many cases in the market where the data are unclear, indefinite in their boundaries, and not very reliable in order to anticipate future interest rate. Examples are subjective interest rate expectations and beliefs of experts, and prices of fixed income securities. Not only that, the dynamics of financial asset prices also should be given attention to prevent information loss. It is also important to ensure that the model chosen to represent the yield curve is capable enough of including a variety of possible shapes. Hence, it is inherent to model the interest rate as a random variable since its future value holds the random outcome property which is not predictable.

Generally, the modeling trend of stochastic interest rate can be seen as developing from unobservable rates such as spot rates, to market rates regularly practised by financial institutions. O. Vasicek was the pioneer of the field when he introduced a general model of interest rate in 1977 by assuming normal distribution for the instantaneous short rate, refer to [107]. Since the normally distributed property might result in negative values for the short rate, Cox et al. [27] (in short, CIR) came up with non-central chi-square distributed short rate model. However, this model might result in imperfect fit when calibrated towards the observed TSIR. Thus, Hull and White [69] (in short, HW) proposed improvements for both the Vasicek and CIR model in 1990. Unfortunately, the extension for the CIR model was regarded as not fully tractable and did not ensure perfect fitting during calibration, as pointed out in [15]. Starting from this point, a methodology which imposed stochastic structure directly on the evolution of the forward rate curve was introduced in 1992 to avoid arbitrage opportunities. Known as the Heath, Jarrow and Morton (in short, HJM) model, it can be used to price and hedge consistently all contingent claims of the term structure, see [65]. Along the general framework of HJM; Brace et al. [12] further analysed a class of term structure models with volatility of lognormal type. This market model possessed the advantage of having observable rates which are quoted by financial markets, compared with spot rate and forward rate models. In addition, it is also consistent with the Black formulas currently being practiced.

In this thesis, we apply some stochastic interest rate models to price variance swaps.

Generally, the maturities of most liquid variance swaps are between three months up to around two years. However, variance swaps traded in indices and more liquid stocks have maturities around three years, or even up to five years and beyond. An example of the related market is the Euro Stoxx 50 which is a stock index of Eurozone stocks designed by STOXX, refer to [2]. Since previous researchers claimed that constant interest rate is only appropriate for short term maturity financial derivatives, it is crucial to use stochastic interest rate models when pricing such variance swaps.

1.2 Literature Review

Researchers working in the field concerning volatility derivatives have been focusing on developing suitable methods for evaluating variance swaps. In this thesis, we shall separate these methods into two main categories : analytical and numerical approaches. For the analytical approaches, Carr and Madan [23] combined static replication using options with dynamic trading in futures to price and hedge certain volatility contracts without specifying the volatility process. The principal assumptions were continuous trading and continuous semi-martingale price processes for the future prices. The selection of a payoff function which diminished the path dependence property ensured that the investor's joint perception regarding volatility and price was also taken into consideration. Further, Demeterfi et al. [29] also produced work in the same area by proving that a variance swap could be reproduced via a portfolio of standard options. The requirements specified were continuity of exercise prices for the options and continuous sampling times for the variance swaps. In addition, incorporation of stochastic volatility into the pricing and hedging models of variance swaps also has been a recent trend in the literatures. Elliott et al. [37] constructed a continuous-time Markov-modulated version of the Heston stochastic volatility model to distinguish the states of a business cycle. Analytical formulas were obtained using the regime switching Esscher transform and price comparisons were made between models with and without switching regimes. Results showed that prices of variance swaps implied by the regime switching Heston stochastic volatility model were significantly higher than those without switching regimes. Grunbichler and Longstaff [54] also developed pricing model for options on variance based on the Heston stochastic volatility model. One important finding was the contrast characteristics between volatility derivatives and options on traded assets. However, it was later noted by Heston and Nandi [67] that specification of the mean-reverting square-root process is difficult to be applied to the real market. Thus, the latter proposed a user-friendly model by working on the discrete-time GARCH

volatility process with parametric specifications. This model had the advantage of real market practicability, as well as the capability to hedge various volatility derivatives using only single asset. Another interesting study by Swishchuk and Xu [103] focused on introducing delay into stochastic volatility models which also involved jumps for pricing variance swaps. This delay is different from the usual filtration definition since the asset price will be determined by the entire information starting from the inception point. They also provided techniques to reduce risk via lower bounds for the delay process. Based on their experiments on the S&P Canada index from 1999-2002, they concluded that their model resulted in higher pricing based on the higher liabilities involved. Overall, all of these researchers assumed continuous sampling time, whereas the discrete sampling is the actual practice in financial markets. In fact, options of discretely-sampled variance swaps were mis-valued when the continuous sampling were used as approximations, and produce huge inaccuracies in certain sampling periods, as discussed in [9, 34, 80, 119].

Recognizing the fact that the continuous sampling evaluation is contrary to the real market, the focus of research has actuated towards discrete sampling. Besides ensuring that this condition is fulfilled, researchers also tried to handle the internal problems commonly occurring in the literature. Carr and Lee [22] addressed the issue of pricing errors in replication strategies up to the third order when at-the-money (ATM) was used for predicting realized variances. They provided a new formulation for the implied volatility along with weight functions, and conditioned on sufficient conditions for approximating volatility swaps which resulted in pricing rates with least errors. Quite recently, Zheng and Kwok [116] highlighted the importance of utilizing the joint moment generating function for assessing prices of third generation volatility products. These gamma swaps, corridor gamma swaps and conditional variance swaps were tested against the ones on continuous sampling to explore the effects on convergence, sampling intervals and sensitivities. Even though these products exhibited convergence towards continuous sampling, linearity was not a must. The period specified in contracts and parameter numbers also influenced the fair strike prices. However, gamma swaps were not affected by the variations in sampling intervals. Furthermore, these authors also discussed the diminishing precision problem for short term volatility derivatives and variance products with nonlinear payoffs. Later on, saddlepoint approximation formulas were derived in [117] along with conditional saddlepoint method based on simulation paths. As predicted, the method of Zheng and Kwok worked well for short time intervals, specially for in-the-money (ITM) options. Also, reasonable accuracy was achievable by varying through various strikes and interval levels.

In addition to the above mentioned analytical approaches, some other authors also conducted researches using numerical approaches. Little and Pant [80] explored the finite-difference method via dimension-reduction approach and obtained high efficiency and accuracy for discretely-sampled variance swaps. The main tool was the assumption of the local volatility as a known function of time and spot price of the underlying asset. Furthermore, Windcliff et al. [109] investigated the effects of employing the partial-integro differential equation on constant volatility, local volatility and jump diffusion-based volatility products using delta-gamma hedging. Large transaction costs involved in constant volatility models may result in inefficiency of their delta-gamma hedging modus. Thus, they suggested that institutions hedging the swaps propose clients as natural counter-parties to reduce the transaction costs. An extension of the approach in [80] was made by Zhu and Lian in [119] through incorporating Heston two-factor stochastic volatility for pricing discretely-sampled variance swaps. Levels of validity for short periods when using the continuous-time sampling were provided through significant errors, along with analytical hedging derivations and numerical simulations. However, a much simpler approach was explored by Rujivan and Zhu [93] who proved that it was not necessary to include the generalized Fourier transform and putting in state variable in the previous framework. Their method for solving the partial differential equations consists of applying the Schwartz solution procedure which fulfilled certain inequalities in order to obtain an affine global solution. Their method was favourable in terms of being directly related to the conditional variance, skewness and kurtosis. Another recent study was conducted by Bernard and Cui [9] on analytical and asymptotic results for discrete sampling variance swaps with three different stochastic volatility models. Their Cholesky decomposition technique exhibited significant simplification compared with the work in the literature. However, the assumption of constant interest rates by these authors as well as other previous authors involving variance swaps was unrealistic with the real market phenomena.

In the past three decades, many authors have concentrated on the issue of modelling interest rate and its application in financial derivatives' pricing using stochastic approaches. Yet, it is undeniable that this goes inextricably with the real scenarios occurring in the financial and economics world. Elliott and Siu [36] pointed out that stochastic interest rate models should be capable of providing a practical realization of the fluctuation property, as well as adequately tractable. They derived exponential-affine form of bond prices with elements of continuous-time Markov chains using enlarged filtration and semi-martingale decompositions. Moreover, Kim et al. [76] showed that incorporation of stochastic interest rate into a stochastic volatility model gave bet-

ter results compared with the constant interest rate case in any maturity. They proposed a model which was a combination of the multi-scale stochastic volatility model in [42] and the Hull-White interest rate model. The call option price approximation for this mixed model was obtained via derivation of the leading order and the first order correction prices using Fouque's multiscale expansion method and operator specifications for the correction terms. In addition, Grzelak and Oosterlee [55] examined correlation issues of European products pricing with the Heston-Hull-White and the Heston-CIR hybrid models. These resulted in big errors for some short-maturity options which was another drawback of the constant interest rate. Another recent study by Shen and Siu [97] showed the important effects of both stochastic interest rates and stochastic volatility with regime switching for pricing variance swaps. Even though their regime switching approach was capable of incorporating inconsistencies between different business steps, they only considered continuous sampling approximation and formulated their variance swap rates in integral form.

In the literature, there has also been a growing number of researchers working in numerical techniques for pricing financial derivatives involving stochastic interest rates. For example, both Guo et al. [60], and Haentjens and In't Hout [62] employed the finite-difference scheme specializing in the amalgamation of the Heston and the Hull-White model. The alternating direction implicit (ADI) time discretization scheme was built up from specifications of boundary conditions and sets of grid points were chosen. The main difference between these two approaches was the number of stochastic differential equations involved, since the former authors were concerned with the change of measure whereas the latter authors did not. Also, the number of experiments considered in [62] involved a wider range.

The model setup featured in this thesis can be categorized in the class of hybrid models, which describe interactions between different asset classes such as stock, interest rate and commodities combined together as new. The main aim of these models is to provide bespoke alternatives for market practitioners and financial institutions, as well as reducing the associated risks between the underlyings. Newly-found stochastic differential equations according to the types of models considered will be derived. The evolving number of complex hybrid models featuring various underlyings can be related to the modernisation of the financial markets today, along with computational advancements. Some popular examples are the hybridization of equity-interest rate products, as well as the combination of equity-FX rate models.

Hybrid models can be generally categorized into two different types, namely hybrid models with full correlations, or hybrid models with partial correlations among the

engaged underlyings. These models are later analyzed using analytical or numerical approaches according to the techniques and complexities involved. The correlation issue can be directly linked to the highlighted importance of imposing correlations, either partially or fully, in the literature. Grzelak et al. [57] and Chen et al. [24] stressed that correlations between equity and interest rate are crucial to ensure that the pricing activities are precise, especially for industrial practice. A study done in [46] on auto-callable securities revealed the effects of correlation between equity and interest rate in terms of the increment or reduction of the final product prices. The essentiality of this property was later illustrated for the Heston-Vasicek and Heston-CIR++ model by imposing indirect correlations through approximations. A detailed description regarding correlation effects among interest rate, volatility and the equity respectively can be found in [61] which provided comparisons in terms of graph shapes and maturity time. According to these authors, the correlation effects between equity and interest rates were more distinct compared with the correlation effects between interest rates and volatility.

Hybrid models with partial correlations between asset classes seem to dominate the field due to less complexity involved. Majority of the researchers focus on either inducing correlation between the stock and interest rate, or between the stock and the volatility. Grzelak and Oosterlee [56] overcame the limitations in [55] regarding interest rate smiles by modeling multi-currency models with smiles for FX rate, domestic and foreign fixed income market, respectively. This was achievable through the Heston-Libor hybrid model which involved the freezing of Libor rates technique due to the non-affine property. Results showed that their model is excellent in terms of producing small errors compared with the original model. Another study focusing on pricing vanilla options using the Heston-multifactor Gaussian hybrid model was done in [57], where comparisons with the Schobel-Zhu-Hull-White (SZHW) hybrid model were made as well. Exploitation on the analytical tractability of Gaussian processes produced decent fitting to ATM volatility structures, along with closed form solutions for the caps and swaptions. In contrast, Ahlip and Rutkowski [1] derived a semi-closed form pricing formula for FX options with Heston stochastic volatility for the exchange rate and the CIR dynamics for the domestic and foreign interest rates. These authors also displayed the effects of incorporating the stochastic interest rates and options' maturity. Furthermore, Ziveyi et al. [121] utilized numerical techniques via sparse grid quadrature for the pricing of deferred annuity options. An interesting observation was the ability to obtain closed form formulas which departed from the usual practice of numerical approximations or Monte Carlo simulation. This was accomplished using

Duhamel's principle and the method of characteristics. Recently, several authors also imposed correlations both between the stock and interest rate, and between the stock and the volatility. An example of analytical technique can be found in [24] where the Stochastic-Alpha-Beta-Rho (SABR)-HW model was proposed. Even though this model enjoyed calibration resolution from its inverse projection formula, the calibrated parameters were only applicable for a single maturity and could not provide a consistent description of the dynamics. As for numerical approach techniques, the finite-difference method was employed in [60] along with Alternating Direction Implicit (ADI) scheme for investigating approximation properties for Heston-Hull-White model. It was concluded that the errors between the solutions of the full scale Heston-Hull-White and its approximation were fairly small. In addition, only the changes in correlation parameters between the interest rate and the stock had impact on those errors.

The hybrid models with full correlations between underlyings also attracted attention for improved model capability. Grzelak et al. [58] and Singor et al. [99] compared their Heston-Hull-White hybrid model with the SZHW hybridization for pricing inflation dependent and European options respectively. Their techniques differ in the way that the former applied square-root approximation method in [55], whereas the latter extended the space vector into one additional dimension. In 2011, some advancements involving numerical techniques were observed through the ADI scheme and the sparse grid approach. The Heston-CIR case for American compound type option studied in [26] and the Heston-Hull-White model used in [62] gave increased accuracy and capability, as well as improved convergence property. Moreover, an exquisite contribution in the literature of Heston-CIR model was attained in [48] where closed form solutions of FX options and basic interest rate derivatives were achieved. Their technique of modeling the involved random factors of interest rate and the volatility of the exchange rate under a process of Wishart matrix promised full analytical tractability.

Despite of the relevance of imposing correlations as described above, the attention should be drawn on the ability of the hybrid models to hold their analytical and computational tractability. This is not surprising based on their expanded ramification, and the fact that this is one of the long standing problems in finance. One possible approach is to implement some modifications in these models' structure so that the property of affine diffusion models could be ensured. This framework which was adopted from [31] guarantees that the state vector would result in closed or nearly closed form expressions. This is applicable with the aid of the characteristic function obtained from Fourier transform techniques. Other advantages of affine diffusion models include the ability to replicate numerous shapes of the term structure, and also provide adequate

fitting either to the whole or initial term structure, refer to Paseka [87].

The supremacies of incorporating the Markov regime switching techniques into financial economics modeling have long been discussed in the past four decades. The concept of regime switching was first introduced by Goldfeld and Quant [49] in 1973 to characterize parameter changes in nonlinear and non-stationary models. Basically, a regime switching process involves an unobservable variable in the time-series that switches among a certain number of states with independent price process for each state. The switching from one state to another results in a switching probability which is combined with the joint conditional probability of the current state to produce joint conditional probability for each future state. These processes are filtered by the transition probability matrix. The notion of regime switching was later developed by Hamilton [63] who considered discrete auto-correlated shifts between positive and negative growth rates in the US post-war business cycles. The author found out that the Markov regime switching approach succeeded in capturing complex dynamic patterns in economic transitions. Based on this realization, incorporation of Markov chains into financial modelling with added adjustability and manageable properties can be seen for some unsolved problems in interest rate modeling. These include sudden jumps in the interest rate dynamics due to unpredictable market events, and the cyclical nature of time-series of interest rates according to the economic cycles.

The first advantage of Markov regime switching could be seen via its efficiency to capture the nonlinear behavior in market trading. Goldfeld and Quant [49] initiated this idea based on the vast quantity of work done on identifying nonlinear parameterizations and their importance. The regime switching models' state dependence on transition probabilities towards lagged level of instantaneous rates, along with the ability to illustrate the unit root traits of those rates assisted in predicting interest rates effectively, see [6, 53]. In addition, regime switching models are also proficient for accommodating financial time-series with time-varying properties through shifting patterns exhibited between recession and growth states, refer to [36, 63, 100]. Roma and Torous [90] claimed that some properties of interest rates such as increment at peak business stages and plunge at trough stages could not be explained by classical interest rate models. In fact, it is important to ensure that the cyclical nature of time-series of interest rates is well described in a model which also allows possible structural changes, such as inclusion of jumps. Moreover, regime switching models also provide the flexibility which might include the mean-reversion, asymmetric distribution or others. Previous studies, e.g., Elliott and Mamon [35], and Elliott and Wilson [38] showed that the dynamics of interest rates was modeled excellently when the mean-reverting

level followed a finite state continuous Markov chain. An interesting observation was noted from the stochastic nature of the business cycle lengths and intensities inherited from the uncertainties involved in the Markov chain. In addition, the present applicability of the regime switching techniques along with their substantial effects should not be ignored. Ang and Bekaert [5] proved empirically that neglecting regime switching for a conditionally risk-free asset incurred more cost which was equivalent to neglecting overseas investment opportunities. Recent work in Liew and Siu [79] and Zhou and Mamon [118] also revealed that regime switching models could display some important observations in the market such as volatility clustering and heavy tail distributions of returns, as well as replication of irregular yield curve shapes. The authors of [118] also certified that regime switching models calibrated better and gave more accurate predictions when compared with their counterparts.

The research work in the literature on option pricing, interest rate modeling as well as volatility derivatives exposed the immense popularity of the Markov regime switching technique. For instance, examples are given in evaluating forward starting options in [89], barrier options in [68], Asian options in [114], American options in [112] and volatility derivatives in [85, 97]. The prevailing issue identified in all of these papers is on overcoming the incomplete market environment induced by the uncertainties of the regime switching. Several techniques proposed in recent work include introduction of additional securities [115], utilization of Esscher transform [34], and implementation of minimal martingale measure approach, see [7, 52, 94]. Zhang et al. [115] added sets of Markov jump assets and ensured that the configuration of the portfolio was equivalent continuously to guarantee that the market was complete. The main objective was to initiate new assets with bigger filtration having its own unique equivalent martingale measure. As for Esscher transform, this time-honored tool can be tracked back to [45] where the applicability of Esscher transform for price processes modulated by stochastic processes with stationary and independent increments was suggested. Pricing formulas for European options were given by characterizing Esscher transform under the risk-neutral probability measure. This concept was later developed by Elliott et al. [33] who justified their choice of equivalent risk-neutral measure for European call options using the Bayes rule and the Girsanov theorem. The essence of incorporating regime switching for pricing variance and volatility swaps under the Heston stochastic volatility model was illustrated in [34] and [37]. In fact, these papers conducted experiments on two regimes representing economies in good and bad states. However, these two papers were distinguished in the sampling type chosen, where the latter concentrated on continuous sampling compared to discrete sampling case by the former.

In addition, Elliott et al. [37] adopted probabilistic and partial differential equation approach to find the conditional price of the volatility derivatives, whereas Elliott and Lian [34] derived forward characteristic functions with the aid of Fourier transform. In contrast, the authors of [7, 52] handled the Markov modulated market by taking benefit of the minimal martingale measure via local risk minimizing strategy. Prices of defaultable bonds, variance and volatility swaps were obtained respectively using the Follmer-Schweizer decomposition which later resulted in solutions of partial differential equations. Working under the same domain, Salvi and Swishchuk [94] took a different approach in tackling the problem by using the Fubini theorem and the property of conditional expectation for pricing covariance and correlation swaps. These result in approximations which were given at the first order, where daily interpolation was taken into account.

Integration of Markov regime switching techniques with stochastic interest rate model is inscribed as a contemporary development in the literature of interest rate modeling. In this direction, the authors of [36, 100] used the regime switching approach to extend the Cox-Ingersoll-Ross (CIR), the Hull-White and the Vasicek models respectively, in order to improve their competencies and practicality. The main difference between these two papers was that the jump-diffusion structure was incorporated in the Vasicek model. An exponential affine form of bond prices with elements of continuous-time Markov chain was derived using enlarged filtration and semi-martingale decompositions. Results showed that regime switching models are capable of incorporating jumps and inconsistencies between different business stages. For the Heath-Jarrow-Morton model, a study was conducted in Elhouar [32] to overcome the model's problem of infinite dimensionality. This was done as an extension from the study in Valchev [105] where only Gaussian models were focused and no finite-dimensional issue was considered. In [32], the volatility was specified as function of continuous-time finite Markov chain which led to proofs of finite-dimensional realizations. It was revealed that a Markov modulated CIR model did not possess any finite-dimensional realization, and there were only little effects of considering different cases for their separable volatility assumption. In addition, Futami [43] analysed the effects of partial information on one-factor Gaussian models for zero-coupon bonds. The regime shift factor was introduced into partial information by calculating averages of yield curves in full information. An interesting observation was the transformation of market price of diffusion risk and the bond's volatility into stochastic variables.

1.3 Research Questions

It is clear from the previous background and literature review sections that the following research questions are still unsolved.

Question 1.1. *Consider the variance swaps pricing model with stochastic volatility in [119]. Can it be extended to a pricing model with stochastic interest rate?*

Question 1.2. *How can we evaluate discretely-sampled variance swaps as a hybrid model of stochastic volatility and stochastic interest rate using an approach similar to that in [119]?*

Question 1.3. *Consider the hybrid model in [55] which determines approximations for non-affine terms and gives approximation formulas for pricing European options. What are the mathematical and economical outcomes for our Heston-CIR hybrid model for pricing variance swaps?*

Question 1.4. *Related to Question 1.3, how do we derive a result for our pricing model with a full correlation structure between the underlyings?*

Question 1.5. *Consider the volatility derivatives pricing model with stochastic volatility and regime switching in [34]. How can our pricing model be extended by incorporating stochastic volatility and stochastic interest rate driven by a continuous-time regime switching Markov chain?*

1.4 Thesis Contributions and Organization

The contributions of this thesis can be expressed in answering the questions given in Section 1.3. These answers are included in subsequent chapters, which are organized as follows.

Chapter 2: This chapter is designed to introduce the mathematical preliminaries and financial terminologies in order to study the research questions proposed previously. Here, we present some notions and results on stochastic calculus, probability theory and others, which will be used frequently in Chapter 3 to Chapter 5. In addition, some financial terminologies are introduced in order to extend simple models into more complicated scenarios, followed by some aspects regarding numerical simulation in the last section.

Chapter 3: This chapter investigates models and pricing formulas for variance swaps with discrete-sampling times. Previous variance swap pricing models did not consider

the incorporation of stochastic interest rate in their pricing evaluations. We extend the foundation provided by Zhu and Lian on pricing variance swaps with stochastic volatility in [119], to a case by incorporating stochastic interest rate. This results in a hybrid model of stochastic volatility and stochastic interest rate, which provides solutions to Question 1.1 and Question 1.2. Semi-closed form solutions for the fair strike values are obtained via derivation of characteristic functions. Numerical experiments are conducted to illustrate the significance of introducing stochastic interest rate into pricing variance swaps.

Chapter 4: The intention of this chapter is to evaluate discretely-sampled variance swaps as a hybrid model of stochastic volatility and stochastic interest rate with a full correlation structure. In the previous work by Grzelak and Oosterlee in [55], deterministic and stochastic projection techniques with nonzero correlation between processes were proposed to guarantee the affine property for hybrid models. We derive an efficient semi-closed form pricing formula for an approximation of the fully correlated pricing model and apply numerical implementation to examine its accuracy. We also discuss the impact of the correlations among the underlying, volatility and interest rate. This analysis provides solutions to Question 1.3 and Question 1.4.

Chapter 5: The aim of this chapter is to address the issue of pricing discretely-sampled variance swaps under stochastic volatility and stochastic interest rate with regime switching. We first extend the framework of [34] by incorporating stochastic interest rate into the Markov-modulated version of the stochastic volatility model. This hybrid model possesses parameters that switch according to a continuous-time observable Markov chain process which can be interpreted as the states of an observable macroeconomic factor. This result yields a solution to Question 1.5. Specifically, we extend the pricing model on variance swaps in Chapter 3 to the regime switching case, and discuss the outcomes related to the variance swaps pricing values. In addition, we also explain the economic consequences of incorporating regime switching into the Heston-CIR model.

Chapter 6: This is the last chapter of the thesis and is devoted to present the conclusion and some potential research directions.

1.5 Bibliographic Notes

PhD research findings are eminent not only for the purpose of contributing to the relevant field itself, but also to delivering the new knowledge to a wider scope of audiences.

This thesis is aimed to achieve those objectives via producing publications in leading journals, as well as participating in local and international conferences. The following papers are also listed as [19, 20, 91, 92] in Bibliography and have been written during Roslan's PhD study.

[A] Cao, J.; Lian, G.; Roslan, T. R. N. Pricing variance swaps under stochastic volatility and stochastic interest rate. *Appl. Math. Comput.* **277** (2016), 72–81.

[B] Cao, J.; Roslan, T. R. N.; Zhang, W. Pricing variance swaps in a hybrid model of stochastic volatility and stochastic interest rate with regime switching, submitted to *J. Comput. Appl. Math.*

[C] Roslan, T. R. N.; Zhang, W.; Cao, J. Valuation of discretely-sampled variance swaps under correlated stochastic volatility and stochastic interest rates. *Proceedings of the 8th International Conference on Applied Mathematics, Simulation and Modelling*, November 2014, Florence, Italy.

[D] Roslan, T. R. N.; Zhang, W.; Cao, J. Evaluation of variance swaps prices with stochastic volatility and stochastic interest rates under full correlation structure, submitted to *Appl. Stoch. Models Bus. Ind.*

In 2013, Roslan attended the 2013 Conference on Quantitative Methods in Finance, which took place in Sydney, Australia. This was her first international conference and she presented the paper [A]. Besides that, she presented paper [C] to the conference participants in the 8th International Conference on Applied Mathematics, Simulation and Modelling in Florence, Italy. Lastly, Roslan exhibited the results of the paper [D] in the 2014 AUT Mathematical Sciences Symposium, held in Auckland, New Zealand. All of these research discoveries provided a platform to enhance the research activities currently conducted at the university.

Chapter 2

Mathematical and Finance Preliminaries

In this chapter, some mathematical preliminaries and financial terminologies are introduced. These include basic notations, definitions and many important facts, which will be used in the subsequent chapters. Section 2.1 gives the mathematical foundations essential for building up the ground knowledge involved. These mathematical foundations include selected topics on stochastic calculus, probability theory, Markov chains and others. In Section 2.2, some key concepts in finance are introduced to assist in developing the necessary tools needed. In addition, some details regarding the Monte Carlo simulation which facilitates model evaluation and validation are described in Section 2.3.

2.1 Mathematical Techniques

In this section, some mathematical concepts and results which are used in our analysis are presented. Most of these are taken from [18, 82, 98]. In addition, the notions related to Markov chains are also described based on [38].

2.1.1 Stochastic Calculus

Stochastic calculus is recognized as the most powerful tool in financial mathematics based on the assumption that asset prices behave randomly under uncertainties. This random property may be affiliated as continuous-time, real-valued stochastic processes with infinite sample paths for each outcome. Each outcome is also assigned with a

number which results in a random variable $X = X(\omega)$ defined for the sample space Ω . First, we introduce some preliminaries of probability theory, starting with the following definitions of a σ -field and a probability measure.

Definition 2.1. Let Ω be a non-empty set, and let \mathcal{F} be a family of subsets of Ω . \mathcal{F} is called a σ -field provided that

- (i) The empty set \emptyset belongs to \mathcal{F} ;
- (ii) If a set $A \in \mathcal{F}$, then its complement $A^c \in \mathcal{F}$;
- (iii) If a sequence of sets $A_1, A_2, \dots \in \mathcal{F}$, then their union $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$.

The ordered pair (Ω, \mathcal{F}) is called a *measurable space*, and elements of \mathcal{F} are called *events*.

Definition 2.2. Let Ω be a non-empty set. The σ -field generated on Ω by a collection of subsets \mathcal{A} of Ω , denoted by $\sigma(\mathcal{A})$, is defined as

$$\sigma(\mathcal{A}) := \bigcap \{ \mathcal{G} : \mathcal{A} \subseteq \mathcal{G} \text{ and } \mathcal{G} \text{ is a } \sigma\text{-field on } \Omega \}.$$

This means that the σ -field generated by \mathcal{A} is the smallest σ -field that contains \mathcal{A} . The σ -field generated by the family of all open intervals on the set of real numbers \mathbb{R} is denoted by $\mathcal{B}(\mathbb{R})$. Sets in $\mathcal{B}(\mathbb{R})$ are called *Borel sets*.

Definition 2.3. A *probability measure* \mathbb{P} on a measurable space (Ω, \mathcal{F}) is a function that assigns a number in $[0, 1]$ to every set $A \in \mathcal{F}$ such that

- (i) $\mathbb{P}(\Omega) = 1$;
- (ii) If $\{A_n : n \geq 1\}$ is a sequence of disjoint sets in \mathcal{F} , then $\mathbb{P}(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mathbb{P}(A_n)$.

For each $A \in \mathcal{F}$, $\mathbb{P}(A)$ is the probability of A , and the triple $(\Omega, \mathcal{F}, \mathbb{P})$ is called a *probability space*. An event A is said to occur *almost surely* (abbreviated as a.s.) whenever $\mathbb{P}(A) = 1$.

Moving on, we present the notion related to probability spaces, random variables and other concepts involved.

Definition 2.4. A function $\xi : \Omega \rightarrow \mathbb{R}$ is called \mathcal{F} -measurable for a σ -field \mathcal{F} on Ω if $\xi^{-1}(B) \in \mathcal{F}$ for every Borel set $B \in \mathfrak{B}(\mathbb{R})$. For a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, an \mathcal{F} -measurable function $\xi : (\Omega, \mathcal{F}, \mathbb{P}) \rightarrow \mathbb{R}$ is called a *random variable*.

The σ -field $\sigma(\xi)$ generated by a random variable ξ is given by

$$\sigma(\xi) = \{\xi^{-1}(B) : B \in \mathfrak{B}(\mathbb{R})\},$$

which implies $\sigma(\xi) \subseteq \mathcal{F}$. Next, consider two events $A, B \in \mathcal{F}$ in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. These events are called *independent* if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B),$$

and the *conditional probability* of A given B is defined as

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}.$$

Two σ -fields \mathcal{G} and \mathcal{H} contained in \mathcal{F} are called *independent* if any two events $A \in \mathcal{G}$ and $B \in \mathcal{H}$ are independent. Two random variables ξ and η are called *independent* if for any Borel sets $A, B \in \mathfrak{B}(\mathbb{R})$, the events $\xi^{-1}(A)$ and $\xi^{-1}(B)$ are independent. If two integrable random variables $\xi, \eta : \Omega \rightarrow \mathbb{R}$ are independent, then they are uncorrelated, i.e., $\mathbb{E}(\xi\eta) = \mathbb{E}(\xi)\mathbb{E}(\eta)$, provided that the product $\xi\eta$ is also integrable. Finally, a random variable ξ is *independent* of a σ -field \mathcal{G} if the σ -fields $\sigma(\xi)$ and \mathcal{G} are independent.

