Free-body and flexural motion of a floating elastic plate under wave maker forcing

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Outline

Elastic plate floating on water

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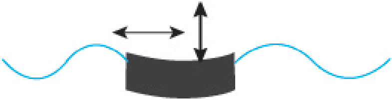
Modelling regimes

Zero-thickness *VS* **Lateral forcing**



Need the velocity potential along the draft: vertical distance between the waterline and the bottom of the plate.

Mathematical Model: Motion of water

Incompressible inviscid fluid

- Time harmonic solution: $\exp(\mathrm{i}\,\omega t)$ dependence
- 2D Linear system
- Laplace's equation for velocity potential

$$\nabla^2 \phi(x, z) = 0$$

- Boundary condition at the flat solid surface is $\phi_z(-h) = 0$
- Boundary condition at the free surface is $\phi_z(0) = \omega^2 \phi$
- For the volume of finite depth h, the solution has discrete modes

$$\cosh k_n(z + h), \text{ is the roots of } k \tanh kh = \omega^2$$

Mathematical Model: Motion of an elastic plate

Floating thin elastic plate

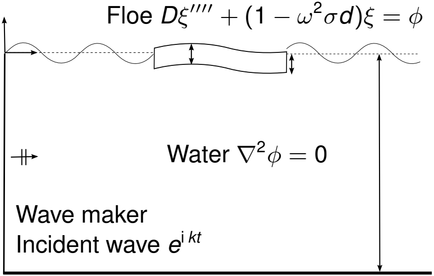
- Time harmonic solution: $\exp(\mathrm{i}\,\omega t)$ dependence
- 1D Linear system
- Floating elastic plate equation for vertical deflection $\xi(x)$ with reference to the draught d

$$\phi = D\xi'''' + (1 - \omega^2 \sigma d)\xi, \quad \partial_z \phi = \omega^2 \sigma \xi, \quad \text{at } z = -d$$

- The solution has discrete modes $\cosh k_n(z + h)$, which are the roots of

$$(k^4 + (1 - \omega^2 \sigma d))k \tanh kh = \omega^2$$

Method of solution: Boundary integral method



Green's function

$$\nabla^2 G(x, z|x_0, z_0) = \delta(x - x_0)\delta(z - z_0)$$

$$\partial_z G = \omega^2 G \text{ on the surface}$$

$$\partial_n G = 0 \text{ on other boundaries.}$$

$$G = \frac{1}{2\,\mathrm{i}} \sum_{n=0}^{\infty} \frac{e^{\mathrm{i}\,k_n(x+x_0)} + e^{\mathrm{i}\,k_n|x-x_0|}}{k_n c_n} w_n(z) w_n(z_0)$$

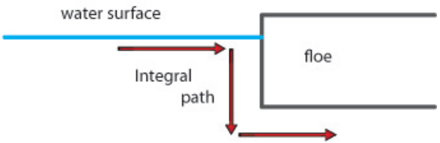
$w_n(z) = \cosh k_n(z + h)$ are eigenfunctions.

Surge motion

The boundary integral equation

$$\epsilon\phi = \phi_{\text{Incident}} - \int_{\text{boundary}} \{(\partial_{n_0} G)\phi - G(\partial_{n_0}\phi)\}\, ds_0,$$

$\epsilon = \theta/2\pi$, where θ is the angle of the corner.



Amplitude of the surge

$$u \propto \int_{-d}^{0} (\phi(b,z) - \phi(a,z))\, dz$$

Green's function and its singularity

Log-like singularity of the Green's function

$$G \sim \log\left(|x - x_0| + |z - z_0|\right) \quad \text{as } x \to x_0, z \to z_0, \ \text{for } (x_0, z_0) \text{ on the corner.}$$

The log-part comes from the series

$$\sum_{n=0}^{\infty} \frac{e^{\mathrm{i}\, k_n |x - x_0|}}{k_n c_n} w_n(z) w_n(z_0)$$

Replace the singular part with a known singular series

$$\sum_{n=1}^{\infty} \frac{e^{-n\gamma |x - x_0|}}{n} \cosh \frac{\mathrm{i}\, n\pi}{h}(z + h) \cosh \frac{\mathrm{i}\, n\pi}{h}(z_0 + h)$$

This sum has the closed form

$$-\frac{1}{4}\left(\log \mathcal{L}_+(x, z) + \log \mathcal{L}_-(x, z)\right)$$

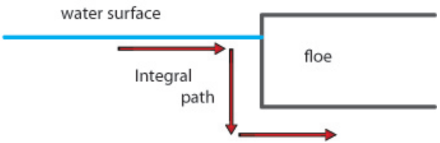
Separating of singular part

Separating the Green's function

$$G = \widetilde{G} + \log \mathcal{R}$$

$\widetilde{G}$ is bounded

The integrals involving $\log \mathcal{R}$ can be evaluated analytically.



