OPTION PRICING UNDER THE HESTON-CIR MODEL WITH STOCHASTIC INTEREST RATES AND TRANSACTION COSTS

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Abstract

The celebrated Black-Scholes model on pricing a European option gives a simple and elegant pricing formula for European options with the underlying price following a geometric Brownian motion. In a realistic market with transaction costs, the option pricing problem is known to lead to solving nonlinear partial differential equations even in the simplest model. The nonlinear term in these partial differential equations (PDE) reflects the presence of transaction costs. Leland developed a modified option replicating strategy which depends on the size of transaction costs and the frequency of revision. In this thesis, we consider the problem of option pricing under the Heston-CIR model, which is a combination of the stochastic volatility model discussed in Heston and the stochastic interest rates model driven by Cox-Ingersoll-Ross (CIR) processes with transaction costs. in this case, the reacted nonlinear PDE with respect to the option price does not have a closed-form solution. We use the finite-difference scheme to solve this PDE and conduct model's performance analysis.

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Attestation of Authorship

I hereby declare that this submission is my own work and that, to the best of my knowledge and belief, it contains no material previously published or written by another person (expect where explicitly defined in the acknowledgements), nor material which to a substantial extent has been submitted for the award of any other degree or diploma of a university or other institution of higher learning.

Signature of student

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Chapter 1

Introduction

1.1 Background and Literature Review

Transaction costs are expenses incurred when buying or selling a good or service. One of assumptions in the Black-Scholes (Black & Scholes, 1973) model assumed no transaction cost in the continuous re-balancing of a hedged portfolio. In real financial markets, this assumption is not valid. The construction of hedging strategies for transaction cost is an important problem. Leland (Leland, 1985) presented a proportional transaction cost based on a method for hedging call option and Black-Scholes assumptions. He assumed the hedge strategy of re-hedging at every time step to give the fundamental of option pricing model with transaction costs. This model assumes that the portfolio of option is re-balanced at every time step, the bid-offer spread and the cost have proportion of the value traded. Then, Hodges and Neuberger (Hodges & Neuberger, 1989) described the replication of a hedged contingent claim under proportional transaction costs. They derived an optimal replicating strategies by considering an alternative and simpler claim which is better than the strategy derived by Leland.

Boyle and Vorst (Boyle & Vorst, 1992) used the long-term price to approximate the Black-Scholes formula with an adjusted variance which is similar to the optimal strategies derived by Leland (Leland, 1985). The results of Leland and Boyle and Vorst are not very effective to the volatility in the Black-Scholes formula. Davis, Panas and Zariphopoulou (Davis, Panas & Zariphopoulou, 1993) considered a framework under which proportional transaction charges are levied on all sales and purchases of stock. In this case, "perfect replication" is no longer possible, and holding an option involves an essential element of risk. They derived a non-linear function as unique viscosity solution with different boundary conditions. Hoggard, Whalley and Wilmott (Hoggard, Wilmott & Whalley, 1994) derived a non-linear parabolic PDE for the option price and gave results for several simple combinations of options, as their results are different from those before. They proposed a portfolio of European options for hedging with transaction costs. This paper assumed the fixed length of time step and reduced the modified variance case presented by Leland. As previously mentioned, the results from Leland need to be proved. Lott (Lott, 1993) provided a rigorous mathematical proof of the footnote remark for Leland's conjecture which claims that the level of transaction costs is a constant. Kabanov and Safarian (Y. M. Kabanov & Safarian, 1997) calculated the hedging error and proved the approximation results for this, because the Leland's constant level of transaction costs is incorrect.

Dewynne, Whalley and Wilmott (Dewynne, Whalley & Wilmott, 1994) considered option pricing with transaction costs to a model, in terms of differential equations. Soner, Shreve and Cvitanić (Soner, Shreve & Cvitanić, 1995) proved that if we are attempting to dominate a European call, then we can use the trivial strategy of buying one share of the underlying stock to dominate the European call and holding to maturity. They derived the least expensive method of dominating a European call in a Black-Scholes model with proportional transaction costs. Mohamed (Mohamed, 1994) considered the issue of hedging options under proportional transaction costs and attempted to evaluate several re-hedging strategies by Monte Carlo simulations. His results found that the best hedging strategy is the Whalley and Wilmott approximation for the Hodges and

Neuberger utility maximization. Whalley and Wilmott (Whalley & Wilmott, 1997) analyzed the different Black–Scholes fair values for pricing European options with re-hedging transaction costs in real financial market. They used the asset price and time in an inhomogeneous diffusion equation to improve the the optimal hedging strategy. Indeed, both Mohamed (Mohamed, 1994) and Whalley and Wilmott (Whalley & Wilmott, 1997) found a dynamic band for the hedging strategy involving with the option's gamma.

Grannan and Swindle (Grannan & Swindle, 1996) illustrated a method for constructing option hedging strategies with transaction costs which contains Leland's discrete time replication scheme. They obtained a strategy using different time intervals between hedging, replication error for a given initial wealth will significantly reduce. Cvitanić and Karatzas (Cvitanić & Karatzas, 1996) derived a formula for the minimal initial wealth needed to hedge strategies with transaction costs and proved an optimal solution to the portfolio optimization problem of maximizing utility from terminal wealth in the same model. Ahn et al. (Ahn, Dayal, Grannan, Swindle et al., 1998) established the concept of diffusion limits for hedging strategies. They obtained the expressions for replication errors of stock price strategies and a variety of "renewal" strategies.

Grandits and Schachinger (Grandits & Schachinger, 2001) proved the limiting hedging error is a removable discontinuity at the exercise price. According to a quantitative result they determined the rate at which that peak becomes narrower as the lengths of revision intervals change. Baran (Baran, 2003) gave a quantile hedging for strategy effectiveness and shortfall risk in a discrete-time market model with transaction costs. Pergamenshchikov (Pergamenshchikov, 2003) proved the limit theorem of the Leland strategy for an approximate hedging and the rate of convergence. Wilmott (Wilmott, 2006) presented a review for the Leland's model (Leland, 1985) for transaction costs and the Hoggard–Whalley–Wilmott model (Hoggard et al., 1994) for option portfolios.

Zhao and Ziemba (Zhao & Ziemba, 2007b) identified that the Leland's claim has mathematical defects. This means that we cannot optimize the option price with transactions costs in the Black–Scholes model (Black & Scholes, 1973). Zhao and Ziemba (Zhao & Ziemba, 2007a) simulated the volatility adjusted by the length of trading interval and the transaction costs. They specified the Leland's model without including the cost of establishing the initial hedge ratio. Leland (Leland, 2007) corrected this problem. The results of Lott (Lott, 1993) and Kabanov and Safarian (Y. M. Kabanov & Safarian, 1997) can be used on the case of more general pay-off functions and unevenness revision intervals, but the terminal values of portfolio do not converge to the non-convex pay-off function. Lépinette (Lépinette, 2008) suggested a modification to Leland's strategy to solve the identification of Kabanov and Safarian. Lépinette (Lépinette-Denis, 2009) showed that the convergence holds for a large class of concave pay-off functions to the Leland strategy. Kabanov and Safarian (Y. Kabanov & Safarian, 2009) considered the hedging errors of Leland's strategies and arbitrage theory for markets with transaction costs. They used a multidimensional HJB equation for the optimal control of portfolios in the presence of market friction. Denis and Kabanov (Denis & Kabanov, 2010) found the convex pay-off function and the first order term of asymptotics for the mean square error. Denis (Denis, 2010) showed that a convex large class of the pay-off functions for the Leland's strategies.

Recently, the most advanced domains of mathematical finance is the arbitrage theory for financial markets with proportional transaction costs. Grépat and Kabanov (Grépat & Kabanov, 2012) established criteria of absence of arbitrage opportunities under small transaction costs for a family of multi-asset models of financial market. Guasoni, Lépinette and Rásonyi (Guasoni, Lépinette & Rásonyi, 2012) proved the Fundamental Theorem of Asset Pricing with transaction costs, when bid and ask prices follow locally bounded cadlag processes. The result of this paper relies on a new notion of admissibility, which reflects future liquidation opportunities. The Robust No Free Lunch with Vanishing Risk condition implies that admissible strategies are predictable processes of a finite variation. Mariani and Sengupta (Mariani & SenGupta, 2012) proposed a particular market completion assumption which asserts the asset is driven by a stochastic volatility process and in the presence of transaction costs and led to solving a nonlinear partial differential equation to find the price of options. Under this paper, Mariani, SenGupta and Bezdek (Mariani, SenGupta & Bezdek, 2012) gave an algorithmic scheme to obtain the solution of the problem by an iterative method and provide numerical solutions using the finite difference method. As we know, if the transaction cost rate does not depend on the number of revisions, the approximation error does not converge to zero as the frequency of revisions tends to infinity. Lépinette (Lépinette, 2012) suggest a modification of Leland strategy ensuring that the approximation error vanishes in the limit.

In particular, transaction costs can be approximately compensated applying the Leland adjusting volatility principle and asymptotic property of the hedging error due to discrete readjustments is characterized. Nguyen (H. Nguyen, 2014) showed that jump risk is approximately eliminated and the results established in continuous diffusion models are recovered. They also confirmed that for constant trading cost rate, the results established by Kabanov and Safarian (Y. M. Kabanov & Safarian, 1997) and Pergamenshchikov (Pergamenshchikov, 2003) are valid in jump-diffusion models with deterministic volatility using the classical Leland parameter. Florescu, Mariani and Sengupta (Florescu, Mariani & Sengupta, 2014) considered the nonlinear term in these partial differential equations (PDE) which reflect an underlying general stochastic volatility model of transaction costs. In this premise, they used a traded proxy for the volatility to obtain a non-linear PDE whose solution provides the option price in the presence of transaction costs. Lépinette and Tran (Lépinette & Tran, 2014) extended the results of Denis (Denis, 2010), Lépinette (Lépinette, 2012) for local volatility models in the market of European options. They proposed an approximation of replication of a European contingent claim when the market is under proportional transaction costs.

In a recent paper, SenGupta (SenGupta, 2014) generalized the nonlinear partial differential equations even when the underlying asset follows a stochastic one-factor interest rate model. The nonlinear term in the resulting PDE corresponding to the presence of transaction costs is modelled using a simple geometric Brownian motion. This paper shows that the model follows a nonlinear parabolic type partial differential equation and proves the existence of classical solution for this model under a particular assumption. Later on, Mariani, SenGupta and Sewell(Mariani, SenGupta & Sewell, 2015) used PDE2D software to solve a complex partial differential equation motivated by applications in finance where the solution of the system gives the price of a European call option, including transaction costs and stochastic volatility. Nguyen and Pergamenshchikov (T. H. Nguyen & Pergamenschchikov, 2015) showed that jump risk is approximately eliminated and the results established in continuous diffusion models are recovered. They described the option replication under constant proportional transaction costs in models where stochastic volatility and jumps are combined to capture market's important features. In particular, transaction costs can be approximately compensated by applying the Leland adjusting volatility principle and asymptotic property of the hedging error due to discrete readjustments is characterized. Later on, Nguyen and Pergamenshchikov (T. H. Nguyen & Pergamenshchikov, 2017) proved several limit theorems for the normalized replication error of Leland's strategy, as well as that of the strategy suggested by Lépinette (Lépinette & Tran, 2014). They fixed the underhedging property pointed out by Kabanov and Safarian (Y. M. Kabanov & Safarian, 1997). Kallsen and Muhle-Karbe,(Kallsen & Muhle-Karbe, 2017) investigated a the general structure of optimal investment and consumption with small proportional transaction costs. For a risk-less asset and a risky asset with general continuous dynamics, traded with random and time-varying but small transaction costs, this paper derives simple formal asymptotics for the optimal policy and welfare.

1.2 Research Questions

Based on the literature review, we considered the following research questions in this thesis.

Question 1.1: How can we derivate Heston stochastic volatility model with transaction costs using an approach similar to that in Mariani, SenGupta and Sewell(Mariani et al., 2015)?

Question 1.2: Whether the estimation of our results under different stochastic volatility models with transaction cost is consistent with the results of Mariani, SenGupta and Sewell(Mariani et al., 2015)?

Question 1.3: How can we derive a solution to the Heston-CIR model with transaction costs and stochastic interest rate using an approach similar to Chapter 3 of this thesis?

1.3 Thesis Contributions and Organization

The contributions of this thesis can be expressed in answering the questions given in Section 1.2. These answers are included in subsequent chapters, which are organized as follows.

Chapter 2: In this chapter, we introduce some mathematical preliminaries and financial terminologies which will be used in the subsequent chapters. Section 2.1 gives the mathematical foundations include probability theory, stochastic processes, Brownian motion and Itô's Lemma. In Section 2.2, we present some financial preliminaries, including the Black-Scholes model, risk neutral measure, the Heston model and the Cox-Ingersoll-Ross (CIR) model.

Chapter 3: In the first section of this chapter, we briefly introduce Leland's (Leland, 1985) classical model on pricing option with transaction costs. In Section 3.2, we extend Leland's model in Section 3.1 by adding transaction costs to Heston's (Heston, 1993) stochastic volatility model. In Section 3.3, we apply the finite-difference method to find an approximate solution to the model derived in Section 3.2. The last section is dedicated to the numerical implementation of the solution obtained in Section 3.3 and comparison our results with these of Mariani, SenGupta and Sewell (Mariani et al., 2015).

Chapter 4: In this chapter, we consider the Heston-CIR model with transaction cost. In Section 4.1, we introduce the Heston-CIR with a partial correlation. In order to analyze the delta hedging portfolio of the Heston-CIR model with transaction cost in Section 4.3, we first derive a pricing formula for zero-coupon bonds in Section 4.2. In Section 4.4, we use the replicating technique to derive the model and substitute the solution of zero-coupon bonds into the PDE. We obtain the numerical solution to the PDE of Heston-CIR model with transaction cost by implementing the finite difference scheme in MATLAB. In the last section of this chapter, we analyze the numerical results of the

PDE.

Chapter 5: This is the last chapter of the thesis and is devoted to present the conclusion and some potential research directions in the future.

Chapter 2

Mathematical and Financial Techniques

In this chapter, we introduce some mathematical preliminaries and financial terminologies which will be used in the subsequent chapters. Section 2.1 gives the mathematical foundations include probability theory, stochastic processes, Brownian motion and Itô's Lemma. In Section 2.2, we present some financial preliminaries the Black-Scholes model, risk neutral measure, the Heston model and the Cox-Ingersoll-Ross (CIR) model.

2.1 Mathematical Techniques

In this chapter, we introduce some mathematical preliminaries and techniques that are applied in this thesis. The majority of the material used in this chapter is taken from textbooks (Shreve, 2004) and (Wilmott, 2006).

2.1.1 Probability Theory

We define a probability space as $(\Omega, \mathcal{F}, \mathbb{P})$ using the terminology of measure theory. The sample space Ω is a set of all possible outcomes $\omega \in \Omega$ of some random experiment. Probability $\mathbb{P}$ is a function, $A \mapsto \mathbb{P}(A)$, which assigns a non negative number $\mathbb{P}(A)$ to A in a subset $\mathcal{F}$ of all possible set of outcomes, where the event space $\mathcal{F}$ is a σ-algebra on Ω. We use 2^{Ω} to denote the set of all possible subset of Ω.

Definition 2.1.1 (σ-algebra)**.** *We say that $\mathcal{F} \subseteq 2^{\Omega}$ is a σ-algebra, if*

- $\Omega \in \mathcal{F}$,
- *If $A \in \mathcal{F}$ then $A^c \in \mathcal{F}$ as well (where $A^c = \Omega \setminus A$).*
- *If $A_i \in \mathcal{F}$ for $i = 1, 2, 3, \ldots$ then also $\bigcup_i A_i \in \mathcal{F}$.*

Definition 2.1.2. *A pair $(\Omega, \mathcal{F})$ with $\mathcal{F}$ a σ-algebra of subsets of Ω is called a measurable space. Given a measurable space $(\Omega, \mathcal{F})$, a measure μ is any countably additive non-negative set function on this space. That is $\mu : \mathcal{F} \to [0, \infty]$, having the properties:*

- $\mu(A) \geq \mu(\varnothing) = 0$ *for all $A \in \mathcal{F}$.*
- $\mu(\bigcup_n A_n) = \sum_n \mu(A_n)$ *for any countable collection of disjoint sets $A_n \in \mathcal{F}$.*

When in addition $\mu(\Omega) = 1$, we call the measure μ a probability measure, and often label it by $\mathbb{P}$ (it is also easy to see that then $\mathbb{P}(A) \leq 1$ for all $A \in \mathcal{F}$).