One of the most important concepts in probability theory is conditional expectation. This concept is explained further by the following definitions and lemma:

Definition 2.5. The *conditional expectation* of any integrable random variable ξ given any event $B \in \mathcal{F}$ such that $\mathbb{P}(B) \neq 0$ is given by

$$\mathbb{E}[\xi|B] = \frac{1}{\mathbb{P}(B)} \int_B \xi \, d\mathbb{P}.$$

Lemma 2.1. [18] *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let \mathcal{G} be a σ -field contained in \mathcal{F} . If ξ is a \mathcal{G} -measurable random variable and for any $B \in \mathcal{G}$,*

$$\int_B \xi \, d\mathbb{P} = 0,$$

then $\xi = 0$ a.s..

Definition 2.6. Let ξ be an integrable random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let \mathcal{G} be a σ -field contained in \mathcal{F} . The conditional expectation of ξ given \mathcal{G} is defined to be a random variable $\mathbb{E}[\xi|\mathcal{G}]$ such that

- (i) $\mathbb{E}[\xi|\mathcal{G}]$ is \mathcal{G} -measurable;
- (ii) for any $A \in \mathcal{G}$, $\int_A \mathbb{E}[\xi|\mathcal{G}] d\mathbb{P} = \int_A \xi d\mathbb{P}$.

Based on Lemma 2.1 and the following Radon-Nikodým Theorem, the existence and uniqueness of the conditional expectation $\mathbb{E}[\xi|\mathcal{G}]$ is ensured.

Theorem 2.1. [18](Radon-Nikodým Theorem). *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let \mathcal{G} be a σ -field contained in \mathcal{F} . Then for any integrable random variable ξ , there exists a \mathcal{G} -measurable random variable ζ such that*

$$\int_A \zeta d\mathbb{P} = \int_A \xi d\mathbb{P}$$

for each $A \in \mathcal{G}$.

Some general properties of conditional expectations are listed below:

- (i) $\mathbb{E}[a\xi + b\zeta|\mathcal{G}] = a\mathbb{E}[\xi|\mathcal{G}] + b\mathbb{E}[\zeta|\mathcal{G}]$ (*linearity*);
- (ii) $\mathbb{E}[\mathbb{E}[\xi|\mathcal{G}]] = \mathbb{E}[\xi]$;
- (iii) $\mathbb{E}[\xi\zeta|\mathcal{G}] = \xi\mathbb{E}[\zeta|\mathcal{G}]$ if ξ is \mathcal{G} -measurable (*taking out what is known*);
- (iv) $\mathbb{E}[\xi|\mathcal{G}] = \mathbb{E}[\xi]$ if ξ is independent of \mathcal{G} (*an independent condition drops out*);
- (v) $\mathbb{E}[\mathbb{E}[\xi|\mathcal{G}]|\mathcal{H}] = \mathbb{E}[\xi|\mathcal{H}]$ if $\mathcal{H} \subseteq \mathcal{G}$ (*tower property*);
- (vi) If $\xi \geq 0$, then $\mathbb{E}[\xi|\mathcal{G}] \geq 0$ (*positivity*).

We now proceed with stochastic process which lies in the core of stochastic calculus.

Definition 2.7. A *stochastic process* on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a family of random variables $X(t)$ parameterized by $t \in T$, where $T \subseteq \mathbb{R}$. For each $\omega \in \Omega$, the function

$$T \ni t \mapsto X(t, \omega)$$

is called a *sample path* of $\{X(t) : t \in T\}$.

When $T = \mathbb{N}$, $\{X(t) : t \in T\}$ is classified as a stochastic process in *discrete time* (i.e., a sequence of random variables). When T is an interval in \mathbb{R} (typically, $T = [0, +\infty)$), $\{X(t) : t \in T\}$ is identified as a stochastic process in *continuous-time*. Brownian motions are continuous-time stochastic processes.

Definition 2.8. A stochastic process $\{W(t) : t \geq 0\}$ is called a *Brownian motion* if the following properties are fulfilled:

- (i) $W(0) = 0$ a.s.;
- (ii) $\{W(t) : t \geq 0\}$ has independent increments; that is, for $0 \leq t_1 < t_2 < \dots < t_n$, $W(t_2) - W(t_1), W(t_3) - W(t_2), \dots, W(t_n) - W(t_{n-1})$ are independent;
- (iii) for all $0 \leq s \leq t$, $W(t) - W(s)$ follows a normal distribution with mean 0 and variance $t - s$;
- (iv) $W(t)$ has continuous sample paths.

Martingales form an important class of stochastic processes. To give a precise definition of a martingale, we need the concept of a filtration.

Definition 2.9. Let (Ω, \mathcal{F}) be a measurable space. A collection $\{\mathcal{F}(t) : t \geq 0\}$ of sub- σ -fields of \mathcal{F} is called a *filtration* if $\mathcal{F}(s) \subseteq \mathcal{F}(t)$ for all $0 \leq s \leq t$.

This means that $\mathcal{F}(t)$ contains all events A that gives us the information at time t regarding the occurrence of A . The number of events A in $\mathcal{F}(t)$ will increase as t increases.

Definition 2.10. A stochastic process $\{X(t) : t \in T\}$ is called a *martingale* with respect to a filtration $\mathbb{F} = \{\mathcal{F}(t) : t \in T\}$ if

- (i) $X(t)$ is integrable for each $t \in T$;
- (ii) $\{X(t) : t \in T\}$ is adapted to \mathbb{F} , that is, $X(t)$ is $\mathcal{F}(t)$ -measurable for each $t \in T$;
- (iii) $X(s) = \mathbb{E}[X(t)|\mathcal{F}(s)]$ for every $s, t \in T$ with $s \leq t$.

A basic paradigm which is important in stochastic calculus is Itô formula pioneered by Kiyoshi Itô in 1940. To study this, we need to specify beforehand the concept of stochastic integrals. Given a fixed time T , let $\{X(t) : 0 \leq t \leq T\}$ be a stochastic process adapted to the filtration up to T , $\mathbb{F} = \{\mathcal{F}(t) : 0 \leq t \leq T\}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, such that

$$\mathbb{E} \left[\int_0^T X(t)^2 dt \right] < \infty. \quad (2.1)$$

By some mathematical manipulation and properties of Brownian motions, one can check that for $0 = t_0^n < t_1^n < \dots < t_n^n = T$,

$$\mathbb{E} \left[\left(\sum_{i=0}^{n-1} X(t_i) (W(t_{i+1}) - W(t_i)) \right)^2 \right] = \mathbb{E} \left[\sum_{i=0}^{n-1} X^2(t_i^n) (t_{i+1}^n - t_i^n) \right], \quad (2.2)$$

which converges to $\mathbb{E} \left[\int_0^T X(t)^2 dt \right]$. For each $n \geq 1$, let $0 = t_0^n < t_1^n < \dots < t_n^n = T$ be a partition of $[0, T]$ and $\delta_n = \max_{0 \leq i \leq n-1} (t_{i+1}^n - t_i^n)$. Assume $\delta_n \rightarrow 0$ as $n \rightarrow \infty$. The stochastic integral of $\{X(t) : 0 \leq t \leq T\}$ with respect to a Brownian motion $\{W(t) : 0 \leq t \leq T\}$ is defined as

$$\int_0^T X(t) dW(t) = \lim_{n \rightarrow \infty} \sum_{i=0}^{n-1} X(t_i^n) (W(t_{i+1}^n) - W(t_i^n)), \quad (2.3)$$

where the convergence in (2.3) is in probability.

From (2.2) we can get

$$\mathbb{E} \left[\left(\int_0^T X(t) dW(t) \right)^2 \right] = \mathbb{E} \left[\int_0^T X^2(t) dt \right]. \quad (2.4)$$

The stochastic integral also satisfies the martingale property, i.e., for any $0 \leq s < t \leq T$

$$\mathbb{E} \left[\int_0^t X(\tau) dW(\tau) \middle| \mathcal{F}(s) \right] = \int_0^s X(\tau) dW(\tau).$$

Theorem 2.2. [82] (Itô formula for Brownian motion). *Let $f(t, x)$ be a function for which the partial derivatives $f_t(t, x)$, $f_x(t, x)$ and $f_{xx}(t, x)$ are defined and continuous,*

and let $\{W(t) : t \geq 0\}$ be a Brownian motion. Then for every $t \geq 0$,

$$df(t, W(t)) = f_t(t, W(t))dt + f_x(t, W(t))dW(t) + \frac{1}{2}f_{xx}(t, W(t))dt. \quad (2.5)$$

Definition 2.11. Let $\{W(t) : t \geq 0\}$ be a Brownian motion, and let $\{\mathcal{F}(t) : t \geq 0\}$ be an associated filtration. An *Itô process* is a stochastic process of the form

$$X(t) = X(0) + \int_0^t \Psi(s)dW(s) + \int_0^t \Theta(s)ds,$$

where $X(0)$ is non-random, and $\Psi(s)$ along with $\Theta(s)$ are adapted stochastic processes.

Theorem 2.3. [82](Itô formula for Itô process). *Let $\{X(t) : t \geq 0\}$ be an Itô process, and let $f(t, x)$ be a function for which the partial derivatives $f_t(t, x)$, $f_x(t, x)$ and $f_{xx}(t, x)$ are defined and continuous. Then for every $t \geq 0$,*

$$\begin{aligned} df(t, X(t)) &= f_t(t, X(t))dt + f_x(t, X(t))\Psi(t)dW(t) + f_x(t, X(t))\Theta(t)dt \\ &\quad + \frac{1}{2}f_{xx}(t, X(t))\Psi^2(t)dt. \end{aligned} \quad (2.6)$$

2.1.2 Markov Chains

Definition 2.12. A stochastic process $\{X(t) : t \geq 0\}$ with a finite state space $S = \{s_1, s_2, \dots, s_N\}$ is said to be a *continuous-time Markov chain* if for all $t, s \geq 0$,

$$\mathbb{P}(X(t+s) = s_j | X(u) : 0 \leq u \leq s) = \mathbb{P}(X(t+s) = s_j | X(s)).$$

Definition 2.13. Let $p_{ij}(t) = \mathbb{P}(X(t) = s_j | X(0) = s_i)$ for all $s_i, s_j \in S$, and $P(t) = [p_{ij}(t)]_{s_i, s_j \in S}$. We call $p_{ij}(t)$ the *transition probability* from state s_i to state s_j at time t , and $P(t)$ is the *transition probability matrix* at time t .

The properties of the transition matrix are listed as follows:

- (i) $p_{ij}(t) \geq 0$ for all $s_i, s_j \in S$ and $t \geq 0$;
- (ii) $\sum_{s_j \in S} p_{ij}(t) = 1$ for all $s_i \in S$ and $t \geq 0$;
- (iii) $p_{ij}(t+s) = \sum_{s_k \in S} p_{ik}(t)p_{kj}(s)$ for all $t \geq 0, s \geq 0$ and $s_i, s_j \in S$.

Since we would like to have only finitely-many jumps in a finite time interval, we assume for small t , and $o(t)$ which is a quantity asymptotically negligible as $t \downarrow 0$ after dividing by t (formally $f(t) = o(t)$ as $t \downarrow 0$ if $f(t)/t \rightarrow 0$ as $t \downarrow 0$),

$$(iv) \quad 0 \leq p_{ij}(t) = o(t) \text{ for } i \neq j;$$

$$(v) \quad 0 \leq 1 - p_{ii}(t) = o(t);$$

so that for $i \neq j$,

$$q_{ij} = \left. \frac{\partial}{\partial t} p_{ij}(t) \right|_{t=0}$$

would be the *transition rate* from s_i to s_j . The transition rate can be defined as number of possible events causing transition in each state of the Markov chain which takes place with parameter q_{ij} for $i \neq j$. The $N \times N$ matrix $Q = [q_{ij}]_{1 \leq i, j \leq N}$ is called the *transition rate matrix*.

2.1.3 Equivalent Probability Measures

Let (Ω, \mathcal{F}) be a measurable space. Recall that two probability measures \mathbb{P} and \mathbb{Q} on (Ω, \mathcal{F}) are said to be equivalent provided that for any $A \in \mathcal{F}$, $\mathbb{P}(A) = 0$ if and only if $\mathbb{Q}(A) = 0$.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\{\mathcal{F}(t) : t \geq 0\}$ be a filtration. Suppose that Z is an almost surely positive random variable such that $\mathbb{E}^{\mathbb{P}}[Z] = 1$. We define \mathbb{Q} by

$$\mathbb{Q}(A) := \int_A Z(\omega) d\mathbb{P}(\omega) \quad \text{for all } A \in \mathcal{F}.$$

Then \mathbb{Q} is a probability measure generated by Z on (Ω, \mathcal{F}) . It can be easily checked that \mathbb{P} and \mathbb{Q} are equivalent probability measures. Moreover, \mathbb{P} and \mathbb{Q} are related by the formula

$$\mathbb{E}^{\mathbb{Q}}[X] = \mathbb{E}^{\mathbb{P}}[XZ].$$

We call Z the *Radon-Nikodým derivative* of \mathbb{Q} with respect to \mathbb{P} , written as

$$Z = \frac{d\mathbb{Q}}{d\mathbb{P}}.$$

The *Radon-Nikodým derivative process* $\{Z(t) : 0 \leq t \leq T\}$ is defined by

$$Z(t) = \mathbb{E}^{\mathbb{P}}[Z | \mathcal{F}(t)], \quad 0 \leq t \leq T.$$

In addition, $\{Z(t) : 0 \leq t \leq T\}$ is a martingale with respect to $\{\mathcal{F}(t) : 0 \leq t \leq T\}$, since for any $0 \leq s \leq t \leq T$,

$$\mathbb{E}^{\mathbb{P}}[Z(t)|\mathcal{F}(s)] = \mathbb{E}^{\mathbb{P}}[\mathbb{E}^{\mathbb{P}}[Z|\mathcal{F}(t)]|\mathcal{F}(s)] = \mathbb{E}^{\mathbb{P}}[Z|\mathcal{F}(s)] = Z(s).$$

Theorem 2.4. [98][One-dimensional Girsanov Theorem] *Let $\{W(t) : 0 \leq t \leq T\}$ be a Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\{\mathcal{F}(t) : 0 \leq t \leq T\}$ be a filtration for this Brownian motion. Let $\{\gamma(t) : 0 \leq t \leq T\}$ be an adapted process with respect to $\{\mathcal{F}(t) : 0 \leq t \leq T\}$. Define*

$$Z(t) = \exp\left(-\int_0^t \gamma(s)dW(s) - \frac{1}{2}\int_0^t \gamma^2(s)ds\right),$$

$$\widetilde{W}(t) = W(t) + \int_0^t \gamma(s)ds,$$

and assume that

$$\mathbb{E}^{\mathbb{P}}\left[\int_0^T \gamma^2(s)Z^2(s)ds\right] < \infty.$$

Set $Z = Z(T)$. Then $\mathbb{E}^{\mathbb{P}}[Z] = 1$, and under the equivalent probability measure \mathbb{Q} generated by Z , the process $\{\widetilde{W}(t) : 0 \leq t \leq T\}$ is a Brownian motion.

Theorem 2.5. [98][Multi-dimensional Girsanov Theorem] *Let $\{\mathbf{W}(t) : 0 \leq t \leq T\}$ be an n -dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\{\mathcal{F}(t) : 0 \leq t \leq T\}$ be its filtration. Let $\gamma(t)$ be an n -dimensional adapted process with respect to $\{\mathcal{F}(t) : 0 \leq t \leq T\}$. Define*

$$Z(t) = \exp\left(-\int_0^t \gamma(s) \cdot d\mathbf{W}(s) - \frac{1}{2}\int_0^t \|\gamma(s)\|^2 ds\right),$$

$$\widetilde{\mathbf{W}}(t) = \mathbf{W}(t) + \int_0^t \gamma(s)ds,$$

and assume that

$$\mathbb{E}^{\mathbb{P}}\left[\int_0^T \|\gamma(s)\|^2 Z(s)^2 ds\right] < \infty.$$

Set $Z = Z(T)$. Then $\mathbb{E}^{\mathbb{P}}(Z) = 1$, and under the probability measure \mathbb{Q} generated by Z , the process $\{\widetilde{\mathbf{W}}(t) : 0 \leq t \leq T\}$ is an n -dimensional Brownian motion.

2.1.4 Feynman-Kac Theorem

In stochastic calculus, the Feynman-Kac Theorem establishes a relationship between stochastic differential equations and partial differential equations. In this subsection, we give two versions of the Feynman-Kac Theorem.

Theorem 2.6. [98][One-dimensional case] *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\{X(t) : t \geq 0\}$ be a stochastic process satisfying the following one-dimensional stochastic differential equation*

$$dX(t) = a(X(t), t)dt + b(X(t), t)dW(t),$$

where $\{W(t) : 0 \leq t \leq T\}$ is a one-dimensional Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$. Let $h(x)$ be a Borel-measurable function and r be a constant representing the interest rate. Fix $T > 0$, for $t \in [0, T]$, $X(t) = x$, define $F(x, t)$ as the following conditional expectation,

$$F(x, t) = \mathbb{E}^{\mathbb{P}} \left[e^{-r(T-t)} h(x(T)) \middle| \mathcal{F}(t) \right]. \quad (2.7)$$

Then $F(x, t)$ satisfies the following partial differential equation

$$\frac{\partial F}{\partial t} + a(x, t) \frac{\partial F}{\partial x} + \frac{1}{2} b(x, t)^2 \frac{\partial^2 F}{\partial x^2} - rF(x, t) = 0, \quad (2.8)$$

subject to the terminal condition $F(x, T) = h(x)$ for all x .

Theorem 2.7. [98][Multi-dimensional case] *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\{\mathbf{X}(t) : t \geq 0\}$ be an n -dimensional stochastic process satisfying the following n -dimensional stochastic differential equation*

$$d\mathbf{X}(t) = \mathbf{a}(\mathbf{X}(t), t)dt + \mathbf{b}(\mathbf{X}(t), t)d\mathbf{W}(t), \quad (2.9)$$

where $\{\mathbf{W}(t) : 0 \leq t \leq T\}$ is an m -dimensional Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$, $\mathbf{a}(\mathbf{X}(t), t)$ is an n -dimensional vector and $\mathbf{b}(\mathbf{X}(t), t)$ is an $n \times m$ matrix. Let $h(\mathbf{x})$ be a Borel-measurable function and r be a constant representing the interest rate. Fix $T > 0$, for $t \in [0, T]$, $\mathbf{X}(t) = \mathbf{x}$, define $F(\mathbf{x}, t)$ as the following conditional expectation,

$$F(\mathbf{x}, t) = \mathbb{E}^{\mathbb{P}} \left[e^{-r(T-t)} h(\mathbf{x}(T)) \middle| \mathcal{F}(t) \right]. \quad (2.10)$$

Then $F(x, t)$ satisfies the following partial differential equation

$$\frac{\partial F}{\partial t} + \sum_{i=1}^n a_i(x, t) \frac{\partial F}{\partial x_i} + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n (\mathbf{b}(\mathbf{x}, t) \mathbf{b}(\mathbf{x}, t)^T)_{ij} \frac{\partial^2 F}{\partial x_i \partial x_j} - rF(\mathbf{x}, t) = 0, \quad (2.11)$$

subject to the terminal condition $F(\mathbf{x}, T) = h(\mathbf{x})$ for all \mathbf{x} .

2.1.5 Fourier Transform

The Fourier transform is an important tool in Mathematics, particularly for solving differential equations. In this subsection we recall the concept of generalized Fourier transform in [13].

Definition 2.14. Let $f(x)$ be a function defined on \mathbb{R} . The *generalized Fourier transform* \hat{f} of f is defined to be

$$\hat{f}(\omega) = \mathfrak{F}[f(x)] = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx,$$

with $i = \sqrt{-1}$ and ω being the Fourier transform variable.

On the other hand, the generalized inverse Fourier transform is useful for retrieving the original function before the transformation.

Definition 2.15. The *generalized inverse Fourier transform* is given by

$$f(x) = \mathfrak{F}^{-1}[\hat{f}(\omega)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x} d\omega.$$

Note that the Fourier transformation of the function $e^{i\xi x}$ is

$$\mathfrak{F}[e^{i\xi x}] = 2\pi \delta_{\xi}(\omega), \quad (2.12)$$

where ξ is any complex number and $\delta_{\xi}(\omega)$ is the generalized delta function satisfying

$$\int_{-\infty}^{\infty} \delta_{\xi}(x) \Phi(x) dx = \Phi(\xi). \quad (2.13)$$

The Fourier transform of a probability density function f for a random variable X is called its *characteristic function* which is represented as an expectation:

$$f(\omega) = \mathbb{E}[e^{i\omega X}]. \quad (2.14)$$

Qualitative properties of a particular distribution such as its volatility, skewness and kurtosis could be retrieved from its characteristic function since it aids in obtaining moments of random variable by taking derivatives at initial point $\omega = 0$.

Some leading numerical techniques in Fourier transforms are known as the Fast Fourier Transform (FFT) and Fourier space time stepping technique. Focusing mostly on Levy processes, the FFT carries the supremacy of incorporating numerous uncertainties in the volatility and correlation, being dependent only on the number of underlying assets. Hence, the common obstacle of increasing dimensionality as the ones in finite-difference and lattice tree methods can be avoided. In addition, FFT also grants higher efficiency since it has vast number of branches, and requires less operation time. In contrast, the Fourier space time stepping technique is convenient for figuring out partial differential integral equation for Levy processes too, which are beneficial for path-dependent asset prices and constraints in option pricing models.

2.1.6 Cholesky Decomposition

The Cholesky decomposition is a tool which is commonly used in linear algebra to factorize a positive definite matrix into a lower triangular matrix and its conjugate transpose. Named after André-Louis Cholesky, given the covariances or correlation between variables, we can perform an invertible linear transformation that de-correlates the variables. Conversely, a set of uncorrelated variables can also be transformed into variables with given covariances. First, we have to ensure that for a given square matrix $A = [a_{ij}]_{1 \leq i, j \leq n}$, A is symmetric meaning that $a_{ij} = a_{ji}$ for all $1 \leq i, j \leq n$. This symmetric matrix is defined as positive definite if $x^T A x > 0$ for any non-zero vector x of real numbers.

The following result can be found in [50].

Proposition 2.1. [50] *Let $A = [a_{ij}]_{1 \leq i, j \leq n}$ be a real symmetric matrix. If A is positive definite, then there is a unique lower triangular matrix C with strictly positive diagonal entries such that $A = C C^T$.*

In Proposition 2.1, CC^\top is called the *Cholesky decomposition* of A . The above proposition can be illustrated through the following matrices, where the coefficients for both sides of the equation are calculated:

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} = \begin{bmatrix} c_{11} & 0 & \dots & 0 \\ c_{21} & c_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \dots & c_{nn} \end{bmatrix} \begin{bmatrix} c_{11} & c_{21} & \dots & c_{n1} \\ 0 & c_{22} & \dots & c_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & c_{nn} \end{bmatrix}.$$

Solving for the unknowns which is the non-zeros results in the following

$$c_{ii} = \sqrt{\left(a_{ii} - \sum_{k=1}^{i-1} c_{ik}^2\right)} \text{ and } c_{ji} = \left(a_{ji} - \sum_{k=1}^{i-1} c_{jk}c_{ik}\right) / c_{ii},$$

where the expression under the square-root is always positive, and all c_{ij} are real since A is symmetric and positive definite.

The Cholesky decomposition can be performed on covariance or correlated matrices based on their positive definiteness property. Let $\mathbf{X} = [X_1, \dots, X_n]^\top$ be an n -dimensional random vector such that $\mathbb{E}[X_i] = 0$ and $\text{Var}[X_i] = \sigma_i^2 > 0$ for all $i = 1, \dots, n$. The correlation coefficient of X_i and X_j for any $1 \leq i, j \leq n$ is given by $\rho_{ij} = \frac{\mathbb{E}[X_i X_j]}{\sigma_i \sigma_j}$. Denote $\Sigma = [\rho_{ij}]_{1 \leq i, j \leq n}$ as the matrix of correlation coefficient. Then it also can be checked that Σ is positive definite.

Applications of the Cholesky decomposition can be seen in solving the normal equations of least squares to produce coefficient estimates in multiple regression analysis. In mathematical finance, the Cholesky decomposition is commonly applied in the Monte Carlo simulation for simulating systems with multiple correlated variables.

2.2 Finance

In this section, some preliminaries regarding option pricing and the Black-Scholes model are presented. Further on, some concepts on forward measure, variance swaps, stochastic volatility and stochastic interest rate which are required in the forthcoming chapters are discussed. Detailed explanations can be found in [15, 108, 113].

2.2.1 The Black-Scholes Model and Risk-Neutral Pricing

Let $\{W(t) : 0 \leq t \leq T\}$ be a Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\{\mathcal{F}(t) : 0 \leq t \leq T\}$ be a filtration for this Brownian motion. Here $T > 0$ is a fixed final time. In the Black-Scholes model, we consider the prices $\{S(t) : 0 \leq t \leq T\}$ of a stock as a stochastic process that satisfies the following stochastic differential equation

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t), \quad 0 \leq t \leq T, \quad (2.15)$$

where μ is the drift rate of $S(t)$ and σ is the volatility of $S(t)$. Both μ and σ are assumed to be constant. Applying Itô's lemma, we can solve equation (2.15) and derive the following formula

$$S(t) = S(0) \exp \left(\left(\mu - \frac{1}{2} \sigma^2 \right) t + \sigma W(t) \right). \quad (2.16)$$

In addition, let $B(t)$ be the value of a bank account at time $t \geq 0$. Assume $B(0) = 1$ and that the bank account evolves according to the following differential equation

$$dB(t) = r(t)B(t)dt, \quad B(0) = 1, \quad (2.17)$$

where $r(t)$ is a positive function of time. Here, $r(t)$ is the instantaneous rate at which the bank account accrues, usually referred to as *instantaneous spot rate*, or briefly as *short rate*. As a consequence,

$$B(t) = \exp \left(\int_0^t r(s)ds \right).$$

Furthermore, the present value of \$1 at time $t > 0$ is given by

$$D(t) = \exp \left(- \int_0^t r(s)ds \right),$$

which is called the *discount factor*.

In finance, a *derivative* can be defined as a security whose value depends on the value of a more basic underlying variable, such as interest rates, commodity prices, stock indices or other traded securities. Options are special types of financial derivatives. Recall that a *European call (put) option* is a contract which gives holder the right, but not obligation, to buy (sell) the underlying asset at an agreed fixed price K (called

the *strike price*) which will be exercised on the expiration date T (called the *maturity time*).

Consider a financial market consisting of a stock (risky investment) whose price follows (2.15), a bank account (risk-free investment) whose value follows (2.17), and an option (on the stock) whose value is $c(t)$. An investor in our Black-Scholes market can form a portfolio from the three investment alternatives: $x(t)$ number of stocks, $y(t)$ number of bonds and $z(t)$ number of options at time t . We call (x, y, z) the *portfolio strategy*, and the value of the portfolio at time t is

$$V(t) = x(t)S(t) + y(t)B(t) + z(t)c(t).$$

Definition 2.16. A portfolio strategy (x, y, z) is called *self-financing* if

$$dV(t) = x(t)dS(t) + y(t)dB(t) + z(t)dc(t),$$

which can be re-written in the integral form as

$$V(t) = V(0) + \int_0^t x(s)dS(s) + \int_0^t y(s)dB(s) + \int_0^t z(s)dc(s).$$

It is clear that if the portfolio is self-financing, then the change in portfolio value is equal to the stock position times the change of stock price added to the bank account position times the change of bank account, plus the option position times the change of option price.

Definition 2.17. A self-financing portfolio strategy is called an *arbitrage opportunity* if $V(0) = 0$, $V(T) \geq 0$ and $\mathbb{E}^{\mathbb{P}}[V(T)] > 0$.

Intuitively, an arbitrage opportunity can be defined as a self-financing trading strategy requiring no initial investment, having zero probability of negative value at expiration, and yet having some possibility of a positive terminal payoff.

Now, we consider the discounted stock price process $\{D(t)S(t) : 0 \leq t \leq T\}$. Its differential is given by

$$\begin{aligned} d(D(t)S(t)) &= (\mu - r(t))D(t)S(t)dt + \sigma D(t)S(t)dW(t) \\ &= \sigma D(t)S(t) \left(\frac{\mu - r(t)}{\sigma} dt + dW(t) \right). \end{aligned} \tag{2.18}$$

Applying the Girsanov theorem, there exists a probability measure \mathbb{Q} equivalent to \mathbb{P} such that $\{\widetilde{W}(t) : 0 \leq t \leq T\}$ is a Brownian motion with respect to \mathbb{Q} , where

$$\widetilde{W}(t) = W(t) + \frac{\mu - r(t)}{\sigma} t.$$

This means that the discounted stock price process $\{D(t)S(t) : 0 \leq t \leq T\}$ is a martingale under \mathbb{Q} . The measure \mathbb{Q} is called the *risk-neutral measure*. Moreover, it can be verified that the discounted value process of a self-financing portfolio $\{D(t)V(t) : 0 \leq t \leq T\}$ is also a martingale under \mathbb{Q} .

A *contingent T-claim* is a financial contract that pays the holder a random amount at time T . Suppose that $f(S(T))$ is the payoff of a European option at the maturity time T . One of the objectives is to find self-financing portfolios which can replicate the claim with an investment in the stock and the bank account. Such portfolios are called *hedging portfolios*, and in this case, the claim is called *attainable*. The value of such portfolios will be denoted by $H(t)$, and for a given strategy (x^H, y^H) , we have

$$H(t) = x^H(t)S(t) + y^H(t)B(t).$$

In order for H to be a hedging portfolio, we must have $V(T) = f(S(T))$ almost surely. We call the market *complete*, if all contingent claims in the market are attainable.

In the Black-Scholes model, it is assumed that the market is complete and arbitrage free. Next, we provide two fundamental results that link arbitrage, risk-neutral probability measure and market completeness.