Modes of the motion

Expansion of the surface deflection

$$\xi(x) = \sum_{m=0}^{M} \xi_m X_m(x)$$

where X_m are the eigenfunctions of

$$X'''' - \kappa_m^4 X = 0, \quad \text{for edges } X'' = 0, X''' = 0$$

$$X_m \propto \{\cos \kappa_m x, \cosh \kappa_m x, \sin \kappa_m x, \sinh \kappa_m x\}$$

Heave and pitch

- ξ_0 represents heave motion
- ξ_1 represents pitch motion

Computation method

Expansion of the potential along the submerged edges of the plate

$$\phi(a,z)=\sum_{n=1}^{N}\alpha_n\mathcal{C}_{2n}(\hat{z}),\quad \phi(b,z)=\sum_{n=1}^{N}\beta_n\mathcal{C}_{2n}(\hat{z})$$

where $\mathcal{C}_{2n}(\hat{z})$ is the Gegenbauer polynomial of the normalized variable

$$\hat{z}=\sqrt{1-\left(\frac{z}{d}\right)^2}$$

Gegenbauer polynomials

Orthogonal polynomials with the desired order of singularity at the corner.

Solving the BIE

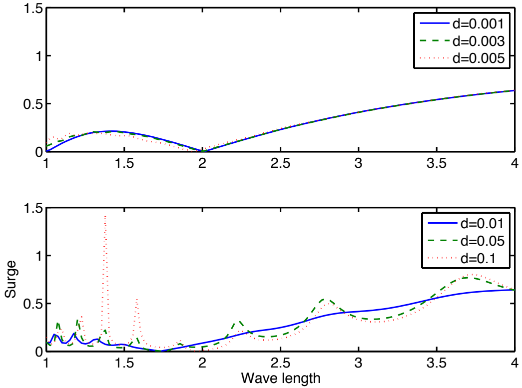
1. Formulate a system of equations with X_n and $\mathcal{C}_{2n}$
2. Solve for the system of equations for $\{\xi_n\}$ and $\{\alpha_n, \beta_n\}$
3. Impose the continuity conditions at the corners $(a, -d)$ and $(b, -d)$

$$\phi(a, -d) = \sum_{m=0}^{M} \kappa_m \xi_m X_m(a), \quad \phi(b, -d) = \sum_{m=0}^{M} \kappa_m \xi_m X_m(b)$$

4. Wave maker con be out in by the forcing term

$$\sum_{n=0}^{\infty} \frac{v_n}{\mathrm{i}\, k_n} e^{\mathrm{i}\, k_n x} w_n(z), \quad v_n = \int_{-h}^{0} v(z) w_n(z)\, dz$$

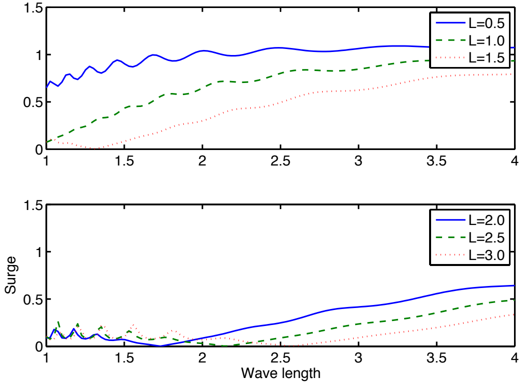
Surge motion



Amplitude of the surge motion of various thicknesses ranging

- from very thin 1mm
- to thick 100mm

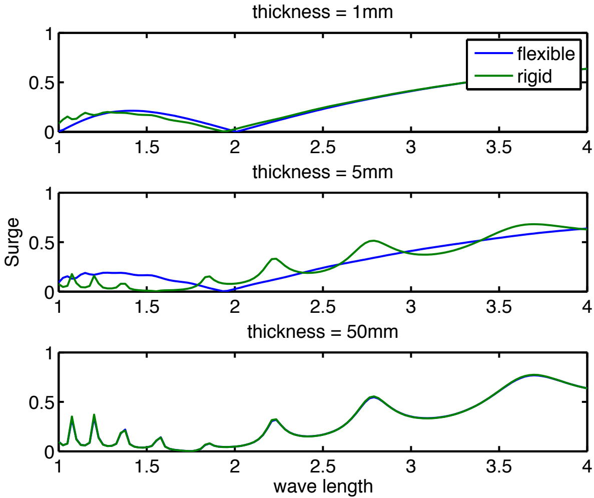
Surge motion



Amplitude of the surge motion of various length ranging

- from short 0.5m
- to long 3m

Surge motion



Comparison between flexible and stiff plates

- very thin 1.0mm
- medium thickness 5.0mm
- thick 50mm

Summary

- Complete description of the hydro-elastic motions of a finite floe.
- Include the draft of the plate and compute the surge motion.
- Analytical treatment of the singularities at the corners of the plate.
- Reduction of computation using the orthogonal polynomials and eigenfunctions.