To summarize, a probability measure space a triple $(\Omega, \mathcal{F}, \mathbb{P})$, with $\mathbb{P}$ a measure on a measurable space $(\Omega, \mathcal{F})$.

Definition 2.1.3 (Random Variable)**.** *A random variable X is a real-valued function $X : \Omega \mapsto \mathbb{R}$ on a probability measure space $(\Omega, \mathcal{F}, \mathbb{P})$ which satisfies the property that for any Borel subset B of $\mathbb{R}$, the subset of Ω given by*

$$X^{-1}(B) := \{\omega : X(\omega) \in B\}. \tag{2.1}$$

belongs to $\mathcal{F}$.

A random variables X is numerical functions $\omega \mapsto X(\omega)$ of the outcome of our random experiment. To define the Borel subsets of $\mathbb{R}$, we first consider the closed intervals $[a, b] \in \mathbb{R}$ and then proceed to add all possible sets that are necessary to have a σ-algebra. Therefore, all possible unions of sequences of closed intervals are Borel sets.

Definition 2.1.4 (Filtration)**.** *A filtration is a family $\{\mathcal{F} : t \geq 0\}$ of sub-σ-algebra such that $\mathcal{F}(s) \subseteq \mathcal{F}(t)$ for all $s \leq t$.*

Theorem 2.1.1 ($\mathcal{G}$-measurable)**.** *Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a random variable X defined on $(\Omega, \mathcal{F}, \mathbb{P})$. Denote $\mathcal{G}$ a σ-algebra of subset of Ω. Then if every set within $\sigma(X)$ is also in $\mathcal{G}$, such that X is $\mathcal{G}$-measurable.*

2.1.2 Expectation

The mean, expected value, or expectation of a random variable X is written as $\mathbb{E}(X)$ or μ_X. The expectation is defined differently for continuous and discrete random variables. Let $f(X)$ be a function of X. We can imagine a long-lerm average of $f(X)$ just as we can imagine a long-term average of X. This average is written as $\mathbb{E}(f(X))$. Imagine observing X many times (N times) to give results $x_1, x_2, ..., x_N$.

Apply the function f to each of these observations, to give $f(x_1), ..., f(x_N)$. The mean of $f(x_1), f(x_2), ..., f(x_N)$ approaches $\mathbb{E}(f(X))$ as the number of observation N tends to infinity.

Theorem 2.1.2. *Let X be a continuous random variable and let f be a function. The expected value of $f(X)$ is defined as*

$$\mathbb{E}(f(X)) = \int_{-\infty}^{\infty} f(x)p(x)dx,$$

where p is th probability density function of X.

Theorem 2.1.3. *Let X be a discrete random variable and let f be a function. The expected value of $f(X)$ is*

$$\mathbb{E}(f(X)) = \sum_x f(x)p(x) = \sum_x f(x)\mathbb{P}(X = x).$$

The expectation of X is an indicator of the mean or first moment of the random variable.

2.1.3 Stochastic Processes

A stochastic process is simply a collection of random variables indexed by time. It will be useful to consider separately the cases of discrete time and continuous time. We will even have occasion to consider indexing the random variables by negative time. A discrete time stochastic process $X = \{X_n, n = 0, 1, 2, ...\}$ is a countable collection of random variables indexed by the non-negative integers, and a continuous time stochastic process $X = \{X_t, 0 \leq t < \infty\}$ is an uncountable collection of random variables indexed by the non-negative real numbers.

Definition 2.1.5. *Suppose that $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, and that $I \subset \mathbb{R}$ is of infinite cardinality. Suppose further that for each $\alpha \in I$, there is a random variable $\{X(\alpha) : \Omega \to \mathbb{R}\}$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$. The function $\{X : I \times \Omega \to \mathbb{R}\}$ defined by $X(\alpha, \omega)$ is called a stochastic process with indexing set I, and it written $X = \{X(\alpha), \alpha \in I\}$.*

In general, we may consider any indexing set $I \subset \mathbb{R}$ having infinite cardinality, so that calling $X = \{X(\alpha), \alpha \in I\}$ a stochastic process simply means that $X(\alpha)$ is a random variable for each $\alpha \in I$. If the cardinality of I is finite, then X is not considered as a stochastic process, but rather a random vector.

2.1.4 Martingales and Markov Process

Definition 2.1.6 (Martingale)**.** *A valued stochastic process $\{X(t) : t \geq 0\}$ is a martingale with respect to a filtration $\{\mathcal{F}(t) : t \geq 0\}$ if it is adapted, that is, $X(t) \in \mathcal{F}(t)$ for all $t \geq 0$, if $\mathbb{E}[X(t)] < \infty$ for all $t \geq 0$, and if*

$$\mathbb{E}[X(t) \mid \mathcal{F}(s)] = X(s).$$

for all $0 \leq s \leq t$.

Definition 2.1.7 (Markov Process)**.** *Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let T denote a fixed positive number and let $\{\mathcal{F}(t) : 0 \leq t \leq T\}$ be a filtration. Let $\{X(t) : 0 \leq t \leq T\}$ denote an adapted stochastic process. Assume that for all s and t, where $0 \leq s \leq t \leq T$, and for every non-negative Borel-measurable function f, there exists another Borel-measurable function g such that*

$$\mathbb{E}[f(X(t)) \mid \mathcal{F}(s)] = g(X(s)).$$

Then we say that $\{X(t) : 0 \leq t \leq T\}$ is a Markov Process.

2.1.5 Brownian Motion

A Brownian motion $\{B(t) : t \geq 0\}$ is a continuous-time stochastic process satisfying the following conditions:

- $B(t)$ is continuous in the parameter t, with $B(0) = 0$.
- For each t, $B(t)$ is normally distributed with expected value 0 and variance t, and they are independent of each other.
- For each t and s the random variables $B(t+s) - B(s)$ and $B(s)$ are independent. Moreover $B(t+s) - B(s)$ has variance t.

However, just because we want something with certain properties does not guarantee that such a thing exists.There is one important fact about Brownian motion,

$$S(t) = e^{\sigma B(t)} e^{(\mu - \sigma^2/2)t}$$

satisfies the stochastic differential equation

$$dS = \mu S dt + \sigma S dB(t). \tag{2.2}$$

The crucial fact about Brownian motion, which we will need, is

$$(dB)^2 = dt,$$

where $(dB)^2$ is determinant, not random and its magnitude is dt. So the amount of change in $(dB)^2$ caused by a change dt in the parameter is equal to dt. To partially justify this statement we compute the expected value of $(B(t+\delta t) - B(t))^2$.

$$\mathbb{E}[(B(t+\delta t) - B(t))^2] = Var[(B(t+\delta t) - B(t))] = \delta t.$$

2.1.6 Itô's Lemma

Theorem 2.1.4. *Let $\{B(t) : t \geq 0\}$ be a Brownian motion and $\{W(t) : t \geq 0\}$ be an Itô's drift-diffusion process which satisfies the stochastic differential equation:*

$$dW(t) = \mu(W(t), t)dt + \sigma(W(t), t)dB(t).$$

If $f(w, t) \in C^2(\mathbb{R}^2, \mathbb{R})$ then $f(W(t), t)$ is also an Ito drift-diffusion process, with its differential given by:

$$d(f(W(t), t)) = \frac{\partial f}{\partial t}(W(t), t)dt + \frac{\partial f}{\partial W}(W(t), t)dW(t) + \frac{1}{2}\frac{\partial^2 f}{\partial W^2}(W(t), t)\sigma^2 dt.$$

For a function $f(x, y)$ of the variable x and y it is not at all hard to justify that the equation below is correct to first order terms.

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy.$$

However, what if we have a function f which depends not only on a real variable t, but also on a stochastic process such as Brownian motion? Suppose that $f = f(t, B(t))$, where $\{B(t) : t \geq 0\}$ denotes a Brownian motion. One is tempted to write as before that

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial B(t)}dB(t).$$

In this case we would be badly mistaken. To see that this is so, we expand df using Taylor's formula and keep the terms involving the second derivatives of f

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial B(t)}dB(t) + \frac{1}{2}\frac{\partial^2 f}{\partial t^2}(dt)^2 + \frac{1}{2}\frac{\partial^2 f}{\partial B(t)^2}(dB(t))^2 + \text{higher order terms}.$$

We discard all terms involving dt to a power higher than 1. Note that the term $dtdB(t)$

has magnitude $(dt)^{3/2}$. This leaves the following expression for df:

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial B(t)}dB(t) + \frac{1}{2}\frac{\partial^2 f}{\partial B(t)^2}(dB(t))^2.$$

We next use the fact that $(dB(t))^2 = dt$.

2.2 Financial Techniques

In this section, we present some financial preliminaries, including the Black-Scholes model and risk neutral measure. Further on, some concepts on stochastic volatility models and stochastic interest rate which are required in the forthcoming chapters are discussed. Detailed explanations can be found in (Wilmott, 2006).

2.2.1 Equivalent Probability Measures

Let $(\Omega, \mathcal{F})$ be a measurable space, $\mathbb{P}$ and $\mathbb{Q}$ are two equivalent probability measures on $(\Omega, \mathcal{F})$ for any $A \in \mathcal{F}, \mathbb{P}(A) = 0$ is and only if $\mathbb{Q}(A) = 0$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\{\mathcal{F}(t) : t \geq 0\}$ be a filtration. Suppose that Z is an positive random variable and $\mathbb{E}^{\mathbb{P}}[Z] = 1$. We define $\mathbb{Q}$ by

$$\mathbb{Q}(A) := \int_A Z(\omega) d\mathbb{P}(\omega) \qquad \text{for all} \quad A \in \mathcal{F}.$$

Then $\mathbb{Q}$ is a probability measure generated by Z on $(\Omega, \mathcal{F})$. It can be easily checked that $\mathbb{P}$ and $\mathbb{Q}$ are equivalent probability measures. Moreover, $\mathbb{P}$ and $\mathbb{Q}$ are related by the formula

$$\mathbb{E}^{\mathbb{Q}}[X] = \mathbb{E}^{\mathbb{P}}[XZ].$$

We call Z the Radon-Nikodým derivative of $\mathbb{Q}$ with respect to $\mathbb{P}$, written as

$$Z = \frac{d\mathbb{Q}}{d\mathbb{P}}.$$

The Radon-Nikodým derivative process $\{Z(t) : 0 \leq t \leq T\}$ is defined by

$$Z(t) = \mathbb{E}^{\mathbb{P}}[Z \mid \mathcal{F}], \qquad 0 \leq t \leq T.$$

In addition, $\{Z(t) : 0 \leq t \leq T\}$ is a martingale with respect to $\{\mathcal{F}(t) : 0 \leq t \leq T\}$, since for any $0 \leq s \leq t \leq T$,

$$\mathbb{E}^{\mathbb{P}}[Z(t) \mid \mathcal{F}(s)] = \mathbb{E}^{\mathbb{P}}[\mathbb{E}^{\mathbb{P}}[Z \mid \mathcal{F}(t)] \mid \mathcal{F}(s)] = \mathbb{E}^{\mathbb{P}}[Z \mid \mathcal{F}(s)] = Z(s).$$

Theorem 2.2.1 (Girsanov's theorem)**.** *Let $\{B(t) : 0 \leq t \leq T\}$ be a Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\{\mathcal{F}(t) : 0 \leq t \leq T\}$ be a filtration for this Brownian motion. Let $\{\alpha(t) : 0 \leq t \leq T\}$ be an adapted process with respect to $\{\mathcal{F}(t) : 0 \leq t \leq T\}$. Define*

$$Z(t) := \exp\left(-\int_0^t \alpha(s)dB(s) - \frac{1}{2}\int_0^t \alpha^2(s)du\right),$$

$$\tilde{B}(t) = B(t) + \int_0^t \alpha(s)ds,$$

and assume that

$$\mathbb{E}^{\mathbb{P}}\left[\int_0^T \alpha^2(s)Z^2(s)ds\right] < \infty.$$

Set $Z = Z(T)$. Then $\mathbb{E}^{\mathbb{P}}[Z] = 1$, and under the equivalent probability measure $\mathbb{Q}$ generated by Z, the process $\{\tilde{B}(t) : 0 \leq t \leq T\}$ is a Brownian motion..

2.2.2 Stock Price Under the Risk-Neutral Measure

Consider a stock whose price is modelled by a generalized geometric Brownian

$$dS(t) = \mu(t)S(t)dt + \sigma(t)S(t)dB(t), \qquad t \in [0, T],$$

with both $\mu(t)$ and $\sigma(t)$ are adapted processes. In the integral form,

$$S(t) = S(0)\exp\left[\int_0^t \sigma(s)dB(s) + \int_0^t \frac{\mu(s) - \sigma^2(s)}{2}ds\right].$$

Let the interest rate $R(t)$ be another adapted process. The discount process

$$D(t) = e^{\int_0^t R(s)ds}$$

satisfies

$$dD(t) = -R(t)D(t)dt,$$

which is a formula from the ordinary calculus, since we can use $d\left(\int_0^t R(s)ds\right) = R(t)dt$. One dollar invested at time 0 in bank becomes $\dfrac{1}{D(t)}$ at time t, or, equivalently, 1 dollar has time 0 value $D(t)$ at time t. We compute

$$d(D(t)S(t)) = (\mu(t) - R(t))D(t)S(t)dt + \sigma(t)D(t)S(t)dB(t)$$

$$= \sigma(t)D(t)S(t)(\alpha(t)dt + dB(t)),$$

where

$$\alpha(t) := \frac{\mu(t) - R(t)}{\sigma(t)}$$

is the market price of risk. Define

$$\tilde{B}(t) = B(t) + \int_0^t \alpha(s)ds.$$

By Girsanov's theorem (Girsanov, 1960), there exists a probability measure $\mathbb{Q}$, equivalent to $\mathbb{P}$, under which $\{\tilde{B} : 0 \le t \le T\}$ is a Brownian motion. Hence,

$$d[D(t)S(t)] = \sigma(t)D(t)S(t)d\tilde{B}(t).$$

follows that $\{D(t)S(t) : 0 \le t \le T\}$ is a martingale under $\mathbb{Q}$. For this reason, $\mathbb{Q}$ is called the risk-neutral measure. The mean rate of return from the stock investment under the risk-neutral measure is the same as that from the bank investment.

2.2.3 The Black Scholes Model and Risk Neutral Pricing

In 1973, Black and Scholes (Black & Scholes, 1973) published a seminal paper on the theory of option pricing. They created a risk-less hedging portfolio by adjusting the proportion of the stock and option in the portfolio. In the Black-Scholes world, they assumed a portfolio with zero risk-less arbitrage in the market, it must have an expected rate of return equal to the risk-less interest rate. Suppose that we have a European call option, whose valued is denoted by $C(S,t)$, where S is the stock price at time t. Assume that the stock price follows the geometric Brownian motion Eq.(2.2). By Itô's Lemma (Theorem 2.1.4) we have

$$dC = \left(\mu S\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2} + \frac{\partial C}{\partial t}\right)dt + \sigma S\frac{\partial C}{\partial S}dB(t). \tag{2.3}$$

If Π is a portfolio of one option and $-\Delta$ shares of the stock, then the value of the portfolio is

$$\Pi = C(S,t) - \Delta S. \tag{2.4}$$

The change in the value of this portfolio in one time-step dt is

$$d\Pi = dC(S,t) - \Delta dS. \tag{2.5}$$

Substituting Eq.(2.2) and (2.3) into (2.5), we have

$$d\Pi = \left(\mu S\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2} + \frac{\partial C}{\partial t} - \mu\Delta S\right)dt + \sigma S\left(\frac{\partial C}{\partial S} - \Delta\right)dB(t). \tag{2.6}$$

If we choose $\Delta = \dfrac{\partial C}{\partial S}$, then the stochastic term is zero. This means the risk of portfolio is reduce to zero and $d\Pi$ becomes

$$d\Pi = \left(\frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2} + \frac{\partial C}{\partial t}\right)dt. \tag{2.7}$$

On the other hand, Under risk-neural probability measure $\mathbb{Q}$, the value of the portfolio is changed by time and the interest rate r as $d\Pi = r\Pi dt$. Then we have

$$r\Pi dt = \left(\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + \frac{\partial C}{\partial t}\right) dt. \tag{2.8}$$

Dividing both sides by dt, we obtain the following Black-Scholes equation

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + \frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} - rC = 0. \tag{2.9}$$

Following Wilmott (Wilmott, 2006), the option value C is a function of the underlying asset price S, the time t and the strike price E. At the maturity, the payoff od option is

$$C(S,T) = \max[S - E, 0].$$

Next, we show the solution of the Black Scholes equation.

