Binary Choice Probabilities on Mixture Sets

MSRG Symposium

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27-28 November 2014

Theoretical context

- Binary stochastic choice.
  - Fechner models: utility difference representation of choice probabilities.
- Stochastic choice between risky or uncertain prospects.
  - New representation theorems: Fechner models with "utility" of non-EU form.

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Binary stochastic choice

- Let A be a set of alternatives.
- Let $P : A \times A \rightarrow [0, 1]$ be a **binary choice probability (BCP)**.
- If $a \neq b$ then $P(a, b)$ is the probability of choosing a from $\{a, b\}$. We leave $P(a, a)$ uninterpreted.

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Binary stochastic choice

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for any $a, b \in A$.

- In particular,

$$P(a, a) = \frac{1}{2}$$

for any $a \in A$.

Fechner models

Definition: The BCP P has a **strong utility representation (SUR)** if there exists a utility function $u : A \to \mathbb{R}$ such that

$$P(a, b) \geq P(c, d) \quad \Leftrightarrow \quad u(a) - u(b) \geq u(c) - u(d)$$

for any $a, b, c, d \in A$.

- This is a standard psychophysical model of choice behaviour: probability of choice depends on the relative stength of stimuli.

Fechner models

What are sufficient conditions (on P) for the existence of a SUR?

Fechner models

- Compact axiomatisations are possible when A is suitably "rich".
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For any $x \in (0,1)$ and any $a, b, c, a', b' \in A$

$$P(a,b) \geq P(a',b') \quad \Leftrightarrow \quad P(a,a') \geq P(b,b') \qquad \textbf{(QC)}$$

$$P(a,b) \geq x \geq P(a,c) \quad \Rightarrow \quad P(a,e) = x \text{ for some } e \in A \qquad \textbf{(S)}$$

Fechner models

The necessity of QC is easy to see:

$$P(a,b) \geq P(a',b') \quad \Leftrightarrow \quad P(a,a') \geq P(b,b')$$

$$u(a) - u(b) \geq u(a') - u(b') \quad \Leftrightarrow \quad u(a) - u(a') \geq u(b) - u(b')$$

Fechner models

A weaker (and more intuitive) property than the QC:

Strong Stochastic Transitivity (SST) For all $a, b, c \in A$

$$P(a,b), P(b,c) \geq \frac{1}{2} \quad \Rightarrow \quad P(a,c) \geq \max\{P(a,b), P(b,c)\}$$

Risk and uncertainty

- If A is a set of *lotteries*, it is natural to require additional structure on the utility function $u : A \to \mathbb{R}$ in a SUR (e.g., expected utility form)

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Risk and uncertainty

- In Dagsvik (2008), A is the unit simplex in $\mathbb{R}^n$ interpreted as lotteries over a fixed set of n possible prizes.

- Given $a, b \in A$ and $\lambda \in [0, 1]$, we write $a\lambda b$ for $\lambda a + (1 - \lambda) b$.
  - Useful to think of $a\lambda b$ as a *compound lottery*.

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Archimedean Property For all $a, b, c \in A$ if

$$P(a, b) > \frac{1}{2} > P(c, b)$$

then there exist $\alpha, \beta \in (0, 1)$ such that

$$P(a\alpha c, b) > \frac{1}{2} > P(a\beta c, b).$$

Strong Independence (SI) For all $a, b, a', b', c \in A$ and all $\lambda \in (0, 1)$

$$P(a, b) \geq P(a', b') \quad \Rightarrow \quad P(a\lambda c, b\lambda c) \geq P(a'\lambda c, b'\lambda c).$$

Risk and uncertainty

- These four axioms suffice for a SUR with linear u.
- Dagsvik's proof uses Debreu (1958).
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Risk and uncertainty

- Define a binary (preference) relation $\geq^*$ on $A \times A$ as follows:[1]

$$(a, d) \geq^* (b, c) \quad \Leftrightarrow \quad P(a, b) \geq P(c, d) \qquad (**)$$

- An ordering on two-state *Anscombe-Aumann (AA) acts*.
- Then P has a SUR iff $\geq^*$ has a *Subjective Expected Utility (SEU)* representation with equi-probable states:

$$(a, d) \geq^* (b, c) \quad \Leftrightarrow \quad P(a, b) \geq P(c, d)$$

$$\tfrac{1}{2}u(a) + \tfrac{1}{2}u(d) \geq \tfrac{1}{2}u(b) + \tfrac{1}{2}u(c) \quad \Leftrightarrow \quad u(a) - u(b) \geq u(c) - u(d)$$

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- Anscombe and Aumann (1963) axiomatise preferences over AA acts which have a SEU representation with a **linear** (EU) utility function.

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Translate Dagsvik's axioms on P into the corresponding restrictions on $\geq^*$ and show (using the techniques of Anscombe and Aumann) that they suffice for a SEU representation with linear utility and subjective probability $\frac{1}{2}$ on each state.

Risk and uncertainty

In particular, the translation of QC ensures equi-probable states:

$$P(a, b) \geq P(c, d) \quad \Leftrightarrow \quad P(a, c) \geq P(b, d)$$

$$(a, d) \geq^* (b, c) \quad \Leftrightarrow \quad (a, d) \geq^* (c, b)$$

New representation theorems

- This proof strategy turns out to be very powerful and very flexible.

- We can:
  - Strengthen Dagsvik's result by weakening QC to SST.
  - Develop new SUR representation theorems that impose alternative restrictions on u (besides linearity).

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New representation theorems

Definition Given some $M \subseteq A$ we say that $u : A \rightarrow \mathbb{R}$ is *M-linear* if

$$u(M) = u(A)$$

and

$$u(a\lambda b) = \lambda u(a) + (1 - \lambda) u(b)$$

for any $a \in A$, any $b \in M$ and any $\lambda \in [0, 1]$.

New representation theorems

Examples of M-linear classes of utility functions:

- Examples with $M = A$:
  - Expected utility for lotteries.
  - Subjective expected utility in an AA environment.
- Maxmin expected utility or Choquet expected utility in an AA environment.
  - $M =$ constant acts.
- Yaari's (1987) Dual Theory for lotteries.
  - $M =$ degenerate lotteries.

New representation theorems

Given an M-linear class $\mathcal{U}$ of utility functions, what are sufficient conditions for a BCP to possess a SUR with respect to some $u \in \mathcal{U}$?

- If we know sufficient conditions for a **preference order** on A to be representable within $\mathcal{U}$, we can provide an answer.
- We give a general "recipe" based on a generalisation of the Anscombe-Aumann approach and specific axiomatisations for all the M-linear utility classes mentioned above.

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Postscript

- Empirical challenges to the (basic) Fechner model: strength of preference versus ease of comparison (e.g., dominance).