New Zealand Cellular Network Base Station Spatial Distribution Analysis by Using Alpha-Stable Distribution Model

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Abstract

Studying the distribution model of cellular network base station not only can optimize the user's wireless communications and Internet experience, but also reveal the patterns of the regional development and human demands for information and communications technologies (ICT). Because the needs of users motivate the deployment of base stations, and the distribution of base stations could be different in various regions with different users' needs. In addition, different speed of regional development could lead to different populations of ICT users. Therefore, the study of the distribution of base stations can direct the way for the future ICT development.

Some previous analysis results of Europe and China have shown that alpha-stable model is suitable to analyze the base station distribution in the urban areas where the population are concentrated. However, in the sparsely populated rural areas, the Weibull model and the Log-normal model are more suitable for analyzing the distribution of base stations. This project is to study the network base stations distribution in New Zealand and explore what distribution models are best fit for New Zealand cases. It could help New Zealand improving its network environment and provide research assistance for future ICT networks and industries that rely on ICT networks.

This project is inspired from the adoption of the alpha-stable distribution model in financial market to study the economical phenomena, thus the alpha-stable model might be applied to study the distribution of cellular network base stations. Firstly, it introduces the alpha-stable distribution model in detail, such as its concept and formulations, as well as how the various parameters in the alpha-stable model affect its performance and shapes. Secondly, we have analyzed the distribution of cellular network base stations in New Zealand. The experimental results show that the distribution of cellular network base stations in New Zealand is presenting the alpha-stable distribution model, Weibull model or Log-normal model according to different regions. It is concluded that the distribution of New Zealand's network base stations is relatively loosely with diverse patterns. Some major cities occupy most base stations, but other places have fewer base stations, which also causes the heavy-tail effect of the distribution model. Finally, this report discusses the impact that network base station distribution may affect other industries, such as autonomous and electric vehicles, unmanned factories and 5G technology because possible ICT infrastructure sharing in the future.

1 Introduction

The French mathematician Levy proposed the alpha-stable distribution, the distribution mostly used in the financial analysis. The financial market has a great trouble to investors because it is innate unpredictable. People are eager to predict the direction of financial markets through a model to reasonably avoid risks, but the introduction of a model does not mean that the model will be restricted and applied in a field. The phenomenon of financial markets can reflect many other phenomena. For example, some economic theories can solve social problems. Therefore, alpha-stable distribution can use in financial market analysis and prediction for the first time. However, the alpha-stable distribution model also can adapt in other areas where random events exist, such as base station location coordinates distribution analyze.

The base station is a part of the mobile network that cannot ignore. However, the installation of the network base station is approximately like a random event. Mobile Internet companies will increase the number of network base stations in an area based on customer needs. This phenomenon is like changes in the financial market, which based on market demand. However, this demand seems to be random. The increase of users in a specific area may bring about an increase in the demand for network base stations. Similarly, the increase in traffic demand for a single user in the same area will also increase the demand for network base stations. The update and iteration of communication technologies may also happen. That can lead to an increase in demand for base stations. Because of the above phenomena, the equal distribution of the network base station seems to be random, but this seemingly random phenomenon hides many regularities that cannot predict accurately. So, this project needs a model to study the deployment of base stations, and it needs to use this model to predict the development of future network base stations. The current alpha-stable distribution model is a useful tool for studying financial markets. However, people can also use the alpha-stable distribution model to study the deployment of network base stations.

1.1 Background

The alpha-stable distribution was first used to study the benefits and risks of financial markets. The study by Frain et al. (2008) describes the implementation of six total reporting income combinations. It found that the prediction based on the alpha-stable distribution model is feasible and better than the traditional measurement. Researchers believe that an alpha-stable distribution model is a powerful tool for risk managers and financial regulators.