Theorem 2.8. [98] (First Fundamental Theorem of Asset Pricing). *A market model is arbitrage-free if and only if it has at least one risk-neutral probability measure.*

Theorem 2.9. [98] (Second Fundamental Theorem of Asset Pricing). *Consider a market model that has a risk-neutral probability measure. The model is complete if and only if the risk-neutral probability measure is unique.*

For a European call option with strike price K and maturity time T , its payoff at maturity is

$$f(S(T)) = \max\{S(T) - K, 0\}.$$

The fact that $\{D(t)H(t) : 0 \leq t \leq T\}$ is a martingale under \mathbb{Q} implies that

$$D(t)H(t) = \mathbb{E}^{\mathbb{Q}}[D(T)H(T)|\mathcal{F}(t)] = \mathbb{E}^{\mathbb{Q}}[D(T)f(S(T)) | \mathcal{F}(t)].$$

The arbitrage-free assumption forces $H(t) = c(t)$. By dividing the discount factor $D(t)$, we may write $c(t)$ as

$$c(t) = \mathbb{E}^{\mathbb{Q}} \left[e^{-\int_t^T r(s)ds} f(S(T)) \mid \mathcal{F}(t) \right], \quad 0 \leq t \leq T.$$

This is the *risk-neutral pricing formula*. In addition, if $r(t)$ is constant, then it can be shown that

$$c(t) = S(T)\mathcal{N}(d_1) - Ke^{-r(T-t)}\mathcal{N}(d_2), \quad (2.19)$$

with

$$d_1 = \frac{\log(S(T)/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{(T-t)}},$$

and

$$d_2 = d_1 - \sigma\sqrt{(T-t)},$$

where $\mathcal{N}(\cdot)$ denotes the cumulative distribution function of standard normal distribution. Equation (2.19) is called the *Black-Scholes pricing formula* for a European call option. Using the same argument, the price $p(t)$ of a European put option can be expressed as

$$p(t) = Ke^{-r(T-t)}\mathcal{N}(-d_2) - S(T)\mathcal{N}(-d_1).$$

Following the Feynman-Kac theorem, the price $c(t)$ of a European call option satisfies the following PDE

$$\frac{\partial c}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 c}{\partial S^2} + rS \frac{\partial c}{\partial S} - rc = 0. \quad (2.20)$$

In the literature, equation (2.20) is called the Black-Scholes PDE.

2.2.2 Numéraire, Forward Measures and Variance Swaps

In finance, a *numéraire* is any positive non-dividend paying asset. The following result, taken from [15] (refer to Proposition 2.2.1 in [15]), provides a fundamental tool for the pricing of derivatives to any numéraire.

Proposition 2.2. [15] *Assume there exists a numéraire N and a probability measure \mathbb{Q}^N , equivalent to the initial \mathbb{Q} , such that the price of any traded asset X (without intermediate payments) relative to N is a martingale under \mathbb{Q}^N , that is,*

$$\frac{X(t)}{N(t)} = \mathbb{E}^{\mathbb{Q}^N} \left[\frac{X(T)}{N(T)} \mid \mathcal{F}(t) \right], \quad 0 \leq t \leq T. \quad (2.21)$$

Let U be an arbitrary numéraire. Then there exists a probability measure \mathbb{Q}^U , equivalent to \mathbb{Q} , such that the price of any attainable claim Y normalized by U is a martingale under \mathbb{Q}^U , that is,

$$\frac{Y(t)}{U(t)} = \mathbb{E}^{\mathbb{Q}^U} \left[\frac{Y(T)}{U(T)} \middle| \mathcal{F}(t) \right], \quad 0 \leq t \leq T. \quad (2.22)$$

Moreover, the Radon-Nikodým derivative defined by \mathbb{Q}^U is given by

$$\frac{d\mathbb{Q}^U}{d\mathbb{Q}^N} = \frac{U(T)N(0)}{U(0)N(T)}.$$

A *zero-coupon bond* with the maturity time T is a contract between two parties, namely the holder and the writer, that guarantees its holder the payment of one unit of currency at time T , with no intermediate payments. The contract value at time $t < T$ is denoted by $P(t, T)$. Clearly, $P(T, T) = 1$ for all $T > 0$.

Under the arbitrage-free assumption, we know that there is a risk-neutral probability measure \mathbb{Q} for the market. Now, take a zero-coupon bond as the numéraire U in Proposition 2.2, and let \mathbb{Q}^T denote the corresponding probability measure, equivalent to \mathbb{Q} . Realizing that $P(T, T) = 1$ eliminates the dependence on the discount process, we obtain

$$Y(t) = P(t, T) \mathbb{E}^T[Y(T) | \mathcal{F}(t)], \quad 0 \leq t \leq T. \quad (2.23)$$

We call \mathbb{Q}^T the *T-forward measure*. It can be verified that the expectation of a future instantaneous spot rate $r(T)$ under \mathbb{Q}^T is equal to the related *instantaneous forward rate* $f(t, T)$, that is,

$$\mathbb{E}^T[r(T) | \mathcal{F}(t)] = f(t, T)$$

for each $0 \leq t \leq T$. For details, refer to [15].

In general, there is more complexity involved in pricing interest rate derivatives because the payoff functions depend on interest rates at multiple time points. Furthermore, the volatilities of these interest rates may differ due to the different periods involved comprising from the short-term rates to the long-term rates. Whenever stochastic interest rates are present, there exists a joint dynamism between the underlying asset price and interest rates in the pricing procedure. The use of the forward measure allows taking out the discounting effect from the joint evolution of the asset price and interest rates, when zero-coupon bonds are used as numéraires.

A *forward contract* is an agreement between two parties, namely the holder and

the writer, where the holder agrees to buy an asset from the writer at delivery time T in the future for a pre-determined delivery price K . In this transaction, no up-front payment occurs. The delivery price is chosen so that the value of the forward contract to both parties is zero when the contract starts. The holder assumes a long position, and the writer assumes a short position.

A variance swap is a forward contract on the future realized variance of the returns of a specified asset. Offering additional purpose in determining the payoff of the financial derivative, this gives extra credit apart from the advantage of avoiding direct exposures to itself. Since the payment of a variance swap is only made in a single fixed payment at maturity, it is defined as a forward contract which is traded over the counter. At maturity time T , a variance swap rate can be evaluated as $V(T) = (RV - K) \times L$, where K is the annualized delivery or strike price for the variance swap and L is the notional amount of the swap in dollars. Roughly speaking, the *realized variance* (RV) is the sum of squared returns. It provides a relatively accurate measure of volatility which is useful for many purposes, including volatility forecasting and forecast evaluation. The formula for the measure of realized variance used in this thesis and several other authors [80, 119] is

$$RV = \frac{AF}{N} \sum_{j=1}^N \left(\frac{S(t_j) - S(t_{j-1})}{S(t_{j-1})} \right)^2 \times 100^2, \quad (2.24)$$

whereas in the market, a typical measure of the realized variance is defined as

$$RV = \frac{AF}{N} \sum_{j=1}^N \left(\ln \frac{S(t_j)}{S(t_{j-1})} \right)^2 \times 100^2. \quad (2.25)$$

The formula in (2.24) is known as the actual return realized variance, and the formula in (2.25) is recognizable as the log return realized variance. The formula in (2.25) had also been used extensively in the literature, such as in [116] and [117]. Several authors also used both definitions in their research, for example [34]. Here, $S(t_j)$ is the closing price of the underlying asset at the j -th observation time t_j , T is the lifetime of the contract and N is the number of observations. AF is the annualized factor which follows the sampling frequency to convert the above evaluation to annualized variance points. Assuming there are 252 business days in a year, $AF = 252$ for every trading day sampling frequency. However if the sampling frequency is every month or every week, then AF will be 12 and 52 respectively. The measure of realized variance requires

monitoring the underlying price path discretely, usually at the end of each business day. For this purpose, we assume equally discrete observations to be compatible with the real market, which reduces to $AF = \frac{1}{\Delta t} = \frac{N}{T}$.

The long position of variance swaps pays a fixed delivery price at the expiration and receives the floating amounts of the annualized realized variance, whereas the short position is the opposite. The notional amount L can be expressed in two terms which are variance notional and vega notional. Variance notional gives the dollar amount of profit or loss obtained from the difference of one point between the realized variance and the delivery price. In contrast, vega notional calculates the profit or loss from one point of change in volatility points. Since it is the market practice to define the variance notional in volatility terms, the notional amount is typically quoted in dollars per volatility point. Even though the vega notional is the common market practice, this does not rise any complication due to the square-root relationship between the variance and volatility.

Generally, short position holders are mostly drawn to the irresistible attributes of variance swaps since the implied volatility is likely to be bigger than the final realized volatility. Moreover, the convexity property allows strike prices to be more expensive than the ones acquired from fair volatility. In addition, variance swaps also offer the capability to record the volatility trends of two correlated indices.

2.2.3 The CIR Model

Even though the Black-Scholes model has been widely recognized as the foundation for practitioners and researchers in the option pricing world, it contains several limitations which induce modelling difficulties. The constant volatility and constant instantaneous interest rate setting are not consistent with real market observations. Previous literatures revealed that implied volatility, volatility clustering and fat tail distribution are common market realities, and these phenomena are absolutely in contrast to the Black-Scholes assumptions ([3],[59],[75]). Starting from this point, many studies have been done on improving the Black-Scholes formula (see [74]) by proposing stochastic interest rate models, introducing jumps components, models with Levy processes and etc.

In this section, a review on the Cox-Ingersoll-Ross (CIR) stochastic interest rate model will be given. Since it is categorized in the class of short-rate models, it also possesses common characteristics such as continuous-time diffusion dynamics as well as the mean-reverting property. Let $\{r(t) : t \geq 0\}$ be the process of interest rate. In 1985,

Cox, Ingersoll and Ross [27] came up with the following model which discussed specifically the effects of anticipation, risk-aversion, investment alternative and consumption timing preferences towards a competitive economy in a continuous-time setting:

$$dr(t) = \alpha[\beta - r(t)]dt + \eta\sqrt{r(t)}dW(t) \quad (2.26)$$

with positive constants α , β and η , where $\{W(t) : t \geq 0\}$ is a standard Brownian motion. One remarkable feature of this model is its capability to ensure positive interest rates due to the nature of square-root term in the short rate dynamics. The condition to avoid possibility of negative rates however can only be achieved by restricting $2\alpha\beta > \eta^2$. This is the main advantage of CIR model compared with the earlier models, besides resembling the tangible condition of interest rates in market. It is also worth to mention that the expectation and the variation for the CIR process can be expressed respectively as

$$\mathbb{E}[r(t)|r(0)] = r(0)e^{-\alpha t} + \beta(1 - e^{-\alpha t}), \quad (2.27)$$

$$\text{Var}[r(t)|r(0)] = r(0)\frac{\eta^2}{\alpha}(e^{-\alpha t} - e^{-2\alpha t}) + \frac{\beta\eta^2}{2\alpha}(1 - e^{-\alpha t})^2. \quad (2.28)$$

Under the arbitrage-free condition, the price $P(t, T)$ of a zero-coupon bond satisfies the following partial differential equation

$$\frac{\partial P}{\partial t} + \frac{1}{2}\eta^2 r \frac{\partial^2 P}{\partial r^2} + \alpha(\beta - r) \frac{\partial P}{\partial r} - rP = 0. \quad (2.29)$$

Assume that $P(t, T)$ has the affine form $P(t, T) = A(t, T)e^{-B(t, T)r(t)}$. Substituting $P(t, T)$ into (2.29), we can derive

$$A(t, T) = \frac{-2a}{\sigma^2} \ln \left[\frac{\gamma e^{1/2b(T-t)}}{\gamma \cosh(\gamma(T-t)) + \frac{1}{2}b \sinh(\gamma(T-t))} \right], \quad (2.30)$$

and

$$B(t, T) = \frac{\sinh(\gamma(T-t))}{\gamma(T-t) + \frac{1}{2}b \sinh(\gamma(T-t))}, \quad (2.31)$$

where $\gamma = \frac{1}{2}\sqrt{b^2 + 2\sigma^2}$, $\sinh u = \frac{e^u - e^{-u}}{2}$, and $\cosh u = \frac{e^u + e^{-u}}{2}$. For details, refer to [98].

Despite the ability to exhibit conditional volatility which depends on levels of short rate, the CIR model is not amenable to a closed form solution. However, the mean

and variance of interest rate at time t can be calculated explicitly if the initial condition of the interest rate is given. Thus, many numerical methods have been proposed in the literature including finite-difference method and discretization schemes. Other flaws of the model include inadequate flexibility to fit the market's term structure and intractability especially when multi-factor cases are involved.

2.2.4 The Heston Model

In general, stochastic volatility models are defined as models which assume the volatility of an asset as a random process. This is particularly important because volatility changes with both time and stock price levels in the real market. One important characteristic of these models is the capability to exhibit the smile curve. In fact, the constant volatility assumption in the Black-Scholes model should be ignored since stochastic volatility models could exhibit skewness property for the implied volatility ([3],[75]). Apart from that, there are also other advantages such as the ability to simulate fat-tail returns which result in proper modelling of intense asset prices as well as incorporation of market jumps.

Despite the attractive superiorities offered by stochastic volatility models, they also possess some impediments such as the difficulties arising in hedging and parameter evaluation problems. The volatility process has an individually independent random component which is imperfectly correlated to the Brownian motion term in its stochastic differential equation and leads to non-existence of a unique equivalent martingale measure. There are a few different driving processes of stochastic volatility models which distinguish them apart such as the lognormal process for the Hull-White model, the Ornstein-Uhlenbeck process for the Stein-Stein and Scott model, and the square-root process for the Heston model. The mean-reversion property held by Ornstein-Uhlenbeck and the square-root process are regarded as very enticing from the financial perspective since the implied volatility process also shares the same property.

The Heston model was introduced to remedy the pitfalls of the Black-Scholes model which did not consider skewness elements and was unsolvable analytically. Derivations of a closed form solution for the price of a European call was done by applying techniques of characteristic functions and imposing correlations between the stock price and volatility. This resulted in the probability density function for a continuously compounded return for a half-year period with prices for in and out-of-the-money options. To be more precise, the stock price $S(t)$ has the dynamics as follows, where the

instantaneous variance $\nu(t)$ follows a CIR process:

$$\begin{aligned} dS(t) &= \mu S(t)dt + \sqrt{\nu(t)}S(t)dW_1(t), \\ d\nu(t) &= \kappa[\theta - \nu(t)]dt + \sigma\sqrt{\nu(t)}dW_2(t), \end{aligned} \tag{2.32}$$

where $\{W_i(t) : t \geq 0\}$, $i = 1, 2$, is a standard Brownian motion, and $(dW_1(t), dW_2(t)) = \rho dt$ for $-1 \leq \rho \leq 1$. Also, the parameters μ represent the rate of return of asset, θ is the long-term variance, κ is the rate which $\nu(t)$ reverts to θ , and σ is the volatility of volatility.

A paramount discovery from this model was the significant effect of the correlation between the volatility and asset price towards the current price of asset consequent to exercise and current price of strike at payment time. These criteria could be measured in terms of skewness in gain and strike prices' intolerance when compared with the Black-Scholes model. Besides that, the parameters in the Heston model which do not exist in the Black-Scholes model assist in better market price predictions via monitoring the shape of the volatility curve, along with its behaviour over time. However, these parameters should be assessed with caution since it would give great impact on model fitting. Carmona and Nadtochiy [21] claimed that fitting performance towards the whole term structure of implied volatility was not adequately guaranteed based on incapability of replicating all market price strikes and maturities. Shamsutdinov [96] elaborated on the calibration procedures of the Heston's model parameters for the European options by utilizing the EURO STOXX 500 Index. Among the steps involved include choosing an appropriate error function and application of various optimization methods such as R for search of parameter setting.

2.3 Monte Carlo Simulation

It is common practice in finance to represent the price of a derivative security using the expected present value of the derivative's payoff according to the asset pricing theory. However, the expectations involved often do not have closed form formulas. The Monte Carlo simulation method comes in handy to find numerical approximations of the expectations. A general introduction about the Monte Carlo simulation method can be found in [47, 108].

Monte Carlo simulation is regarded as an ideal tool for pricing European style financial derivatives involving integrations which could not be derived analytically. This

is based on its capability to sample paths of different types of models of stochastic differential equations. Many authors, e.g. [17, 58, 78], have utilized the Monte Carlo simulation to price financial derivatives for the jump-diffusion, hybrid and two-factor interest rate models. The Monte Carlo simulation is a generic algorithm which generates a large number of sample paths according to the model under consideration, then computes the derivative payoff for each path in the sample. The average is then taken to find an approximation to the expected present value of the derivative. The Monte Carlo simulation result converges to the derivative value in the limit as the number of paths in the sample goes to infinity.

The main advantage of the Monte Carlo simulation is that it is usually easy to implement and can be used to evaluate a large range of European style derivative securities. Also, if enough sample paths are taken, then Monte Carlo simulation is reliable to give a good approximation of the value of the derivatives. Therefore, it is often used as a benchmark value for many complicated European style derivatives.

However, many authors, e.g. [77, 106], criticized the Monte Carlo simulation in terms of long computational time, high expense, and difficulty in calculating the Greeks. In addition, it is impractical for American options since American options involve option valuation at intermediate times between simulation start time and expiry time. Some breakthroughs have been discovered to improve the performance of Monte Carlo methods including implementing antithetic variables and control variate technique. These techniques contribute to increasing accuracy levels but do not guarantee any improvement in convergence speed.

In this thesis, we mainly use two schemes of the Monte Carlo simulation to generate sample paths of stochastic variables. Let $\{X(t) : t \geq 0\}$ be a general stochastic process which follows an autonomous stochastic differential equation:

$$dX(t) = \mu(X(t))dt + \sigma(X(t))dW(t), \quad X(0) = X_0,$$

where $\{W(t) : t \geq 0\}$ is a standard Brownian motion, $\mu(X(t))$ is the drift term of the process and $\sigma(X(t))$ is the volatility term of the process.

The Euler-Maruyama and Milstein schemes are constructed within the Itô integral framework. In particular, consider a basic discretization $\mathfrak{T} : \{t_0, t_1, \dots, t_N = T\}$ for interval $[t_0, T]$ with N sub-intervals of equal length and denote the interval length $\Delta t = \frac{T-t_0}{N}$. For the Euler-Maruyama scheme, we approximate the realization of the

stochastic variable $X(t)$ at time t_j as X_j , where

$$X_j = X_{j-1} + \mu(X_{j-1})\Delta t + \sigma(X_{j-1})(W(t_j) - W(t_{j-1})), \quad X_0 = X(0).$$

Even though the Euler-Maruyama scheme is user-friendly and relevant for most scenarios, its convergence speed is quite low since the error of the diffusion term is $o(\sqrt{\Delta t})$. This urges the need for a refinement of the scheme with emphasis on the diffusion term. The Milstein scheme reduces the error of the diffusion terms to $o(\Delta t)$ by adding additional terms to the Euler-Maruyama scheme. In particular, the realization of the stochastic variable at time t_j in the Milstein scheme is approximated by:

$$\begin{aligned} X_j &= X_{j-1} + \mu(X_{j-1})\Delta t + \sigma(X_{j-1})(W(t_j) - W(t_{j-1})) \\ &\quad + \frac{1}{2}\sigma(X_{j-1})\sigma'(X_{j-1})((W(t_j) - W(t_{j-1}))^2 - \Delta t), \quad X_0 = X(0). \end{aligned}$$

This results in increasing the rate of convergence for the Milstein scheme.

Chapter 3

Pricing Variance Swaps under Stochastic Factors : Partial Correlation Case

In this chapter, we investigate the problem on pricing discretely-sampled variance swaps under the framework of stochastic interest rate and stochastic volatility based on the Heston-CIR hybrid model. In Section 3.1, we first present some details regarding the Heston-CIR hybrid model. In Section 3.2, we extend the hybrid model in Section 3.1 to the specific case of pricing variance swaps. We do this by first implementing the change of measure to the model setup, followed by two computation steps to solve the corresponding equations. Finally, in Section 3.3, we discuss the effects of stochastic interest rate on the price of variance swaps and show some comparisons for the purpose of formula validation.

3.1 The Heston-CIR Model

In this section, we present a hybridization of the Heston stochastic volatility model in [66] and the Cox-Ingersoll-Ross (CIR) interest rate model in [27], which is widely known as the Heston-CIR model. The risk-neutral probability measure is being considered in this section.

As revealed by many empirical studies [59, 75], the classical Black-Scholes model in [10] may fail to reflect certain features of financial markets due to some unrealistic assumptions including the constant volatility and constant interest rate. To remedy these drawbacks of the Black-Scholes model, many models have been proposed by

academic researchers and practitioners to incorporate stochastic interest rate, jump diffusion and stochastic volatility [27, 31, 101]. Among stochastic volatility models, the one proposed by [66] has received a lot of attentions, since it gives a satisfactory description of the underlying asset dynamics [34, 37]. Recently, Zhu and Lian [119, 120] used Heston model to derive a closed form exact solution to the price of variance swaps. In this thesis, we move a step further by taking advantage of the hybridization between the Heston stochastic volatility model and the CIR interest rate model. Note that Heston-CIR hybrid models have been discussed and applied to the studies of pricing American options, the affine approximation pricing techniques with correlations, and the convergence of approximated prices using discretization methods, refer to [26, 55, 77]. The Heston-CIR hybrid model that we shall use in our framework can be described as follows

$$\begin{cases} dS(t) = \mu S(t)dt + \sqrt{\nu(t)}S(t)dW_1(t), 0 \leq t \leq T, \\ d\nu(t) = \kappa(\theta - \nu(t))dt + \sigma\sqrt{\nu(t)}dW_2(t), 0 \leq t \leq T, \\ dr(t) = \alpha(\beta - r(t))dt + \eta\sqrt{r(t)}dW_3(t), 0 \leq t \leq T, \end{cases} \quad (3.1)$$

where $r(t)$ is the stochastic instantaneous interest rate in which α determines the speed of mean reversion for the interest rate process, β is the long-term mean of the interest rate and η controls the volatility of the interest rate. In the stochastic instantaneous variance process $\nu(t)$, κ is its mean-reverting speed parameter, θ is its long-term mean and σ is its volatility. In order to ensure that the square root processes are always positive, it is required that $2\kappa\theta \geq \sigma^2$ and $2\alpha\beta \geq \eta^2$ respectively [27, 66]. Throughout this chapter, we assume that correlations involved in the above model are given by $(dW_1(t), dW_2(t)) = \rho dt$, $(dW_1(t), dW_3(t)) = 0$ and $(dW_2(t), dW_3(t)) = 0$, where $0 \leq t \leq T$, and ρ is a constant with $-1 < \rho < 1$. In Chapter 4, we shall consider the case with full correlation structure between all underlyings.

For any $0 \leq t \leq T$, let

$$\begin{aligned} Z(t) = & \exp \left[-\frac{1}{2} \int_0^t (\gamma_1(s))^2 ds - \int_0^t \gamma_1(s) dW_1(s) - \frac{1}{2} \int_0^t (\gamma_2(s))^2 ds \right. \\ & \left. - \int_0^t \gamma_2(s) dW_2(s) - \frac{1}{2} \int_0^t (\gamma_3(s))^2 ds - \int_0^t \gamma_3(s) dW_3(s) \right], \end{aligned}$$

where $\gamma_1(t) = \frac{\mu - r(t)}{\sqrt{\nu(t)}}$, $\gamma_2(t) = \frac{\lambda_1 \sqrt{\nu(t)}}{\sigma}$ and $\gamma_3(t) = \frac{\lambda_2 \sqrt{r(t)}}{\eta}$ are the market prices of risk (risk premium) of Brownian processes $\{W_1(t) : 0 \leq t \leq T\}$, $\{W_2(t) : 0 \leq t \leq T\}$ and $\{W_3(t) : 0 \leq t \leq T\}$, respectively. Here, λ_j ($j = 1, 2$) is the premium of volatility risk as illustrated in [66], where Breeden's consumption-based model is applied to yield

a volatility risk premium of the form $\lambda_j(t, S(t), \nu(t)) = \lambda_j \nu$ for the CIR square-root process, see [14].

Similar to that in [24, 26, 62], we define three processes $\widetilde{W}_1(t)$, $\widetilde{W}_2(t)$ and $\widetilde{W}_3(t)$ such that

$$\begin{cases} d\widetilde{W}_1(t) = dW_1(t) + \gamma_1(t)dt, 0 \leq t \leq T, \\ d\widetilde{W}_2(t) = dW_2(t) + \gamma_2(t)dt, 0 \leq t \leq T, \\ d\widetilde{W}_3(t) = dW_3(t) + \gamma_3(t)dt, 0 \leq t \leq T. \end{cases}$$

According to Girsanov theorem, $\mathbb{E}^{\mathbb{P}}[Z(T)] = 1$ and there exists a risk-neutral probability measure \mathbb{Q} equivalent to the real world probability measure \mathbb{P} such that $Z(t) = \frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}(t)}$ for all $0 \leq t \leq T$. In what follows, the conditional expectation at time t with respect to \mathbb{Q} is denoted by $\mathbb{E}^{\mathbb{Q}}[\cdot | \mathcal{F}(t)]$, where $\mathcal{F}(t)$ is the filtration up to time t . Under \mathbb{Q} , the system of equations (3.1) is transformed into the following form

$$\begin{cases} dS(t) = r(t)S(t)dt + \sqrt{\nu(t)}S(t)d\widetilde{W}_1(t), 0 \leq t \leq T, \\ d\nu(t) = \kappa^*(\theta^* - \nu(t))dt + \sigma\sqrt{\nu(t)}d\widetilde{W}_2(t), 0 \leq t \leq T, \\ dr(t) = \alpha^*(\beta^* - r(t))dt + \eta\sqrt{r(t)}d\widetilde{W}_3(t), 0 \leq t \leq T, \end{cases} \quad (3.2)$$

where $\kappa^* = \kappa + \lambda_1$, $\theta^* = \frac{\kappa\theta}{\kappa + \lambda_1}$, $\alpha^* = \alpha + \lambda_2$ and $\beta^* = \frac{\alpha\beta}{\alpha + \lambda_2}$ are the risk-neutral parameters, $\{\widetilde{W}_i(t) : 0 \leq t \leq T\}$ ($1 \leq i \leq 3$) is a Brownian motion process under \mathbb{Q} .

3.2 Pricing Variance Swaps under the Heston-CIR Model with Partial Correlation

In this section, we will derive a semi-closed form solution for the delivery price of variance swaps in a Heston-CIR hybrid model. The first part of this section consists of our solution techniques which involve two steps of computation and change of measure. For this purpose, a decomposition of the hybrid model will be demonstrated. The second and third parts of this section deal with solutions for two steps of computation. Our solutions extend the corresponding results in [119], where only stochastic volatility in the pricing model was considered.

3.2.1 Outline of the Solution Approach

Here, we demonstrate our techniques for pricing discretely-sampled variance swaps. Note that this solution outline is also applicable to the pricing case with full correlation

in Chapter 4. However, for the case with full correlation, we will need to make some adjustments in the part involving the change of measure. We will show how to handle these changes in the next chapter.

Previously in Chapter 2, it has been defined that at maturity time T , a variance swap rate can be evaluated as

$$V(T) = (RV - K) \times L, \quad (3.3)$$

where K is the annualized delivery price or strike price for the variance swap and L is the notional amount of the swap in dollars. In the risk-neutral world, the value of a variance swap with stochastic interest rate at time t is the expected present value of its future payoff with respect to \mathbb{Q} , that is,

$$V(t) = \mathbb{E}^{\mathbb{Q}} \left[e^{-\int_t^T r(s)ds} (RV - K) \times L | \mathcal{F}(t) \right]. \quad (3.4)$$

This value should be zero at $t = 0$ since it is defined in the class of forward contracts. The above expectation calculation involves the joint distribution of the interest rate and the future payoff which is complicated to be evaluated. Thus, it would be more convenient to use the bond price as the numeraire since the joint dynamics can be diminished by taking advantage of the property $P(T, T) = 1$.

Since the price of a T -maturity zero-coupon bond at $t = 0$ is given by

$$P(0, T) = \mathbb{E}^{\mathbb{Q}} \left[e^{-\int_0^T r(s)ds} | \mathcal{F}(0) \right], \quad (3.5)$$

we can determine the value of K by changing the risk-neutral measure \mathbb{Q} to the T -forward measure \mathbb{Q}^T . It follows that

$$\mathbb{E}^{\mathbb{Q}} \left[e^{-\int_0^T r(s)ds} (RV - K) \times L | \mathcal{F}(0) \right] = P(0, T) \mathbb{E}^T [(RV - K) \times L | \mathcal{F}(0)], \quad (3.6)$$

where $\mathbb{E}^T[(\cdot) | \mathcal{F}(0)]$ denotes the expectation under \mathbb{Q}^T with respect to $\mathcal{F}(0)$ at $t = 0$. Thus, the fair delivery price or the strike price of the variance swap is defined as $K = \mathbb{E}^T[RV | \mathcal{F}(0)]$, and we shall focus in finding this value for the rest of the chapter.

Under the T -forward measure, the valuation of the fair delivery price for a variance swap is reduced to calculating the N expectations expressed in the form of

$$\mathbb{E}^T \left[\left(\frac{S(t_j) - S(t_{j-1})}{S(t_{j-1})} \right)^2 \middle| \mathcal{F}(0) \right] \quad (3.7)$$

**3.2. PRICING VARIANCE SWAPS UNDER THE HESTON-CIR MODEL WITH
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for $t_0 = 0$, some fixed equal time period Δt and N different tenors $t_j = j\Delta t$ ($j = 1, \dots, N$). It is important to note that we have to consider two cases $j = 1$ and $j > 1$ separately. For the case $j = 1$, we have $t_{j-1} = 0$, and $S(t_{j-1}) = S(0)$ which is a known value at time $t_0 = 0$, instead of an unknown value of $S(t_{j-1})$ for any other cases with $j > 1$. In the process of calculating this expectation, j , unless otherwise stated, is regarded as a constant. Hence both t_j and t_{j-1} are regarded as known constants.