Theorem 2.2.2. *The value of the European call option is given by*

$$c(S,t) = SN(d_1) - Ee^{-r(T-t)}N(d_2), \tag{2.10}$$

where

$$N(d) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} e^{-\frac{1}{2}s^2} ds$$

is the cumulative distribution function for the standard normal distribution,

$$d_1 = \frac{\ln(S/E) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \quad \textit{and} \quad d_2 = \frac{\ln(S/E) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}. \tag{2.11}$$

Theorem 2.2.3. *The value of the European put option is given by*

$$p(S,t) = Ee^{-r(T-t)}N(-d_2) - SN(-d_1), \tag{2.12}$$

where $N(d)$, d_1 and d_2 are given in Theorem 2.2.1.

2.2.4 The CIR Model

The Cox-Ingersoll-Ross (CIR) model (Cox, Ingersoll Jr & Ross, 1985) is a well-known short-rate model that describes the interest rate movements driven by one source of market risk. This model has been widely used to describe the dynamics of the short rate interest because it has some fundamental features like intuitive parametrization, non-negativity and pricing formulas. According to the CIR model, the dynamics of interest rate can described as follows:

$$dr(t) = \lambda(\theta - r(t))dt + \sigma\sqrt{r(t)}dX_t^3, \tag{2.13}$$

where $r(t)$ is the short rate interest, λ is the speed of mean reversion, θ is the long-run mean and σ is the volatility process.

2.2.5 The Heston Model

The Heston model was published in 1993 (Heston, 1993) to value European options under stochastic volatility. Under the Heston model, the dynamics of the stock price and the variance processes under the risk-neutral measure, and described as follows:

$$dS(t) = rS(t)dt + \sqrt{V(t)}S(t)dX_t^1, \tag{2.14}$$

$$dV(t) = \lambda(\theta - V(t))dt + \sigma\sqrt{V(t)}dX_t^2, \tag{2.15}$$

where the two Brownian motions $\{X_t^1 : t \geq 0\}$ and $\{X_t^2 : t \geq 0\}$ are correlated with a correlation coefficient $\rho(-1 \leq \rho \leq 1)$, this is,

$$\langle dX_t^1, dX_t^2 \rangle = \rho dt, \tag{2.16}$$

r is the rate of return, the parameters λ, θ and σ represent the speed of mean reversion, the long run mean variance and the volatility of the variance, respectively.

Chapter 3

Option Pricing Under the Heston Model with Transaction Costs

In the first section of this chapter, we briefly introduce Leland's (Leland, 1985) classical model on pricing options with transaction costs. In Section 3.2, we extend Leland's model in Section 3.1 by adding transaction costs to Heston's (Heston, 1993) stochastic volatility model. In Section 3.3, we apply the finite-difference method to find an approximate solution to the model derived in Section 3.2. The last section is dedicated to the numerical implementation of the solution obtained in Section 3.3 and comparison between our results and these of Mariani, SenGupta and Sewell (Mariani et al., 2015).

3.1 The Leland Model

Local volatility models are popular as they can be calibrated to the market of European options by the simple Dupire formula. Leland (Leland, 1985) developed a modified option replicating strategy which depends on the size of transaction costs and the frequency of revision. In this paper, Leland introduced the idea of using expected transaction costs over a small interval. SenGupta (SenGupta, 2014) proposed a modified Leland's method which allows to approximately replicate a European contingent claim when the market is under proportional transaction costs. Horsky and Sayer (Horsky & Sayer, 2015) presented an innovative hybrid model for the valuation of equity options. Their model is of affine structure, allows for correlations between the stock, the short rate and the volatility processes and can be fitted perfectly to the initial term structure. In their paper, they determined the zero bond price formula and derived the analytic solution for European type options in terms of characteristic functions needed for fast calibration. Mariani, SenGupta and Sewell (Mariani et al., 2015) solved a complex partial differential equation motivated by applications in finance where the solution of the system gives the price of European options, including transaction costs and stochastic volatility. Since the seminal work by Leland (Leland, 1985), people have been developing strategies of option pricing with transactions costs. The main contributions of Leland (Leland, 1985) are based on the Black-Scholes and Merton (Black & Scholes, 1973) assumptions and model. One of key assumptions is that the value of stock follows a stationary log-normal diffusion process, this is,

$$\frac{dS}{S} = dt + \sigma Z\sqrt{dt}, \tag{3.1}$$

where Z is a standard normal random variable. Using the Taylor expression theorem, we can show

$$\frac{\delta S}{S} = \mu\delta t + \sigma Z\sqrt{\delta t} + O(\delta t^{3/2}), \tag{3.2}$$

where S is the current stock price, σ is the volatility of underlying asset.

Let $C(S; E, t, r, \sigma^2)$ be the value of a European call option when the current stock price is S, the striking price is E, the time to maturity is $T - t$, the interest rate is r and the rate of return have variance σ^2. In the absence of transaction costs with possible continuous trading, Black and Scholes (Black & Scholes, 1973) showed that

$$C = SN(d_1) - Ee^{-rT}N(d_1 - \sigma\sqrt{T-t}), \tag{3.3}$$

where $d_1 = \dfrac{\ln\left(\dfrac{S}{E}\right) + (r + \dfrac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$. It follows that C satisfies the partial differential equation,

$$\frac{1}{2}C_{SS}S^2\sigma^2 + C_t - r[C - C_S S] = 0, \tag{3.4}$$

where the boundary condition is

$$C[S; E, T, r, \sigma^2] = \max[S - E, 0]. \tag{3.5}$$

Consider now holding a fixed portfolio of D shares of stock and Q dollars of the risk-free security over the interval $[t, t + \delta t]$. The length of the interval, δt, will be termed the revision interval. The return to this portfolio will be

$$\delta P = DS\left(\frac{\delta S}{S}\right) + rQ\delta t + O(\delta t^2). \tag{3.6}$$

The change in value of a call option $C[S;K,t,r,\sigma^2]$ will be

$$\delta C = C_S S\left(\frac{\delta S}{S}\right) + C_t\delta t + \frac{1}{2}C_{SS}S^2\left(\frac{\delta S}{S}\right)^2 + O(\delta t^{3/2}). \tag{3.7}$$

The different between the change in value of the portfolio and the call option, δH

$$\delta H = \delta P - \delta C = (DS - C_S S)\left(\frac{\delta S}{S}\right) + (rQ - C_t)\delta t - \frac{1}{2}C_{SS}S^2\left(\frac{\delta S}{S}\right)^2 + O(\delta t^{3/2}). \tag{3.8}$$

We define a replicating portfolio such that

$$D = C_S \tag{3.9}$$

and

$$Q = C - C_S S. \tag{3.10}$$

Since $P = DS + Q = C$ at each time period, this portfolio yield the option return $\max[S - E, 0]$ at T. Substituting (3.9) and (3.10) into (3.8) and using (3.4) yields

$$\delta H = \frac{1}{2}C_{SS}S^2\left[\sigma^2\delta t - \left(\frac{\delta S}{S}\right)^2\right] + O(\delta t^{3/2}). \tag{3.11}$$

The expectation of δH will be

$$\mathbb{E}[\delta H] = \frac{1}{2}C_{SS}S^2\left[\sigma^2\delta t - \left(\frac{\delta S}{S}\right)^2\right] = 0. \tag{3.12}$$

Following Leland (Leland, 1985), we use k to denote the proportional transaction cost rate, the adjusted volatility of short positions on option pricing is given as follows

$$\hat{\sigma}^2(\sigma^2, k, \delta t) = \sigma^2\left[1 + k\mathbb{E}\left|\frac{\delta S}{S}\right| / \sigma^2\delta t\right] \tag{3.13}$$

$$= \sigma^2[1 + \sqrt{2/\pi}k\sigma\sqrt{\delta t}],$$

since

$$\mathbb{E}\left[\frac{\delta S}{S}\right] = \sqrt{(2/\pi)}\sigma\sqrt{\delta t}.$$

Let

$$\hat{C}[S, E, \sigma^2, r, t, k, \delta t] = S_t N(\hat{d}_1) - Ee^{-r(T-t)}N(\hat{d}_1 - \hat{\sigma}\sqrt{T-t}), \tag{3.14}$$

where

$$\hat{d}_1 = \frac{\ln\left(\frac{S}{E}\right) + (r + \frac{1}{2}\hat{\sigma}^2)(T-t)}{\hat{\sigma}\sqrt{T-t}}. \tag{3.15}$$

That is, $\hat{C}$ is the Black-Scholes option price based on the modified volatility (3.13). Since the augmented volatility does not depend on the strike price, one might be skeptical about the truth of the theory in the first place.

Theorem 3.1.1 ((Leland, 1985))**.** *Following the modified delta hedging strategy, the Black-Scholes price* $\hat{C}$ *will yield* $\max[S - E, 0]$ *almost inclusive of transaction costs, as* $\delta t \to 0$.

3.2 Heston's Stochastic Model with Transaction Costs

In this section, we present a general stochastic volatility model for which a valuation formula can be derived. The Heston model is one of the most widely used stochastic volatility models today. From the Heston (Heston, 1993) model, under the risk-neutral probability measure $\mathbb{Q}$ we assume the spot index and the volatility are given as follows

$$dS(t) = rS(t)dt + \sqrt{V(t)}S(t)dX_t^1, \tag{3.16}$$

$$dV(t) = \lambda(\theta - V(t))dt + \sigma\sqrt{V(t)}dX_t^2, \tag{3.17}$$

where the two Brownian motions X_t^1 and X_t^2 are correlated with a correlation coefficient $\rho(-1 \leq \rho \leq 1)$, this is,

$$\langle dX_t^1, dX_t^2 \rangle = \rho dt. \tag{3.18}$$

We can now proceed the hedging argument in order to form a risk-free portfolio. If we consider a portfolio Π consisting of one European call option, with value $C(S, V, t)$ on the current price S and variance V at time t, quantities $-\Delta$ and $-\Delta_1$ of S and V, respectively. The value of hedging portfolio is

$$\Pi = C - \Delta S - \Delta_1 V. \tag{3.19}$$

According to the self-financing argument and considering the transaction cost, we know an expression for the change in value of the portfolio Π

$$d\Pi = dC - \Delta dS - \Delta_1 dV - kS \mid \nu \mid -k_1 V \mid \nu_1 \mid . \tag{3.20}$$

Applying Itô's formula to get the dynamics of C, we obtain

$$dC = \frac{\partial C}{\partial t}dt + \frac{\partial C}{\partial S}dS + \frac{\partial C}{\partial V}dV + \frac{1}{2}VS^2\frac{\partial^2 C}{\partial S^2}dt + \frac{1}{2}V\sigma^2\frac{\partial^2 C}{\partial V^2}dt + \rho\sigma VS\frac{\partial^2 C}{\partial S\partial V}dt.$$

Substituting dC into Eq. (3.20), we can now deduce a valuation formula for the change in value of the portfolio Π

$$d\Pi = \left(\frac{\partial C}{\partial t} + \frac{1}{2}VS^2\frac{\partial^2 C}{\partial S^2} + \frac{1}{2}V\sigma^2\frac{\partial^2 C}{\partial V^2} + \rho\sigma VS\frac{\partial^2 C}{\partial S\partial V}\right)dt \tag{3.21}$$

$$+\left(\frac{\partial C}{\partial S} - \Delta\right)dS + \left(\frac{\partial C}{\partial V} - \Delta_1\right)dV - kS\mid\nu\mid - k_1V\mid\nu_1\mid.$$

The risk can be hedged away to leading order by setting the coefficients of dS and dV to zero. Following Mariani, SenGupta and Sewell (Mariani et al., 2015), we let

$$\Delta = \frac{\partial C}{\partial S},$$

and

$$\Delta_1 = \frac{\partial C}{\partial V}.$$

Substituting Δ and Δ_1 into Eq. (3.21,)we can eliminate the dS and dV terms and the dynamics of Π becomes

$$d\Pi = \left(\frac{\partial C}{\partial t} + \frac{1}{2}VS^2\frac{\partial^2 C}{\partial S^2} + \frac{1}{2}V\sigma^2\frac{\partial^2 C}{\partial V^2} + \rho\sigma VS\frac{\partial^2 C}{\partial S\partial V}\right)dt - kS\mid\nu\mid - k_1V\mid\nu_1\mid. \tag{3.22}$$

Now in principle options depending on the underlying asset S and possibly even the variance V can be priced by developing a numerical scheme for the PDE and working backward in time from the payoff at maturity. However, in the real financial market this price is not readily justified, since the variance V is not a tradable asset in the marketplace and must be dynamically hedged in other way. This model also has many parameters to be estimated in order to model the market. Therefore we consider adding transaction costs and this addition may influence the prices obtained for the options. In this section we investigate the costs associated with trading the asset. If the number of asset held at time t is

$$\Delta_t = \frac{\partial C}{\partial S}(S, V, t), \tag{3.23}$$

we assume that after a time step δt and re-hedging, the number of assets we hold in the small time interval $[t, t + \delta t]$ is

$$\Delta_{t+\delta t} = \frac{\partial C}{\partial S}(S + \delta S, V + \delta V, t + \delta t).$$

According to the conditions given above, the time step δt is small and the changes in asset and the interest rate are also small. Now, we applying Taylor's formula to expend $\Delta_{t+\delta t}$ yields,

$$\Delta_{t+\delta t} \simeq \frac{\partial C}{\partial S} + \delta t \frac{\partial^2 C}{\partial t \partial S} + \delta S \frac{\partial^2 C}{\partial S^2} + \delta V \frac{\partial^2 C}{\partial S \partial V} + \cdots \tag{3.24}$$

If we set up $\delta S = \sqrt{V} S \delta X^1 + \mathcal{O}(\delta t)$ and $\delta V = \sigma \sqrt{V} \delta X^2 + \mathcal{O}(\delta t)$, substitute δS and δV into Eq. (3.24), by neglecting all terms proportional to δt or with higher order in δt, we will have an expression as follow,

$$\Delta_{t+\delta t} \simeq \frac{\partial C}{\partial S} + \sqrt{V} S \delta X^1 \frac{\partial^2 C}{\partial S^2} + \sigma \sqrt{V} \delta X^2 \frac{\partial^2 C}{\partial S \partial V}. \tag{3.25}$$

Next, we substitute (3.23) into (3.25) to get the number of assets trading during a time step:

$$\nu = \sqrt{V} S \delta X^1 \frac{\partial^2 C}{\partial S^2} + \sigma \sqrt{V} \delta X^2 \frac{\partial^2 C}{\partial S \partial V}. \tag{3.26}$$

Because X^1 and X^2 are correlated Brownian motions, we consider Z_1 and Z_2 are independent normal variables with mean 0 and variance 1. Then, we have

$$\delta X^1 = Z_1\sqrt{\delta t}$$

and

$$\delta X^2 = \rho Z_1\sqrt{\delta t} + \sqrt{1-\rho^2}Z_2\sqrt{\delta t}.$$

Substituting expressions of δX^1 and δX^2 in Eq. (3.26) and denoting:

$$\alpha_1 = \sqrt{V}S\sqrt{\delta t}\frac{\partial^2 C}{\partial S^2} + \sigma\sqrt{V}\rho\sqrt{\delta t}\frac{\partial^2 C}{\partial S\partial V}, \tag{3.27}$$

$$\beta_1 = \sigma\sqrt{V}\sqrt{1-\rho^2}\sqrt{\delta t}\frac{\partial^2 C}{\partial S\partial V},$$

then we can rewrite the change in the number of shares over a time step δt as:

$$\nu = \alpha_1 Z_1 + \beta_1 Z_2.$$

In a very similar way we can express ν_1 as follows:

$$\nu_1 = \sqrt{V}S\delta X^1\frac{\partial^2 C}{\partial S\partial V} + \sigma\sqrt{V}\delta X^2\frac{\partial^2 C}{\partial V^2}. \tag{3.28}$$