Financial markets are full of risks. With the globalization of financial markets and the rapid development of financial derivatives markets, financial markets have shown unprecedented volatility, coupled with constant financial innovation, financial institutions and investors are facing with increasing and complex financial risks, there are numerous examples of losses or corporate failures in the market due to poor risk management. For example, the collapse of the US investment bank triggered by the US subprime mortgage crisis in 2007 was the result of poor risk management. It caused the bankruptcy of the five major investment banks on Wall Street, which eventually led to a global financial crisis. Therefore, for financial institutions, investors, risk regulators, and even the government, it is imperative to strengthen the measurement and prevention of risks. The risk measurement of financial assets is the foundation and core of financial market risk management. It has always been the focus of risk researchers and investment managers, and they are making unremitting efforts to find suitable risk measurement tools. As early as the 1950s, Markowitz proposed the mean-variance model (stability model) (Markowitz, 1952), which uses the variance of the rate of return to describe the risk of securities investment. Because the variance has excellent statistical properties, and the measurement risk is simple and easy, it can use in investment management widely. However, with the deepening of empirical research, many scholars have found that the rate of return does not obey the normal distribution but has the fractal feature of "spike and thick tail," and generally has a certain degree of skewness and positional parameters. Agarwal and Naik believe that the mean-variance model underestimates the heavy-tail risk (Agarwal& Naik, 2004). In 1996, JP.

Morgan introduced VaR (value at risk) and risk value in risk measurement (Marshall & Siegel, 1996). Academics and practitioners have widely recognized it. Many US rating companies and securities trading commissions have announced and supported VaR as a risk measure.

Moreover, the main methods of managing risk, the Basel Committee on Banking Supervision, and the Federal Reserve Bank of the United States also recognize VaR as one of the accepted measures of financial risk. However, with the deepening of research, VaR has been criticized by many people because it does not meet the criteria of consistent compatibility risk measurement, and cannot measure the heavy-tail risk, let alone measure the risk of extreme events. In the research of Artzner et al. (1999) introduced CVaR (conditional value at risk) as a risk measurement tool to compensate for the above disadvantages of VaR. It satisfies the consistent compatibility risk measurement condition and can measure the risk of heavy-tail occurrence. Rockafellar and Uryasevb believe that CVaR measures financial market risk better than VaR. CVaR satisfies sub-quantitative additivity, can measure risk beyond VaR part and can solve the practical boundary problem of large-scale portfolio optimization through linear programming (Rockafellar & Uryasev, 2002). However, when the density function of the portfolio loss is not a continuous function, CVaR is no longer a consistency risk measurement model. On this basis, Tasche proposed the ES (expected shortfall) model, which is an improvement of the CVaR model (Tasche, 2002). It is a measure of consistency risk. When the loss density function is continuous, the results of the Es model and the CVaR model obtained. The result is the same. When the loss distribution is discontinuous, the results of the two calculations are different. The distribution function satisfies the consistency risk measurement condition, whether it is continuous or discontinuous. Therefore, it can use to test extreme risk models.

The alpha-stable distribution is used in the financial field to study the regularity of random distribution. The random distribution of events in the financial market is like other random distributions in the real world. Take the stock exchange market as an example. The random distribution in the stock market mainly depends on the benefits and risks that a single stock may bring. However, these stock income and risks are random, because the rise and fall of stock prices caused by various reasons, which are often unpredictable. In society, things such as store location are also due to many random events. A business will select a store based on the traffic of the street, but this does not guarantee the store's revenue. Random events like this can also be studied using the alpha-stable distribution.

The rise and fall distribution of the stock price is like the coordinate position of the network base station. The longitude and latitude of the network base station could be the rise and fall of the stock price. Although the alpha-stable distribution is mostly used to study financial markets, this distribution model can use in more professions and fields. It is a good idea to apply the alpha-stable distribution model to study the distribution of network base station coordinate.

Kabašinskas et al. (2009) studied historical data of financial securities, statistical models of stock returns, parameter estimation methods, effects of self-similarity and multifractal, and algorithms for financial portfolio selection. Establish numerical methods for estimating model parameters and compare their efficiencies.