Based on the tower property of conditional expectations, the calculation of expectation (3.7) can be separated into two steps in the following form

$$\mathbb{E}^T \left[\left(\frac{S(t_j)}{S(t_{j-1})} - 1 \right)^2 \middle| \mathcal{F}(0) \right] = \mathbb{E}^T \left[\mathbb{E}^T \left[\left(\frac{S(t_j)}{S(t_{j-1})} - 1 \right)^2 \middle| \mathcal{F}(j-1) \right] \middle| \mathcal{F}(0) \right]. \quad (3.8)$$

For notational convenience, we denote the term

$$\mathbb{E}^T \left[\left(\frac{S(t_j)}{S(t_{j-1})} - 1 \right)^2 \middle| \mathcal{F}(j-1) \right] = E_{j-1}, \quad (3.9)$$

meaning that in the first step, the computation involved is to find E_{j-1} , and in the second step, we need to compute

$$\mathbb{E}^T [E_{j-1} | \mathcal{F}(0)]. \quad (3.10)$$

To this purpose, we implement the measure change from \mathbb{Q} to the T -forward measure \mathbb{Q}^T . In order to obtain the new dynamics for our SDEs in (3.2) under \mathbb{Q}^T , we need to find the volatilities for both numeraires respectively (refer [15]). Note that the numeraire under \mathbb{Q} is $N_{1,t} = e^{\int_0^t r(s) ds}$ and the numeraire under \mathbb{Q}^T is $N_{2,t} = P(t, T) = A(t, T)e^{-B(t, T)r(t)}$, where

$$A(t, T) = \left(\frac{2 \left(e^{(\alpha^* + \sqrt{(\alpha^*)^2 + 2\eta^2}) \frac{(T-t)}{2}} \right) \sqrt{(\alpha^*)^2 + 2\eta^2}}{2\sqrt{(\alpha^*)^2 + 2\eta^2} + \left(\alpha^* + \sqrt{(\alpha^*)^2 + 2\eta^2} \right) \left(e^{(T-t)\sqrt{(\alpha^*)^2 + 2\eta^2}} - 1 \right)} \right)^{\frac{2\alpha^* \beta^*(t)}{\eta^2}},$$

and

$$B(t, T) = \frac{2 \left(e^{(T-t)\sqrt{(\alpha^*)^2 + 2\eta^2}} - 1 \right)}{2\sqrt{(\alpha^*)^2 + 2\eta^2} + \left(\alpha^* + \sqrt{(\alpha^*)^2 + 2\eta^2} \right) \left(e^{(T-t)\sqrt{(\alpha^*)^2 + 2\eta^2}} - 1 \right)},$$

(see [15]). Differentiating $\ln N_{1,t}$ yields

$$d \ln N_{1,t} = r(t)dt = \left(\int_0^t \alpha^*(\beta^* - r(s))ds \right) dt + \left(\int_0^t \eta \sqrt{r(s)} d\widetilde{W}_3(s) \right) dt,$$

whereas the differentiation of $\ln N_{2,t}$ gives

$$d \ln N_{2,t} = \left[\frac{A'(t, T)}{A(t, T)} - B'(t, T)r(t) - B(t, T)\alpha^*(\beta^*(t) - r(t)) \right] dt - B(t, T)\eta \sqrt{r(t)} d\widetilde{W}_3(t).$$

It can be observed that now we have obtained the volatilities for both numeraires. We proceed by applying the Cholesky decomposition to our SDEs in (3.2), which can be re-written as

$$\begin{bmatrix} \frac{dS(t)}{S(t)} \\ d\nu(t) \\ dr(t) \end{bmatrix} = \mu^{\mathbb{Q}} dt + \Sigma \times C \times \begin{bmatrix} dW_1^*(t) \\ dW_2^*(t) \\ dW_3^*(t) \end{bmatrix}, \quad 0 \leq t \leq T, \quad (3.11)$$

with

$$\mu^{\mathbb{Q}} = \begin{bmatrix} r(t) \\ \kappa^*(\theta^* - \nu(t)) \\ \alpha^*(\beta^* - r(t)) \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \sqrt{\nu(t)} & 0 & 0 \\ 0 & \sigma \sqrt{\nu(t)} & 0 \\ 0 & 0 & \eta \sqrt{r(t)} \end{bmatrix}$$

and

$$C = \begin{bmatrix} 1 & 0 & 0 \\ \rho & \sqrt{1 - \rho^2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

such that

$$CC^T = \begin{bmatrix} 1 & \rho & 0 \\ \rho & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and $dW_1^*(t)$, $dW_2^*(t)$ and $dW_3^*(t)$ are mutually independent under \mathbb{Q} satisfying

$$\begin{bmatrix} d\widetilde{W}_1(t) \\ d\widetilde{W}_2(t) \\ d\widetilde{W}_3(t) \end{bmatrix} = C \times \begin{bmatrix} dW_1^*(t) \\ dW_2^*(t) \\ dW_3^*(t) \end{bmatrix}, \quad 0 \leq t \leq T.$$

Next, we show how to find μ^T which is the new drift for our SDEs under \mathbb{Q}^T by utilizing

the formula below

$$\mu^T = \mu^{\mathbb{Q}} - \left[\Sigma \times C \times C^T \times (\Sigma^{\mathbb{Q}} - \Sigma^T) \right], \quad (3.12)$$

with $\Sigma^{\mathbb{Q}}$ and Σ^T given by

$$\Sigma^{\mathbb{Q}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \Sigma^T = \begin{bmatrix} 0 \\ 0 \\ -B(t, T)\eta\sqrt{r(t)} \end{bmatrix},$$

along with Σ and CC^T as defined in (3.11). This results in the transformation of (3.11)

under \mathbb{Q} to the following system under \mathbb{Q}^T

$$\begin{bmatrix} \frac{dS(t)}{S(t)} \\ d\nu(t) \\ dr(t) \end{bmatrix} = \begin{bmatrix} r(t) \\ \kappa^*(\theta^* - \nu(t)) \\ \alpha^*\beta^* - [\alpha^* + B(t, T)\eta^2]r(t) \end{bmatrix} dt + \Sigma \times C \times \begin{bmatrix} dW_1^*(t) \\ dW_2^*(t) \\ dW_3^*(t) \end{bmatrix}, \quad 0 \leq t \leq T. \quad (3.13)$$

3.2.2 The First Step of Computation

As described in the previous subsection, our solution techniques involve finding solution for two steps. We shall first calculate E_{j-1} and consider a contingent claim $U_j(S(t), \nu(t), r(t), t)$ over $[t_{j-1}, t_j]$, whose payoff at expiry t_j is $\left(\frac{S(t_j)}{S(t_{j-1})} - 1\right)^2$. Applying the Feynman-Kac theorem, we can obtain a PDE for U_j over $[t_{j-1}, t_j]$ as

$$\begin{aligned} & \frac{\partial U_j}{\partial t} + \frac{1}{2}\nu S^2 \frac{\partial^2 U_j}{\partial S^2} + \frac{1}{2}\sigma^2 \nu \frac{\partial^2 U_j}{\partial \nu^2} + \frac{1}{2}\eta^2 r \frac{\partial^2 U_j}{\partial r^2} + \rho\sigma\nu S \frac{\partial^2 U_j}{\partial S \partial \nu} + rS \frac{\partial U_j}{\partial S} \\ & + \kappa^*(\theta^* - \nu) \frac{\partial U_j}{\partial \nu} + [\alpha^*\beta^* - (\alpha^* + B(t, T)\eta^2)r] \frac{\partial U_j}{\partial r} = 0 \end{aligned} \quad (3.14)$$

with the terminal condition

$$U_j(S(t_j), \nu, r, t_j) = \left(\frac{S(t_j)}{S(t_{j-1})} - 1\right)^2. \quad (3.15)$$

If the underlying asset follows the dynamic process (3.13), we can find the solution of PDE (3.14) with condition (3.15) by the generalized Fourier transform method. Its solution can be derived by the following general proposition.

Proposition 3.1. *If the underlying asset follows the dynamic process (3.13) and a European-style derivative written on this underlying has a payoff function $U(S, \nu, r, T) = H(S)$ at expiry T , then the solution to the associated PDE system of the derivative value*

$$\begin{cases} \frac{\partial U}{\partial t} + \frac{1}{2}\nu S^2 \frac{\partial^2 U}{\partial S^2} + \frac{1}{2}\sigma^2 \nu \frac{\partial^2 U}{\partial \nu^2} + \frac{1}{2}\eta^2 r \frac{\partial^2 U}{\partial r^2} + \rho\sigma\nu S \frac{\partial^2 U}{\partial S \partial \nu} + rS \frac{\partial U}{\partial S} \\ + \kappa^*(\theta^* - \nu) \frac{\partial U}{\partial \nu} + [\alpha^*\beta^* - (\alpha^* + B(t, T)\eta^2)r] \frac{\partial U}{\partial r} = 0, \\ U(S, \nu, r, T) = H(S) \end{cases} \quad (3.16)$$

can be expressed in semi-closed form as

$$U(x, \nu, r, \tau) = \mathfrak{F}^{-1} \left[e^{C(\omega, \tau) + D(\omega, \tau)\nu + E(\omega, \tau)r} \mathfrak{F}[H(e^x)] \right], \quad (3.17)$$

in terms of the generalized Fourier transform (see [88]), where $x = \ln S$, $\tau = T - t$, $i = \sqrt{-1}$, ω is the Fourier transform variable,

$$\begin{cases} D(\omega, \tau) = \frac{a + b}{\sigma^2} \frac{1 - e^{b\tau}}{1 - ge^{b\tau}}, \\ a = \kappa^* - \rho\sigma\omega i, \quad b = \sqrt{a^2 + \sigma^2(\omega^2 + \omega i)}, \quad g = \frac{a + b}{a - b}, \end{cases} \quad (3.18)$$

and $E(\omega, \tau)$ along with $C(\omega, \tau)$ satisfy the following system of ODEs

$$\begin{cases} \frac{dE}{d\tau} = \frac{1}{2}\eta^2 E^2 - (\alpha^* + B(T - \tau, T)\eta^2)E + \omega i, \\ \frac{dC}{d\tau} = \kappa^*\theta^* D + \alpha^*\beta^* E, \end{cases} \quad (3.19)$$

with the initial conditions

$$C(\omega, 0) = 0, \quad E(\omega, 0) = 0. \quad (3.20)$$

We now present a proof of Proposition 3.1. Applying the following transform

$$\begin{cases} \tau = T - t, \\ x = \ln S, \end{cases} \quad (3.21)$$

we can convert (3.16) to the following

$$\left\{ \begin{array}{l} \frac{\partial U}{\partial \tau} = \frac{1}{2}\nu \frac{\partial^2 U}{\partial x^2} + \frac{1}{2}\sigma^2\nu \frac{\partial^2 U}{\partial \nu^2} + \frac{1}{2}\eta^2 r \frac{\partial^2 U}{\partial r^2} + \rho\sigma\nu \frac{\partial^2 U}{\partial x \partial \nu} + \left(r - \frac{1}{2}\nu\right) \frac{\partial U}{\partial x} \\ \quad + \kappa^* (\theta^* - \nu) \frac{\partial U}{\partial \nu} + [\alpha^* \beta^* - (\alpha^* + B(T - \tau, T)\eta^2)r] \frac{\partial U}{\partial r}, \\ U(x, \nu, r, 0) = H(e^x). \end{array} \right. \quad (3.22)$$

Performing the Fourier transform to (3.22) with respect to the variable x , we obtain the following equation for $\tilde{U}(\omega, \nu, r, \tau) = \mathfrak{F}[U(x, \nu, r, \tau)]$

$$\left\{ \begin{array}{l} \frac{\partial \tilde{U}}{\partial \tau} = \frac{1}{2}\sigma^2\nu \frac{\partial^2 \tilde{U}}{\partial \nu^2} + \frac{1}{2}\eta^2 r \frac{\partial^2 \tilde{U}}{\partial r^2} + [\kappa^* \theta^* + (\rho\sigma\omega j - \kappa^*)\nu] \frac{\partial \tilde{U}}{\partial \nu} \\ \quad + [\alpha^* \beta^* - (\alpha^* + B(T - \tau, T)\eta^2)r] \frac{\partial \tilde{U}}{\partial r} + \left[-\frac{1}{2}(\omega i + \omega^2)\nu + r\omega i\right] \tilde{U}, \\ \tilde{U}(\omega, \nu, r, 0) = \mathfrak{F}[H(e^x)]. \end{array} \right. \quad (3.23)$$

Following the solution procedure of Heston in [66], the solution to the above PDE can be assumed to be of the following form

$$\tilde{U}(\omega, \nu, r, \tau) = e^{C(\omega, \tau) + D(\omega, \tau)\nu + E(\omega, \tau)r} \tilde{U}(\omega, \nu, r, 0). \quad (3.24)$$

We can then substitute (3.24) into (3.23) to reduce it to the following system of three ODEs

$$\left\{ \begin{array}{l} \frac{dD}{d\tau} = \frac{1}{2}\sigma^2 D^2 + (\rho\sigma\omega i - \kappa^*)D - \frac{1}{2}(\omega^2 + \omega i), \\ \frac{dE}{d\tau} = \frac{1}{2}\eta^2 E^2 - (\alpha^* + B(T - \tau, T)\eta^2)E + \omega i, \\ \frac{dC}{d\tau} = \kappa^* \theta^* D + \alpha^* \beta^* E, \end{array} \right. \quad (3.25)$$

with the initial conditions

$$C(\omega, 0) = 0, \quad D(\omega, 0) = 0, \quad E(\omega, 0) = 0. \quad (3.26)$$

Based on the ODEs, only the function D can be solved analytically as

$$D(\tau) = \frac{a + b}{\sigma^2} \frac{1 - e^{b\tau}}{1 - g e^{b\tau}}, \quad a = \kappa^* - \rho\sigma\omega i, \\ b = \sqrt{a^2 + \sigma^2(\omega^2 + \omega i)}, \quad g = \frac{a + b}{a - b},$$

whereas numerical integration is used to obtain the solutions of the functions E and C using standard mathematical software package, e.g., MATLAB. We list out the MATLAB codes used to perform this integration in the Appendix of this chapter.

It can be observed that the Fourier transform variable ω appears as a parameter in function C , D and E . After performing the inverse Fourier transform, we obtain the solution as in the original form of our PDE as follows

$$\begin{aligned} U(x, \nu, r, \tau) &= \mathfrak{F}^{-1} \left[\tilde{U}(\omega, \nu, r, \tau) \right] \\ &= \mathfrak{F}^{-1} \left[e^{C(\omega, \tau) + D(\omega, \tau)\nu + E(\omega, \tau)r} \mathfrak{F}[H(e^x)] \right]. \end{aligned} \quad (3.27)$$

Note that Proposition 3.1 is applicable to most derivatives whose payoffs depend on spot price S in the framework of the Heston-CIR hybrid model under our assumptions. However, in some cases, it is hard to handle the general Fourier transform. Next, we apply the terminal condition $H(S(t_j)) = \left(\frac{S(t_j)}{S(t_{j-1})} - 1 \right)^2$ to Proposition 3.1, the Fourier inverse transform could be explicitly worked out and hence the solution to (3.14) can be written in an elegant form. For convenience, define the variable $I(t) = \int_0^t \delta(t_{j-1} - \tau) S(\tau) d\tau$, where $\delta(\cdot)$ is the Dirac delta function. Note that for $t \geq t_{j-1}$, the variable $I(t)$ is equal to the constant $S(t_{j-1})$. Thus for $[t_{j-1}, t_j]$, we can simplify the notation $I(t)$ as I . Denoting $x = \ln S$, we perform the generalized Fourier transform to the payoff function $H(e^x)$ with respect to x and derive

$$\mathfrak{F} \left[\left(\frac{e^x}{I} - 1 \right)^2 \right] = 2\pi \left[\frac{\delta_{-2i}(\omega)}{I^2} - 2 \frac{\delta_{-i}(\omega)}{I} + \delta_0(\omega) \right]. \quad (3.28)$$

Thus, the solution to (3.14) is given by

$$\begin{aligned} U_j(S, \nu, r, \tau) &= \mathfrak{F}^{-1} \left[e^{C(\omega, \tau) + D(\omega, \tau)\nu + E(\omega, \tau)r} 2\pi \left[\frac{\delta_{-2i}(\omega)}{I^2} - 2 \frac{\delta_{-i}(\omega)}{I} + \delta_0(\omega) \right] \right] \\ &= \int_{-\infty}^{\infty} e^{C(\omega, \tau) + D(\omega, \tau)\nu + E(\omega, \tau)r} \left[\frac{\delta_{-2i}(\omega)}{I^2} - 2 \frac{\delta_{-i}(\omega)}{I} + \delta_0(\omega) \right] e^{x\omega i} d\omega \\ &= \frac{e^{2x}}{I^2} e^{\tilde{C}(\tau) + \tilde{D}(\tau)\nu + \tilde{E}(\tau)r} - \frac{2e^x}{I} e^{\hat{C}(\tau) + \hat{E}(\tau)r} + 1, \end{aligned} \quad (3.29)$$

where $t_{j-1} \leq t \leq t_j$ and $\tau = t_j - t$. $\tilde{C}(\tau)$, $\tilde{D}(\tau)$ and $\tilde{E}(\tau)$ are the notations $C(-2i, \tau)$, $D(-2i, \tau)$ and $E(-2i, \tau)$ respectively, whereas $\hat{C}(\tau)$ and $\hat{E}(\tau)$ are equal to $C(-i, \tau)$ and $E(-i, \tau)$ respectively.

Finally, let $\tau = \Delta t$ in $U_j(S, \nu, r, \tau)$, and by noting that $\ln S(t_{j-1}) = \ln I(t)$ in (3.29),

we obtain

$$E_{j-1} = e^{\tilde{C}(\Delta t) + \tilde{D}(\Delta t)\nu(t_{j-1}) + \tilde{E}(\Delta t)r(t_{j-1})} - 2e^{\hat{C}(\Delta t) + \hat{E}(\Delta t)r(t_{j-1})} + 1. \quad (3.30)$$

3.2.3 The Second Step of Computation

In the previous subsection, we had obtained the solution for the first computation step. In this subsection, we shall proceed by finding the expectation of this solution, as defined in our solution outline. In particular, we need to calculate $\mathbb{E}^T[E_{j-1}|\mathcal{F}(0)]$ and K .

Since E_{j-1} is an exponential function of the stochastic variables $\nu(t_{j-1})$ and $r(t_{j-1})$ in affine form, it is possible for us to carry out the expectation with a semi-closed form solution, by using the characteristic functions of $\nu(t_{j-1})$ and $r(t_{j-1})$. We assume that $\nu(t_{j-1})$ and $r(t_{j-1})$ are independent. Thus,

$$\begin{aligned} \mathbb{E}^T[E_{j-1}|\mathcal{F}(0)] &= e^{\tilde{C}(\Delta t)} \cdot \mathbb{E}^T \left[e^{\tilde{D}(\Delta t)\nu(t_{j-1})} | \mathcal{F}(0) \right] \cdot \mathbb{E}^T \left[e^{\tilde{E}(\Delta t)r(t_{j-1})} | \mathcal{F}(0) \right] \\ &\quad - 2e^{\hat{C}(\Delta t)} \cdot \mathbb{E}^T \left[e^{\hat{E}(\Delta t)r(t_{j-1})} | \mathcal{F}(0) \right] + 1. \end{aligned} \quad (3.31)$$

If we put

$$h(\phi, \nu, \tau) = \mathbb{E}^T \left[e^{\phi\nu(t+\tau)} | \mathcal{F}(t) \right] \quad (3.32)$$

and

$$f(\phi, r, \tau) = \mathbb{E}^T \left[e^{\phi r(t+\tau)} | \mathcal{F}(t) \right], \quad (3.33)$$

then we can express $\mathbb{E}^T[E_{j-1}]$ as follows

$$\begin{aligned} \mathbb{E}^T[E_{j-1}|\mathcal{F}(0)] &= e^{\tilde{C}(\Delta t)} \cdot h(\tilde{D}(\Delta t), \nu(0), t_{j-1}) \cdot f(\tilde{E}(\Delta t), r(0), t_{j-1}) \\ &\quad - 2e^{\hat{C}(\Delta t)} \cdot f(\hat{E}(\Delta t), r(0), t_{j-1}) + 1. \end{aligned} \quad (3.34)$$

Here, we show how to derive expressions for f and h by solving the corresponding PDEs. First, we define

$$f(\phi, r, \tau) = \mathbb{E}^T \left[e^{\phi r(t+\tau)} | \mathcal{F}(t) \right]. \quad (3.35)$$

Given that the stochastic process of $r(t)$ follows the equation in (3.13) under the T -forward probability measure \mathbb{Q}^T , applying the Feynman-Kac formula, we can derive

that $f(\phi, r, \tau)$ satisfies the following PDE

$$\begin{cases} \frac{\partial f}{\partial \tau} = \frac{1}{2}\eta^2 r \frac{\partial^2 f}{\partial r^2} + [\alpha^* \beta^* - (\alpha^* + B(t_{j-1} - \tau, T)\eta^2)r] \frac{\partial f}{\partial r}, \\ f(\phi, r, \tau = 0) = e^{\phi r}, \end{cases} \quad (3.36)$$

whose solution has the form $f(\phi, r, \tau) = e^{F(\phi, \tau) + H(\phi, \tau)r}$. Substituting this function into (3.36), we obtain the following system of ODEs

$$\begin{cases} \frac{dH}{d\tau} = \frac{1}{2}\eta^2 H^2 - (\alpha^* + B(t_{j-1} - \tau, T)\eta^2)H, \\ \frac{dF}{d\tau} = \alpha^* \beta^* H, \end{cases} \quad (3.37)$$

with the initial conditions

$$H(\phi, 0) = \phi, \quad F(\phi, 0) = 0. \quad (3.38)$$

The solution to this system is retrieved via numerical integration in the MATLAB software, and the codes used are displayed in Appendix. Next, we define the function

$$h(\phi, \nu, \tau) = \mathbb{E}^T \left[e^{\phi \nu(t+\tau)} | \mathcal{F}(t) \right]$$

in order to derive an expression of $\mathbb{E}^T \left[e^{\tilde{D}(\Delta t)\nu(t_{j-1})} | \mathcal{F}(0) \right]$. Then, h satisfies the following PDE

$$\begin{cases} \frac{\partial h}{\partial \tau} = \frac{1}{2}\sigma^2 \nu \frac{\partial^2 h}{\partial \nu^2} + \kappa^* (\theta^* - \nu) \frac{\partial h}{\partial \nu}, \\ h(\phi, \nu, \tau = 0) = e^{\phi \nu}, \end{cases}$$

whose solution has the form $h(\phi, \nu, \tau) = e^{L(\phi, \tau) + M(\phi, \tau)\nu}$. We will later obtain the following system of ODEs

$$\begin{cases} \frac{dM}{d\tau} = \frac{1}{2}\sigma^2 M^2 - \kappa^* M, \\ \frac{dL}{d\tau} = \kappa^* \theta^* M, \end{cases}$$

with initial conditions $M(\phi, 0) = \phi$ and $L(\phi, 0) = 0$. We obtain the solution to the

above ODEs as

$$\begin{cases} M(\phi, \tau) = \frac{\phi e^{-\kappa^* \tau}}{1 - \frac{\sigma^2 \phi (1 - e^{-\kappa^* \tau})}{2\kappa^*}}, \\ L(\phi, \tau) = \frac{-2\kappa^* \theta^*}{\sigma^2} \ln \left(1 - \frac{\sigma^2 \phi (1 - e^{-\kappa^* \tau})}{2\kappa^*} \right). \end{cases}$$

3.2.4 Delivery Price of Variance Swaps

In the previous two subsections, we demonstrate our solution techniques for pricing variance swaps by separating them into two computation steps. Now, by referring back to the formula of RV as given in (2.24), we have the fair delivery price of variance swaps as

$$K = \mathbb{E}^T[RV|\mathcal{F}(0)] = \frac{100^2}{T} \sum_{j=1}^N \mathbb{E}^T[E_{j-1}|\mathcal{F}(0)]. \quad (3.39)$$

Using (3.34), the summation in (3.39) for the whole period of $j = 1$ to $j = N$ can now be carried out all the way except for the very first period with $j = 1$. We need to treat the case $j = 1$, separately, because in this case we have $t_{j-1} = 0$ and $S(t_{j-1}) = S(0)$ is a known value, instead of an unknown value of $S(t_{j-1})$ for any other cases with $j > 1$. For the case $j = 1$, we put

$$G(\nu(0), r(0)) = \mathbb{E}^T \left[\left(\frac{S(t_1)}{S(0)} - 1 \right)^2 \middle| \mathcal{F}(0) \right], \quad (3.40)$$

and for any other cases with $j > 1$, we put

$$G_j(\nu(0), r(0)) = \mathbb{E}^T[E_{j-1}|\mathcal{F}(0)].$$

Then, $G(\nu(0), r(0))$ can be derived from Proposition 3.1 directly. Finally, we obtain the fair delivery price of a variance swap as

$$K = \mathbb{E}^T[RV|\mathcal{F}(0)] = \frac{100^2}{T} \left[G(\nu(0), r(0)) + \sum_{j=2}^N G_j(\nu(0), r(0)) \right]. \quad (3.41)$$

This formula is obtained by solving the associated PDEs in two steps. Since we have managed to express the solution of the associated PDEs in both steps, we are able to write the fair delivery price of a variance swap with discretely-sampled realized variance

defined in its payoff in a simple and semi-closed form.

3.3 Numerical Examples and Simulation

In this section, we perform a numerical analysis for pricing variance swaps under our model and utilize our semi-closed form pricing formula. This involves comparison of our formula with the Monte Carlo simulation and the continuously-sampled variance swaps model. In addition, we also investigate the effects of stochastic interest rate in our pricing formulation.

We use the parameters in Table 3.1, unless otherwise stated, in all our numerical examples. This set of parameters for the hybrid Heston-CIR model was also adopted by [55].

Table 3.1: Model parameters of the Heston-CIR hybrid model

Parameters	S_0	ρ	V_0	θ^*	κ^*	σ	r_0	α^*	β^*	η	T
Values	1	-0.40	$(22.36\%)^2$	$(22.36\%)^2$	2	0.1	5%	1.2	5%	0.01	1

3.3.1 Monte Carlo Simulation

We firstly have implemented Monte Carlo (MC) simulations to obtain numerical results as references for comparisons. The stochastic processes of the model are discretized by using the simple Euler-Milstein scheme.

Figure 3.1 shows the comparison among the numerical results obtained from our semi-closed form formula (3.41), those from Monte Carlo simulations, and the numerical calculation of the continuously-sampled realized variance. Model parameters are presented in Table 3.1, and the time to maturity is 1 year. It is clearly seen that the results from our solution perfectly match the results from the MC simulations which serve as benchmark values. For example, for the weekly-sampled variance swaps (the sampling frequency is 52 in the figure), the relative difference between numerical results obtained from formula (3.41) and the MC simulations is less than 0.05% already, when the number of paths reaches 200,000 in MC simulations. Such a relative difference is further reduced when the number of paths is increased; demonstrating the convergence of the MC simulations towards our semi-closed form solution and hence to a certain extent providing a verification of our solution.

Figure 3.1: Comparison of our formula for variance swaps with MC simulation in different sampling frequency.

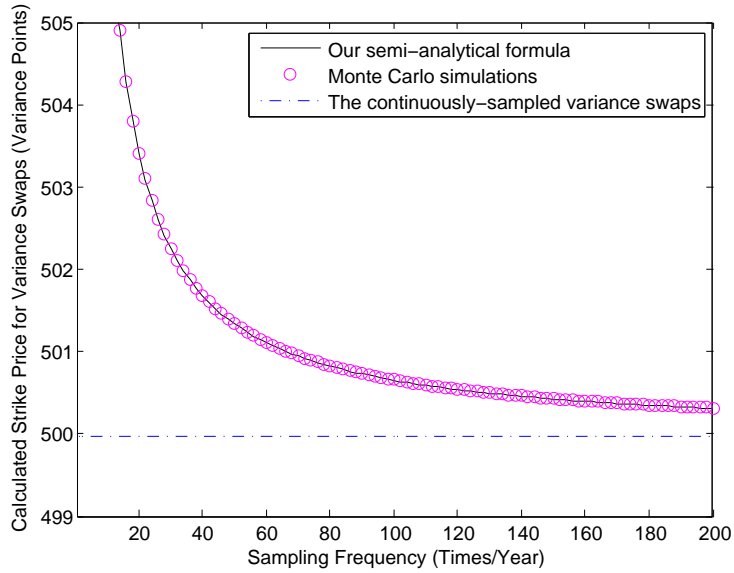


Figure 3.2: Values of variance swaps with different β^* in the Heston-CIR hybrid model.

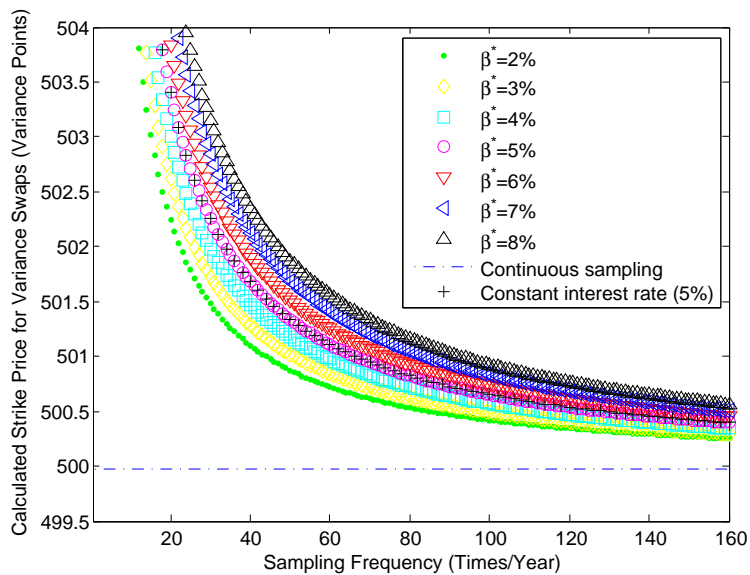
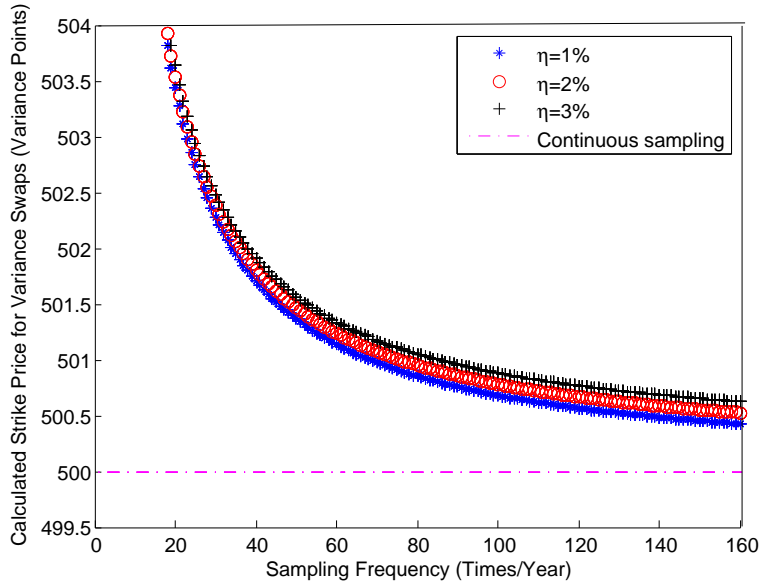


Figure 3.3: Values of variance swaps with different η in the Heston-CIR hybrid model.