We know the expectation of the change in value of the portfolio is

$$\mathbb{E}[d\Pi] = \left(\frac{\partial C}{\partial t} + \frac{1}{2}VS^2\frac{\partial^2 C}{\partial S^2} + \frac{1}{2}V\sigma^2\frac{\partial^2 C}{\partial V^2} + \rho\sigma VS\frac{\partial^2 C}{\partial S\partial V}\right)dt - kS\mathbb{E}[\mid\nu\mid] - k_1V\mathbb{E}[\mid\nu_1\mid]. \tag{3.29}$$

Under the risk-neutral measure $\mathbb{Q}$,

$$\mathbb{E}[d\Pi] = r\Pi dt. \tag{3.30}$$

Hence, we have

$$\left(\frac{\partial C}{\partial t}+\frac{1}{2}VS^2\frac{\partial^2 C}{\partial S^2}+\frac{1}{2}V\sigma^2\frac{\partial^2 C}{\partial V^2}+\rho\sigma VS\frac{\partial^2 C}{\partial S\partial V}\right)dt-kS\mathbb{E}[\mid\nu\mid]-k_1V\mathbb{E}[\mid\nu_1\mid$$

$$=r\left(C-S\frac{\partial C}{\partial S}-V\frac{\partial C}{\partial V}\right)dt.$$

Dividing each side by dt and re-arranging yield

$$\frac{\partial C}{\partial t}+\frac{1}{2}VS^2\frac{\partial^2 C}{\partial S^2}+\frac{1}{2}V\sigma^2\frac{\partial^2 C}{\partial V^2}+\rho\sigma VS\frac{\partial^2 C}{\partial S\partial V}-\frac{kS}{dt}\mathbb{E}[\mid\nu\mid]-\frac{k_1V}{dt}\mathbb{E}[\mid\nu_1\mid$$

$$=r\left(C-S\frac{\partial C}{\partial S}-V\frac{\partial C}{\partial V}\right). \tag{3.31}$$

Next, we calculate $\mathbb{E}[\mid\nu\mid]$ and $\mathbb{E}[\mid\nu_1\mid]$. By Eq.(3.28), we have

$$\mathbb{E}[\mid\nu\mid]=\sqrt{\frac{2}{\pi}}\sqrt{\alpha_1^2+\beta_1^2}=\sqrt{\frac{2\delta t}{\pi}}\times\sqrt{VS^2\left(\frac{\partial^2 C}{\partial S^2}\right)^2+\sigma^2V\left(\frac{\partial^2 C}{\partial S\partial V}\right)^2+2\rho V\sigma S\frac{\partial^2 C}{\partial S^2}\frac{\partial^2 C}{\partial S\partial V}}. \tag{3.32}$$

Similarly, we can obtain

$$\mathbb{E}[\mid\nu_1\mid]=\sqrt{\frac{2\delta t}{\pi}}\times\sqrt{VS^2\left(\frac{\partial^2 C}{\partial S\partial V}\right)^2+\sigma^2V\left(\frac{\partial^2 C}{\partial V^2}\right)^2+2\rho V\sigma S\frac{\partial^2 C}{\partial V^2}\frac{\partial^2 C}{\partial S\partial V}}. \tag{3.33}$$

If we substitute Eq.(3.32) and Eq.(3.33) into Eq. (3.31) and note that $dt=\delta t$, we get a partial differential equation of stochastic volatility model with transaction costs,

$$\frac{\partial C}{\partial t}+\frac{1}{2}VS^2\frac{\partial^2 C}{\partial S^2}+\frac{1}{2}V\sigma^2\frac{\partial^2 C}{\partial V^2}+\rho\sigma VS\frac{\partial^2 C}{\partial S\partial V}+rS\frac{\partial C}{\partial S}+rV\frac{\partial C}{\partial V}-rC \tag{3.34}$$

$$-kS\sqrt{\frac{2}{\pi\delta t}}\times\sqrt{VS^2\left(\frac{\partial^2 C}{\partial S^2}\right)^2+\sigma^2V\left(\frac{\partial^2 C}{\partial S\partial V}\right)^2+2\rho V\sigma S\frac{\partial^2 U}{\partial S^2}\frac{\partial^2 U}{\partial S\partial V}}$$

$$-k_1V\sqrt{\frac{2}{\pi\delta t}}\times\sqrt{VS^2\left(\frac{\partial^2 C}{\partial S\partial V}\right)^2+\sigma^2V\left(\frac{\partial^2 C}{\partial V^2}\right)^2+2\rho V\sigma S\frac{\partial^2 C}{\partial V^2}\frac{\partial^2 C}{\partial S\partial V}}=0.$$

We assume a European call option with the strike price E and time to maturity time T satisfies the PDE (3.34) subject to the following terminal condition:

$$C(S,V,T) = \max[S - E, 0], \tag{3.35}$$

and boundary conditions

$$\begin{aligned} C(0,V,t) = 0, \quad & \frac{\partial C}{\partial S}(S_{\max},V,t) = 1, \\ \frac{\partial C}{\partial t}(S,0,t) + rS\frac{\partial C}{\partial S}&(S,0,t) - rC(S,0,t) = 0, \\ C(S,V_{\max},t) &= S. \end{aligned} \tag{3.36}$$

3.3 Finite Difference Scheme for the Heston Model with Transaction Costs

In this section, we explain how to build the finite difference schemes for solving Eq. (3.34). We assume that the stock price S is between 0 and $S_{\max}$, the volatility V is between 0 and $V_{\max}$, and the time t is in the interval $0 \le t \le T$. In practice, $S_{\max}$ does not have to be too large. Typically, it should be three or four times the value of the exercise price. In the next section, we take $V_{\max} = 1$. To derive the finite difference scheme, we first transform the domain of the continuous problem

$$\{(S, V, t) : 0 \le S \le S_{\max}, 0 \le V \le V_{\max}, 0 \le t \le T\}$$

into a discretized domain with a uniform system of meshes or node points $(i\delta S, j\delta V, n\delta t)$, where $i = 1, 2, ..., I$, $j = 1, 2, ..., J$ and $n = 1, 2, ..., N$ so that $I\delta S = S_{\max}$, $J\delta V = V_{\max}$ and $N\delta t = T$. Let $C_{i,j}^n$ denote the numerical approximation of $C(i\delta S, j\delta V, n\delta t)$. The continuous temporal and spatial derivatives in (3.34) are approximated by the following finite difference operators

$$\frac{\partial C}{\partial t} \approx \frac{C_{i,j}^{n+1} - C_{i,j}^n}{\delta t},$$

$$\frac{\partial C}{\partial S} \approx \frac{C_{i+1,j}^n - C_{i-1,j}^n}{2\delta S},$$

$$\frac{\partial^2 C}{\partial S^2} \approx \frac{C_{i+1,j}^n - 2C_{i,j}^n + C_{i-1,j}^n}{(\delta S)^2},$$

$$\frac{\partial C}{\partial V} \approx \frac{C_{i,j+1}^n - C_{i,j-1}^n}{2\delta V},$$

$$\frac{\partial^2 C}{\partial V^2} \approx \frac{C_{i,j+1}^n - 2C_{i,j}^n + C_{i,j-1}^n}{(\delta V)^2},$$

$$\frac{\partial^2 C}{\partial S \partial V} \approx \frac{C_{i+1,j+1}^n + C_{i-1,j-1}^n - C_{i-1,j+1}^n - C_{i+1,j-1}^n}{4\delta S\delta V}.$$

Applying these approximations to Eq. (3.34), we obtain the following explicit Forward-Time-Centered-Space finite difference scheme,

$$C_{i,j}^{n+1} = C_{i,j}^{n} + \frac{1}{2}VS^2\frac{\delta t}{(\delta S)^2}(C_{i+1,j}^{n} - 2C_{i,j}^{n} + C_{i-1,j}^{n}) + \frac{1}{2}V\sigma^2\frac{\delta t}{(\delta V)^2}(C_{i,j+1}^{n} - 2C_{i,j}^{n} + C_{i,j-1}^{n})$$

$$+ rS\frac{\delta t}{2\delta S}(C_{i+1,j}^{n} - C_{i-1,j}^{n}) + rV\frac{\delta t}{2\delta V}(C_{i,j+1}^{n} - C_{i,j-1}^{n}) \tag{3.37}$$

$$+\rho\sigma VS\frac{\delta t}{4\delta S\delta V}(C_{i+1,j+1}^{n} + C_{i-1,j-1}^{n} - C_{i-1,j+1}^{n} - C_{i+1,j-1}^{n}) - r\delta t C_{i,j}^{n} - \mathcal{F}_1 - \mathcal{F}_2,$$

where

$$\mathcal{F}_1 = kS\delta t\sqrt{\frac{2}{\pi\delta t}}\sqrt{VS^2\left(\frac{(C_{i+1,j}^{n} - 2C_{i,j}^{n} + C_{i-1,j}^{n})}{(\delta S)^2}\right)^2 + V\sigma^2\left(\frac{(C_{i+1,j+1}^{n} + C_{i-1,j-1}^{n} - C_{i-1,j+1}^{n} - C_{i+1,j-1}^{n})}{4\delta S\delta V}\right)^2}$$

$$\overline{+2\rho V\sigma S\frac{(C_{i+1,j+1}^{n} + C_{i-1,j-1}^{n} - C_{i-1,j+1}^{n} - C_{i+1,j-1}^{n})}{4\delta S\delta V}\frac{(C_{i+1,j}^{n} - 2C_{i,j}^{n} + C_{i-1,j}^{n})}{(\delta S)^2}},$$

$$\mathcal{F}_2 = k_1\sigma\delta t\sqrt{\frac{2}{\pi\delta t}}\sqrt{VS^2\left(\frac{(C_{i+1,j+1}^{n} + C_{i-1,j-1}^{n} - C_{i-1,j+1}^{n} - C_{i+1,j-1}^{n})}{4\delta S\delta V}\right)^2 + \sigma^2 V\left(\frac{(C_{i,j+1}^{n} - 2C_{i,j}^{n} + C_{i,j-1}^{n})}{(\delta V)^2}\right)^2}$$

$$\overline{+2\rho V\sigma S\frac{(C_{i+1,j+1}^{n} + C_{i-1,j-1}^{n} - C_{i-1,j+1}^{n} - C_{i+1,j-1}^{n})}{4\delta S\delta V}\frac{(C_{i,j+1}^{n} - 2C_{i,j}^{n} + C_{i,j-1}^{n})}{(\delta V)^2}},$$

and $i = 1, 2, ..., I$, $j = 1, 2, ..., J$ and $n = 1, 2, ..., N$.

The terminal condition (3.35) becomes

$$C_{i,j}^{n} = \max[i\delta S - E, 0],$$

and the boundary conditions (3.36) becomes

$$C_{1,j}^{n} = 0, \quad C_{I,j}^{n+1} = \delta S + C_{I-1,j}^{n+1},$$

$$C_{i,J}^{n+1} = i\delta S,$$

$$C_{i,1}^{n+1} = ri\delta t(C_{i+1,1}^{n} - C_{i,1}^{n}) - C_{i,1}^{n}(r\delta t + 1).$$

3.4 Numerical Results and Model's Performance Analysis

In this section, we solve Eq.(3.34) numerically by implementing the finite difference scheme in MATLAB. The parameters are set up as follows: strike price $E = 100$, $S_{\max} = 200$, $V_{\max} = 1$, interest rate $r = 0.05$, the correlation factor $\rho = 0.8$, $\sigma = 0.4$ and maturity time $T = 1$. Figure 3.1 shows the option C at time $t = 0$, for the case $k = k_1 = 0$. Figure 3.2 shows the option C at time $t = 0$, for the case $k = k_1 = 0.02$.

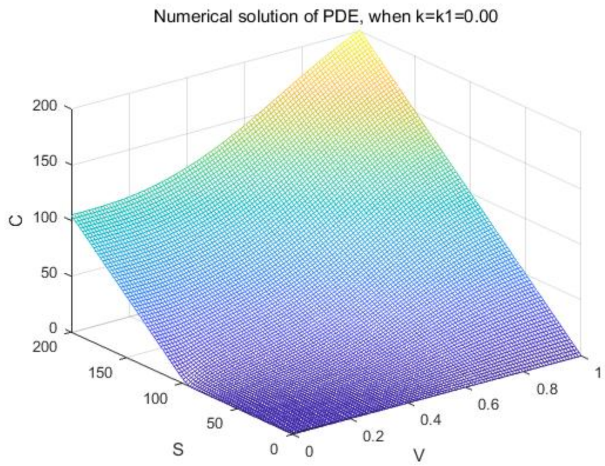

Figure 3.1: Solution of Eq.(3.34), when $k = 0$ and $k_1 = 0$

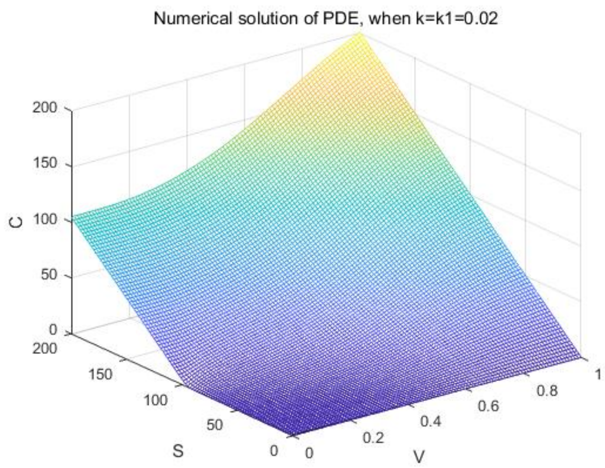

Figure 3.2: Solution of Eq.(3.4), when k = 0.02 and k_1 = 0.02

Tabulated results when $k = k_1 = 0$ at $V = 0.05$ and $V = 0.6$, S between 80 to 120 are shown in Table 3.1. Tabulated results when $k = k_1 = 0.02$ at $V = 0.05$ and $V = 0.6$, S between 80 to 120 are shown in Table 3.2. In Tables 3.1 and 3.2, the second and the third columns from left are our numerical results.

Mariani, SenGupta and Sewell (Mariani et al., 2015) considered a stochastic volatility model, similar to that of (Wiggins, 1987). In this model, they considered the following stochastic volatility model:

$$dS_t = \mu S_t dt + \sigma_t S_t X_t^1,$$

$$d\sigma_t = \alpha\sigma_t dt + \beta\sigma_t dX_t^2,$$

where the two Brownian motions X_t^1 and X_t^2 are correlated with correlation coefficient ρ:

$$\langle dX_t^1, dX_t^2 \rangle = \rho dt.$$

Applying the same approach, they derived a PDE for a European call option with transaction cost under the risk-neutral probability measure. Their PDE is solved numerically by a software package PDE2D. The two columns from right are results of Mariani, SenGupta and Sewell (Mariani et al., 2015). We compare our results in two tables with those of Mariani, SenGupta and Sewell (Mariani et al., 2015), and find there is no significant difference when $V = 0.05$. When $V = 0.6$, we can observe the difference between our results and those of Mariani, SenGupta and Sewell (Mariani et al., 2015), but no significant effect in data analysis. After we compare results between Table 3.1 and Table 3.2, we find that when k and k_1 increase to 0.02, the values of option C decrease and all changes of $C(V = 0.05)$ are less than 0.3. The changes of C are more significant when $V = 0.6$.