The purpose of the research of Komaty et al. (2015) is to analyze the case of EMD (Empirical Mode Decomposition) and MEMD (Multivariate EMD) in a random case involving non-Gaussian noise, more specifically, Komaty et al. (2015) The case of symmetric alpha stabilized (SαS) noise. The results of the report show that, unlike EMD, MEMD can align multiple channels in the same index IMF in the standard frequency mode. Besides, the simulation shows that, contrary to EMD, the stability of MEMD satisfies the lower index mode well and uses this result to estimate the stability index of the SαS input signal.

Gunay and Khaki attempt to simulate the volatility of energy futures under different distributions (Gunay & Khaki, 2018). In the empirical analysis, the researchers used the GARCH and APARCH models to predict the volatility of natural gas futures, Brent crude oil futures and heating oil futures under the premise of using a stable distribution. The researchers also applied various Value and Risk (VaR) analyses, Gauss, History, and Correction (Cornish-Fisher) VaR for each variable. The results show that the APARCH model is better than the GARCH model and performs better in terms of the heavy-tail effect in the simulated return.

As the research progresses, the researchers found that the alpha-stable distribution model can be used not only in the financial industry but also in other industries. For example, Cuomo et al. (2016) found that the alpha-stable distribution is suitable for study the distribution of network base stations in urban environments.

In the study of Zhou et al. (2013), the researchers study delays in mobile cellular networks, delay distribution, and other movement patterns. It also finds that the Power-Regularity distribution is more suitable for the arrival interval and the dwell time. Besides, the network delay motion patterns during the day, night, country, and urban areas further prove the feasibility of the power regularity model. A new transmission time distribution model and the number of receiving users are also proposed to characterize other motion patterns. The research shows that the corresponding simulation results are very consistent with the empirical model results.

In the research of Lee et al. (2014), researchers first introduced the analysis of traffic volume measurements collected from commercial cellular networks in China and demonstrated that the spatial distribution of traffic density (traffic load per unit area) could be Log-normal or Weibull distribution. Then a spatial traffic model is proposed, which can be directly used to generate real-world spatial traffic patterns for cellular network simulation, such as network planning and load balancing performance evaluation.

In the project of Zhou et al. (2015), researchers re-examined the statistical model of base stations in cellular networks and tried to find the spatial density distribution that is most suitable for network base stations. By studying the deployment information of many actual cellular network base stations, the researchers found that the widely used Poisson distribution causes severe deviations in the actual base station density distribution. Zhou et al. (2014) pointed out that Poisson distribution has used widely as a useful model for the spatial distribution of base stations in cellular networks. However, due to the impact of the environment on actual site planning, real base station deployments are rarely entirely random. Poisson distribution is not the best spatial distribution model of network base stations.

The project of Cuomo et al. (2016) studied the spatial distribution data of network base stations in different countries in Europe. The purpose of this is to find the model closest to the actual data distribution from the distribution models of Poisson, Weibull, Generalized Pareto, Log-normal, and α-Stable. Cuomo et al. (2016) concluded that the alpha-stable distribution is suitable for studying the distribution of network base stations in urban environments, while the Log-normal distribution is suitable for studying the distribution of network base stations in non-regions.

The project of Chen et al. (2018) pointed out that α-Shape is a powerful algebraic geometry tool for the analysis of actual base station position data in six Asian countries and six European countries. First, the base station deployment in Asian and European countries represents fractal features based on two different proof indicators, namely the Betti number and the Hurst coefficient. Secondly, the actual base station deployment characterized by Euler features, the Log-normal distribution exhibits the best match with the cellular network topology.

1.2 Motivation

With the development of technology, driverless technology, and remote-control technology will be widely used in the future, but these technologies are more demanding on the network environment. Network delays can cause severe problems with driverless technology. The most common standards for mobile networks today are 2G, 3G, 4G. Among them, the fastest network transmission technology is 4G technology. However, if the driverless car equipped with 4G technology, the network delay problem will affect the safety of the driverless car. If the International Telecommunication Union can download more than 100 Mbps for 4G mobile networks, it is 25 times faster than home broadband ADSL (4Mbs) (Ameen, Ismail, & Idrus, 2011), and can meet the requirements of most users for wireless services. The network delay of 4G technology cannot be to meet the needs of driverless cars, such as China Mobile's

4G network delay is 60-90ms, while China Mobile and China Telecom's 4G network delay is 20-40ms (Chang & Mason, 2012).