3.3.2 Effect of Stochastic Interest Rate

To test the effects of the stochastic interest rate, we now calculate the fair strike values of variance swaps with stochastic interest rate and deterministic interest rate. So we implement the semi-analytical pricing formula (3.41) with the parameters tabulated in Table 3.1 (unless otherwise stated) to obtain numerical values of variance swaps with stochastic interest rate. For the variance swaps with constant deterministic interest rate, we implement the formula in [119]. Of course, since the semi-closed form pricing formula (3.41) is derived based on a more general Heston-CIR hybrid model, we can obtain values of variance swaps with constant deterministic interest rate by setting $\alpha^* = 0$, $\beta^* = 0$ and $\eta = 0$. The time to maturity in all the numerical examples below is $T = 1$.

Figure 3.2 depicts the fair strike values of variance swaps in different sampling frequencies, ranging from 15 sampling times in total per year to 160 times per year. Model parameters are presented in Table 3.1, except for β^* that can take different values as indicated in the figure. Time to maturity is 1 year. We notice that with the increase

of sampling frequency, the value of a discrete variance swap decreases, converging to the continuous sampling counterpart. This is consistent with the convergence pattern of constant interest rate as shown in [16, 119]. We can also observe that, when the spot interest rate ($r_0 = 5\%$) is equal to the long-term interest rate (β^* in our notation), the value of a variance swap with stochastic interest rate coincides with the value in the case of constant interest rate which remains unchanging as 5%. This implies that the parameters α^* and η have little effect on the values of variance swaps. To confirm this, we also examine the pricing behaviour of variance swaps with respect to η which is displayed in Figure 3.3. Increasing values of η would lead to increasing prices of a variance swap. There exists some slight difference for small sampling frequencies, but the difference grows larger as the number of sampling frequencies increases. However, the same convergence pattern towards the continuous sampling counterpart can also be seen for these different η parameters.

Finally, we can see that, when β^* increases, the value of a variance swap increases correspondingly. This implies that the interest rate can impact and change the value of a variance swap, ignoring the effect of interest rate will result in mispricing. Since interest rate changes and is modeled by a stochastic process (such as a CIR process), working out the semi-closed form pricing formula for discretely-sampled variance swaps in the Heston-CIR hybrid model can help pricing variance swaps more accurately.

3.4 Appendix

As described in Subsection 3.2.2, we use MATLAB to perform the numerical integration for the ODEs of functions E and C . Here, we exhibit the codes used during the implementation process. We start with the ODE of function $\tilde{E}(\tau)$ which represents $E(-2i, \tau)$ by writing the codes in the file named *myode_ewidetilde.m* as follows

```
function dydt = myode_ewidetilde(t,y,Bt,B)

B = interp1(Bt,B,t);

[ K S nu0 rho rho_sqvr rho_xr rho_vr theta_star lambda.1 lambda.2 sigma
kapa T t alpha beta_star r0 eta]=InputPara.2( );
kapa_star= kapa + lambda.1; alpha_star = alpha + lambda.2;
alpha_star = 1.2;
beta_star = 0.05;
```

```
dydt = ((1/2).* eta.^2 .* y.^2) -((alpha_star + B.*eta.^2).*y) +2;
```

Next, we embed this file under a new file named *call_ewidetilde.m* to obtain the solution of numerical integration of $\tilde{E}(\tau)$

```
function [E_tilde] = call_ewidetilde( t, T, delT, r0, nu0, rho, rho_sqvr,
rho_xr, rho_vr, sigma, alpha_star, kapa_star, lambda_1, lambda_2, beta_star,
eta, theta_star)

[ K S nu0 rho rho_sqvr rho_xr rho_vr theta_star lambda_1 lambda_2 sigma
kapa T t alpha beta_star r0 eta]=InputPara_2( );
kapa_star=kapa + lambda_1; alpha_star = alpha + lambda_2;

alpha_star = 1.2;
beta_star = 0.05;
eta = 0.01;
r0 = 0.05;

delTend=delT;

delT=0:delTend/100:delTend;

b_function = B_function( t, T, delT, r0, nu0, rho, rho_sqvr, rho_xr, rho_vr,
sigma, alpha_star, kapa_star, lambda_1, lambda_2, beta_star, eta, theta_star);

B=b_function;
Bt=delT;

Tspan = delT; % Solve from t=1 to t=5
IC = 0; % y(t=0) = 1
[Tn Y] = ode45(@ (t,y) myode_ewidetilde(t,y,Bt,B),Tspan,IC); % Solve ODE
E_tilde=Y;
```

Following the same steps as above, we can write the codes for the function $\hat{E}(\tau)$ by substituting $\omega = -i$ into its ODE in (3.25) to obtain its numerical integration solution.

Moving on to function C , we have to find the solutions for $\tilde{C}(\tau)$ and $\hat{C}(\tau)$ which are the notations for $C(-2i, \tau)$ and $C(-i, \tau)$ respectively. The codes are listed below

```
function dydt = myode_cwidetilde(t,y,Dt,D,Et,E)
D = interp1(Dt,D,t); % Interpolate the data set (gt,g) at time t
E = interp1(Et,E,t); % Interpolate the data set (gt,g) at time t
```

```

[ K S nu0 rho theta_star lambda.1 lambda.2 sigma kapa T t alpha beta_star
r0 eta]=InputPara_2( );
kapa_star= kapa + lambda.1; alpha_star = alpha + lambda.2;

alpha_star = 1.2;
beta_star = 0.05;
eta = 0.01;
r0 = 0.05 ;
T=1;

dydt = kapa_star*theta_star*D+alpha_star*beta_star*E;

```

We can later obtain the solution for $\tilde{C}(\tau)$ using the following codes

```

function [C_tilde] = call_cwidetilde( t, T, delT, r0, nu0, rho, sigma,
alpha_star, kapa_star, lambda.1, lambda.2, beta_star, eta, theta_star)
[ K S nu0 rho theta_star lambda.1 lambda.2 sigma kapa T t alpha beta_star
r0 eta]=InputPara_2( );
kapa_star= kapa + lambda.1; alpha_star = alpha + lambda.2;

alpha_star = 1.2;
beta_star = 0.05;
eta = 0.01;
r0 = 0.05 ;
T=1;

delTend=delT;
delT=0:delTend/100:delTend;

d_wildetilde = D_widetilde( t, T, delT, r0, nu0, rho, sigma,
alpha_star, kapa_star, lambda.1, lambda.2, beta_star, eta,
theta_star);
e_wildetilde = call_ewidetilde( t, T, delTend, r0, nu0, rho, sigma,
alpha_star, kapa_star, lambda.1, lambda.2, beta_star, eta,
theta_star);

D=d_wildetilde;
Dt=delT;
Et =delT;
E = e_wildetilde;

```

```
Tspan = delT; % Solve from t=1 to t=5
IC = 0; % y(t=0) = 1
[Tn Y] = ode45(@ (t,y) myode_cwidetilde(t,y,Dt,D,Et,E), Tspan, IC); % Solve ODE
C_tilde=Y;
```

The solution for $\widehat{C}(\tau)$ is attained by replacing the functions *d_wildetilde* and *e_wildetilde* in the codes above with *e_widehat*, then following the same manner as outlined.

Finally, we proceed in finding the solutions of functions H and F as defined in (3.37) under Subsection 3.2.3. Note that we have to deal with two functions since the variable ϕ in $f(\phi, r, \tau)$ consists of $\widetilde{E}(\Delta)$ and $\widehat{E}(\Delta)$. The codes for the ODE of $H(\widetilde{E}(\Delta), \tau)$ can be written as

```
function dydt = myode_H_function_ewidetilde(t,y,Bt,B)

% Interpolate the data set (gt,g) at time t
B = interp1(Bt,B,t);

[ K S nu0 rho theta_star lambda_1 lambda_2 sigma kapa T t alpha beta_star
r0 eta]=InputPara_2( );
kapa_star= kapa + lambda_1; alpha_star = alpha + lambda_2;

alpha_star = 1.2;
beta_star = 0.05;
eta = 0.01;
r0 = 0.05 ;
T=1;

dydt = ((1/2).* eta.^2 .* y.^2) -((alpha_star + B.*eta.^2).*y);
```

which is later solved numerically by

```
function [H_func] = call_H_function_ewidetilde( t, T, delT, r0, nu0, rho,
sigma, alpha_star, kapa_star, lambda_1, lambda_2, beta_star, eta,
theta_star, e_wildetilde)

[ K S nu0 rho theta_star lambda_1 lambda_2 sigma kapa T t alpha beta_star
r0 eta]=InputPara_2( );
kapa_star= kapa + lambda_1; alpha_star = alpha + lambda_2;

alpha_star = 1.2;
beta_star = 0.05;
```

```

eta = 0.01;
r0 = 0.05 ;
T=1;

tjMinus1=delT;
delTend=delT;
delT=0:delTend/1000:delTend;

        b_function = B_function( t, T, T-(tjMinus1-delT), r0, nu0, rho, sigma,
        alpha_star, kapa_star, lambda_1, lambda_2, beta_star, eta, theta_star);

B=b_function;
Bt =delT;

Tspan = delT; % Solve from t=1 to t=5
IC = e_wildetilde; % y(t=0) = 1
[Tn Y] = ode45(@ (t,y) myode_H_function_ewidetilde(t,y,Bt,B),Tspan,IC);
H_func=Y;

```

One can later find the numerical integration of $H(\widehat{E}(\Delta), \tau)$ by substituting $e_widehat$ into the initial condition denoted as IC in the codes above. Once the solutions for functions $H(\widetilde{E}(\Delta), \tau)$ and $H(\widehat{E}(\Delta), \tau)$ are obtained, we can find the solutions for $F(\widetilde{E}(\Delta), \tau)$ and $F(\widehat{E}(\Delta), \tau)$ utilizing the same procedures as outlined for finding $\widetilde{C}(\tau)$ and $\widehat{C}(\tau)$.

Chapter 4

Pricing Variance Swaps under Stochastic Factors : Full Correlation Case

The evolving number of complex hybrid models featuring various underlyings in the financial world nowadays brings attention to the correlation issue in the models. This issue can be directly linked to the highlighted importance of imposing correlations, either partially or fully, in the literature. In this chapter, the hybrid model of stochastic volatility and stochastic interest rate for pricing variance swaps given in Chapter 3 is further extended. First, an extension of the model in Subsection 3.2.1 to a pricing model with full correlation among the asset, interest rate as well as the volatility is established in Section 4.1. In Section 4.2, we approach this pricing problem via approximations since this model is incompliant with the analytical tractability property. We first determine the approximations for the non-affine terms, then solve the corresponding equations using approximations of normally distributed random variables. Section 4.3 presents some numerical results, along with the study of the correlation impacts on the delivery price of a variance swap.

4.1 The Heston-CIR Model with Full Correlation

In this section, we study the problem of pricing variance swaps under the Heston-CIR Model with full correlation. We will deal with an extension of the model in Subsection 3.2.1 to the case where full correlation structure is involved.

Assume that the correlations involved in model (3.1) are given by $(dW_1(t), dW_2(t)) =$

$\rho_{12}dt = \rho_{21}dt$, $(dW_1(t), dW_3(t)) = \rho_{13}dt = \rho_{31}dt$ and $(dW_2(t), dW_3(t)) = \rho_{23}dt = \rho_{32}dt$, where $0 \leq t \leq T$ and $-1 < \rho_{ij} < 1$ for all $i, j = 1, 2, 3$ which are constants. The system in (3.11) under the risk-neutral measure, \mathbb{Q} can be adjusted as follows

$$\begin{bmatrix} \frac{dS(t)}{S(t)} \\ d\nu(t) \\ dr(t) \end{bmatrix} = \mu^{\mathbb{Q}}dt + \Sigma \times C \times \begin{bmatrix} dW_1^*(t) \\ dW_2^*(t) \\ dW_3^*(t) \end{bmatrix}, \quad 0 \leq t \leq T, \quad (4.1)$$

where

$$\mu^{\mathbb{Q}} = \begin{bmatrix} r(t) \\ \kappa^*(\theta^* - \nu(t)) \\ \alpha^*(\beta^* - r(t)) \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \sqrt{\nu(t)} & 0 & 0 \\ 0 & \sigma\sqrt{\nu(t)} & 0 \\ 0 & 0 & \eta\sqrt{r(t)} \end{bmatrix}$$

and

$$C = \begin{bmatrix} 1 & 0 & 0 \\ \rho_{12} & \sqrt{1 - \rho_{12}^2} & 0 \\ \rho_{13} & \frac{\rho_{23} - \rho_{13}\rho_{12}}{\sqrt{1 - \rho_{12}^2}} & \sqrt{1 - \rho_{13}^2 - \left(\frac{\rho_{23} - \rho_{13}\rho_{12}}{\sqrt{1 - \rho_{12}^2}}\right)^2} \end{bmatrix}$$

such that

$$CC^{\top} = \begin{bmatrix} 1 & \rho_{12} & \rho_{13} \\ \rho_{21} & 1 & \rho_{23} \\ \rho_{31} & \rho_{32} & 1 \end{bmatrix}.$$

Here, $\{\widetilde{W}_1(t) : 0 \leq t \leq T\}$, $\{\widetilde{W}_2(t) : 0 \leq t \leq T\}$ and $\{\widetilde{W}_3(t) : 0 \leq t \leq T\}$ are three Brownian motions under \mathbb{Q} such that $d\widetilde{W}_1(t)$, $d\widetilde{W}_2(t)$ and $d\widetilde{W}_3(t)$ are mutually independent and satisfy the following relation

$$\begin{bmatrix} d\widetilde{W}_1(t) \\ d\widetilde{W}_2(t) \\ d\widetilde{W}_3(t) \end{bmatrix} = C \times \begin{bmatrix} dW_1^*(t) \\ dW_2^*(t) \\ dW_3^*(t) \end{bmatrix}, \quad 0 \leq t \leq T.$$

Following the techniques for the change of measure same as those in Subsection 3.2.1, we demonstrate how to find μ^T which is the new drift for our SDEs under \mathbb{Q}^T by utilizing the formula below as defined in (3.12)

$$\mu^T = \mu^{\mathbb{Q}} - \left[\Sigma \times C \times C^{\top} \times (\Sigma^{\mathbb{Q}} - \Sigma^T) \right],$$

with $\Sigma^{\mathbb{Q}}$ and Σ^T given by

$$\Sigma^{\mathbb{Q}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \Sigma^T = \begin{bmatrix} 0 \\ 0 \\ -B(t, T)\eta\sqrt{r(t)} \end{bmatrix},$$

along with Σ and CC^T as defined in (4.1). This gives us the new dynamics for (4.1) under the forward measure, \mathbb{Q}^T for $0 \leq t \leq T$ as follows

$$\begin{bmatrix} \frac{dS(t)}{S(t)} \\ d\nu(t) \\ dr(t) \end{bmatrix} = \begin{bmatrix} r(t) - \rho_{13}B(t, T)\eta\sqrt{r(t)}\sqrt{\nu(t)} \\ \kappa^*(\theta^* - \nu(t)) - \rho_{23}\sigma B(t, T)\eta\sqrt{r(t)}\sqrt{\nu(t)} \\ \alpha^*\beta^* - [\alpha^* + B(t, T)\eta^2]r(t) \end{bmatrix} dt + \Sigma \times C \times \begin{bmatrix} dW_1^*(t) \\ dW_2^*(t) \\ dW_3^*(t) \end{bmatrix}. \quad (4.2)$$

4.2 Solution Techniques for Pricing Variance Swaps with Full Correlation

As described in Subsection 3.2.1, our solution outline involves finding solution for two computation steps. In this section, we exhibit our approach in obtaining a semi-closed form approximation pricing formula for variance swaps with full correlation structure. The first part of this section presents an extension of the method given in Subsection 3.2.2 to the case where full correlation is involved. Here, we demonstrate how to deal with the deterministic approximation proposed in [55]. The second part of this section presents the solution for the second step which utilizes approximation properties of normally distributed random variables.

4.2.1 Solution for the First Step

As illustrated in (3.9), we shall focus on a contingent claim $U_j(S(t), \nu(t), r(t), t)$, whose payoff at expiry t_j is $\left(\frac{S(t_j)}{S(t_{j-1})} - 1\right)^2$. Applying standard techniques in the general asset

valuation theory, the PDE for U_j over $[t_{j-1}, t_j]$ can be described as

$$\begin{aligned}
 & \frac{\partial U_j}{\partial t} + \frac{1}{2} \nu S^2 \frac{\partial^2 U_j}{\partial S^2} + \frac{1}{2} \sigma^2 \nu \frac{\partial^2 U_j}{\partial \nu^2} + \frac{1}{2} \eta^2 r \frac{\partial^2 U_j}{\partial r^2} + \rho_{12} \sigma \nu S \frac{\partial^2 U_j}{\partial S \partial \nu} \\
 & + \left[rS - \rho_{13} B(t, T) \eta \sqrt{r(t)} \sqrt{\nu(t)} S \right] \frac{\partial U_j}{\partial S} + \left[\kappa^* (\theta^* - \nu) - \rho_{23} \sigma B(t, T) \eta \sqrt{r(t)} \sqrt{\nu(t)} \right] \frac{\partial U_j}{\partial \nu} \\
 & + \left[\alpha^* \beta^* - (\alpha^* + B(t, T) \eta^2) r \right] \frac{\partial U_j}{\partial r} + \rho_{23} \sigma \eta \sqrt{\nu(t)} \sqrt{r(t)} \frac{\partial^2 U_j}{\partial \nu \partial r} + \rho_{13} \eta \sqrt{\nu(t)} \sqrt{r(t)} S \frac{\partial^2 U_j}{\partial S \partial r} \\
 & = 0
 \end{aligned} \tag{4.3}$$

with the terminal condition

$$U_j(S(t_j), \nu, r, t_j) = \left(\frac{S(t_j)}{S(t_{j-1})} - 1 \right)^2. \tag{4.4}$$

Proposition 4.1. *If the underlying asset follows the dynamic process (4.2) and a European-style derivative written on this asset has a payoff function $U(S, \nu, r, T) = H(S)$ at expiry T , then the solution of the associated PDE system of the derivative value*

$$\left\{ \begin{aligned}
 & \frac{\partial U}{\partial t} + \frac{1}{2} \nu S^2 \frac{\partial^2 U}{\partial S^2} + \frac{1}{2} \sigma^2 \nu \frac{\partial^2 U}{\partial \nu^2} + \frac{1}{2} \eta^2 r \frac{\partial^2 U}{\partial r^2} + \rho_{12} \sigma \nu S \frac{\partial^2 U}{\partial S \partial \nu} \\
 & + \left[rS - \rho_{13} B(t, T) \eta \sqrt{r(t)} \sqrt{\nu(t)} S \right] \frac{\partial U}{\partial S} + \left[\kappa^* (\theta^* - \nu) - \rho_{23} \sigma B(t, T) \eta \sqrt{r(t)} \sqrt{\nu(t)} \right] \frac{\partial U}{\partial \nu} \\
 & + \left[\alpha^* \beta^* - (\alpha^* + B(t, T) \eta^2) r \right] \frac{\partial U}{\partial r} + \rho_{23} \sigma \eta \sqrt{\nu(t)} \sqrt{r(t)} \frac{\partial^2 U}{\partial \nu \partial r} + \rho_{13} \eta \sqrt{\nu(t)} \sqrt{r(t)} S \frac{\partial^2 U}{\partial S \partial r} \\
 & = 0 \\
 & U(S, \nu, r, T) = H(S),
 \end{aligned} \right. \tag{4.5}$$

can be expressed in semi-closed form as

$$U(x, \nu, r, t) = \mathfrak{F}^{-1} \left[e^{C(\omega, T-t) + D(\omega, T-t)\nu + E(\omega, T-t)r} \mathfrak{F}[H(e^x)] \right], \tag{4.6}$$

using generalized Fourier transform method [88], where $x = \ln S$, $i = \sqrt{-1}$, $\tau = T - t$,

ω is the Fourier transform variable,

$$\begin{cases} D(\omega, \tau) = \frac{a+b}{\sigma^2} \frac{1-e^{b\tau}}{1-ge^{b\tau}}, \\ a = \kappa^* - \rho_{12}\sigma\omega i, \quad b = \sqrt{a^2 + \sigma^2(\omega^2 + \omega i)}, \quad g = \frac{a+b}{a-b}, \end{cases} \quad (4.7)$$

and $E(\omega, \tau)$ along with $C(\omega, \tau)$ satisfy the following ODEs

$$\begin{cases} \frac{dE}{d\tau} = \frac{1}{2}\eta^2 E^2 - (\alpha^* + B(T-\tau, T)\eta^2)E + \omega i, \\ \frac{dC}{d\tau} = \kappa^*\theta^*D + \alpha^*\beta^*E - \rho_{13}\eta\mathbb{E}^T(\sqrt{\nu(T-\tau)}\sqrt{r(T-\tau)})\omega i B(T-\tau, T) \\ \quad + \rho_{13}\eta\mathbb{E}^T(\sqrt{\nu(T-\tau)}\sqrt{r(T-\tau)})\omega i E \\ \quad - \rho_{23}\sigma\eta\mathbb{E}^T(\sqrt{\nu(T-\tau)}\sqrt{r(T-\tau)})DB(T-\tau, T) \\ \quad + \rho_{23}\eta\sigma\mathbb{E}^T(\sqrt{\nu(T-\tau)}\sqrt{r(T-\tau)})DE, \end{cases} \quad (4.8)$$

with the initial conditions

$$C(\omega, 0) = 0, \quad E(\omega, 0) = 0. \quad (4.9)$$

The derivation leading to the solution in Proposition 4.1 can be referred from the proof below. We denote $\tau = T - t$ and $x = \ln S$, then the PDE system (4.3) is transformed into the following

$$\begin{cases} \frac{\partial U}{\partial \tau} = \frac{1}{2}\nu \frac{\partial^2 U}{\partial x^2} + \frac{1}{2}\sigma^2\nu \frac{\partial^2 U}{\partial \nu^2} + \frac{1}{2}\eta^2 r \frac{\partial^2 U}{\partial r^2} + \rho_{12}\sigma\nu \frac{\partial^2 U}{\partial x \partial \nu} \\ \quad + \left[r - \rho_{13}B(T-\tau, T)\eta\sqrt{\nu(T-\tau)}\sqrt{r(T-\tau)} - \frac{1}{2}\nu \right] \frac{\partial U}{\partial x} \\ \quad + \left[\kappa^*(\theta^* - \nu) - \rho_{23}\sigma B(T-\tau, T)\eta\sqrt{\nu(T-\tau)}\sqrt{r(T-\tau)} \right] \frac{\partial U}{\partial \nu} \\ \quad + \left[\alpha^*\beta^* - (\alpha^* + B(T-\tau, T)\eta^2)r \right] \frac{\partial U}{\partial r} + \rho_{13}\eta\sqrt{\nu(T-\tau)}\sqrt{r(T-\tau)} \frac{\partial^2 U}{\partial x \partial r} \\ \quad + \rho_{23}\sigma\eta\sqrt{\nu(T-\tau)}\sqrt{r(T-\tau)} \frac{\partial^2 U}{\partial \nu \partial r}. \\ U(x, \nu, r, 0) = H(e^x). \end{cases} \quad (4.10)$$

Following the technique of Fourier transform same as that in (3.23), the PDE system

is converted to the following

$$\left\{ \begin{array}{l} \frac{\partial \tilde{U}}{\partial \tau} = \frac{1}{2} \sigma^2 \nu \frac{\partial^2 \tilde{U}}{\partial \nu^2} + \frac{1}{2} \eta^2 r \frac{\partial^2 \tilde{U}}{\partial r^2} \\ + \left[\kappa^* \theta^* + (\rho_{12} \sigma \omega i - \kappa^*) \nu - \rho_{23} \sigma B(T - \tau, T) \eta \sqrt{\nu(T - \tau)} \sqrt{r(T - \tau)} \right] \frac{\partial \tilde{U}}{\partial \nu} \\ + \left[\alpha^* \beta^* - (\alpha^* + B(T - \tau, T) \eta^2) r + \rho_{13} \eta \sqrt{\nu(T - \tau)} \sqrt{r(T - \tau)} \omega i \right] \frac{\partial \tilde{U}}{\partial r} \\ + \rho_{23} \sigma \eta \sqrt{\nu(T - \tau)} \sqrt{r(T - \tau)} \frac{\partial^2 \tilde{U}}{\partial \nu \partial r} \\ + \left[-\frac{1}{2} (\omega i + \omega^2) \nu + r \omega i - \rho_{13} B(T - \tau, T) \eta \sqrt{\nu(T - \tau)} \sqrt{r(T - \tau)} \omega i \right] \tilde{U}, \\ \tilde{U}(\omega, \nu, r, 0) = \mathfrak{F}[H(e^x)], \end{array} \right. \quad (4.11)$$

where $i = \sqrt{-1}$ and ω is the Fourier transform variable.

As seen above, the partial differential equation contains the non-affine term of $\sqrt{\nu(t)} \sqrt{r(t)}$, where $t = T - \tau$ for writing convenience. Note that standard techniques to find characteristic functions as in [31] could not be applied in this case, thus we need to find an approximation for this non-affine term. Following an approach employed in [55], the expectation $\mathbb{E}^T(\sqrt{\nu(t)})$ with the CIR-type process can be approximated by

$$\mathbb{E}^T(\sqrt{\nu(t)}) \approx \sqrt{q_1(t)(\varphi_1(t) - 1) + q_1(t)l_1 + \frac{q_1(t)l_1}{2(l_1 + \varphi_1(t))}} =: \Lambda_1(t), \quad (4.12)$$

with

$$q_1(t) = \frac{\sigma^2(1 - e^{-\kappa^* t})}{4\kappa^*}, \quad l_1 = \frac{4\kappa^* \theta^*}{\sigma^2}, \quad \varphi_1(t) = \frac{4\kappa^* \nu(0) e^{-\kappa^* t}}{\sigma^2(1 - e^{-\kappa^* t})}. \quad (4.13)$$

This first-order approximation is obtained using delta method, where a function $f(X)$ can be approximated by a first-order Taylor expansion at $\mathbb{E}^T(X)$, for a given random variable X and its first two moments exist. It is assumed that f and its first-order derivative f' are sufficiently smooth.

In order to avoid further complication during derivation of the characteristic function and present a more efficient computation, the above approximation is further simplified and given by

$$\mathbb{E}^T(\sqrt{\nu(t)}) \approx m_1 + p_1 e^{-Q_1 t} =: \tilde{\Lambda}_1(t), \quad (4.14)$$

where

$$m_1 = \sqrt{\theta^* - \frac{\sigma^2}{8\kappa^*}}, \quad p_1 = \sqrt{\nu(0)} - m_1, \quad Q_1 = -\ln[p_1^{-1}(\Lambda_1(1) - m_1)]. \quad (4.15)$$

The same procedure can be applied to find the expectation of $\mathbb{E}^T(\sqrt{r(t)})$ with the square-root type of stochastic process as follows

$$\mathbb{E}^T(\sqrt{r(t)}) \approx \sqrt{q_2(t)(\varphi_2(t) - 1) + q_2(t)l_2 + \frac{q_2(t)l_2}{2(l_2 + \varphi_2(t))}} =: \Lambda_2(t), \quad (4.16)$$

with

$$q_2(t) = \frac{\eta^2(1 - e^{-\alpha^*t})}{4\alpha^*}, \quad l_2 = \frac{4\alpha^*\beta^*}{\eta^2}, \quad \varphi_2(t) = \frac{4\alpha^*r(0)e^{-\alpha^*t}}{\eta^2(1 - e^{-\alpha^*t})}, \quad (4.17)$$

and

$$\mathbb{E}^T(\sqrt{r(t)}) \approx m_2 + p_2e^{-Q_2t} =: \widetilde{\Lambda}_2(t), \quad (4.18)$$

where

$$m_2 = \sqrt{\beta^* - \frac{\eta^2}{8\alpha^*}}, \quad p_2 = \sqrt{r(0)} - m_2, \quad Q_2 = -\ln[p_2^{-1}(\Lambda_2(1) - m_2)]. \quad (4.19)$$

Utilizing the above expectations of both stochastic processes, we are able to obtain $\mathbb{E}^T(\sqrt{\nu(t)}\sqrt{r(t)})$ numerically. Here we present the derivation of $\mathbb{E}^T(\sqrt{\nu(t)}\sqrt{r(t)})$ by employing properties of dependent random variables and instantaneous correlation. Based on the dependence property between the variables $\sqrt{\nu(t)}$ and $\sqrt{r(t)}$, we can obtain the following

$$\mathbb{E}^T(\sqrt{\nu(t)}\sqrt{r(t)}) = \text{Cov}^T(\sqrt{\nu(t)}, \sqrt{r(t)}) + \mathbb{E}^T(\sqrt{\nu(t)})\mathbb{E}^T(\sqrt{r(t)}). \quad (4.20)$$

In order to figure out $\text{Cov}^T(\sqrt{\nu(t)}, \sqrt{r(t)})$, we utilize the definition of instantaneous correlations

$$\rho_{\sqrt{\nu(t)}\sqrt{r(t)}} = \frac{\text{Cov}^T(\sqrt{\nu(t)}, \sqrt{r(t)})}{\sqrt{\text{Var}^T(\sqrt{\nu(t)})\text{Var}^T(\sqrt{r(t)})}}. \quad (4.21)$$

Substitution of the following

$$\text{Var}^T(\sqrt{\nu(t)}) \approx \frac{\text{Var}^T(\nu(t))}{4\mathbb{E}^T(\nu(t))} \approx \left[q_1(t) - \frac{q_1(t)l_1}{2(l_1 + \varphi_1(t))} \right], \quad (4.22)$$

and

$$\mathbb{V}ar^T(\sqrt{r(t)}) \approx \frac{\mathbb{V}ar^T(r(t))}{4\mathbb{E}^T(r(t))} \approx \left[q_2(t) - \frac{q_2(t)l_2}{2(l_2 + \varphi_2(t))} \right] \quad (4.23)$$

into (4.21) gives us

$$\mathbb{C}ov^T(\sqrt{\nu(t)}, \sqrt{r(t)}) \approx \rho_{\sqrt{\nu(t)}\sqrt{r(t)}} \left(\sqrt{\left(q_1(t) - \frac{q_1(t)l_1}{2(l_1 + \varphi_1(t))} \right) \left(q_2(t) - \frac{q_2(t)l_2}{2(l_2 + \varphi_2(t))} \right)} \right). \quad (4.24)$$

Finally, combining (4.14), (4.18), and (4.24), an approximation of $\mathbb{E}^T(\sqrt{\nu(t)}\sqrt{r(t)})$ is given by

$$\begin{aligned} \mathbb{E}^T(\sqrt{\nu(t)}\sqrt{r(t)}) &\approx \rho_{\sqrt{\nu(t)}\sqrt{r(t)}} \left(\sqrt{\left(q_1(t) - \frac{q_1(t)l_1}{2(l_1 + \varphi_1(t))} \right) \left(q_2(t) - \frac{q_2(t)l_2}{2(l_2 + \varphi_2(t))} \right)} \right) \\ &\quad + (m_1 + p_1 e^{-Q_1 t})(m_2 + p_2 e^{-Q_2 t}). \end{aligned} \quad (4.25)$$

Now that we have obtained the expressions for $\mathbb{E}^T(\sqrt{\nu(t)}\sqrt{r(t)})$ in (4.11), we can adopt Heston's assumption in [66] as follows

$$\tilde{U}(\omega, \nu, r, \tau) = e^{C(\omega, \tau) + D(\omega, \tau)\nu + E(\omega, \tau)r} \tilde{U}(\omega, \nu, r, 0). \quad (4.26)$$

By substituting (4.26) into (4.11), we can obtain the following ODE

$$\frac{dD}{d\tau} = \frac{1}{2}\sigma^2 D^2 + (\rho_{12}\omega\sigma i - \kappa^*)D - \frac{1}{2}(\omega^2 + \omega i), \quad (4.27)$$

with the initial condition

$$D(\omega, 0) = 0, \quad (4.28)$$

and the other two ODEs with initial conditions as described in (4.8) and (4.9).