Table 3.1: Solution of Eq.(3.37),when $k = k_1 = 0.0$ and $k = k_1 = 0.02$

S	C ($V = 0.05$)	C ($V = 0.6$)	C ($V = 0.05$)	C ($V = 0.6$)
	($k = k_1 = 0$)	($k = k_1 = 0$)	($k = k_1 = 0.02$)	$k = k_1 = 0.02$)
	our results	our results	(Mariani et al., 2015)	(Mariani et al., 2015)
80	2.658	40.111	2.349	38.627
82	2.989	41.361	2.654	39.857
84	3.351	42.626	2.989	41.103
86	3.747	43.908	3.357	42.367
88	4.177	45.207	3.760	43.648
90	4.647	46.521	4.201	44.947
92	5.159	47.853	4.685	46.262
94	5.722	49.201	5.221	47.595
96	6.348	50.565	5.823	48.944
98	7.056	51.946	6.513	50.311
100	7.872	53.343	7.321	51.695
102	8.827	54.756	8.282	53.096
104	9.940	56.186	9.418	54.514
106	11.219	57.631	10.734	55.948
108	12.651	59.093	12.214	57.399
110	14.216	60.571	13.831	58.866
112	15.887	62.064	15.555	60.350
114	17.642	63.573	17.359	61.849
116	19.459	65.097	19.221	63.365
118	21.322	66.637	21.124	64.896
120	23.220	68.192	23.056	66.443

Table 3.2: Solution of Eq.(3.37),when k = k_1 = 0.0 and k = k_1 = 0.02

S	C (σ = 0.05)	C (σ = 0.6)	C (σ = 0.05)	C (σ = 0.6)
	(k = k_1 = 0)	(k = k_1 = 0)	(k = k_1 = 0.02)	k = k_1 = 0.02)
	our results	our results	(Mariani et al., 2015)	(Mariani et al., 2015)
80	0.032	17.794	0.007	16.980
82	0.057	18.806	0.015	17.978
84	0.102	19.846	0.030	19.004
86	0.181	20.912	0.059	20.057
88	0.316	22.005	0.116	21.138
90	0.544	23.125	0.227	22.247
92	0.917	24.270	0.440	23.382
94	1.503	25.442	0.842	24.544
96	2.378	26.639	1.582	25.732
98	3.594	27.861	2.861	26.946
100	5.145	29.108	4.754	28.185
102	6.950	30.380	6.800	29.450
104	8.894	31.676	8.834	30.739
106	10.883	32.995	10.855	32.053
108	12.879	34.338	12.867	33.391
110	14.876	35.704	14.873	34.752
112	16.875	37.092	16.875	36.136
114	18.876	38.503	18.877	37.543
116	20.877	39.936	20.877	38.972
118	22.877	41.389	22.877	40.423
120	24.877	42.864	24.877	41.896

Chapter 4

Heston-CIR Model with Transaction Cost

In this chapter, we consider the Heston-CIR model with transaction cost. In Section 4.1, we introduce the Heston-CIR with a partial correlation. In order to analyze the delta hedging portfolio of the Heston-CIR model with transaction cost in Section 4.3, we first derive a pricing formula for zero-coupon bonds in Section 4.2. In Section 4.4, we use the replicating technique to derive the model and substitute the solution of zero-coupon bonds into the PDE. We obtain the numerical solution to the PDE of Heston-CIR model with transaction cost by implementing the finite difference scheme in MATLAB. In the last section of this chapter, we analyze the numerical results of the PDE.

4.1 The Heston-CIR Model with Transaction costs

The Cox-Ingersoll-Ross (CIR) (Cox et al., 1985) model is a diffusion process suitable for modeling the term structure of interest rates. The simplest version of this model describes the dynamics of the interest rate $r(t)$ as a solution of the following stochastic differential equation (SDE):

$$dr(t) = \alpha(\beta - r(t))dt + \eta\sqrt{r(t)}dX_t^3$$

for constants $\alpha > 0$, $\beta > 0$, $\eta > 0$ and a standard Brownian motion $\{X_t^3 : t \geq 0\}$. We consider the Heston-CIR hybrid model as follows

$$dS(t) = \mu S(t)dt + \sqrt{V(t)}S(t)dX_t^1, \tag{4.1}$$

$$dV(t) = k(\theta - V(t))dt + \sigma\sqrt{V(t)}dX_t^2, \tag{4.2}$$

$$dr(t) = \alpha(\beta - r(t))dt + \eta\sqrt{r(t)}dX_t^3. \tag{4.3}$$

We assume that correlations involved in the above model are given by

$$\langle dX_t^1, dX_t^2\rangle = \rho dt, \qquad \langle dX_t^1, dX_t^3\rangle = 0, \qquad \langle dX_t^2, dX_t^3\rangle = 0.$$

The random variables $S(t)$, $V(t)$ and $r(t)$ represent, respectively, the asset price, its variance and the interest rate at time $t > 0$. The parameters k, θ, σ, η, α and β are given positive real constants. The $\{X_t^1 : t \geq 0\}$, $\{X_t^2 : t \geq 0\}$ and $\{X_t^3 : t \geq 0\}$ are Brownian motions under a risk-neutral measure.

4.2 Pricing Zero-Coupon Bonds

The CIR process has some appealing properties from an applied point of view, for example, the interest rate stays non-negative, and is elastically pulled towards the long-term constant value β at a speed controlled by mean-reverting α. We ensure that $r(t)$ remains positive. Intuitively, when the rate is at a low level, the standard deviation $\eta\sqrt{r(t)}$ also becomes close to zero, which dampens the effect of the random shock on the rate. Consequently, when the rate gets close to zero, its evolution becomes dominated by the drift factor, which pushes the rate upwards and towards equilibrium. For the general stochastic interest rate model, we consider

$$dr = u(r,t)dt + w(r,t)dX_t^3. \tag{4.4}$$

The functional forms of $u(r,t)$ and $w(r,t)$ determine the behavior of the spot rate r. We denote λ to be the market price of risk. The risk-neutral interest rate follows

$$dr = [u(r,t) - \lambda w(r,t)]dt + w(r,t)d\tilde{X}_t^3, \tag{4.5}$$

where $\{\tilde{X}_t^3 : t \leq 0\}$ is a Brownian motion under the risk-neutral probability measure $\mathbb{Q}$. Next, if we assume that the discount process is

$$D(t) = e^{-\int_0^t r(\tau)d\tau}$$

and the money market account price process should be

$$\frac{1}{D(t)} = e^{\int_0^t r(\tau)d\tau}.$$

We will have the differential formula for $D(t)$ and $\frac{1}{D(t)}$:

$$dD(t) = -r(t)e^{-\int_0^t r(\tau)d\tau}dt = -r(t)D(t)dt,$$

$$d\frac{1}{D(t)} = \frac{r(t)}{D(t)}dt.$$

Let $P(t,T)$ be the price of a zero-coupon bond with maturity at time T, as seen at time t. We assume $P(T,T) = 1$. Since the discounted price of this bond should be a martingale under the risk-neutral measure, we have

$$\mathbb{E}_t^{\mathbb{Q}}[P(T,T)D(T)] = \mathbb{E}_t^{\mathbb{Q}}[D(T)] = D(t)P(t,T).$$

Following this formula, we have

$$P(t,T) = \mathbb{E}_t^{\mathbb{Q}}[e^{-\int_t^T r(\tau)d\tau}].$$

Since dr is a Markov process and we have

$$P(t,T) = Z(t,r(t)).$$

To find the PDE for $Z(t,r(t))$, we apply Itô's lemma to differentiate $D(t)P(t,T) = D(t)Z(t,r(t))$ to get,

$$d(D(t)Z(t,r(t))) = Z(t,r(t))dD(t) + D(t)dZ(t,r(t)) \tag{4.6}$$

$$= D(t)\left[-rZdt + \frac{\partial Z}{\partial t}dt + \frac{\partial Z}{\partial r}dr + \frac{1}{2}\frac{\partial^2 Z}{\partial r^2}dr^2\right].$$

Substituting (4.5) into (4.6) yields

$$d[D(t)Z] = D(t)\left[-rz + \frac{\partial Z}{\partial t} + (\mu - \lambda w)\frac{\partial Z}{\partial r} + \frac{1}{2}w^2\frac{\partial^2 Z}{\partial r^2}\right]dt + D(t)w\frac{\partial Z}{\partial r}d\tilde{X}_t^3. \quad (4.7)$$

Since $D(t)Z$ is a martingale, we can set the dt term equal to zero, which yields a PDE for the bond price:

$$\frac{\partial Z}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 Z}{\partial r^2} + (u - \lambda w)\frac{\partial Z}{\partial r} - rZ = 0, \quad (4.8)$$

with the final condition $Z(r,t;T) = 1$. In the Heston-CIR model, we consider

$$dr = \alpha(\beta - r)dt + \eta\sqrt{r}dX_3. \quad (4.9)$$

If we choose $\lambda\sqrt{r}$ to be the market price of risk, then for the Heston-CIR model, Eq. (4.8) becomes

$$\frac{\partial Z}{\partial t} + \frac{1}{2}\eta^2 r\frac{\partial^2 Z}{\partial r^2} + (\alpha(\beta - r) - \lambda\eta)\frac{\partial Z}{\partial r} - rZ = 0, \quad (4.10)$$

with the final condition $Z(r,t;T) = 1$. The term $\alpha(\beta - r) - \lambda\eta$ can be re-written as

$$\alpha(\beta - \frac{\lambda\eta}{\alpha} - r) := a(b - r),$$

and the PDE is re-written as

$$\frac{\partial Z}{\partial t} + \frac{1}{2}\eta^2 r\frac{\partial^2 Z}{\partial r^2} + a(b - r)\frac{\partial Z}{\partial r} - rZ = 0. \quad (4.11)$$

In the final step, we look for a solution in the form of

$$Z(r,t;T) = e^{-B(t,T)r + A(t,T)}. \quad (4.12)$$

Differentiating each parameters in (4.12), we have

$$\frac{\partial Z}{\partial t} = (-rB'(t,T) - A'(t,T))Z(t,r),$$

$$\frac{\partial Z}{\partial r} = -B(t,T)Z(t,r),$$
$$\frac{\partial^2 Z}{\partial r^2} = B^2(t,T)Z(t,r).$$

Substituting the above equations into the PDE (4.11) gives

$$[-rB'(t,T) - A'(t,T) + (ab - ar)B(t,T) + \frac{1}{2}\eta^2 rB^2(t,T) - r]Z(t,r) = 0.$$

$$\Rightarrow [(-B'(t,T) + \frac{1}{2}\eta^2 B^2(t,T) + aB(t,T) - 1)r - A'(t,T) - abB(t,T)]Z(t,r) = 0. \tag{4.13}$$

The Eq. (4.13) yields the following two ODEs

$$A'(t,T) = -abB(t,T), \tag{4.14}$$

$$B'(t,T) = \frac{1}{2}\eta^2 B^2(t,T) + aB(t,T) - 1. \tag{4.15}$$

Solving Eq. (4.14) and Eq. (4.15), we obtain

$$B(t,T) = \frac{2(e^{\gamma(T-t)} - 1)}{(a-\gamma)(e^{\gamma(T-t)} - 1) + 2\gamma}, \tag{4.16}$$

$$A(t,T) = \ln\left(\frac{2\gamma e^{(\gamma+a)(T-t)/2}}{(\gamma+a)(e^{\gamma(T-t)} - 1) + 2\gamma}\right)^{\frac{2ab}{\eta^2}}, \tag{4.17}$$

where $\gamma = \sqrt{a^2 + 2\eta^2}$.

4.3 Delta Hedging Portfolio of the Heston-CIR Model with Transaction costs

We consider a portfolio Π that contains one option, with value $C(S,V,r,t)$, and quantities $-\Delta$,$-\Delta_1$ and $-\Delta_2$ of S, V and Z, respectively, where $Z \in \mathbb{B}$. Thus we hedge the stochastic interest rate with a zero coupon bond Z maturing at the same time as the option C. That is

$$\Pi = C - \Delta S - \Delta_1 V - \Delta_2 Z. \tag{4.18}$$

According to the self-financing argument, considering the transaction cost and stochastic interest rate, we obtain

$$d\Pi = dC - \Delta dS - \Delta_1 dV - \Delta_2 dZ, \tag{4.19}$$

where we apply Itô's formula to get the dynamics of C,

$$dC = \frac{\partial C}{\partial t}dt+\frac{\partial C}{\partial S}dS+\frac{\partial C}{\partial r}dr+\frac{\partial C}{\partial V}dV+\frac{1}{2}VS^2\frac{\partial^2 C}{\partial S^2}dt+\rho\sigma VS\frac{\partial^2 C}{\partial S\partial V}dt+\frac{1}{2}\sigma^2 V\frac{\partial^2 C}{\partial V^2}dt+\frac{1}{2}\eta^2 r\frac{\partial^2 C}{\partial r^2}dt,$$

and

$$dZ = \frac{\partial Z}{\partial t}dt + \frac{\partial Z}{\partial r}dr + \frac{1}{2}\eta^2 r\frac{\partial^2 Z}{\partial r^2}dt.$$

Substituting dC to $d\Pi$, we have the change in value of the portfolio Π:

$$d\Pi = \left(\frac{\partial C}{\partial t} + \frac{1}{2}VS^2\frac{\partial^2 C}{\partial S^2} + \rho\sigma VS\frac{\partial^2 C}{\partial S\partial V} + \frac{1}{2}\sigma^2 V\frac{\partial^2 C}{\partial V^2} + \frac{1}{2}\eta^2 r\frac{\partial^2 C}{\partial r^2} - \Delta_2\frac{\partial Z}{\partial t} - \frac{\Delta_2}{2}\eta^2 r\frac{\partial^2 Z}{\partial r^2}\right)dt$$

$$+\left(\frac{\partial C}{\partial S} - \Delta\right)dS+\left(\frac{\partial C}{\partial V} - \Delta_1\right)dV+\left(\frac{\partial C}{\partial r} - \Delta_2\frac{\partial Z}{\partial r}\right)dr-k_0 S\mid \nu\mid -k_1 V\mid \nu_1\mid -k_2 Z\mid \nu_2\mid, \tag{4.20}$$

where $k_0 S\mid \nu\mid$, $k_1 V\mid \nu_1\mid$ and $k_2 Z\mid \nu_2\mid$ represent the transaction costs associated with trading ν of the main asset S and ν_1 of the volatility index V and ν_2 of the zero coupon bond Z during the time step dt.