In ordinary cars, the reaction time for the driver to find out the situation and take measures is usually 0.75 to 1 second, and the braking distance is 8.33 meters (Johansson & Rumar, 1971). For example, the braking distance on the asphalt road at a speed of 30 km / h is 5.9 m, the braking distance on the snowy road is 17.7 m, and the braking distance on the ice road is 35. 4 meters. If the driverless car is driving at a speed of 30 kilometers per hour, the delay between the discovery of the car and the response is about 100ms, which means that the driverless program is about 100ms slower than the driver. If the driverless car needs emergency braking, it will be about one meter longer than the regular car Rothenbücher et al. (2016). The distance of this meter is enough to capture a person's life. Enterprises and governments cannot allow such driverless cars to obtain permission to go on the road. Therefore, the current major auto companies adopt a semi-automatic driving scheme, which requires a driver and autopilot program.

However, with the development of 5G communication technology, the above problems can be solved. According to the currently published 5G communication technology standard, the transmission speed of the 5G will be ten times (1000 Mbase station) of 4G, and the delay of 5G is smaller than 4G Ding et al. (2016). For example, in the eMBB scenario, the 5G user plane delay is 4ms. Such a delay can substantially meet the network delay requirements of a crewless vehicle. Not only do driverless cars use 5G, but other technologies such as teleoperation require 5G. Remote surgery is the best solution in some remote and physiotherapy areas and countries, and doctors can perform complex operations on patients by remote manipulation. Therefore, the innovation of 5G technology is not only the progress of the communication industry, but the progress of other industries will also realize through the commercialization of 5G technology. However, the deployment of 5G base stations also needs to learn from the experience of 4G base station deployment. Therefore, the research motivation of this paper is to use alpha-stable Distribution model to analyze the current 2G，3G，4G the coordinate position data of the network base station to predict the future through the current network base station deployment5GThe distribution form of the base station.

Moreover, then by using the5GBase station predictions for the future to help industries such as a driverless car. Alternatively, predict industrial development such as driverless through the current distribution of base stations.

1.3 Contributions

First, this report will explain the alpha-stable in detail and understand the changes of the alpha-stable model by understanding the dependence of each parameter in the alpha-stable distribution on the alpha-stable distribution model. After that, the Property and unique form of the alpha-stable distribution model can explain, which helps to understand the distribution model after using the alpha-stable distribution model.

Then, using the coordinate location data of the network base station, the network base station coordinate data of New Zealand is following the alpha-stable distribution model, but the network base station distribution in New Zealand also conforms to other candidate distribution models. By comparing the experimental results in Italy, the distribution of network base stations in New Zealand is different from the distribution of Italian-to-network base stations. The distribution of network base stations in Italy is only the model of alpha-stable distribution, but the network base stations in New Zealand can follow other candidate distribution models in some area. By analyzing the experimental results, researcher concludes that New Zealand has caused the current distribution of network base stations due to its conditions such as area topography and population distribution. we found that the distribution of network base stations in New Zealand has become extremely uneven. The number of network base stations in select cities is significantly larger than the number of network base stations in other regions. These cities are multi-dimensional New Zealand's more developed large cities such as Christchurch, Wellington, Auckland. We expect to help New Zealand build a complete network environment through the experiments and help the development of the upcoming 5G network and other industries.