Note that only the function D has analytical form as

$$\begin{aligned} D(\tau) &= \frac{a+b}{\sigma^2} \frac{1 - e^{b\tau}}{1 - g e^{b\tau}}, \quad a = \kappa^* - \rho_{12}\sigma\omega i, \\ b &= \sqrt{a^2 + \sigma^2(\omega^2 + \omega i)}, \quad g = \frac{a+b}{a-b}. \end{aligned}$$

The solutions of the functions E and C can be found by numerical integration using standard mathematical software package, e.g., MATLAB. Note that the ODE for func-

tion E is the same as the one in Chapter 3. Thus, here we only discuss the algorithm to find an approximate for function C . The MATLAB codes used to perform this integration are given in the Appendix at the end of this chapter.

Since the Fourier transform variable ω appears as a parameter in functions C , D and E , the inverse Fourier transform is conducted to retrieve the solution as in its initial setup

$$\begin{aligned} U(x, \nu, r, \tau) &= \mathfrak{F}^{-1} \left[\tilde{U}(\omega, \nu, r, \tau) \right] \\ &= \mathfrak{F}^{-1} \left[e^{C(\omega, \tau) + D(\omega, \tau)\nu + E(\omega, \tau)r} \mathfrak{F}[H(e^x)] \right]. \end{aligned} \quad (4.29)$$

Following the techniques same as those used in (3.28) to (3.30), the solution of the PDE (4.3) is derived as follows

$$U_j(S, \nu, r, I, \tau) \approx \frac{e^{2x}}{I^2} e^{\tilde{C}(\tau) + \tilde{D}(\tau)\nu + \tilde{E}(\tau)r} - \frac{2e^x}{I} e^{\hat{C}(\tau) + \hat{E}(\tau)r} + 1, \quad (4.30)$$

where $t_{j-1} \leq t \leq t_j$ and $\tau = t_j - t$. We use $\tilde{C}(\tau)$, $\tilde{D}(\tau)$ and $\tilde{E}(\tau)$ to denote $C(-2i, \tau)$, $D(-2i, \tau)$ and $E(-2i, \tau)$ respectively. In addition, $\hat{C}(\tau)$ and $\hat{E}(\tau)$ represent $C(-i, \tau)$ and $E(-i, \tau)$ respectively.

4.2.2 Solution for the Second Step

In this subsection, we shall continue to carry out the second step in finding out the expectation (3.7) as described in Subsection 3.2.1 with full correlation case. In particular, we aim to calculate the expectation $\mathbb{E}^T[E_{j-1}|\mathcal{F}(0)]$, which will finally lead us to obtain the fair delivery price of a variance swap.

The computation of E_{j-1} has been worked out by our first step. Following (4.30) and by letting $\tau = \Delta t$ in $U_j(S, \nu, r, I, \tau)$, the solution of the first step is

$$E_{j-1} \approx e^{\tilde{C}(\Delta t) + \tilde{D}(\Delta t)\nu(t_{j-1}) + \tilde{E}(\Delta t)r(t_{j-1})} - 2e^{\hat{C}(\Delta t) + \hat{E}(\Delta t)r(t_{j-1})} + 1. \quad (4.31)$$

Now, (4.31) implies that

$$\begin{aligned}
\mathbb{E}^T[E_{j-1}|\mathcal{F}(0)] &\approx \mathbb{E}^T \left[e^{\tilde{C}(\Delta t) + \tilde{D}(\Delta t)\nu(t_{j-1}) + \tilde{E}(\Delta t)r(t_{j-1})} - 2e^{\hat{C}(\Delta t) + \hat{E}(\Delta t)r(t_{j-1})} + 1 | \mathcal{F}(0) \right] \\
&\approx \mathbb{E}^T \left[e^{\tilde{C}(\Delta t) + \tilde{D}(\Delta t)\nu(t_{j-1}) + \tilde{E}(\Delta t)r(t_{j-1})} | \mathcal{F}(0) \right] \\
&\quad - 2\mathbb{E}^T \left[e^{\hat{C}(\Delta t) + \hat{E}(\Delta t)r(t_{j-1})} | \mathcal{F}(0) \right] + 1.
\end{aligned} \tag{4.32}$$

Based on the assumption that $\nu(t_{j-1})$ and $r(t_{j-1})$ are dependent, and the approximations of $\mathbb{E}^T(\sqrt{\nu(t)})$ and $\mathbb{E}^T(\sqrt{r(t)})$, we give an approximation of $\mathbb{E}^T[E_{j-1}|\mathcal{F}(0)]$ as follows

$$\begin{aligned}
\mathbb{E}^T[E_{j-1}|\mathcal{F}(0)] &\approx e^{\tilde{C}(\Delta t)} \cdot \mathbb{E}^T \left[e^{\tilde{D}(\Delta t)\nu(t_{j-1}) + \tilde{E}(\Delta t)r(t_{j-1})} | \mathcal{F}(0) \right] \\
&\quad - 2e^{\hat{C}(\Delta t)} \cdot \mathbb{E}^T \left[e^{\hat{E}(\Delta t)r(t_{j-1})} | \mathcal{F}(0) \right] + 1 \\
&\approx e^{\tilde{C}(\Delta t)} \cdot \exp[\tilde{D}(\Delta t)(q_1(t_{j-1})(l_1 + \varphi_1(t_{j-1}))) \\
&\quad + \tilde{E}(\Delta t)(q_2(t_{j-1})(l_2 + \varphi_2(t_{j-1}))) \\
&\quad + \frac{\tilde{D}(\Delta t)^2}{2}(q_1(t_{j-1})^2(2l_1 + 4\varphi_1(t_{j-1}))) \\
&\quad + \frac{\tilde{E}(\Delta t)^2}{2}(q_2(t_{j-1})^2(2l_2 + 4\varphi_2(t_{j-1}))) \\
&\quad + \tilde{D}(\Delta t)\tilde{E}(\Delta t)\rho_{23}\sqrt{q_1(t_{j-1})^2(2l_1 + 4\varphi_1(t_{j-1}))} \\
&\quad \sqrt{q_2(t_{j-1})^2(2l_2 + 4\varphi_2(t_{j-1}))}] \\
&\quad - 2e^{\hat{C}(\Delta t)} \cdot \exp[\hat{E}(\Delta t)(q_2(t_{j-1})(l_2 + \varphi_2(t_{j-1}))) \\
&\quad + \frac{\hat{E}(\Delta t)^2}{2}(q_2(t_{j-1})^2(2l_2 + 4\varphi_2(t_{j-1})))]) + 1.
\end{aligned} \tag{4.33}$$

Now, we show how to derive expressions for $\mathbb{E}^T \left[e^{\tilde{D}(\Delta t)\nu(t_{j-1}) + \tilde{E}(\Delta t)r(t_{j-1})} | \mathcal{F}(0) \right]$ and $\mathbb{E}^T \left[e^{\hat{E}(\Delta t)r(t_{j-1})} | \mathcal{F}(0) \right]$ by approximations using normally distributed random variables. The variables $\nu(t_{j-1})$ and $r(t_{j-1})$ can be well approximated by normally distributed random variables as follows

$$\nu(t_{j-1}) \approx \mathcal{N} \left(q_1(t_{j-1})(l_1 + \varphi_1(t_{j-1})), q_1(t_{j-1})^2(2l_1 + 4\varphi_1(t_{j-1})) \right), \tag{4.34}$$

and

$$r(t_{j-1}) \approx \mathcal{N}\left(q_2(t_{j-1})(l_2 + \varphi_2(t_{j-1})), q_2(t_{j-1})^2(2l_2 + 4\varphi_2(t_{j-1}))\right). \quad (4.35)$$

Let $Y(0, t_{j-1}) = \tilde{D}(\Delta t)\nu(t_{j-1}) + \tilde{E}(\Delta t)r(t_{j-1})$. With those normal random variable approximations in (4.34) and (4.35), we can find the characteristic function of $Y(0, t_{j-1})$ as follows

$$\mathbb{E}^T \left[e^{Y(0, t_{j-1})} | \mathcal{F}(0) \right] \approx \exp[\mathbb{E}^T(Y(0, t_{j-1})) + \frac{1}{2}\text{Var}^T(Y(0, t_{j-1}))], \quad (4.36)$$

where

$$\mathbb{E}^T(Y(0, t_{j-1})) \approx \tilde{D}(\Delta t)(q_1(t_{j-1})(l_1 + \varphi_1(t_{j-1}))) + \tilde{E}(\Delta t)(q_2(t_{j-1})(l_2 + \varphi_2(t_{j-1}))), \quad (4.37)$$

and

$$\begin{aligned} \text{Var}^T(Y(0, t_{j-1})) &\approx \tilde{D}(\Delta t)^2(q_1(t_{j-1})^2(2l_1 + 4\varphi_1(t_{j-1}))) \\ &\quad + \tilde{E}(\Delta t)^2(q_2(t_{j-1})^2(2l_2 + 4\varphi_2(t_{j-1}))) \\ &\quad + 2\tilde{D}(\Delta t)\tilde{E}(\Delta t)\rho_{23}\sqrt{q_1(t_{j-1})^2(2l_1 + 4\varphi_1(t_{j-1}))} \\ &\quad \sqrt{q_2(t_{j-1})^2(2l_2 + 4\varphi_2(t_{j-1}))}. \end{aligned} \quad (4.38)$$

We can apply the same procedure to find the expression for $\mathbb{E}^T \left[e^{\hat{E}(\Delta t)r(t_{j-1})} | \mathcal{F}(0) \right]$, which is given by

$$\begin{aligned} \mathbb{E}^T \left[e^{\hat{E}(\Delta t)r(t_{j-1})} | \mathcal{F}(0) \right] &\approx \exp[\mathbb{E}^T(\hat{E}(\Delta t)r(t_{j-1})) + \frac{1}{2}\text{Var}^T(\hat{E}(\Delta t)r(t_{j-1}))] \\ &\approx \exp[\hat{E}(\Delta t)(q_2(t_{j-1})(l_2 + \varphi_2(t_{j-1}))) \\ &\quad + \frac{\hat{E}(\Delta t)^2}{2}(q_2(t_{j-1})^2(2l_2 + 4\varphi_2(t_{j-1})))]. \end{aligned} \quad (4.39)$$

Following the steps in Subsection 3.2.4, we can substitute (4.33) into (3.41) which will finally give us the fair delivery price of a variance swap. Our solution technique involves derivation of the characteristic function using approximations in order to fulfill the affinity property for fully correlated state variables. Since this approach proposes two steps of PDE solving problem, the utmost complexity is embedded in finding the final semi-analytical solution. Using some numerical results, the next section conducts analysis for the performance of our formulation for pricing variance swaps.

4.3 Numerical Results

In this section, we present some numerical tests in order to analyze the performance of our approximation formula for evaluating prices of variance swaps. In Subsection 4.3.1, comparisons are made among the results calculated by our formula, the Monte Carlo (MC) simulation which resembles the real market, and the continuous-sampling variance swaps model in [102]. We also use the finite difference method to obtain a numerical solution. In addition, in Subsection 4.3.2, we investigate the impact of full correlations among the state variables in our model on the delivery prices of variance swaps. For the base parameter setting, we follow these parameter values, where $S(0) = 1$, $\rho_{12} = -0.615$, $\rho_{13} = 0.20$, $\rho_{23} = 0.15$, $\nu(0) = (0.2045)^2$, $\theta^* = (0.2874)^2$, $\kappa^* = 0.3$, $\sigma = 0.4921$, $r(0) = 0.04$, $\alpha^* = 0.501$, $\beta^* = 0.04$, $\eta = 0.005$ and $T = 1$.

4.3.1 Comparison among Our Formula, MC Simulation and Continuous-Sampling Model

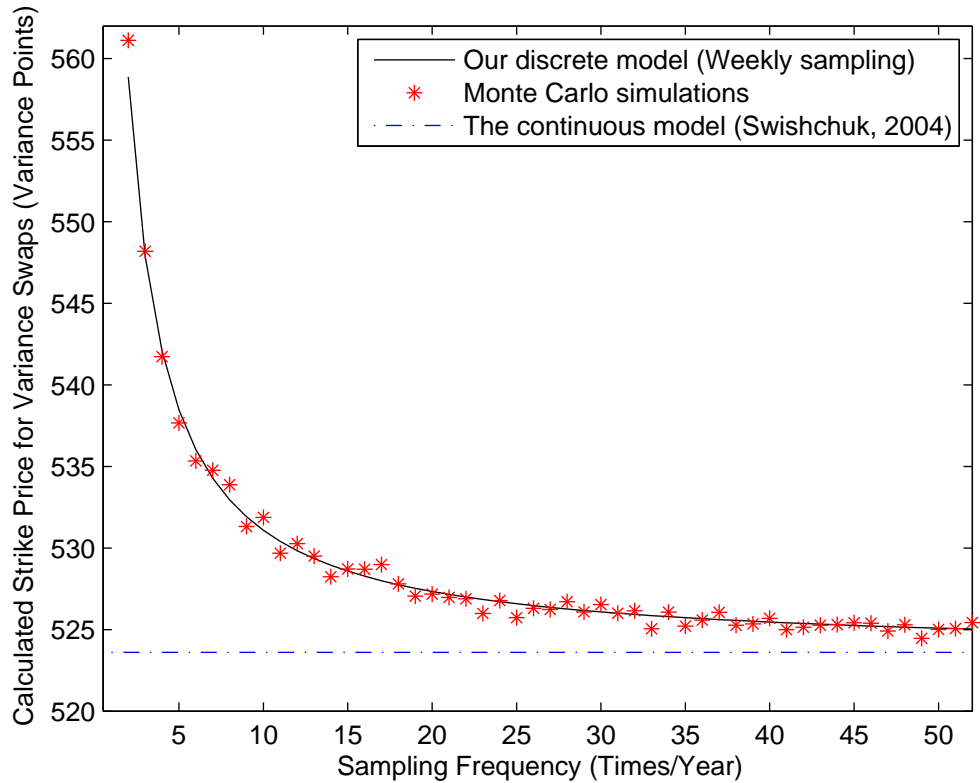
We perform our MC simulation in this paper using the Euler-Maruyama scheme with 200,000 path numbers and $S(0) = 1$. The comparison results made between numerical implementation of our formula, along with the MC simulation and the continuous-sampling model are presented in Figure 4.1 and in Table 4.1. All values for the fair delivery prices are depicted in variance points, which are the squares of volatility points.

Table 4.1: Comparing prices of variance swaps among our formula, continuous-sampling model and MC simulation.

Sampling Frequency	Our formula	Continuous-sampling model	MC simulation
N=4	542.06	524.00	541.73
N=12	529.84	524.00	530.27
N=26	526.47	524.00	526.30
N=52	525.03	524.00	525.43

In Figure 4.1 the graphs of all three compared models are plotted against each other, with the MC simulation taken as benchmark. It could be clearly seen that our approximation formula matches the MC simulation very well, whereas the continuous-sampling model does not provide a satisfactory fit. To gain some insights of the relative difference between our formula and the MC simulation, we compare their relative percentage error. By taking $N = 52$ which is the weekly sampling frequency for market practice and 200,000 path numbers, we discover that the error produced is 0.07%, with further reduction of this quantity as path numbers reach 500,000. In fact, even for

Figure 4.1: Comparison of the delivery price of variance swaps produced among our model, the continuous-sampling model and MC simulation.



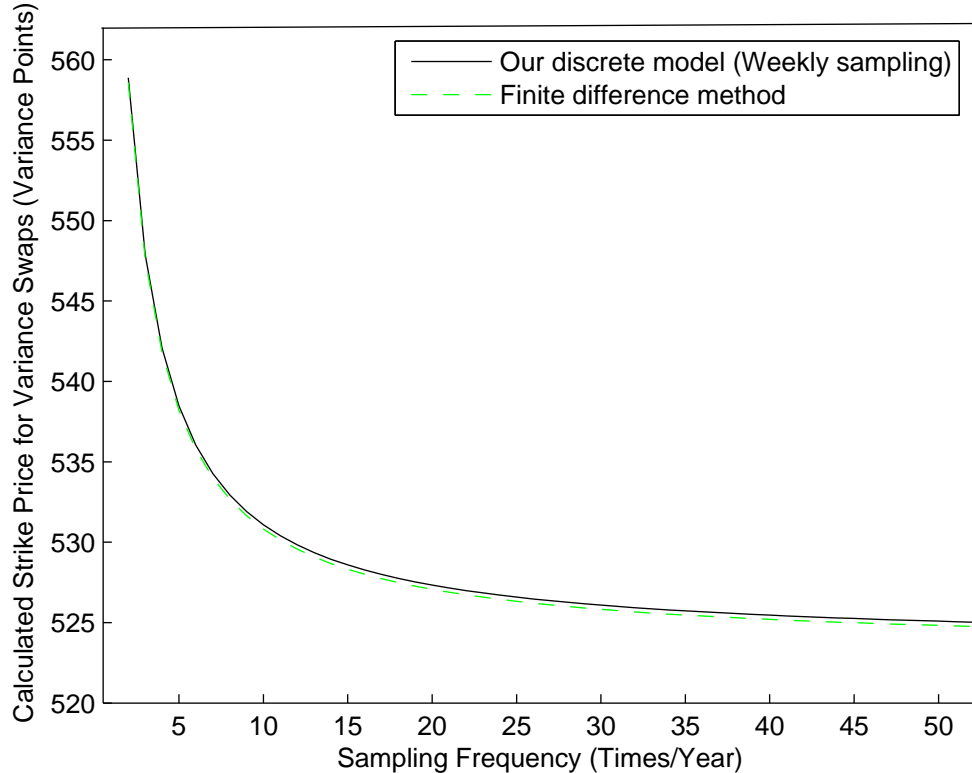
small sampling frequency such as the quarterly sampling frequency when $N = 4$, our formula can be executed just in 0.49 seconds compared to 27.7 seconds needed by the MC simulation. These findings verify the accuracy and efficiency of our formula.

In addition, we conduct a comparison between our approximation formula and the numerical solution of the partial differential equation in (4.3) using the finite difference method. Using the same set of parameter values, we found out in Figure 4.2 that our approximation formula resembles the numerical solution very closely. This finding supports the performance quality of our pricing formula.

4.3.2 Impact of Correlation among Asset Classes

Next, we investigate the impact of the correlation coefficients among the interest rate with the underlying and the volatility respectively in our model. For this purpose,

Figure 4.2: Comparison of the delivery price of variance swaps produced among our model and the finite difference method.

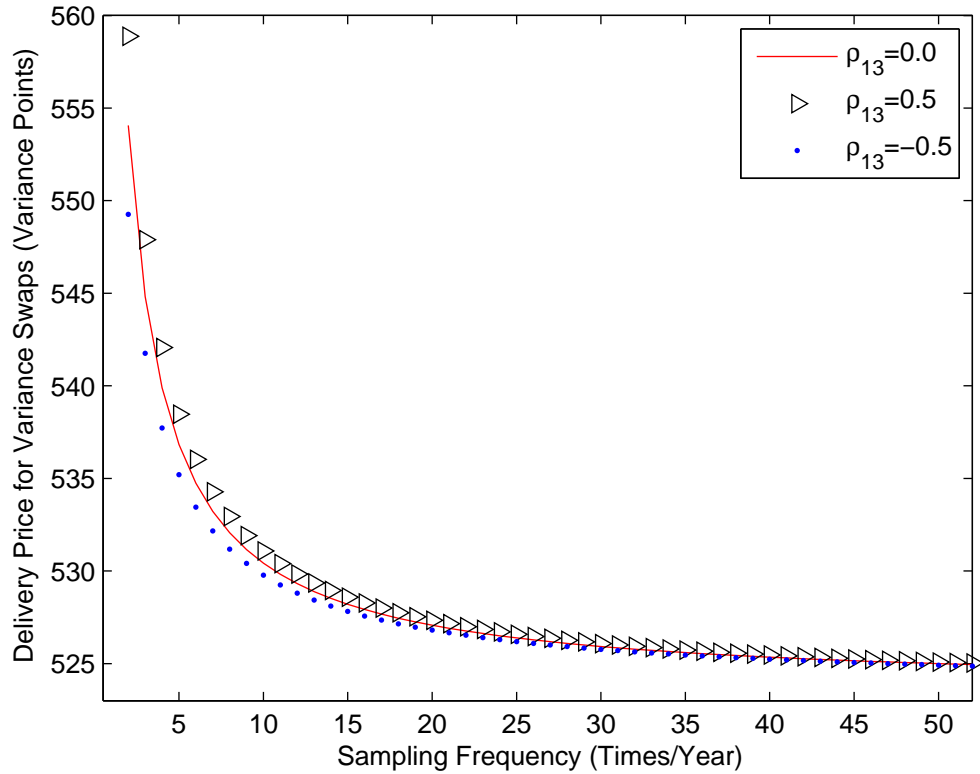


we examine the relationship produced by the delivery price of the variance swap when computed against different correlation settings. The impact of the correlations among the interest rate with the underlying can be found in Figure 4.3.

From Figure 4.3, we can see that the values of variance swaps are increasing as the correlation values increase. Ignoring the correlation coefficient between the interest rate and the underlying might result in mispricing of about 5 basis points in the delivery price. This is very crucial since a relative error of even 2% might produce considerable amount of loss due to the nature of the notional amount and size of contract traded per order. However, it is also observed that the impact of these correlation coefficients becomes less apparent as the number of sampling frequencies increases.

The effects of the correlation coefficient among the interest rate and the volatility are given in Figure 4.4. In contrast to the correlation effects in Figure 4.3, smaller impacts are observed for these asset classes. The variance swap values produced by all three coefficients are almost overlapping with each other. For example, for $N = 12$

Figure 4.3: Impact of different ρ_{13} values on delivery prices of variance swaps in the Heston-CIR hybrid model.

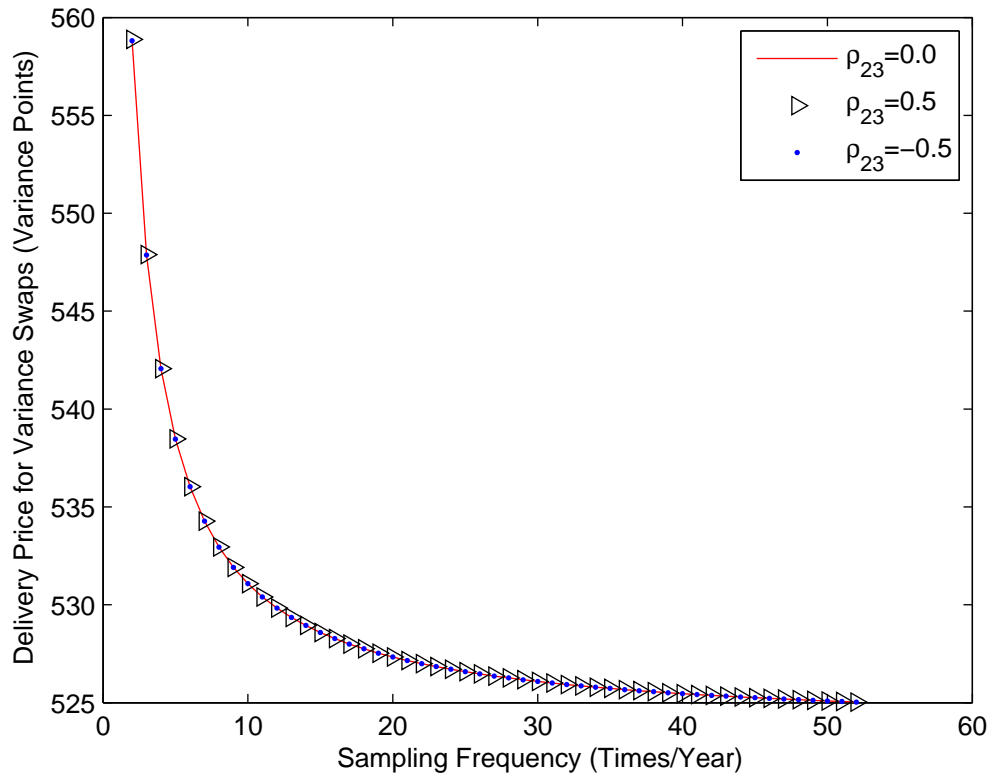


which is the monthly sampling frequency, the delivery price is 529.834 for $\rho_{23} = 0$, with only a slight increase to 529.836 for $\rho_{23} = 0.5$, and a slight reduction to 529.833 for $\rho_{23} = -0.5$. Besides that, this graph also displays the same trend of reducing impact of the correlation coefficients as the number of sampling frequencies increases.

4.4 Appendix

In this appendix, we show the codes implemented in MATLAB to find the approximate solution for function C . Note that the codes for function E in this chapter are the same as given in Appendix Chapter 3. For function C , we have to find solutions for $\tilde{C}(\tau)$ and $\hat{C}(\tau)$ which are the notations for $C(-2i, \tau)$ and $C(-i, \tau)$ respectively. We represent $\mathbb{E}^T(\sqrt{\nu(T-\tau)}\sqrt{r(T-\tau)})$ as M in our code for the ODE of $\tilde{C}(\tau)$ below

Figure 4.4: Impact of different ρ_{23} values on delivery prices of variance swaps in the Heston-CIR hybrid model.



```

function dydt = myodefull_cwidetilde(t,y,Dt,D,Et,E,Bt,B,Mt,M)
D = interp1(Dt,D,t);
E = interp1(Et,E,t);
B = interp1(Bt,B,t);
M = interp1(Mt,M,t);

[ K S nu0 rho rho_sqvr rho_xr rho_vr theta_star lambda_1 lambda_2 sigma
kapa T t alpha beta_star r0 eta]=InputPara_2( );
kapa_star= kapa + lambda_1; alpha_star = alpha + lambda_2;
alpha_star = 0.501;
beta_star = 0.04;

dydt = (kapa_star*theta_star.*D)+(alpha_star*beta_star.*E)
-(2.*(rho_xr.*eta.*B.*M))+ (2.*(rho_xr.*eta.*E.*M)
-(rho_vr.*sigma.*eta.*B.*D.*M )+(rho_vr.*eta.*sigma.*D.*E.*M );

```

We can later obtain the approximate solution for $\tilde{C}(\tau)$ using the following codes

```

function [C_tilde] = call_cwidetildefull( t, T, delT, r0, nu0, rho, rho_sqvr,
rho_xr, rho_vr, sigma, alpha_star, kapa_star, lambda_1, lambda_2, beta_star,
eta, theta_star)

[ K S nu0 rho rho_sqvr rho_xr rho_vr theta_star lambda_1 lambda_2 sigma
kapa T t alpha beta_star r0 eta]=InputPara.2( );
kapa_star= kapa + lambda_1; alpha_star = alpha + lambda_2;

alpha_star = 0.501;
beta_star = 0.04;
eta = 0.005;
r0 = 0.04;
T=1;

delTend=delT;
delT=0:delTend/100:delTend;

d_wildetilde = D_widetilde( t, T, delT, r0, nu0, rho, rho_sqvr,
rho_xr, rho_vr, sigma, alpha_star, kapa_star, lambda_1, lambda_2,
beta_star, eta, theta_star);
e_wildetildefull = call_ewidetildefull( t, T, delTend, r0, nu0,
rho, rho_sqvr, rho_xr, rho_vr, sigma, alpha_star, kapa_star,
lambda_1, lambda_2, beta_star, eta, theta_star);
exp_sqvr = E_sqvr( t, T, delT, r0, nu0, rho, rho_sqvr, rho_xr,
rho_vr, sigma, alpha_star, kapa_star, lambda_1, lambda_2,
beta_star, eta, theta_star);
b_function = B_function( t, T, delT, r0, nu0, rho, rho_sqvr,
rho_xr, rho_vr, sigma, alpha_star, kapa_star, lambda_1, lambda_2,
beta_star, eta, theta_star);

D=d_wildetilde;
Dt=delT;
Et =delT;
E = e_wildetildefull;
M= exp_sqvr;
Mt=delT;
B=b_function;
Bt=delT;

```

```
Tspan = delT; % Solve from t=1 to t=5
IC = 0; % y(t=0) = 1
[Tn Y] = ode45(@ (t,y) myodefull.cwidetilde(t,y,Dt,D,Et,E,Bt,B,Mt,M),
Tspan,IC);
C_tilde=Y;
```

Finally, the approximate solution for $\widehat{C}(\tau)$ is attained by replacing $\omega = -i$ in its ODE in (4.13) and following the coding steps as outlined for $\widetilde{C}(\tau)$.

Chapter 5

Pricing Variance Swaps under Stochastic Factors with Regime Switching

In this chapter, the hybrid pricing model given in Chapter 3 is further considered. We address the issue of pricing discretely-sampled variance swaps under stochastic volatility and stochastic interest rate with regime switching. This regime switching hybrid model is presented in Section 5.1 and possesses parameters that switch according to a continuous-time observable Markov chain process. This switching phenomena can be interpreted as the responsiveness of the underlying asset towards the states of an observable macroeconomic factor. To accomplish this, we first derive the dynamics for this model under the T -forward measure in Subsection 5.1.2. Here, we demonstrate how to handle the change of measure with the existence of regime switching. In Section 5.2, we derive the forward characteristic function in order to obtain the analytical formula for the price of variance swaps. The last section demonstrates some numerical examples and discussion on our findings, including the observed impacts of regime switching.

5.1 Modeling Framework

In this section, we consider the Heston stochastic volatility model which is combined with the one-factor Cox-Ingersoll-Ross (CIR) stochastic interest rate model with regime switching to describe the valuation of variance swaps. A regime switching model for pricing volatility derivatives was first considered by Elliot et al. in [37]. Our aim is to

achieve a better characterization of the market by incorporating stochastic interest rate into the modeling framework. This extends the work of Elliot and Lian in [34] which only focuses on regime switching effects on the Heston stochastic volatility model. In the literature, [97] presented a pricing formula for continuously-sampled variance swaps under the hybridization of the Schöbel-Zhu and the Hull-White models. However, positive interest rate or positive volatility could not be guaranteed in this hybrid model. In fact, their formula was represented in an integral form as a continuous approximation to the discrete sampling practice in the market. The change of measure technique proposed by these authors was also different from the technique used in this thesis. In what follows, we shall give some description regarding the formulation of this hybrid model. Furthermore, we shall demonstrate how to deal with an extension to the model in Section 3.1 to the case where regime switching is incorporated.