If we let

$$\left(\frac{\partial C}{\partial S} - \Delta\right) = 0,$$

$$\left(\frac{\partial C}{\partial V} - \Delta_1\right) = 0,$$

and

$$\left(\frac{\partial C}{\partial r} - \Delta_2 \frac{\partial Z}{\partial r}\right) = 0,$$

we can eliminate the dS and dV and dr terms in Eq. (4.20) and the dynamics of Π becomes

$$d\Pi = \left(\frac{\partial C}{\partial t} + \frac{1}{2} V S^2 \frac{\partial^2 C}{\partial S^2} + \rho\sigma V S \frac{\partial^2 C}{\partial S \partial V} + \frac{1}{2}\sigma^2 V \frac{\partial^2 C}{\partial V^2} + \frac{1}{2}\eta^2 r \frac{\partial^2 C}{\partial r^2} - \Delta_2 \frac{\partial Z}{\partial t} - \frac{\Delta_2}{2}\eta^2 V \frac{\partial^2 Z}{\partial r^2}\right) dt$$
$$- k_0 S \mid \nu \mid - k_1 V \mid \nu_1 \mid - k_2 Z \mid \nu_2 \mid . \qquad (4.21)$$

4.4 The PDE derivation of the Heston-CIR Model

In the last two sections, we derive the PDE of the Heston-CIR system and give numerical solutions. In this section we investigate the costs associated with trading the asset and set up the PDE for the Heston-CIR model with transaction cost.. If the number of asset held short at time t is

$$\Delta_t = \frac{\partial C}{\partial S}(S, V, r, t), \tag{4.22}$$

we assume after a time step δt and re-hedging, the number of assets we hold over a small time interval δt is

$$\Delta_{t+\delta t} = \frac{\partial C}{\partial S}(S + \delta S, V + \delta V, r + \delta r, t + \delta t).$$

According to the conditions given above, the time step δt is small, the changes in asset and the interest rate are also small. Applying the Taylor's formula to expend $\Delta_{t+\delta t}$ yields,

$$\Delta_{t+\delta t} \simeq \frac{\partial C}{\partial S} + \delta S \frac{\partial^2 C}{\partial S^2} + \delta V \frac{\partial^2 C}{\partial S \partial V} + \delta r \frac{\partial^2 C}{\partial S \partial r} + \delta t \frac{\partial^2 C}{\partial S \partial t}. \tag{4.23}$$

In Section 4.1 we assume that the correlations of this model are $\langle dX_t^1, dX_t^2 \rangle = \rho dt$, $\langle dX_t^1, dX_t^3 \rangle = 0$ and $\langle dX_t^2, dX_t^3 \rangle = 0$. So, we can neglect the term $\frac{\partial^2 C}{\partial S \partial r}$. In addition, for small δt, $\delta t \frac{\partial^2 C}{\partial S \partial t}$ is also negligible. If we set up $\delta S = \sqrt{V} S \delta X^1 + \mathcal{O}(\delta t)$ and $\delta V = \sigma \sqrt{V} \delta X^2 + \mathcal{O}(\delta t)$, substitute δS and δV into Eq. (4.23) and neglect the last two terms in (4.23) and any other term proportional to δt or with higher order in δt, we have an expression as follow,

$$\Delta_{t+\delta t} \simeq \frac{\partial C}{\partial S} + \sqrt{V} S \delta X^1 \frac{\partial^2 C}{\partial S^2} + \sigma \sqrt{V} \delta X^2 \frac{\partial^2 C}{\partial S \partial V}. \tag{4.24}$$

Next, we subtract (4.22) from (4.24), we obtain the number of assets trading during the time step δt as follows:

$$\nu = \sqrt{V} S \delta X^1 \frac{\partial^2 C}{\partial S^2} + \sigma \sqrt{V} \delta X^2 \frac{\partial^2 C}{\partial S \partial V}. \tag{4.25}$$

Because X_t^1 and X_t^2 are correlated Brownian motions, we consider two independent normal variables Z_1 and Z_2 with mean 0 and variance 1. Then, we have

$$\delta X^1 = Z_1 \sqrt{\delta t}$$

and

$$\delta X^2 = \rho Z_1 \sqrt{\delta t} + \sqrt{1 - \rho^2} Z_2 \sqrt{\delta t}.$$

Substituting these expressions of δX^1 and δX^2 in Eq (4.25) and denoting

$$\alpha_1 = \sqrt{V} S \sqrt{\delta t} \frac{\partial^2 C}{\partial S^2} + \sigma \sqrt{V} \rho \sqrt{\delta t} \frac{\partial^2 C}{\partial S \partial V}, \tag{4.26}$$

$$\beta_1 = \sigma \sqrt{V} \sqrt{1 - \rho^2} \sqrt{\delta t} \frac{\partial^2 C}{\partial S \partial V},$$

then we can rewrite the change in the number of shares over the time step δt as:

$$\nu = \alpha_1 Z_1 + \beta_1 Z_2.$$

In a very similar way we can express ν_1 as follows:

$$\nu_1 = \sqrt{V} S \delta X^1 \frac{\partial^2 C}{\partial S \partial V} + \sigma \sqrt{V} \delta X^2 \frac{\partial^2 C}{\partial V^2}. \tag{4.27}$$

Since X^3 is independent from X^1 and X^2, we have

$$\nu_2 = \frac{\eta \sqrt{r}}{\zeta(r,t)} \frac{\partial^2 C}{\partial r^2} \delta X^3, \tag{4.28}$$

where $\zeta(r,t) = \frac{\partial Z}{\partial r}$. Let $\Delta_2 = \frac{1}{\zeta(r,t)}\frac{\partial C}{\partial r}$. We know the expectation of the change in value of the portfolio is

$$\mathbb{E}[d\Pi] = \left(\frac{\partial C}{\partial t} + \frac{1}{2}VS^2\frac{\partial^2 C}{\partial S^2} + \rho\sigma VS\frac{\partial^2 C}{\partial S\partial V} + \frac{1}{2}\sigma^2 V\frac{\partial^2 C}{\partial V^2} + \frac{1}{2}\eta^2 r\frac{\partial^2 C}{\partial r^2} - \Delta_2\frac{\partial Z}{\partial t} - \frac{\Delta_2}{2}\eta^2 r\frac{\partial^2 Z}{\partial r^2}\right)dt$$

$$- k_0 S\mathbb{E}[\mid \nu \mid] - k_1 V\mathbb{E}[\mid \nu_1 \mid] - k_2 Z\mathbb{E}[\mid \nu_2 \mid]. \tag{4.29}$$

Under the risk-neutral measure $\mathbb{Q}$,

$$\mathbb{E}[d\Pi] = r\Pi dt. \tag{4.30}$$

Hence, we have

$$\left(\frac{\partial C}{\partial t} + \frac{1}{2}VS^2\frac{\partial^2 C}{\partial S^2} + \rho\sigma VS\frac{\partial^2 C}{\partial S\partial V} + \frac{1}{2}\sigma^2 V\frac{\partial^2 C}{\partial V^2} + \frac{1}{2}\eta^2 r\frac{\partial^2 C}{\partial r^2} - \Delta_2\frac{\partial Z}{\partial t} - \frac{\Delta_2}{2}\eta^2 r\frac{\partial^2 Z}{\partial r^2}\right)dt$$

$$-k_0 S\mathbb{E}[\mid \nu \mid] - k_1 V\mathbb{E}[\mid \nu_1 \mid] - k_2 Z\mathbb{E}[\mid \nu_2 \mid]$$

$$= \left(-rS\frac{\partial C}{\partial S} - rV\frac{\partial C}{\partial V} - r\frac{Z}{\zeta}\frac{\partial C}{\partial r} + rC\right)dt.$$

Dividing each side by dt and re-arranging yield

$$\frac{\partial C}{\partial t} + \frac{1}{2}VS^2\frac{\partial^2 C}{\partial S^2} + \rho\sigma VS\frac{\partial^2 C}{\partial S\partial V} + \frac{1}{2}\sigma^2 V\frac{\partial^2 C}{\partial V^2} + \frac{1}{2}\eta^2 r\frac{\partial^2 C}{\partial r^2} - \Delta_2\frac{\partial Z}{\partial t} - \frac{\Delta_2}{2}\eta^2 r\frac{\partial^2 Z}{\partial r^2}$$

$$-\frac{k_0 S}{dt}\mathbb{E}[\mid \nu \mid] - \frac{k_1 V}{dt}\mathbb{E}[\mid \nu_1 \mid] - \frac{k_2 Z}{dt}\mathbb{E}[\mid \nu_2 \mid]$$

$$= -rS\frac{\partial C}{\partial S} - rV\frac{\partial C}{\partial V} - r\frac{Z}{\zeta}\frac{\partial C}{\partial r} + rC. \tag{4.31}$$

Next, we calculate $\mathbb{E}[\mid \nu \mid]$, $\mathbb{E}[\mid \nu_1 \mid]$ and $\mathbb{E}[\mid \nu_2 \mid]$. By Eq.(4.26), we have

$$\mathbb{E}[\mid \nu \mid] = \sqrt{\frac{2\delta t}{\pi}}\sqrt{\alpha_1^2 + \beta_1^2} = \sqrt{\frac{2\delta t}{\pi}} \times \sqrt{VS^2(\frac{\partial^2 C}{\partial S^2})^2 + \sigma^2 V(\frac{\partial^2 C}{\partial S\partial V})^2 + 2\rho V\sigma S\frac{\partial^2 C}{\partial S^2}\frac{\partial^2 C}{\partial S\partial V}}. \tag{4.32}$$

Similarly, we can obtain

$$\mathbb{E}[\mid \nu_1 \mid] = \sqrt{\frac{2\delta t}{\pi}} \times \sqrt{VS^2(\frac{\partial^2 C}{\partial S \partial V})^2 + \sigma^2 V(\frac{\partial^2 C}{\partial V^2})^2 + 2\rho V\sigma S\frac{\partial^2 C}{\partial S^2}\frac{\partial^2 C}{\partial S\partial V}}, \quad (4.33)$$

and

$$\mathbb{E}[\mid \nu_2 \mid] = \sqrt{\frac{2\delta t}{\pi}} \times \frac{\eta}{\mid \zeta \mid}\sqrt{r}\left|\frac{\partial^2 C}{\partial r^2}\right|. \quad (4.34)$$

Substituting Eq.(4.32), Eq.(4.33) and Eq. (4.34) into Eq. (4.31) and note that $dt = \delta t$, we get a partial differential equation of stochastic volatility model with transaction costs,

$$\frac{\partial C}{\partial t}+\frac{1}{2}VS^2\frac{\partial^2 C}{\partial S^2}+\frac{1}{2}V\sigma^2\frac{\partial^2 C}{\partial V^2}+\rho\sigma VS\frac{\partial^2 C}{\partial S\partial V}+\frac{1}{2}\eta^2 r\frac{\partial^2 C}{\partial r^2}+rS\frac{\partial C}{\partial S}+rV\frac{\partial C}{\partial V}+a(b-r)\frac{\partial C}{\partial r}-rC$$

$$-\mathcal{F}_1-\mathcal{F}_2-\mathcal{F}_3=0, \quad (4.35)$$

where

$$\mathcal{F}_1 = k_0 S\sqrt{\frac{2}{\pi\delta t}} \times \sqrt{VS^2(\frac{\partial^2 C}{\partial S^2})^2 + \sigma^2 V(\frac{\partial^2 C}{\partial S\partial V})^2 + 2\rho V\sigma S\frac{\partial^2 C}{\partial S^2}\frac{\partial^2 C}{\partial S\partial V}},$$

$$\mathcal{F}_2 = k_1 V\sqrt{\frac{2}{\pi\delta t}} \times \sqrt{VS^2(\frac{\partial^2 C}{\partial S\partial V})^2 + \sigma^2 V(\frac{\partial^2 C}{\partial V^2})^2 + 2\rho V\sigma S\frac{\partial^2 C}{\partial S^2}\frac{\partial^2 C}{\partial S\partial V}},$$

$$\mathcal{F}_3 = k_2\frac{Z}{\mid \zeta \mid}\sqrt{\frac{2}{\pi\delta t}}\eta\sqrt{r}\left|\frac{\partial^2 C}{\partial r^2}\right|.$$

We assume that a European call option with the strike price E and maturity at time T satisfies the PDE (4.35) subject to the following terminal condition:

$$C(S,V,r,T) = \max[S-E,0], \quad (4.36)$$

and boundary conditions

$$C(0,V,r,t) = 0,$$

$$\frac{\partial C}{\partial S}(S_{\max},V,r,t) = 1,$$

$$\frac{\partial C}{\partial t}(S,0,r,t)+rS\frac{\partial C}{\partial S}(S,0,r,t)+a(b-r)\frac{\partial C}{\partial r}-rC(S,0,r,t)-\mathcal{F}_3(S,0,r,t) = 0, \quad (4.37)$$

$$C(S,V_{\max},r,t) = S,$$

$$\frac{\partial C}{\partial r}(S,V,0,t) = 0, \quad \frac{\partial U}{\partial r}(S,V,r_{max},t) = 0.$$

4.5 Finite Difference Scheme

Next, we will explain how to build the finite difference schemes. We assume that the stock price S is between 0 and $S_{\max}$, the volatility V is between 0 and $V_{\max}$, the interest rate r is between 0 and $r_{\max}$ and the time t is in the interval $0 \leq t \leq T$. In practice, $S_{\max}$ does not have to be too large. Typically, it should be three or four times the value of the exercise prise. In the next section, we take $V_{\max} = 1$. To derive the finite difference scheme, we first transform the domain of the continuous problem

$$\{(S, V, t) : 0 \leq S \leq S_{\max}, 0 \leq V \leq V_{\max}, 0 \leq r \leq r_{\max}, 0 \leq t \leq T\}$$

into a discretized domain with a uniform system of meshes or node points $(i\delta S, j\delta V, n\delta t)$, where $i = 1, 2, ..., I$, $j = 1, 2, ..., J$, $k = 1, 2, ..., K$ and $n = 1, 2, ..., N$ so that $I\delta S = S_{\max}$, $J\delta V = V_{\max}$, $K\delta r = r_{\max}$ and $N\delta t = T$. Let $C^n_{i,j,k}$ denote the numerical approximation of $C(i\delta S, j\delta V, k\delta r, n\delta t)$. The continuous temporal and spatial derivatives in (4.35) are approximated by the following finite difference operators

$$\frac{\partial C}{\partial t} \approx \frac{C^{n+1}_{i,j,k} - C^n_{i,j,k}}{\delta t},$$

$$\frac{\partial C}{\partial S} \approx \frac{C^n_{i+1,j,k} - C^n_{i-1,j,k}}{2\delta S},$$

$$\frac{\partial^2 C}{\partial S^2} \approx \frac{C^n_{i+1,j,k} - 2C^n_{i,j,k} + C^n_{i-1,j,k}}{(\delta S)^2},$$

$$\frac{\partial C}{\partial V} \approx \frac{C^n_{i,j+1,k} - C^n_{i,j-1,k}}{2\delta V},$$

$$\frac{\partial^2 C}{\partial V^2} \approx \frac{C^n_{i,j+1,k} - 2C^n_{i,j,k} + C^n_{i,j-1,k}}{(\delta V)^2},$$

$$\frac{\partial^2 C}{\partial S \partial V} \approx \frac{C^n_{i+1,j+1,k} + C^n_{i-1,j-1,k} - C^n_{i-1,j+1,k} - C^n_{i+1,j-1,k}}{4\delta S \delta V},$$

$$\frac{\partial C}{\partial r} \approx \frac{C^n_{i,j,k+1} - C^n_{i,j,k-1}}{2\delta r},$$

$$\frac{\partial^2 C}{\partial r^2} \approx \frac{C^n_{i,j,k+1} - 2C^n_{i,j,k} + C^n_{i,j,k-1}}{(\delta r)^2}.$$

We obtain the following explicit Forward-Time-Centered-Space finite difference scheme for the (4.35):

$$C^{n+1}_{i,j,k} = C^n_{i,j,k} + \frac{1}{2}VS^2\frac{\delta t}{(\delta S)^2}(C^n_{i+1,j,k} - 2C^n_{i,j,k} + C^n_{i-1,j,k}) + \frac{1}{2}V\sigma^2\frac{\Delta t}{(\delta V)^2}(C^n_{i,j+1,k} - 2C^n_{i,j,k} + C^n_{i,j-1,k})$$

$$+\frac{1}{2}\eta^2 r\frac{\delta t}{(\delta r)^2}(C^n_{i,j,k+1} - 2C^n_{i,j,k} + U^n_{i,j,k-1}) + \rho\sigma VS\frac{\delta t}{4\Delta S\delta V}(C^n_{i+1,j+1,k} + C^n_{i-1,j-1,k} - C^n_{i-1,j+1,k} - C^n_{i+1,j-1,k})$$

$$+rS\frac{\delta t}{2\delta S}(C^n_{i+1,j,k} - C^n_{i-1,j,k}) + rV\frac{\delta t}{2\delta V}(C^n_{i,j+1,k} - C^n_{i,j-1,k}) + a(b-r)\frac{\delta t}{2\delta r}(C^n_{i,j,k+1} - C^n_{i,j,k-1}) - r\delta t C^n_{i,j,k} \tag{4.38}$$

$$-\mathcal{F}_1 - \mathcal{F}_2 - \mathcal{F}_3,$$

where

$$\mathcal{F}_1 = k_0 S\delta t\sqrt{\frac{2}{\pi\delta t}}\sqrt{2\rho V\sigma S\frac{(C^n_{i+1,j+1,k} + C^n_{i-1,j-1,k} - C^n_{i-1,j+1,k} - C^n_{i+1,j-1,k})}{4\delta S\delta V}\frac{(C^n_{i+1,j,k} - 2C^n_{i,j,k} + C^n_{i-1,j,k})}{(\delta S)^2}}$$

$$\overline{+VS^2\left(\frac{(C^n_{i+1,j,k} - 2C^n_{i,j,k} + C^n_{i-1,j,k})}{(\delta S)^2}\right)^2 + V\sigma^2\left(\frac{(C^n_{i+1,j+1,k} + C^n_{i-1,j-1,k} - C^n_{i-1,j+1,k} - C^n_{i+1,j-1,k})}{4\delta S\delta V}\right)^2},$$

$$\mathcal{F}_2 = k_1 V\delta t\sqrt{\frac{2}{\pi\delta t}}\sqrt{2\rho V\sigma S\frac{(C^n_{i+1,j+1,k} + C^n_{i-1,j-1,k} - C^n_{i-1,j+1,k} - C^n_{i+1,j-1,k})}{4\delta S\delta V}\frac{(C^n_{i,j+1,k} - 2C^n_{i,j,k} + C^n_{i,j-1,k})}{(\delta V)^2}}$$

$$\overline{+VS^2\left(\frac{(C^n_{i+1,j+1,k} + C^n_{i-1,j-1,k} - C^n_{i-1,j+1,k} - C^n_{i+1,j-1,k})}{4\delta S\delta V}\right)^2 + \sigma^2 V\left(\frac{(C^n_{i,j+1,k} - 2C^n_{i,j,k} + C^n_{i,j-1,k})}{(\delta V)^2}\right)^2},$$

$$\mathcal{F}_3 = k_2\frac{Z}{|\zeta|}\delta t\sqrt{\frac{2}{\pi\delta t}}\eta\sqrt{r}\frac{C^n_{i,j,k+1} - 2C^n_{i,j,k} + C^n_{i,j,k-1}}{(\delta r)^2},$$

and $i = 1, 2, ..., I$, $j = 1, 2, ..., J$, $k = 1, 2, ..., K$ and $n = 1, 2, ..., N$.