1.4 Project report structure

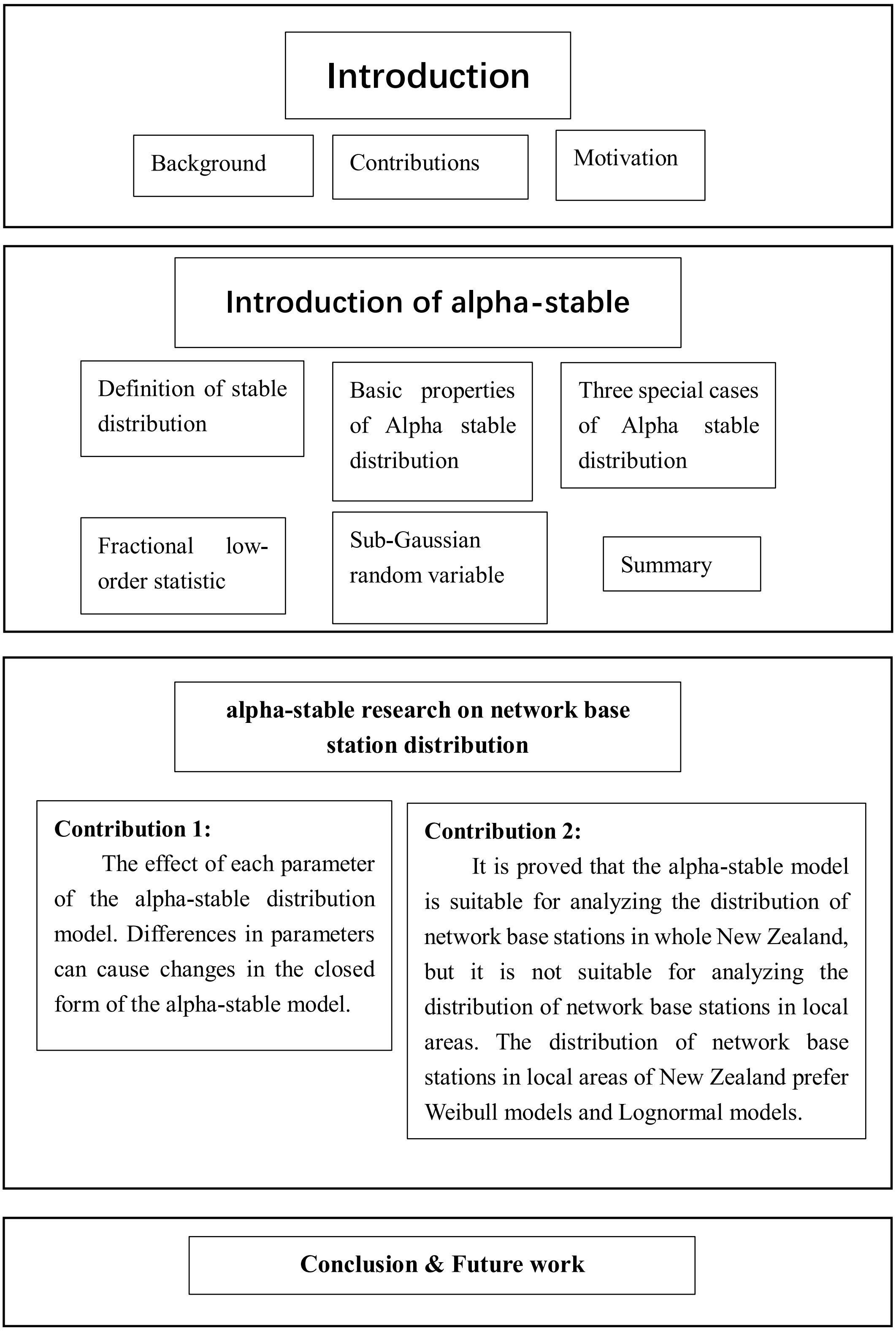

Figure 1.1 Report structure

This project report has four major parts. The first part introduces the overall content of the project, including the motivation of this study and the contribution of this research.

The second part is to introduce the alpha-stable distribution model. The second part introduces the alpha-stable model from the definition of the alpha-stable model itself and the meaning of each parameter in the alpha-stable model. Introduce the property and unique circumstances of the alpha-stable model. Help with better results in later experiments by introducing an alpha-stable model

The third part of this project is the experimental part. Analyze the coordinates of the New Zealand network base station to find out whether the network base station distribution in New Zealand conforms to the alpha-stable distribution model. This project also explores whether the distribution of network base stations in New Zealand meets other candidate distributions. Model. After comparing the experimental results with the experimental results in Italy, and there is a difference between the distribution of network base stations in New Zealand and the distribution of network base stations in Italy. This project also made the reason for the distribution of network base stations in the third part. At the same time, this project also analyzes the impact of New Zealand's network base station distribution on other industries.