5.1.1 The Heston-CIR Model with Regime Switching

The Heston-CIR model adjusted by the Markov chain described in this section is capable of capturing several macroeconomic issues such as alternating business cycles which affect the asset price as well as the dynamics of volatility and interest rate. In other words, specific parameters of the dynamics involved in the asset price, volatility and interest rate switch over time according to the observable states of an economy. Let $[0, T]$ be a finite time interval. Since we will deal with an extension of the model in Section 3.1 to the case where regime switching is involved, we shall consider a risk-neutral probability measure \mathbb{Q} . We model the market dynamics by a continuous-time finite-state observable Markov chain $\mathbf{X} = \{X(t) : 0 \leq t \leq T\}$ with different states of an economy denoted by state space $S = \{s_1, s_2, \dots, s_N\}$. Without loss of generality, the state space can be identified with the set of unit vectors $\{e_1, e_2, \dots, e_N\}$, where $e_i = (0, \dots, 1, \dots, 0)^\top \in \mathbb{R}^N$. An N -by- N rate matrix $Q = [q_{ij}]_{1 \leq i, j \leq N}$ is used to generate the evolution of the chain under \mathbb{Q} . Here for $i \neq j$, q_{ij} is the (constant) intensity of the transition of the chain \mathbf{X} from state e_i to state e_j in a small interval of time, for each $i, j = 1, 2, \dots, N$, satisfying $q_{ij} \geq 0$ for $i \neq j$ and $\sum_{i=1}^N q_{ij} = 0$ for all $j = 1, 2, \dots, N$. According to [38], a semi-martingale representation holds for the process \mathbf{X} as follows

$$X(t) = X(0) + \int_0^t QX(s)ds + M(t), \quad (5.1)$$

where $\{M(t) : 0 \leq t \leq T\}$ is a \mathbb{R}^N -valued martingale with respect to the filtration generated by \mathbf{X} under \mathbb{Q} .

As mentioned before, the regime switching effect is captured in our Heston-CIR model via assuming that the stock, the volatility and the interest rate are dependent on market trends or other economic factors indicated by the regime switching Markov chain \mathbf{X} . In particular, it is assumed that the long-term mean of variance $\theta^*(t)$ of the risky stock depends on the states of the economic indicator $X(t)$, that is

$$\theta^*(t) = \langle \theta^*, X(t) \rangle, \quad (5.2)$$

where $\theta^* = (\theta_1^*, \theta_2^*, \dots, \theta_N^*)^\top$ with $\theta_i^* > 0$, for each $i = 1, 2, \dots, N$. $\langle \cdot, \cdot \rangle$ denotes the scalar product in \mathbb{R}^N . In addition, the long-term mean of the interest rate $\beta^*(t)$ is also assumed to be influenced by the states of the economic indicator described by $X(t)$, i.e.

$$\beta^*(t) = \langle \beta^*, X(t) \rangle, \quad (5.3)$$

where $\beta^* = (\beta_1^*, \beta_2^*, \dots, \beta_N^*)^\top$ with $\beta_i^* > 0$, for each $i = 1, 2, \dots, N$. Thus, we adjust the system in (3.2) under the risk-neutral measure \mathbb{Q} as follows

$$\begin{cases} dS(t) = r(t)S(t)dt + \sqrt{\nu(t)}S(t)d\widetilde{W}_1(t), & 0 \leq t \leq T, \\ d\nu(t) = \kappa^*(\theta^*(t) - \nu(t))dt + \sigma\sqrt{\nu(t)}d\widetilde{W}_2(t), & 0 \leq t \leq T, \\ dr(t) = \alpha^*(\beta^*(t) - r(t))dt + \eta\sqrt{r(t)}d\widetilde{W}_3(t), & 0 \leq t \leq T. \end{cases} \quad (5.4)$$

Working on the decomposition technique same as that in (3.11), we can re-write the SDEs in (5.4) under \mathbb{Q} in the form of

$$\begin{bmatrix} \frac{dS(t)}{S(t)} \\ d\nu(t) \\ dr(t) \end{bmatrix} = \mu^{\mathbb{Q}}dt + \Sigma \times C \times \begin{bmatrix} dW_1^*(t) \\ dW_2^*(t) \\ dW_3^*(t) \end{bmatrix}, \quad 0 \leq t \leq T, \quad (5.5)$$

with

$$\mu^{\mathbb{Q}} = \begin{bmatrix} r(t) \\ \kappa^*(\theta^*(t) - \nu(t)) \\ \alpha^*(\beta^*(t) - r(t)) \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \sqrt{\nu(t)} & 0 & 0 \\ 0 & \sigma\sqrt{\nu(t)} & 0 \\ 0 & 0 & \eta\sqrt{r(t)} \end{bmatrix} \quad (5.6)$$

and

$$C = \begin{bmatrix} 1 & 0 & 0 \\ \rho & \sqrt{1-\rho^2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

such that

$$CC^\top = \begin{bmatrix} 1 & \rho & 0 \\ \rho & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and $dW_1^*(t)$, $dW_2^*(t)$ and $dW_3^*(t)$ are mutually independent under \mathbb{Q} satisfying

$$\begin{bmatrix} d\widetilde{W}_1(t) \\ d\widetilde{W}_2(t) \\ d\widetilde{W}_3(t) \end{bmatrix} = C \times \begin{bmatrix} dW_1^*(t) \\ dW_2^*(t) \\ dW_3^*(t) \end{bmatrix}, \quad 0 \leq t \leq T.$$

5.1.2 Model Dynamics under the T -Forward Measure with Regime Switching

In this subsection, we shall focus on obtaining the new dynamics for our model under the T -forward measure when the regime switching is incorporated. Hence, we shall adjust the numeraire under \mathbb{Q}^T in Subsection 3.2.1 due to the enlarged filtration generated by the short rate and the Markov process.

First, we assume that the bond price is having a regime switching exponential affine form representation which is denoted by

$$P(t, T, r(t), X(t)) = e^{A(t, T, X(t)) - B(t, T)r(t)}, \quad (5.7)$$

where $A(t, T, X(t))$ and $B(t, T)$ are to be determined. Note that under \mathbb{Q} , we have to take the discounted price of this bond to ensure the martingale property. The discounted bond price is given by

$$\widetilde{P}(t, T, r(t), X(t)) = e^{-\int_0^t r(s)ds} P(t, T, r(t), X(t)). \quad (5.8)$$

Next, applying Itô formula to $\widetilde{P}(t, T, r(t), X(t))$ and realizing that the non-martingale terms must sum to zero, we obtain

$$\frac{\partial P}{\partial t} + \alpha^*(\beta^*(t) - r) \frac{\partial P}{\partial r} + \langle \mathbf{P}, QX(t) \rangle + \frac{1}{2} \frac{\partial^2 P}{\partial r^2} \eta^2 r - rP = 0, \quad (5.9)$$

with terminal condition $P(T, T, r(T), X(T)) = 1$, $\mathbf{P} = (P_1, P_2, \dots, P_N)^\top$, $P_i = P(t, T, r, e_i)$ and $P_i = e^{\int_0^t r(s)ds} \widetilde{P}_i$ for all $i = 1, 2, \dots, N$. Realizing the fact that $X(t)$ takes one of the

values from the set of unit vectors $\{e_1, e_2, \dots, e_N\}$, we can write

$$\begin{aligned} X(t) &= e_i \quad (i = 1, 2, \dots, N), \\ \theta^*(t) &= \langle \theta^*, X(t) \rangle = \theta_i^*, \\ \beta^*(t) &= \langle \beta^*, X(t) \rangle = \beta_i^*, \\ P(t, T, r, X(t)) &= P(t, T, r, e_i) = P_i. \end{aligned}$$

As a result, equation (5.9) becomes the following N coupled PDEs for all $i = 1, 2, \dots, N$

$$\frac{\partial P_i}{\partial t} + \alpha^*(\beta_i^* - r) \frac{\partial P_i}{\partial r} + \langle \mathbf{P}, Qe_i \rangle + \frac{1}{2} \frac{\partial^2 P_i}{\partial r^2} \eta^2 r - rP_i = 0, \quad (5.10)$$

with terminal condition $P_i(T, T, r) = 1$. We later substitute the expressions of $\frac{\partial P}{\partial t}$, $\frac{\partial P}{\partial r}$ and $\frac{\partial^2 P}{\partial r^2}$ into the above PDE to obtain the following differential equations

$$\begin{cases} \frac{dB(t, T)}{dt} = \frac{1}{2} \eta^2 B(t, T)^2 + \alpha^* B(t, T) - 1, \\ \frac{dA_i}{dt} = \alpha^* \beta_i^* B(t, T) - e^{-A_i} \langle \tilde{\mathbf{A}}, Qe_i \rangle, \end{cases} \quad (5.11)$$

where $A_i = A(t, T, e_i)$, $\tilde{A}_i = e^{A_i}$, $\mathbf{A} = (A_1, A_2, \dots, A_N)^\top$ and $\tilde{\mathbf{A}} = (\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_N)^\top$ for all $i = 1, 2, \dots, N$. Also, $B(T, T) = 0$, $A(T, T, X(T)) = 0$ and $A_i(T, T) = 0$ respectively. Solution of the first equation of (5.11) is similar to that of the same type of ODE for the CIR model as given in [27] using characteristic equations, and

$$B(t, T) = \frac{2 \left(e^{(T-t)\sqrt{(\alpha^*)^2 + 2\eta^2}} - 1 \right)}{2\sqrt{(\alpha^*)^2 + 2\eta^2} + \left(\alpha^* + \sqrt{(\alpha^*)^2 + 2\eta^2} \right) \left(e^{(T-t)\sqrt{(\alpha^*)^2 + 2\eta^2}} - 1 \right)}.$$

Next, we show the steps to obtain the function A_i . Let $\Upsilon_i(t) = \alpha^* \beta_i^* B(t, T)$ for $i = 1, 2, \dots, N$ and consider the diagonal matrix

$$\text{diag}(\Upsilon(t)) = \text{diag}(\Upsilon_1(t), \Upsilon_2(t), \dots, \Upsilon_N(t)).$$

Denote $\tilde{A}_i = e^{A_i}$ for $i = 1, 2, \dots, N$ and substituting this into the ODE of function A_i in (5.11) satisfies

$$\frac{d\tilde{\mathbf{A}}}{dt} = [\text{diag}(\Upsilon(t)) - Q^\top] \tilde{\mathbf{A}},$$

with $\tilde{\mathbf{A}}(T, T) = \mathbf{1}$ and $\mathbf{1} = (1, 1, \dots, 1)^\top \in \mathbb{R}^N$. Suppose $\Phi(t)$ is the fundamental matrix solution of

$$\frac{d\Phi(t)}{dt} = [\text{diag}(\Upsilon(t)) - Q^\top]\Phi(t), \quad \Phi(T) = I,$$

then we can write

$$\tilde{\mathbf{A}}(t, T) = \Phi(t)\tilde{\mathbf{A}}(T, T) = \Phi(t)\mathbf{1}.$$

Now, $\tilde{\mathbf{A}}(t, T) = \Phi(t)\mathbf{1}$ implies that $\tilde{A}_i(t, T, e_i) = \langle \Phi(t)\mathbf{1}, e_i \rangle$, and utilizing $\tilde{A}_i = e^{A_i}$ for $i = 1, 2, \dots, N$ gives us $A_i(t, T, e_i) = \ln(\langle \Phi(t)\mathbf{1}, e_i \rangle)$. Thus,

$$A(t, T, X(t)) = \langle \mathbf{A}, X(t) \rangle. \quad (5.12)$$

Moving on, we implement the measure change from the risk-neutral measure \mathbb{Q} to the T -forward measure \mathbb{Q}^T . Following the techniques same as those in Subsection 3.2.1, differentiating $\ln N_{1,t}$ which is $e^{\int_0^t r(s)ds}$ yields

$$d \ln N_{1,t} = r(t)dt = \left(\int_0^t \alpha^*(\beta^* - r(s))ds \right) dt + \left(\int_0^t \eta \sqrt{r(s)} d\tilde{W}_3(s) \right) dt,$$

whereas the differentiation of $\ln N_{2,t}$ which is $P(t, T, r(t), X(t)) = \tilde{A}(t, T, X(t))e^{-B(t, T)r(t)}$ gives

$$\begin{aligned} d \ln N_{2,t} = & \left[\frac{\frac{\partial \tilde{A}(t, T, X(t))}{\partial t}}{\tilde{A}(t, T, X(t))} - \frac{\partial B(t, T)}{\partial t} r(t) - B(t, T) \alpha^*(\beta^*(t) - r(t)) \right. \\ & \left. + \left\langle \frac{\tilde{\mathbf{A}}(t, T)}{\tilde{A}(t, T, X(t))}, QX(t) \right\rangle \right] dt - B(t, T) \eta \sqrt{r(t)} d\tilde{W}_3(t) \\ & + \left\langle \frac{\tilde{\mathbf{A}}(t, T)}{\tilde{A}(t, T, X(t))}, dM(t) \right\rangle. \end{aligned}$$

As mentioned in the previous chapters, the above steps assist in obtaining the volatilities for both numeraires. Next, we proceed to find μ^T which is the new drift for our SDEs under \mathbb{Q}^T with regime switching by using the formula in (3.12)

$$\mu^T = \mu^{\mathbb{Q}} - \left[\Sigma \times C \times C^\top \times (\Sigma^{\mathbb{Q}} - \Sigma^T) \right],$$

with $\Sigma^{\mathbb{Q}}$ and Σ^T given by

$$\Sigma^{\mathbb{Q}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ and } \Sigma^T = \begin{bmatrix} 0 \\ 0 \\ -B(t, T)\eta\sqrt{r(t)} + \left\langle \frac{\tilde{\mathbf{A}}(t, T)}{\tilde{A}(t, T, X(t))}, dM(t) \right\rangle \end{bmatrix},$$

along with Σ and CC^T as defined in (5.6). However, due to the independence between $d\tilde{W}_3(t)$ and $M(t)$, we obtain the new dynamics for (5.4) under the forward measure \mathbb{Q}^T as

$$\begin{bmatrix} \frac{dS(t)}{S(t)} \\ d\nu(t) \\ dr(t) \end{bmatrix} = \begin{bmatrix} r(t) \\ \kappa^*(\theta^*(t) - \nu(t)) \\ \alpha^*\beta^*(t) - [\alpha^* + B(t, T)\eta^2]r(t) \end{bmatrix} dt + \Sigma \times C \times \begin{bmatrix} dW_1^*(t) \\ dW_2^*(t) \\ dW_3^*(t) \end{bmatrix}, 0 \leq t \leq T. \quad (5.13)$$

In addition, under \mathbb{Q}^T , the semi-martingale decomposition of the Markov chain \mathbf{X} is given by

$$X(t) = X(0) + \int_0^t Q^T(s)X(s)ds + M^T(t), \quad (5.14)$$

with the rate matrix $Q^T(t) = [q_{ij}^T(t)]_{i,j=1,2,\dots,N}$ of the chain \mathbf{X} defined as

$$q_{ij}^T(t) = \begin{cases} q_{ij} \frac{\tilde{A}(t, T, e_j)}{\tilde{A}(t, T, e_i)}, & i \neq j, \\ -\sum_{k \neq i} q_{ik} \frac{\tilde{A}(t, T, e_k)}{\tilde{A}(t, T, e_i)}, & i = j. \end{cases}$$

Note that by defining $\gamma(T)$ as the Radon-Nikodým derivative of \mathbb{Q}^T with respect to \mathbb{Q} , we obtain

$$\begin{aligned} \frac{d\gamma(t)}{\gamma(t)} &= \frac{d\gamma_1(t)}{\gamma_1(t)} \cdot \frac{d\gamma_2(t)}{\gamma_2(t)} \\ &= -B(t, T)\eta\sqrt{r(t)}d\tilde{W}_3(t) \cdot \left\langle \frac{\tilde{\mathbf{A}}(t, T)}{\tilde{A}(t, T, X(t))}, dM(t) \right\rangle \end{aligned}$$

which later gives us

$$\gamma_1(t) = \exp \left(\int_0^t \eta\sqrt{r(s)}B(s, T)d\tilde{W}_3(s) - 1/2 \int_0^t \eta^2 r(s)B^2(s, T)ds \right)$$

due to the Girsanov theorem, and

$$\gamma_2(t) = \frac{\tilde{A}(t, T, X(t))}{\tilde{A}(0, T, X(0))} \exp \left(- \int_0^t \frac{\frac{d\tilde{A}}{ds} + Q\tilde{A}(s, T, X(s))}{\tilde{A}(s, T, X(s))} ds \right)$$

which is obtained by taking integration on $\frac{d\gamma_2(t)}{\gamma_2(t)}$. Finally, the rate matrix of $Q^T(t)$ is obtained by following the result from [84].

5.2 Derivation of Pricing Formula

In this section, we will find a semi-closed form solution for pricing variance swaps under stochastic volatility and stochastic interest rate with regime switching using characteristic function. The incorporation of regime switching results in an enlarged filtration. Basically, we still work on finding the expectation (3.7) in Subsection 3.2.1, but we change the time notations in this chapter. Let $y(T) = \ln S(T + \Delta) - \ln S(T)$. In this case we have to evaluate the conditional price given the information about the sample path of the Markov chain \mathbf{X} from time 0 up to time $T + \Delta$ for $0 \leq t \leq T + \Delta$.

First, define $\mathcal{F}_1(t)$, $\mathcal{F}_2(t)$ and $\mathcal{F}_3(t)$ as the natural filtrations generated by the three Brownian motions $\{W_1^*(t) : 0 \leq t \leq T\}$, $\{W_2^*(t) : 0 \leq t \leq T\}$ and $\{W_3^*(t) : 0 \leq t \leq T\}$ up to time t respectively, and denote $\mathcal{F}_X(t)$ as the filtration for the regime switching Markov chain $X(t)$ up to time t , that is

$$\begin{aligned} \mathcal{F}_1(t) &= \sigma\{W_1^*(u) : u \leq t\}, \\ \mathcal{F}_2(t) &= \sigma\{W_2^*(u) : u \leq t\}, \\ \mathcal{F}_3(t) &= \sigma\{W_3^*(u) : u \leq t\}, \\ \mathcal{F}_X(t) &= \sigma\{X(u) : u \leq t\}. \end{aligned} \tag{5.15}$$

To obtain the characteristic function of $y(T)$, we consider the evaluation of its conditional value given the information about $\mathcal{F}_X(T + \Delta)$ which is the sample path of the Markov chain \mathbf{X} from time 0 to time $T + \Delta$. This means we shall concentrate on the

following form

$$\begin{aligned}
& \mathbb{E}^T[e^{\phi y(T)} | \mathcal{F}_1(t) \vee \mathcal{F}_2(t) \vee \mathcal{F}_3(t) \vee \mathcal{F}_X(t)] \\
&= \mathbb{E}^T[\mathbb{E}^T[e^{\phi y(T)} | \mathcal{F}_1(t) \vee \mathcal{F}_2(t) \vee \mathcal{F}_3(t) \vee \mathcal{F}_X(T + \Delta)] \\
&\quad | \mathcal{F}_1(t) \vee \mathcal{F}_2(t) \vee \mathcal{F}_3(t) \vee \mathcal{F}_X(t)].
\end{aligned} \tag{5.16}$$

To achieve this, we shall separate our derivation process into two subsections. In the first subsection, we shall compute the characteristic function for $y(T)$ given path $\mathcal{F}_X(T + \Delta)$, that is $\mathbb{E}^T[e^{\phi y(T)} | \mathcal{F}_1(t) \vee \mathcal{F}_2(t) \vee \mathcal{F}_3(t) \vee \mathcal{F}_X(T + \Delta)]$. In the second subsection, we use the result in the first subsection to compute the characteristic function for $y(T)$.

5.2.1 Characteristic Function for Given Path $\mathcal{F}_X(T + \Delta)$

Denote the current time as t , where $t < T$. Under this pricing model with regime switching, we consider an enlarged filtration in which we define the forward characteristic function $f(\phi; t, T, \Delta, \nu(t), r(t))$ of the stochastic variable $y(T) = \ln S(T + \Delta) - \ln S(T)$ as

$$\begin{aligned}
& f(\phi; t, T, \Delta, \nu(t), r(t)) \\
&= \mathbb{E}^T[e^{\phi y(T)} | \mathcal{F}_1(t) \vee \mathcal{F}_2(t) \vee \mathcal{F}_3(t) \vee \mathcal{F}_X(t)] \\
&= \mathbb{E}^T[\exp(\phi(\ln S(T + \Delta) - \ln S(T))) | \mathcal{F}_1(t) \vee \mathcal{F}_2(t) \vee \mathcal{F}_3(t) \vee \mathcal{F}_X(t)].
\end{aligned} \tag{5.17}$$

As mentioned earlier, our main aim is to obtain the characteristic function given the information about the sample path of the Markov chain \mathbf{X} from time 0 to time $T + \Delta$. In particular, given $\mathcal{F}_X(T + \Delta)$, the conditional characteristic function is given by the following proposition.

Proposition 5.1. *If the underlying asset follows the dynamics (5.13), then the forward characteristic function of the stochastic variable $y(T) = \ln S(T + \Delta) - \ln S(T)$ conditional on $\mathcal{F}_X(T + \Delta)$ is given by*

$$\begin{aligned}
f(\phi; t, T, \Delta, \nu(t), r(t) | \mathcal{F}_X(T + \Delta)) &= \mathbb{E}^T[e^{\phi y(T)} | \mathcal{F}_1(t) \vee \mathcal{F}_2(t) \vee \mathcal{F}_3(t) \vee \mathcal{F}_X(T + \Delta)] \\
&= e^{C(\phi, T)} j(D(\phi, T); t, T, \nu(t)) \\
&\quad \cdot k(E(\phi, T); t, T, r(t)),
\end{aligned} \tag{5.18}$$

where $D(\phi, t), j(\phi; t, T, \nu(t))$ and $k(\phi; t, T, r(t))$ are given by

$$D(\phi, t) = \frac{a + b}{\sigma^2} \frac{1 - e^{b(T+\Delta-t)}}{1 - ge^{b(T+\Delta-t)}}, \quad (5.19)$$

$$a = \kappa^* - \rho\sigma\phi, \quad b = \sqrt{a^2 + \sigma^2(\phi - \phi^2)}, \quad g = \frac{a + b}{a - b},$$

with

$$j(\phi; t, T, \nu(t)) = e^{F(\phi, t) + G(\phi, t)\nu(t)},$$

$$F(\phi, t) = \int_t^T \langle \kappa^* \theta^* G(\phi, s), X(s) \rangle ds, \quad (5.20)$$

$$G(\phi, t) = \frac{2\kappa^* \phi}{\sigma^2 \phi + (2\kappa^* - \sigma^2 \phi)e^{\kappa^*(T-t)}},$$

$$k(\phi; t, T, r(t)) = e^{L(\phi, t) + M(\phi, t)r(t)}, \quad (5.21)$$

and $C(\phi, t), E(\phi, t), L(\phi, t)$ and $M(\phi, t)$ are determined by the following ODEs

$$-\frac{dE}{dt} = \frac{1}{2}\eta^2 E^2 - (\alpha^* + B(t, T)\eta^2)E + \phi,$$

$$-\frac{dC}{dt} = \kappa^* \theta^*(t)D + \alpha^* \beta^*(t)E, \quad (5.22)$$

$$-\frac{dM}{dt} = \frac{1}{2}\eta^2 M^2 - (\alpha^* + B(t, T)\eta^2)M,$$

$$-\frac{dL}{dt} = \alpha^* \beta^*(t)M.$$

Here, we give a brief proof for Proposition 5.1. Following (3.8), we can represent the conditional forward characteristic function for $y(T)$ in two computation steps as

$$f(\phi; t, T, \Delta, \nu(t), r(t) | \mathcal{F}_X(T + \Delta)) = \mathbb{E}^T[\mathbb{E}^T[e^{\phi y(T)} | \mathcal{F}_1(T) \vee \mathcal{F}_2(T) \vee \mathcal{F}_3(T) \vee \mathcal{F}_X(T + \Delta)]$$

$$| \mathcal{F}_1(t) \vee \mathcal{F}_2(t) \vee \mathcal{F}_3(t) \vee \mathcal{F}_X(T + \Delta)]. \quad (5.23)$$

We focus first on solving the inner expectation

$$\mathbb{E}^T[e^{\phi y(T)} | \mathcal{F}_1(T) \vee \mathcal{F}_2(T) \vee \mathcal{F}_3(T) \vee \mathcal{F}_X(T + \Delta)].$$

By defining function

$$U(\phi; t, X, \nu, r) = \mathbb{E}^T[e^{\phi y(T)} | \mathcal{F}_1(t) \vee \mathcal{F}_2(t) \vee \mathcal{F}_3(t) \vee \mathcal{F}_X(T + \Delta)]$$

with $T \leq t \leq T + \Delta$, and applying the Feynman-Kac theorem, we obtain

$$\begin{aligned} \frac{\partial U}{\partial t} + \frac{1}{2}\nu \frac{\partial^2 U}{\partial X^2} + \frac{1}{2}\sigma^2\nu \frac{\partial^2 U}{\partial \nu^2} + \frac{1}{2}\eta^2 r \frac{\partial^2 U}{\partial r^2} + \rho\sigma\nu \frac{\partial^2 U}{\partial X \partial \nu} + \left[r - \frac{1}{2}\nu \right] \frac{\partial U}{\partial X} \\ + [\kappa^*(\theta^*(t) - \nu)] \frac{\partial U}{\partial \nu} + [\alpha^*\beta^*(t) - (\alpha^* + B(t, T)\eta^2)r] \frac{\partial U}{\partial r} = 0, \end{aligned} \quad (5.24)$$

$$U(\phi; t = T + \Delta, X, \nu, r) = e^{\phi X},$$

where $X = \ln S(t) - \ln S(T)$ in $(T \leq t \leq T + \Delta)$. In order to solve the above PDE system, we adopt Heston's (1993) assumption as follows

$$U(\phi; t, X, \nu(t), r(t)) = e^{C(\phi, t) + D(\phi, t)\nu + E(\phi, t)r + \phi X}. \quad (5.25)$$

We can later obtain three ODEs by substituting the above function into the previous PDE

$$\begin{cases} -\frac{dD}{dt} = \frac{1}{2}\phi(\phi - 1) + (\rho\sigma\phi - \kappa^*)D + \frac{1}{2}\sigma^2 D^2, \\ -\frac{dE}{dt} = \frac{1}{2}\eta^2 E^2 - (\alpha^* + B(t, T)\eta^2)E + \phi, \\ -\frac{dC}{dt} = \kappa^*\theta^*(t)D + \alpha^*\beta^*(t)E \end{cases} \quad (5.26)$$

with the initial conditions

$$C(\phi, T + \Delta) = 0, \quad D(\phi, T + \Delta) = 0, \quad E(\phi, T + \Delta) = 0. \quad (5.27)$$

Finally, we can write the solution of function D as

$$\begin{cases} D(\phi, t) = \frac{a + b}{\sigma^2} \frac{1 - e^{b(T+\Delta-t)}}{1 - g e^{b(T+\Delta-t)}}, \\ a = \kappa^* - \rho\sigma\phi, \quad b = \sqrt{a^2 + \sigma^2(\phi - \phi^2)}, \quad g = \frac{a + b}{a - b}. \end{cases} \quad (5.28)$$

Numerical integration is performed to obtain the solutions of the functions E and C as outlined in Chapter 3.

Now that we had obtained the solution for the inner expectation, we shall move on

to solve the outer expectation ($0 \leq t \leq T$). At time $t = T$, $X = \ln S(T) - \ln S(T) = 0$. Substituting this back into the inner expectation of (5.23) means

$$\begin{aligned} & \mathbb{E}^T[e^{\phi y(T)} | \mathcal{F}_1(T) \vee \mathcal{F}_2(T) \vee \mathcal{F}_3(T) \vee \mathcal{F}_X(T + \Delta)] \\ &= U(\phi; T, X, \nu(T), r(T)) \\ &= e^{C(\phi, T) + D(\phi, T)\nu(T) + E(\phi, T)r(T)}. \end{aligned}$$

By defining the following functions

$$j(\phi; t, T, \nu(t)) = \mathbb{E}^T[e^{\phi \nu(T)} | \mathcal{F}_1(t) \vee \mathcal{F}_2(t) \vee \mathcal{F}_3(t)],$$

and

$$k(\phi; t, T, r(t)) = \mathbb{E}^T[e^{\phi r(T)} | \mathcal{F}_1(t) \vee \mathcal{F}_2(t) \vee \mathcal{F}_3(t)],$$

we obtain the respective PDEs as

$$\begin{aligned} \frac{\partial j}{\partial t} + \frac{1}{2}\sigma^2\nu\frac{\partial^2 j}{\partial \nu^2} + [\kappa^*(\theta^*(t) - \nu)]\frac{\partial j}{\partial \nu} &= 0, \\ j(\phi, t = T, T, \nu) &= e^{\phi\nu}, \end{aligned} \tag{5.29}$$

and

$$\begin{aligned} \frac{\partial k}{\partial t} + \frac{1}{2}\eta^2 r \frac{\partial^2 k}{\partial r^2} + [\alpha^*\beta^*(t) - (\alpha^* + B(t, T)\eta^2)r]\frac{\partial k}{\partial r} &= 0, \\ k(\phi, t = T, T, r) &= e^{\phi r}. \end{aligned} \tag{5.30}$$

Taking advantage of the affine-form solution techniques as those in [31, 66], we assume the solution to (5.29) is in the form of

$$j(\phi; t, T, \nu(t)) = e^{F(\phi, t) + G(\phi, t)\nu(t)}. \tag{5.31}$$

The functions $F(\phi, t)$ and $G(\phi, t)$ can be found by solving two Riccati ODEs

$$\begin{aligned} -\frac{dG}{dt} &= \frac{1}{2}\sigma^2 G^2 - \kappa^* G, \\ -\frac{dF}{dt} &= \kappa^*\theta^*(t)G, \end{aligned} \tag{5.32}$$

with the initial conditions

$$F(\phi, T) = 0, \quad G(\phi, T) = \phi. \tag{5.33}$$

The solutions are

$$\begin{cases} F(\phi, t) = \int_t^T \kappa^* \theta^*(s) G(\phi, s) ds, \\ G(\phi, t) = \frac{2\kappa^* \phi}{\sigma^2 \phi + (2\kappa^* - \sigma^2 \phi) e^{\kappa^*(T-t)}}. \end{cases} \quad (5.34)$$

Next, the function $k(\phi; t, T, r(t)) = e^{L(\phi, t) + M(\phi, t)r(t)}$ is defined in order to derive a solution to (5.30). The initial conditions are $L(\phi, T) = 0$ and $M(\phi, T) = \phi$. Then, L and M satisfy the following ODEs

$$\begin{cases} -\frac{dM}{dt} = \frac{1}{2} \eta^2 M^2 - (\alpha^* + B(t, T) \eta^2) M, \\ -\frac{dL}{dt} = \alpha^* \beta^*(t) M, \end{cases} \quad (5.35)$$

whose solutions can be obtained numerically.