The terminal condition (3.36) becomes

$$C^{n}_{i,j,k} = \max[i\Delta S - E, 0],$$

and the boundary conditions (4.37) become

$$C^{n}_{1,j,k} = 0, \quad C^{n+1}_{I,j,k} = \Delta S + C^{n+1}_{I-1,j,k},$$

$$C^{n+1}_{i,J,k} = i\Delta S,$$

$$C^{n+1}_{i,1,k} = \Delta t\left(ri(C^{n}_{i+1,1,k} - C^{n}_{i,1,k}) + \frac{a(b-r)}{\Delta r}(C^{n}_{i,1,k+1} - C^{n}_{i,1,k}) - rC^{n}_{i,1,k} - \mathcal{F}'_3\right) + C^{n}_{i,1,k},$$

$$C^{n+1}_{i,j,1} = C^{n+1}_{i,j,2}, \quad C^{n+1}_{i,j,K} = C^{n+1}_{i,j,K-1},$$

where

$$\mathcal{F}'_3 = k_2\frac{Z}{|\zeta|}\Delta t\sqrt{\frac{2}{\pi\delta t}}\eta\sqrt{r}\frac{C^{n}_{i,1,k+1} - 2U^{n}_{i,1,k} + C^{n}_{i,1,k-1}}{(\Delta r)^2}.$$

4.6 Numerical Analysis

In this section, we solve the PDE numerically in MATLAB. We assume that the strike price $E = 100$, the range of stock prices S is $[0, 200]$, the range of volatility V is $[0, 1]$ and interest rate $r \in [0, 1]$. The parameters ρ, σ η and T table values as follows: $\rho = 0.8$, $\sigma = 0.05$, $\eta = 0.2$ and $T = 1$. The figure 4.1 shows that at time $t = 0$ how the option value C changes with S and V, when $k_0 = k_1 = k_2 = 0$. The figure 4.2 shows that at time $t = 0$ how the option value C changes with S and V, when $k_0 = k_1 = k_2 + 0.02$.

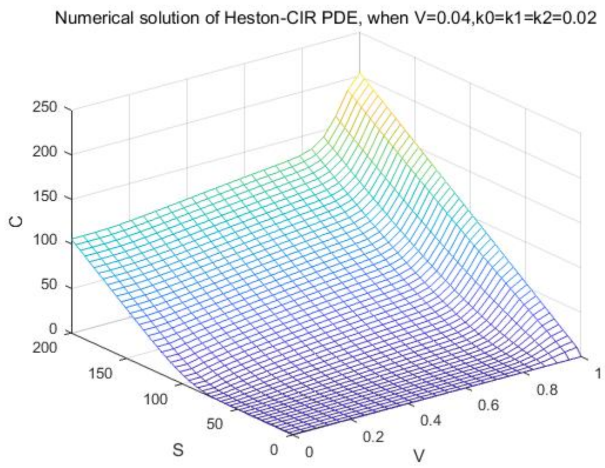

Figure 4.1: Solution of PDE, when $k_0 = 0$, $k_1 = 0$, $k_2 = 0$

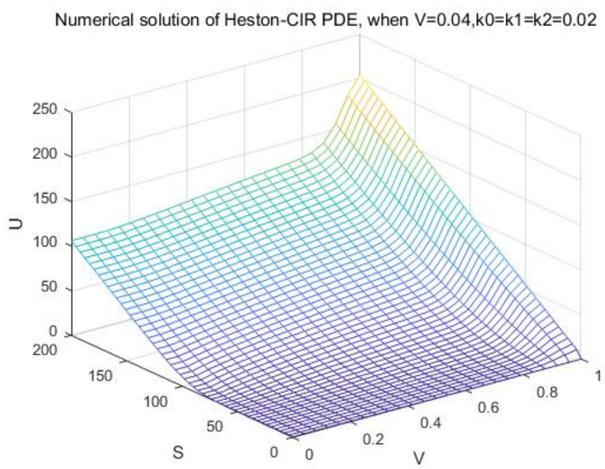

Figure 4.2: Solution of Heston-CIR PDE, when $k_0 = 0.02$, $k_1 = 0.02$, $k_2 = 0.02$

Tabulated results when $k_0 = k_1 = k_2 = 0$ at $V = 0.04$ and $V = 0.6$, S between 80 to 120 are shown in Table 4.1. Table 4.1 shows us when $V = 0.04$, there are more obvious changes from $r = 0.04$ to $r = 0.08$, all values of option C increase between 0.5 to 2.5. However, these changes are not significant when $V = 0.6$.

Table 4.1: Solution of Heston-CIR PDE, when $k_0 = 0$, $k_1 = 0$, $k_2 = 0$

S	C ($V = 0.04$)	C ($V = 0.6$)	C ($V = 0.04$)	C ($V = 0.6$)
	($r = 0.04$)	($r = 0.04$)	($r = 0.08$)	$r = 0.08$)
80	1.712	21.221	2.200	21.692
84	2.654	23.741	3.335	24.242
88	3.912	26.366	4.813	26.896
92	5.512	29.089	6.650	29.647
96	7.461	31.906	8.842	32.491
100	9.751	34.811	11.369	35.422
104	12.359	37.798	14.199	38.435
108	15.253	40.865	17.293	41.525
112	18.396	44.005	20.612	44.688
116	21.749	47.216	24.113	47.921
120	25.274	50.493	27.762	51.219

Tabulated results when $k_0 = k_1 = k_2 = 0.02$ at $V = 0.05$ and $V = 0.6$, S between 80 to 120 are shown in Table 4.2. Table 4.2 shows us when $V = 0.04$, there are more obvious changes from $r = 0.04$ to $r = 0.08$, all values of option C increase between 0.4 to 2.2. In addition, these changes are also significant when $V = 0.6$, from $r = 0.04$ to $r = 0.08$, all values of option C are increase by 1.

Table 4.2: Solution of Heston-CIR PDE, when k_0 = 0.02, k_1 = 0.02, k_2 = 0.02

S	C (V = 0.04)	C (V = 0.6)	C (V = 0.04)	C (V = 0.6)
	(r = 0.04)	(r = 0.04)	(r = 0.08)	(r = 0.08)
80	1.445	20.660	1.8526	21.118
84	2.328	23.162	2.913	23.650
88	3.545	25.770	4.335	26.288
92	5.130	28.479	6.142	29.025
96	7.094	31.283	8.333	31.856
100	9.427	34.177	10.885	34.776
104	12.102	37.155	13.762	37.779
108	15.081	40.213	16.919	40.861
112	18.318	43.346	20.307	44.017
116	21.766	46.551	23.880	47.243
120	25.384	49.822	27.597	50.536

After we compare Table 4.1 with Table 4.2, we find when k_0 $k_1$and k_2 increase to 0.02, the value of option C will decrease and all changes of $C(V = 0.04)$ are less than 0.3. The changes of C are more significant when $V = 0.6$. These results are very similar to those in Chapter 3. Since S increases from 80 to 120, the changes will be increasingly obscure.

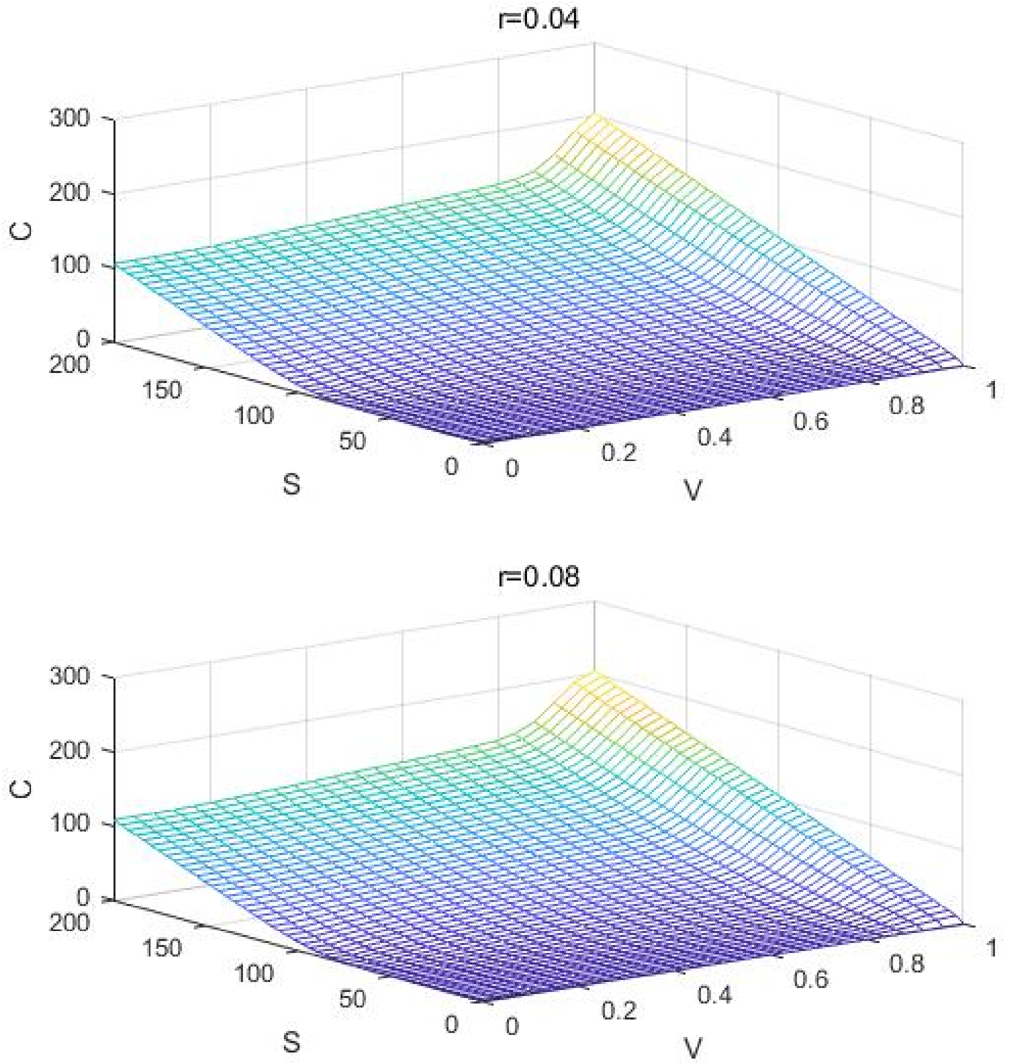

Figure 4.3: Solution of Heston-CIR PDE, when k_0 = 0.02, k_1 = 0.02, k_2 = 0.02

Chapter 5

Discussion and Conclusion

The main goal of this thesis is to consider the problem of option pricing under the Heston-CIR model, which is a combination of the stochastic volatility model discussed in Heston and the stochastic interest rates model driven by Cox-Ingersoll-Ross (CIR) processes. We obtain the numerical solution to the PDE of Heston-CIR model with transaction cost by implementing the finite difference scheme in MATLAB. In this chapter, we summarise our results in Chapter 3 and Chapter 4 and suggest some possible directions for future work. Section 5.1 is devoted to discuss the significance and conclusion of our main results. In Section 5.2, we discuss the limitations of our findings and propose some future research directions which may be worth of pursuing.

5.1 Conclusion

The Black-Scholes (Black & Scholes, 1973) model assumed no transaction cost in the continuous re-balancing of a hedged portfolio. In real financial markets, this assumption is not valid. The construction of hedging strategies for transaction cost is an important problem. Based on a method for hedging call option and Black-Scholes assumptions, Leland (Leland, 1985) presented a hedging strategy with a proportional transaction cost. Mariani, SenGupta and Sewell (Mariani et al., 2015) considered a stochastic volatility model, similar to that of (Wiggins, 1987). Applying the same approach, this thesis focuses on the Heston-CIR model with transaction cost and stochastic interest rate.

In this thesis, we study the Heston-CIR model in the framework of transaction cost and stochastic interest rate. In Chapter 3, we extend Leland's model by adding transaction costs to Heston's (Heston, 1993) stochastic volatility model and derive a PDE for a general class of stochastic volatility models. We apply the finite-difference method to find an approximate solution to this model and compare our numerical results with these of Mariani, SenGupta and Sewell (Mariani et al., 2015). We find that the results of our stochastic volatility models are no significant difference with these of Mariani, SenGupta and Sewell (Mariani et al., 2015). Moreover, we also discuss the impact of transaction cost under this stochastic volatility model. We discover that the change of the transaction cost have little impact on the value of option price.

Following the study in Chapter 3, we consider the Heston-CIR model with partial correlation in Chapter 4. To do this, we derive a pricing formula for zero-coupon bonds and analyze the Delta hedging portfolio of the Heston-CIR model with transaction cost. We use replicating technique to derive the model and substitute the solution of zero-coupon bonds into the PDE. We obtain the numerical solution to the PDE of Heston-CIR model with transaction cost by implementing the finite difference scheme in MATLAB. We analyze the impact of the interest rate change under the Heston-CIR model. Furthermore, we also discuss the impact of transaction cost under the Heston-CIR model. When the Heston-CIR model with non transaction cost, We discover that if volatility is a small value (V = 0.04), all values of option price increase between 0.5 to 2.5 and these changes are not significant when V = 0.6. However, when the Heston-CIR model with transaction cost, all values of option price increase becomes less apparent when V = 0.04 and these changes are more significant when V = 0.6. These results are very similar to those in Chapter 3.

5.2 Future Research

In this section, we discuss some limitations of our findings and potential directions for future research. In the Black and Scholes world, where the volatility of asset returns is assumed to be a constant, pure delta hedging suffices to solve the hedging problem. An option can be perfectly hedged by dynamically trading the underlying stock. Delta is the rate of change of the option value with respect to the underlying asset price. The delta measure is most important in hedging the exposure of a portfolio of options to the market risk. In our work we did not use the real market data to test our results. In reality, due to many factors, financial markets may experience jumps from time to time. However, our model does not capture jumps in either volatility or underlying stock prices. In Chapters 3 and 4, when we set up a risk-less portfolio to hedge the option, we needed an asset whose value depends on volatility. We followed the approach used by Mariani, SenGupta and Sewell (Mariani et al., 2015), and did not consider the market risk of this asset due to the stock price and volatility movements. However, Gatheral (Gatheral, 2011) considered this risk factor for option pricing under stochastic volatility models. In Chapter 4, we consider the Heston-CIR model only with a partial correlation rather than full correlation. It would be of interests to consider how to overcome the above three limitations. These will be possible research directions in the future.