5.2.2 Characteristic Function for Given Path $\mathcal{F}_X(t)$

In this subsection, we shall proceed by finding the expectation of the computations obtained in the previous subsection. In particular, our focus is to derive the characteristic function for given path $\mathcal{F}_X(t)$. We need to evaluate the equation (5.18), where $\theta^*(t)$ and $\beta^*(t)$ depend on the path of the Markov chain process up to time $T + \Delta$, by

considering

$$\begin{aligned}
 & f(\phi; t, T, \Delta, \nu(t), r(t)) \\
 &= \mathbb{E}^T [\mathbb{E}^T [e^{\phi y(T)} | \mathcal{F}_1(t) \vee \mathcal{F}_2(t) \vee \mathcal{F}_3(t) \vee \mathcal{F}_X(T + \Delta)] | \mathcal{F}_1(t) \vee \mathcal{F}_2(t) \vee \mathcal{F}_3(t) \vee \mathcal{F}_X(t)] \\
 &= \mathbb{E}^T [e^{C(\phi, T)} \cdot j(D(\phi, T); t, T, \nu(t)) \cdot k(E(\phi, T); t, T, r(t)) | \mathcal{F}_1(t) \vee \mathcal{F}_2(t) \vee \mathcal{F}_3(t) \vee \mathcal{F}_X(t)] \\
 &= \mathbb{E}^T \left[\exp \left(\int_T^{T+\Delta} \langle \alpha^* \beta^* E(\phi, s) + \kappa^* \theta^* D(\phi, s), X(s) \rangle ds \right. \right. \\
 &\quad \left. \left. + \int_t^T \langle \kappa^* \theta^* G(D(\phi, T), s), X(s) \rangle ds + \frac{2\kappa^* D(\phi, T)}{\sigma^2 D(\phi, T) + (2\kappa^* - \sigma^2 D(\phi, T)) e^{\kappa^*(T-t)}} \nu(t) \right. \right. \\
 &\quad \left. \left. + \int_t^T \langle \alpha^* \beta^* M(E(\phi, T), s), X(s) \rangle ds \right. \right. \\
 &\quad \left. \left. + r(t) \int_t^T \left(\frac{1}{2} \eta^2 M^2(E(\phi, T), s) - (\alpha^* + B(s, T) \eta^2) M(E(\phi, T), s) \right) ds \right) \right. \\
 &\quad \left. \left| \mathcal{F}_1(t) \vee \mathcal{F}_2(t) \vee \mathcal{F}_3(t) \vee \mathcal{F}_X(t) \right] \right. \\
 &= \mathbb{E}^T [\exp(\int_t^{T+\Delta} \langle J(s), X(s) \rangle ds) | \mathcal{F}_1(t) \vee \mathcal{F}_2(t) \vee \mathcal{F}_3(t) \vee \mathcal{F}_X(t)] \\
 &\quad \times \exp(\nu(t) G(D(\phi, T), t)) \times \exp(r(t) M(E(\phi, T), t)).
 \end{aligned} \tag{5.36}$$

Here, the function $J(t) \in \mathbb{R}^N$ is given by

$$\begin{aligned}
 J(t) &= [\kappa^* \theta^* G(D(\phi, T), t) + \alpha^* \beta^* M(E(\phi, T), t)] (1 - H_T(t)) \\
 &\quad + [\alpha^* \beta^* E(\phi, t) + \kappa^* \theta^* D(\phi, t)] H_T(t)
 \end{aligned} \tag{5.37}$$

along with $H_T(t)$ which is a Heaviside unit step function defined as

$$H_T(t) = \begin{cases} 1, & \text{if } t \geq T, \\ 0, & \text{else.} \end{cases}$$

Hence, the final conditional expectation in (5.36) can be written as

$$\mathbb{E}^T \left[\exp \left(\int_t^T \langle \tilde{v}, X(s) \rangle u(s) ds \right) \left| \mathcal{F}_1(t) \vee \mathcal{F}_2(t) \vee \mathcal{F}_3(t) \vee \mathcal{F}_X(t) \right. \right], \tag{5.38}$$

where \tilde{v} is an \mathbb{R}^N vector and $u(s)$ is a general deterministic integrable function. The following proposition shows that we can compute this value by evaluating the characteristic function of $\int_t^T \langle \tilde{v}, X(s) \rangle u(s) ds$.

Proposition 5.2. *If X is a regime switching Markov chain with dynamics (5.14), then the characteristic function, $f(\psi, t)$ of the stochastic variable $\int_t^T \langle \tilde{v}, X(s) \rangle u(s) ds \in \mathbb{R}$ is given by*

$$\begin{aligned} f(\psi, t) &= \mathbb{E}^T \left[\exp \left(\psi \int_t^T \langle \tilde{v}, X(s) \rangle u(s) ds \right) \middle| \mathcal{F}_1(t) \vee \mathcal{F}_2(t) \vee \mathcal{F}_3(t) \vee \mathcal{F}_X(t) \right] \\ &= \langle \Phi(t, T; \tilde{v}) X(t), I \rangle, \end{aligned} \quad (5.39)$$

where the function $\Phi(t, T; \tilde{v})$ is an N -by- N \mathbb{R} -valued matrix given by

$$\Phi(t, T; \tilde{v}) = \exp \left(\int_t^T (Q^T(s) + \psi u(s) \text{diag}[\tilde{v}]) ds \right), \quad (5.40)$$

with $I = (1, 1, \dots, 1) \in \mathbb{R}^N$.

The proof of Proposition 5.2 is given as follows. Note that the problem of finding the expectation in (5.36) reduces into finding the characteristic function of $\int_t^T \langle \tilde{v}, X(s) \rangle u(s) ds$ by writing (5.38) in the following form

$$f(\psi, t) = \mathbb{E}^T \left[\exp \left(\psi \int_t^T \langle \tilde{v}, X(s) \rangle u(s) ds \right) \middle| \mathcal{F}_1(t) \vee \mathcal{F}_2(t) \vee \mathcal{F}_3(t) \vee \mathcal{F}_X(t) \right]. \quad (5.41)$$

Consider the function $Z(t, T) = X(T) \exp \left(\psi \int_t^T \langle \tilde{v}, X(s) \rangle u(s) ds \right)$. By taking differentiation and using (5.14), we obtain

$$\begin{aligned} dZ(t, T) &= \exp \left(\psi \int_t^T \langle \tilde{v}, X(s) \rangle u(s) ds \right) (Q^T(T) X(T) dT + dM^T(T)) \\ &\quad + \psi \langle \tilde{v}, X(T) \rangle u(T) X(T) \exp \left(\psi \int_t^T \langle \tilde{v}, X(s) \rangle u(s) ds \right) dT \\ &= \exp \left(\psi \int_t^T \langle \tilde{v}, X(s) \rangle u(s) ds \right) dM^T(T) \\ &\quad + \exp \left(\psi \int_t^T \langle \tilde{v}, X(s) \rangle u(s) ds \right) Q^T(T) X(T) dT \\ &\quad + \psi \langle \tilde{v}, X(T) \rangle u(T) X(T) \exp \left(\psi \int_t^T \langle \tilde{v}, X(s) \rangle u(s) ds \right) dT \\ &= \exp \left(\psi \int_t^T \langle \tilde{v}, X(s) \rangle u(s) ds \right) dM^T(T) + X(T) \exp \left(\psi \int_t^T \langle \tilde{v}, X(s) \rangle u(s) ds \right) \\ &\quad \times (Q^T(T) + \psi u(T) \text{diag}[\tilde{v}]) dT. \end{aligned} \quad (5.42)$$

Integrating each side of (5.42) gives

$$\begin{aligned} \int_t^T dZ(t, s) &= \int_t^T (Q^T(s) + \psi u(s) \text{diag}[\tilde{v}])Z(t, s)ds \\ &+ \int_t^T \exp\left(\psi \int_t^w \langle \tilde{v}, X(y) \rangle u(y) dy\right) dM^T(s). \end{aligned} \quad (5.43)$$

Let the function $\Psi(t, T; \tilde{v}) = \mathbb{E}^T \left[Z(t, T) \middle| \mathcal{F}_1(t) \vee \mathcal{F}_2(t) \vee \mathcal{F}_3(t) \vee \mathcal{F}_X(t) \right]$. Taking expectations in both sides of (5.43) results in

$$\Psi(t, T; \tilde{v}) = X(t) + \int_t^T (Q^T(s) + \psi u(s) \text{diag}[\tilde{v}])\Psi(t, s; \tilde{v})ds. \quad (5.44)$$

Suppose $\Phi(t, s; \tilde{v})$ is the $N \times N$ matrix solution of the linear system of ordinary differential equation

$$\begin{aligned} \frac{d\Phi(t, s; \tilde{v})}{ds} &= (Q^T(s) + \psi u(s) \text{diag}[\tilde{v}])\Phi(t, s; \tilde{v}), \\ \Phi(t, t; \tilde{v}) &= \text{diag}[I]. \end{aligned} \quad (5.45)$$

Comparing with (5.44) results in $\Psi(t, T; \tilde{v}) = \Phi(t, T; \tilde{v})X(t)$, which finally gives us

$$f(\psi, t) = \langle \Phi(t, T; \tilde{v})X(t), I \rangle. \quad (5.46)$$

Finally, we substitute Proposition 5.2 back into (5.36) which gives us the characteristic function of the stochastic variable $y(T) = \ln S(T + \Delta) - \ln S(T)$ for the Heston-CIR model with regime switching, as in the Proposition 5.3 below.

Proposition 5.3. *If the underlying asset follows the dynamics (5.13), then the forward characteristic function of the stochastic variable $y(T) = \ln S(T + \Delta) - \ln S(T)$ is given by*

$$\begin{aligned} f(\phi; t, T, \Delta, \nu(t), r(t)) &= \mathbb{E}^T[e^{\phi y(T)} | \mathcal{F}_1(t) \vee \mathcal{F}_2(t) \vee \mathcal{F}_3(t) \vee \mathcal{F}_X(t)] \\ &= \exp(\nu(t)G(D(\phi, T), t)) \times \exp(r(t)M(E(\phi, T), t)) \\ &\quad \times \langle \Phi(t, T + \Delta; J)X(t), I \rangle, \end{aligned} \quad (5.47)$$

where $D(\phi, t)$, $G(\phi, t)$, $J(t)$ and $\Phi(t, T + \Delta; J)$ are given by

$$\begin{aligned}
 D(\phi, t) &= \frac{a + b}{\sigma^2} \frac{1 - e^{b(T+\Delta-t)}}{1 - ge^{b(T+\Delta-t)}}, \\
 a &= \kappa^* - \rho\sigma\phi, \quad b = \sqrt{a^2 + \sigma^2(\phi - \phi^2)}, \quad g = \frac{a + b}{a - b}, \\
 G(\phi, t) &= \frac{2\kappa^*\phi}{\sigma^2\phi + (2\kappa^* - \sigma^2\phi)e^{\kappa^*(T-t)}}, \\
 J(t) &= [\kappa^*\theta^*G(D(\phi, T), t) + \alpha^*\beta^*M(E(\phi, T), t)](1 - H_T(t)) \\
 &\quad + [\alpha^*\beta^*E(\phi, t) + \kappa^*\theta^*D(\phi, t)]H_T(t), \\
 \Phi(t, T + \Delta; J) &= \exp\left(\int_t^{T+\Delta} (Q^T(s) + \text{diag}[J(s)])ds\right),
 \end{aligned} \tag{5.48}$$

and $E(\phi, t)$ along with $M(\phi, t)$ are determined by the following ODEs

$$\begin{aligned}
 -\frac{dE}{dt} &= \frac{1}{2}\eta^2 E^2 - (\alpha^* + B(t, T)\eta^2)E + \phi, \\
 -\frac{dM}{dt} &= \frac{1}{2}\eta^2 M^2 - (\alpha^* + B(t, T)\eta^2)M.
 \end{aligned} \tag{5.49}$$

Now, by using the valuation of the fair delivery price for a variance swap as given in (3.7), and summarizing the whole procedure above, we can write the forward characteristic function for a variance swap as

$$\begin{aligned}
 &\mathbb{E}^T \left[\left(\frac{S(t_j)}{S(t_{j-1})} - 1 \right)^2 \middle| \mathcal{F}_1(0) \vee \mathcal{F}_2(0) \vee \mathcal{F}_3(0) \vee \mathcal{F}_X(0) \right] \\
 &= \mathbb{E}^T [e^{2y(t_{j-1}, t_j)} - 2e^{y(t_{j-1}, t_j)} + 1 | \mathcal{F}_1(t) \vee \mathcal{F}_2(t) \vee \mathcal{F}_3(t) \vee \mathcal{F}_X(t)] \\
 &= f(2; 0, t_{j-1}, \Delta t, \nu(t), r(t)) - 2f(1; 0, t_{j-1}, \Delta t, \nu(t), r(t)) + 1,
 \end{aligned} \tag{5.50}$$

where $y(t_{j-1}) = \ln S(t_j) - \ln S(t_{j-1})$, $\Delta t = t_j - t_{j-1}$, and function $f(\phi; t, T, \Delta, \nu(t), r(t))$ is given in equation (5.18). Finally, by utilizing (3.41), the fair strike price for a variance swap in terms of the spot variance $\nu(0)$ and the spot interest rate $r(0)$ is given as

$$RV = \frac{100^2}{T} \sum_{j=1}^N [f(2; 0, t_{j-1}, \Delta t, \nu(0), r(0)) - 2f(1; 0, t_{j-1}, \Delta t, \nu(0), r(0)) + 1]. \tag{5.51}$$

5.3 Formula Validation and Results

In this section, we assess the performance of our semi-closed form pricing formula as defined in the previous subsection. Our main focus will be on investigating the effects of incorporating regime switching into pricing discretely-sampled variance swaps with stochastic volatility and stochastic interest rate. For illustration purposes, we consider three regimes named as $X(t) = 1$, $X(t) = 2$ and $X(t) = 3$, which represent the states *Contraction*, *Trough* and *Expansion* of the business cycle respectively. The *Contraction* state can be defined as the situation when the economy starts slowing down, whereas the *Trough* state happens when the economy hits bottom, usually in a recession. In addition, *Expansion* is identified as the situation when the economy starts growing again. Here, we assume that the Heston-CIR model without regime switching corresponds to the first regime and it will switch to the other two regimes over time. This will be discussed in Subsection 5.3.2.

However, to start off, we first demonstrate the validation of our pricing formula against the Monte Carlo (MC) simulation in Subsection 5.3.1. The following Table 5.1 shows the set of parameters that we use to implement all the numerical experiments, unless otherwise stated.

Table 5.1: Model parameters of the Heston-CIR hybrid model with regime switching.

Parameters	S_0	ρ	V_0	θ^*	κ^*	σ	r_0	α^*	β^*	η	T
Values	1	-0.40	0.05	$[0.05, 0.075, 0.04]^T$	2	0.1	0.05	1.2	$[0.05, 0.04, 0.075]^T$	0.01	1

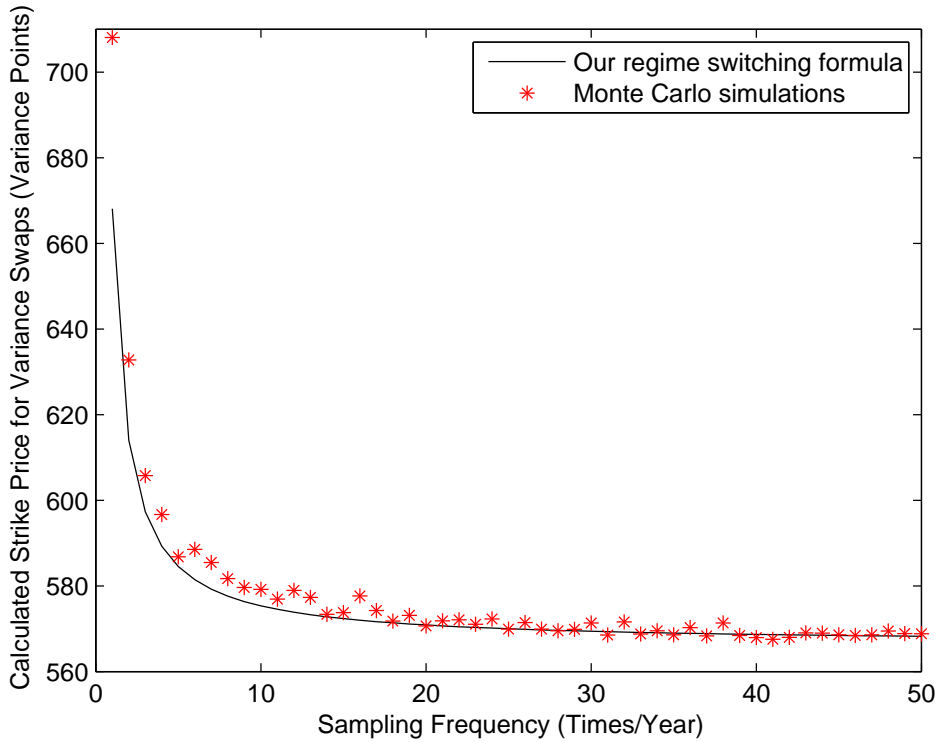
In addition, the transition rate matrix for the Markov chain is given by

$$Q = \begin{bmatrix} -1 & 0.1 & 0.9 \\ 0.9 & -1 & 0.1 \\ 0.5 & 0.5 & -1 \end{bmatrix}.$$

5.3.1 Validation of Our Pricing Formula

Here, we compare the prices of variance swaps from our pricing formula in (5.51) against those obtained from the Markov Chain Monte Carlo simulation. The sampling frequency varies from $N = 1$ up to $N = 52$, and the MC simulation is conducted using the Euler discretization with 200,000 sample paths. The comparison is displayed in Figure 5.1.

Figure 5.1: Strike prices of variance swaps for the Heston-CIR model with regime switching and Monte Carlo simulation.



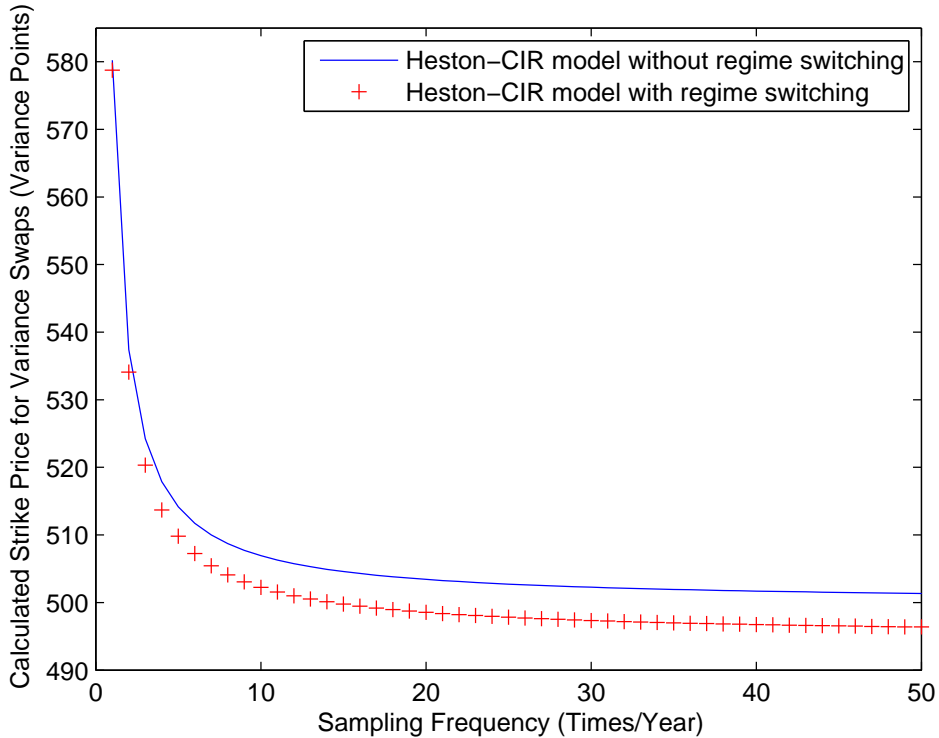
We see from the graph plot in Figure 5.1 that our pricing formula compares very well and provides a satisfactory fit to the MC simulation for $N = 52$ which is the weekly sampling. In fact, the error calculated between our pricing formula and the MC simulation is less than 0.077% for $N = 52$, and this error will be reduced as the number of sample paths increases. In addition, it should be emphasized that for $N = 4$, the run time of our pricing formula is only 3.28 seconds, whereas the MC simulation took about 8200 seconds. It is clear that our pricing formula attains almost the same accuracy in far less time compared to the MC simulation which serves as benchmark values.

5.3.2 Effect of Regime Switching

In order to explore the economic consequences of incorporating regime switching into the Heston-CIR model, we present Figure 5.2 which shows the difference between our pricing formula in (5.51) versus the Heston-CIR model without regime switching in [19]. For the Heston-CIR model without regime switching, we let the parameter values of

State 1 be $\theta_1^* = 0.05$ and $\beta_1^* = 0.05$. For the Heston-CIR model with regime switching, all parameter values are given in Table 5.1.

Figure 5.2: Strike prices of variance swaps for the Heston-CIR model with and without regime switching.



We observe that the prices of variance swaps obtained from the Heston-CIR model with regime switching are significantly lower than those from the corresponding model without regime switching. For example, for $N = 52$, the difference between variance swaps prices calculated from the two models is 1.001%. This can be explained from the values of θ_1^* and β_1^* which remain constant, whereas the values of θ^* and β^* in the Heston-CIR model with regime switching vary according to the changing states. Besides that, for the weekly sampling case, the difference in variance swaps prices between the two models becomes larger and stabilizes as the sampling frequency reaches 52. One possible explanation for this is the number of transitions between states in the Heston-CIR model with regime switching increases as the sampling frequency increases.

In addition, we also examine the economic aftermath for the prices of variance swaps by allowing the Heston-CIR model to switch across three regimes. In particular, we

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denote $\theta_1^* = 0.05$ and $\beta_1^* = 0.05$ for the *Contraction* state, $\theta_2^* = 0.075$ and $\beta_2^* = 0.04$ for the *Trough* state, and $\theta_3^* = 0.04$ and $\beta_3^* = 0.075$ for the *Expansion* state respectively. These values are assumed by noting that a good (resp. bad) economy is identified by high (resp. low) interest rate and low (resp. high) volatility. We provide the variance swaps pricing outcome for these three regimes in Table 5.2.

Table 5.2: Comparing prices of variance swaps among three different states in our pricing formula.

Sampling Frequency	State <i>Contraction</i>	State <i>Trough</i>	State <i>Expansion</i>
N=4	517.89	661.93	464.79
N=12	505.74	648.32	450.21
N=26	502.61	644.83	446.42
N=52	501.28	643.37	444.82

From Table 5.2, we discover that the price of a variance swap is highest in the *Trough* state, followed by the *Contraction* state, and found lowest in the *Expansion* state. This trend is consistent throughout all sampling frequencies from $N = 4$ to $N = 52$. We can relate this finding to the economic condition of each of the states. In particular, the *Trough* state is the state with the worst economy among the three, whereas the *Expansion* state resembles the best economy. Thus, the price of a variance swap is cheapest in the best economy among the three, and most expensive in the worst economy among all. This implies that regime switching has an important impact in capturing the economic changes on the prices of variance swaps.

Chapter 6

Conclusion and Future Work

The main goal of this thesis is to develop some techniques for pricing variance swaps under stochastic volatility and stochastic interest rate. To do this, several mathematical techniques and financial concepts are employed. In this chapter, we summarise the main outcomes from our analysis and suggest some possible directions for future work. More precisely, Section 6.1 is devoted to discuss the conclusion of our work in three major aspects. In Section 6.2, we propose some future research directions which may be worth of pursuing.

6.1 Conclusion

In this thesis, we study the pricing of discretely-sampled variance swaps in the framework of stochastic volatility and stochastic interest rate. In Chapter 3, we present the Heston-CIR model for pricing variance swaps with partial correlation imposed between the asset price and the volatility. We derive a semi-closed form pricing formula for the fair delivery price of a variance swap via dimension-reduction technique and derivation of characteristic functions. We demonstrate the practical implementation of our pricing formula through numerical experiments. We compare the numerical results obtained from our pricing formula with those from Monte Carlo (MC) simulation and the numerical calculation of the continuously-sampled variance swaps model. We find that the results from our pricing formula perfectly match the results from the MC simulation. In addition, the relative difference is further reduced as the number of paths increased. This provides a verification of our pricing formula since the MC simulation resembles the real market. Moreover, we also discuss the impact of interest rate on the values of variance swaps. First, we notice that the value of a discrete variance swap decreases and converges to the continuous sampling counterpart as the sampling frequency in-

creases. This is consistent with the convergence pattern of constant interest rate which had been explored by other researchers. Secondly, we discover that the speed α^* of mean-reversion, and the volatility η of the stochastic interest rate have little impact on the value of variance swaps. Finally, the impact of the long-term mean β^* of the interest rate on the value of variance swaps is also discussed. Our results show that the value of a variance swap increases accordingly as β^* increases. This highlights the importance of incorporating stochastic interest rate in pricing variance swaps.

Following the study in Chapter 3, we discuss the pricing of discretely-sampled variance swaps with full correlation among the asset price, interest rate as well as the volatility in Chapter 4. This full correlation model is incompliant with the analytical tractability property due to the presence of non-affine terms in its structure. One possible way to deal with this issue is to define this model in the class of affine diffusion processes to obtain its characteristic function. Thus, we determine the approximations for the non-affine terms and present a semi-closed form approximation formula for the fair delivery price of a variance swap. From theoretical results and numerical examples, we show that our pricing formula is efficient in terms of reducing the computational time, and significantly accurate compared with the continuous sampling model. Furthermore, we also investigate the impact of the correlation coefficients between the interest rate with the underlying and the volatility respectively on our pricing model. In particular, we examine how the delivery price of a variance swap changes when the correlation parameter takes values from -0.5, 0 to 0.5. We discover that the value of a variance swap increases as the correlation value between the asset price and the interest rate increases. However, the impact of these correlation coefficients becomes less apparent as the number of sampling frequencies increases. In contrast, we discover that the correlation coefficient between the volatility and the interest rate has little impact on the price of a variance swap. The impact of the correlation coefficient on the prices gets smaller as the number of sampling frequencies increases.

Finally, we investigate the pricing of discretely-sampled variance swaps under stochastic volatility and stochastic interest rate with regime switching. In Chapter 5, we describe the Heston-CIR model with regime switching which is capable of capturing several macroeconomic issues such as alternating business cycles. In particular, we assume that the long-term mean $\theta^*(t)$ of variance of the risky stock, and the long-term mean $\beta^*(t)$ of the interest rate depend on the states of the economy indicated by the regime switching Markov chain. We demonstrate our solution techniques and derive a semi-closed form formula for pricing variance swaps. Numerical experiments reveal that our pricing formula attains almost the same accuracy in far less time compared

with the MC simulation which serves as benchmark values. To analyse the effects of incorporating regime switching into pricing variance swaps, we first compare the variance swaps prices calculated from the regime switching Heston-CIR model with the corresponding model without regime switching. We find that the prices of variance swaps obtained from the regime switching Heston-CIR model are significantly lower than those of its non-regime switching counterparts. In fact, for the weekly sampling case, the difference in variance swaps prices between the two models becomes larger and stabilizes towards the end. We relate this finding to the number of transitions between states in the Heston-CIR model with regime switching which increases as the sampling frequency becomes larger. Next, we explore the economic consequence for the prices of variance swaps by allowing the Heston-CIR model to switch across three regimes. These three regimes represent three economic conditions of the business cycle which reflects the best, moderate and worst economy. We notice that the price of a variance swap is cheapest in the best economy among the three, and most expensive in the worst economy among all. This confirms the essentiality of incorporating regime switching in pricing variance swaps, since the price of a variance swap changes according to the respective economic condition.

6.2 Remaining Problems with the Existing Models

In what follows, we shall discuss some potential directions for future research. Overall, this thesis focuses on pricing variance swaps with stochastic volatility and stochastic interest rate under the Heston-CIR model. However, we can also consider many other stochastic processes to represent the dynamics of factors driving the model. For example, we can consider the Heston-LIBOR model since there is not much existing literature concerning the hybrid model of these two processes, especially those involving the LIBOR model due to increased complexity. In fact, the structure of the hybrid model can be modified to include more factors or larger dimensions, as well as additional random jumps. These would offer interesting ways to evaluate the variance swaps in different conditions compared with other existing literatures. Furthermore, the techniques presented in this thesis can be generalised for pricing other derivatives, for example the third generation volatility products, as well as options on volatility and VIX futures. Besides that, we believe that the techniques used in this thesis can also be applied to the problem of pricing variance swaps using the definition of realized variance given in equation (2.25) which is more popular in the market. This will be one of the research directions in the future.

In terms of numerical results, a possible extension would be to consider calibration and parameter estimation techniques of our pricing model to the volatility index. In addition, we can also improve the computational efficiency and robustness by implementing effective algorithm schemes for time discretization, and variance reduction techniques for the MC simulation.

Regarding pricing discretely-sampled variance swaps with full correlation among all asset classes, we may perform some calibration procedure to obtain the correlation parameters as explained in [99]. Besides that, an empirical study on comparison between the relative performance of the Heston-CIR model and other hybridizations of stochastic volatility and stochastic interest rate models with full correlations may also be investigated. Other techniques may also be implemented to deal with the full correlation structure, such as the Wishart process proposed in [48].

Finally, for incorporating regime switching in pricing discretely-sampled variance swaps, Futami [43] estimated the business cycle using observable information obtained from previous short rate history in predicting the term structure of interest rates. This helps to filter the current regime from observable short rate, which affects the market price of diffusion and volatility respectively. Thus, a possible extension would be to analyze our pricing formula under this partial information. Also, it is of practical interest to explore some approaches to estimate the number of states in the Markov chain, as well as the parameters for the transition matrix. Another interesting future research direction is to investigate the inclusion of rare states in the framework of our pricing model. This notion proposed in [7] is claimed to capture tight liquidity conditions, which definitely gives extra credit in terms of documenting economic consequences.

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