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Appendix A

MATLAB code

```
%For chapter 3, when k=k1=0
%parameter values
Smax=200;Smin=0;Vmax=1;Vmin=0;K=100;
I=100;J=100;r=0.05;
rho=0.8;k=0.0;k1=0.0;
T=1; deltat=1; dt=1/10000;
sigma=0.4;

N=1+ceil(T/dt);% dt = T/nt;
% the lower bounds of s and v are both 0
ds = (Smax-Smin)/(I-1); % step length of s
dv = (Vmax+Vmin)/(J-1); % step length of v

U=zeros(I,J,N);
for i=1:I
for j=1:J
U(i,j,1)=max(0,(i-1)*ds-K);
end
end

for n=1:N-1 % the interior elements. Cross term part.
for j=2:J-1
for i = 2:I-1
F1=0; F2=0;
U(i,j,n+1)=U(i,j,n)+((j-1)*dv)*((i-1)*ds)^2*dt/(2*ds^2)*(U(i+1,j,n)-2*U(i,j,n)+U(i-1,j,n))...
+sigma^2*((j-1)*dv)*dt/(2*dv^2)*(U(i,j+1,n)-2*U(i,j,n)+U(i,j-1,n))...
+r*(i-1)*ds*dt/(2*ds)*(U(i+1,j,n)-U(i-1,j,n))...
+r*(j-1)*dv*dt/(2*dv)*(U(i,j+1,n)-U(i,j-1,n))...
+rho*sigma*((j-1)*dv)*((i-1)*ds)*dt/(4*ds*dv)*(U(i+1,j+1,n)+U(i-1,j-1,n)-U(i-1,j+1,n)-U(i+1,j-1,n))...
-r*dt*U(i,j,n)-F1-F2;
end
end

for j=1:J
U(1,j,n+1)=0;
```

```
end
for i=2:I-1
U(i,1,n+1)= r*(i-1)*dt*(U(i+1,1,n)-U(i,1,n))+U(i,1,n)*(1-r*dt);
end
for i=2:I-1
U(i,J,n+1)=(i-1)*ds;
end
for j=1:J
U(I,j,n+1)=ds+U(I-1,j,n+1);
end
end

figure(1)
S=0:ds:200;
V=0:dv:1;
mesh(V,S,U(:,:,n))
xlabel('V')
ylabel('S')
zlabel('C')
title('Numerical solution of PDE, when k=k1=0.00')
T1=(table(U(39:61,5,n),U(39:61,60,n)));
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%For chapter 3, when k=k1=0.02
%parameter values
Smax=200;Smin=0; Vmax=1;Vmin=0;K=100;
I=100;J=100;r=0.05;
rho=0.8;k=0.02;k1=0.02;
T=1; deltat=1; dt=1/10000;
sigma=0.4;

N=1+ceil(T/dt);% dt = T/nt;
% the lower bounds of s and v are both 0
ds = (Smax-Smin)/(I-1); % step length of s
dv = (Vmax+Vmin)/(J-1); % step length of v

U=zeros(I,J,N);
for i=1:I
for j=1:J
U(i,j,1)=max(0,(i-1)*ds-K);
end
end

for n=1:N-1 % the interior elements. Cross term part.
for j=2:J-1
for i = 2:I-1
f1=(U(i+1,j,n)-2*U(i,j,n)+U(i-1,j,n))/(ds)^2;
f2=(U(i+1,j+1,n)+U(i-1,j-1,n)-U(i-1,j+1,n)-U(i+1,j-1,n))/(4*ds*dv);
f3=(U(i,j+1,n)-2*U(i,j,n)+U(i,j-1,n))/(dv)^2;
F1=k*((i-1)*ds)*dt*sqrt(2/(pi*deltat))*sqrt(((j-1)*dv)*((i-1)*ds)^2*(f1)^2 +...
sigma^2*((j-1)*dv)*(f2)^2+2*rho*sigma*((j-1)*dv)*((i-1)*ds)*(f2*f1));
F2=k1*((j-1)*dv)*dt*sqrt(2/(pi*deltat))*sqrt(((j-1)*dv)*((i-1)*ds)^2*(f2)^2+...
sigma^2*((j-1)*dv)*(f3)^2+2*rho*sigma*((j-1)*dv)*((i-1)*ds)*(f2*f3));
U(i,j,n+1)=U(i,j,n)+((j-1)*dv)*((i-1)*ds)^2*dt/(2*ds^2)*(U(i+1,j,n)-2*U(i,j,n)+U(i-1,j,n))...
```

```
+sigma^2*((j-1)*dv)*dt/(2*dv^2)*(U(i,j+1,n)-2*U(i,j,n)+U(i,j-1,n))...
+r*(i-1)*ds*dt/(2*ds)*(U(i+1,j,n)-U(i-1,j,n))...
+r*(j-1)*dv*dt/(2*dv)*(U(i,j+1,n)-U(i,j-1,n))...
+rho*sigma*((j-1)*dv)*((i-1)*ds)*dt/(4*ds*dv)*(U(i+1,j+1,n)+U(i-1,j-1,n)-U(i-1,j+1,n)-U(i+1,j-1,n))...
-r*dt*U(i,j,n)-F1-F2;
end
end

for j=1:J
U(1,j,n+1)=0;
end
for i=2:I-1
U(i,1,n+1)= r*(i-1)*dt*(U(i+1,1,n)-U(i,1,n))+U(i,1,n)*(1-r*dt);
end
for i=2:I-1
U(i,J,n+1)=(i-1)*ds;
end
for j=1:J
U(I,j,n+1)=ds+U(I-1,j,n+1);
end
end

figure(2)
S=0:ds:200;
V=0:dv:1;
mesh(V,S,U(:,:,n))

xlabel('V')
ylabel('S')
zlabel('C')
title('Numerical solution of PDE, when k=k1=0.00')
T2=(table(U(39:61,5,n),U(39:61,60,n)))
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%For chapter 4 when k0=k1=k2=0
%parameter values
Smax=200;Smin=0;Vmax=1;Vmin=0;rmax=1;rmin=0;
E=100;% strike price
I=50;J=25;K=25;
eta=0.2;rho=0.8;k0=0.02;k1=0.02;k2=0.02;
T=1; dt=1/10000;a=0.5;b=0.1;
sigma=0.05;%or sigma=0.6
N=1+ceil(T/dt);% dt = T/nt; %N=100
deltat=1;
% the lower bounds of s and v are both 0
ds = (Smax-Smin)/(I-1); % step length of s
dr = (rmax-rmin)/(K-1); % step length of r
dv = (Vmax+Vmin)/(J-1); % step length of v

%initial conditions
U=zeros(I,J,K,N);
for i=1:I
for j=1:J
for k=1:K
```

```
U(i,j,k,1)=max(0,(i-1)*ds-E);
end
end
end

for n=1:N-1 % equation 2.
for j=2:J-1
for k = 2:K-1
for i = 2:I-1
%bonds price
kappa=0.5;alpha=0.5;
h = sqrt(kappa^2 + 2*sigma^2);
A  = (2*h*exp((kappa+h)*T/2)/(2*h + (kappa+h)*(exp(T*h)-1)))^((2*kappa*alpha)/sigma^2);
B  = 2*(exp(T*h)-1)/(2*h + (kappa+h)*(exp(T*h)-1));
Z =  exp(-B*((k-1)*dr)+A); %equation 4.14
Zeta = -B*Z; %Zeta=dZ/dr

F1=0;
F2=0;
F3=0;

U(i,j,k,n+1)=U(i,j,k,n)+((j-1)*dv)*((i-1)*ds)^2*dt/(2*ds^2)*(U(i+1,j,k,n)-2*U(i,j,k,n)+U(i-1,j,k,n))...
+sigma^2*((j-1)*dv)*dt/(2*dv^2)*(U(i,j+1,k,n)-2*U(i,j,k,n)+U(i,j-1,k,n))...
+(eta^2)*((k-1)*dr)*dt/(2*dr^2)*(U(i,j,k+1,n)-2*U(i,j,k,n)+U(i,j,k-1,n))...
+((k-1)*dr)*(i-1)*ds*dt/(2*ds)*(U(i+1,j,k,n)-U(i-1,j,k,n))...
+((k-1)*dr)*(j-1)*dv*dt/(2*dv)*(U(i,j+1,k,n)-U(i,j-1,k,n))...
+(a*(b-(k-1)*dr))*dt/(2*dr)*(U(i,j,k+1,n)-U(i,j,k-1,n))...
+rho*sigma*((j-1)*dv)*((i-1)*ds)*dt/(4*ds*dv)*(U(i+1,j+1,k,n)+U(i-1,j-1,k,n)-U(i-1,j+1,k,n)-U(i+1,j-1,k,n))...
-(k-1)*dr*dt*U(i,j,k,n)-F1-F2-F3;

F33=dt*k2*Z/Zeta*sqrt(2/(pi*deltat))*eta*sqrt((k-1)*dr)/(dr^2)*(U(i,1,k+1,n)-2*U(i,1,k,n)+U(i,1,k-1,n));

end
end
end

%boundary conditions
%when S=min
for k=1:K
for j=1:J
U(1,j,k,n+1)=0;
end
end
%when r=min
for i=2:I-1
for j=2:J-1
U(i,j,1,n+1)=U(i,j,2,n+1);
end
end
%when r=max
for i=2:I-1
for j=2:J-1
U(i,j,K,n+1)= U(i,j,K-1,n+1);
```

```
end
end
%when V=max
for i=2:I-1
for k=2:K-1
U(i,J,k,n+1)=(i-1)*ds;
end
end
%when V=min
for i=2:I-1
for k=2:K-1
U(i,1,k,n+1)=U(i,1,k,n)...
+((k-1)*dr)*(i-1)*ds*dt/(2*ds)*(U(i+1,1,k,n)-U(i-1,1,k,n))...
+(a*(b-(k-1))*dr)*dt/(2*dr)*(U(i,1,k+1,n)-U(i,1,k-1,n))...
-(k-1)*dr*dt*U(i,1,k,n)-F33;
end
end
%when S=max
for k=1:K
for j=1:J
U(I,j,k,n+1)=ds+U(I-1,j,k,n+1);
end
end
end

figure(3)
S=0:ds:200;
V=0:dv:1;
mesh(V,S,U(:,:,1,n))
xlabel('V')
ylabel('S')
zlabel('C')
title('Numerical solution of Heston-CIR PDE, when V=0.04,k0=k1=k2=0.02')
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%For chapter 4 when k0=k1=k2=0.02
%parameter values
Smax=200;Smin=0;Vmax=1;Vmin=0;rmax=1;rmin=0;
E=100;% strike price
I=50;J=25;K=25;
eta=0.2;rho=0.8;k0=0.02;k1=0.02;k2=0.02;
T=1; dt=1/10000;a=0.5;b=0.1;
sigma=0.05;%or sigma=0.6 deltat=1;
N=1+ceil(T/dt);% dt = T/nt; %N=100
% the lower bounds of s and v are both 0
ds = (Smax-Smin)/(I-1); % step length of s
dr = (rmax-rmin)/(K-1); % step length of r
dv = (Vmax+Vmin)/(J-1); % step length of v

%initial conditions
U=zeros(I,J,K,N);
for i=1:I
for j=1:J
for k=1:K
```

```
U(i,j,k,1)=max(0,(i-1)*ds-E);
end
end
end

for n=1:N-1 % equation 2.
for j=2:J-1
for k = 2:K-1
for i = 2:I-1
%bonds price
kappa=0.5;alpha=0.5;
h = sqrt(kappa^2 + 2*sigma^2);
A  = (2*h*exp((kappa+h)*T/2)/(2*h + (kappa+h)*(exp(T*h)-1)))^((2*kappa*alpha)/sigma^2);
B  = 2*(exp(T*h)-1)/(2*h + (kappa+h)*(exp(T*h)-1));
Z =  exp(-B*((k-1)*dr)+A); %equation 4.14
Zeta = -B*Z; %Zeta=dZ/dr

f1= (U(i+1,j,k,n)-2*U(i,j,k,n)+U(i-1,j,k,n))/(ds^2);
f2= (U(i,j+1,k,n)-2*U(i,j,k,n)+U(i,j-1,k,n))/(dv^2);
f3= (U(i+1,j+1,k,n)+U(i-1,j-1,k,n)-U(i-1,j+1,k,n)-U(i+1,j-1,k,n))/(4*ds*dv);

F1=k0*((i-1)*ds)*dt*sqrt(2/(pi*deltat))...
*sqrt(2*rho*(j-1)*dv*sigma*(i-1)*ds*f1*f3+(j-1)*dv*((i-1)*ds)^2*(f1)^2+(j-1)*dv*sigma^2*(f3)^2);

F2=k1*(j-1)*dv*dt*sqrt(2/(pi*deltat))...
*sqrt(2*rho*(j-1)*dv*sigma*(i-1)*ds*f2*f3+(j-1)*dv*((i-1)*ds)^2*(f3)^2+(j-1)*dv*sigma^2*(f2)^2);

F3=dt*k2*Z/Zeta*sqrt(2/(pi*deltat))*eta*sqrt((k-1)*dr)/(dr^2)*(U(i,j,k+1,n)-2*U(i,j,k,n)+U(i,j,k-1,n));

U(i,j,k,n+1)=U(i,j,k,n)+((j-1)*dv)*((i-1)*ds)^2*dt/(2*ds^2)*(U(i+1,j,k,n)-2*U(i,j,k,n)+U(i-1,j,k,n))...
+sigma^2*((j-1)*dv)*dt/(2*dv^2)*(U(i,j+1,k,n)-2*U(i,j,k,n)+U(i,j-1,k,n))...
+(eta^2)*((k-1)*dr)*dt/(2*dr^2)*(U(i,j,k+1,n)-2*U(i,j,k,n)+U(i,j,k-1,n))...
+((k-1)*dr)*(i-1)*ds*dt/(2*ds)*(U(i+1,j,k,n)-U(i-1,j,k,n))...
+((k-1)*dr)*(j-1)*dv*dt/(2*dv)*(U(i,j+1,k,n)-U(i,j-1,k,n))...
+(a*(b-(k-1)*dr))*dt/(2*dr)*(U(i,j,k+1,n)-U(i,j,k-1,n))...
+rho*sigma*((j-1)*dv)*((i-1)*ds)*dt/(4*ds*dv)*(U(i+1,j+1,k,n)+U(i-1,j-1,k,n)-U(i-1,j+1,k,n)-U(i+1,j-1,k,n))...
-(k-1)*dr*dt*U(i,j,k,n)-F1-F2-F3;

F33=dt*k2*Z/Zeta*sqrt(2/(pi*deltat))*eta*sqrt((k-1)*dr)/(dr^2)*(U(i,1,k+1,n)-2*U(i,1,k,n)+U(i,1,k-1,n));

end
end
end
%boundary conditions

%when S=min
for k=1:K
for j=1:J
U(1,j,k,n+1)=0;
end
end
%when r=min
for i=2:I-1
for j=2:J-1
```

```
U(i,j,1,n+1)=U(i,j,2,n+1);
end
end
%when r=max
for i=2:I-1
for j=2:J-1
U(i,j,K,n+1)= U(i,j,K-1,n+1);
end
end
%when V=max
for i=2:I-1
for k=2:K-1
U(i,J,k,n+1)=(i-1)*ds;
end
end
%when V=min
for i=2:I-1
for k=2:K-1
U(i,1,k,n+1)=U(i,1,k,n)...
+((k-1)*dr)*(i-1)*ds*dt/(2*ds)*(U(i+1,1,k,n)-U(i-1,1,k,n))...
+(a*(b-(k-1))*dr)*dt/(2*dr)*(U(i,1,k+1,n)-U(i,1,k-1,n))...
-(k-1)*dr*dt*U(i,1,k,n)-F33;
end
end
%when S=max
for k=1:K
for j=1:J
U(I,j,k,n+1)=ds+U(I-1,j,k,n+1);
end
end
end

figure(4)

S=0:ds:200;
V=0:dv:1;
mesh(V,S,U(:,:,1,n))
xlabel('V')
ylabel('S')
zlabel('C')
title('Numerical solution of Heston-CIR PDE, when V=0.04,k0=k1=k2=0.02')
```