The fourth part of the project is the summary. This project report also summarizes the shortcomings in this experiment and the results of the experiment. At the same time, this project report also made corresponding plans for the next experiment to make up for the problems caused by the lack of experiments.

2 Introduction of alpha-stable distribution

Alpha-stable distribution (Prokhorov, 1965) (Samoradnitsky, 2017) (Zolotarev, 1989), also known as non-Gaussian stable distribution, heavy-tail distribution, was initially proposed by P. Lévy in 1925 when studying the generalized central limit theorem, and since then, alpha-stable distribution theory in mathematics Community has been widely valued and developed. alpha-stable distribution is the only one that satisfies the distribution of the generalized central limit theorem, that is, the sum of the random variables with an infinite number of independent distributions with infinite variation, and the limit distribution is alpha-stable distribution (Prokhorov, 1965). alpha-stable distribution is the generalization of Gaussian distribution; that is, Gaussian distribution is only a case of it. Many data that do not satisfy the classical central limit theorem described by stable distribution, so it has a more general scope of application. Unlike the trailing tail with Gaussian distribution with exponential attenuation, the fat tail of the alpha-stable distribution attenuated by square regularity, and the attenuation speed is related to exponential alpha. The smaller the alpha, the slower the attenuation of the distribution. The more substantial the distribution of the heavy- tail. At present, there are three main types of estimation methods for alpha-stable distribution model parameters: Maximum Likelihood method (Maximum Likelihood, ML) (DuMouchel, 1973) (Bodenschatz & Nikias, 1999), Division number method (Quantile) (McCulloch, 1986) and Feature function method (Characteristic Function, CF) (Koutrouvelis, 1980). Theoretically, the maximum likelihood method has good accuracy, but the substantial computation results in significant computation time overhead, so it is not practical. The decimal method needs to be estimated based on the calculated decimal digits and is limited to the estimation of $\alpha \in$,[0.6-0.2]. The feature function method is quick and straightforward to calculate and maintains good estimation accuracy, so the estimation method based on feature function method has been paid more and more attention by researchers at present (Nolan, 1997) (Koyon & Williams, 1995) (Kozubowski, 1999). However, this method is challenging to determine the regression estimation interval of σ, β, and μ of different scale parameters. Because the estimation of Alpha and Sigma needs to use in β and μ estimation, the cumulative propagation of error makes the estimation error of β and μ significant, and because the characteristic function of alpha-stable distribution is discontinuous at $\alpha=1$, this error is particularly severe when $\alpha\rightarrow1$ and $\beta\neq0$. Therefore, it is still a challenging task to study the combined estimation method of four parameters with high precision, small computation, and extensive application range.

The probability density function of alpha-stable distribution (Probability Density Function, PDF) does not have closed expression to the practical application. However, for stable stochastic sequences, Nolan uses the sample feature function to find the inverse Fourier transform (Inverse Fast Fourier Transform, IFFT) method (Nolan, 1997), you can use a sufficient precision numerical calculation method to approximate the PDF, this method is not only applicable to one-dimensional data but also applicable to multidimensional data (Nolan & Rajput, 1995). The other is a progressive estimation method for Symmetric Alpha Stable (SaS), with infinite series approximation method (Shao & Nikias, 1993) and sub-Gaussian (Sub-gaussian) theory (Samoradnitsky, 2017), using a mixture of infinite multiple Gaussian distributions to achieve stable noise PDF progressive estimation method (Kuruoglu, Fitzgerald & Rayner, 1998). Stable distribution and process is an essential branch of probability and stochastic process theory. Stable distribution can describe the effects of small random factors in the distribution of multiple independent identical distribution (IID) random variables (Samoradnitsky, 2017). P. Lévy initially established the invariant theory of stable distribution, and Khinchine in comedies were and further developed by (Gnedenko & Kolmogorovy 1954) and (Prokhorov, 1965). The concept of stable distribution is closely related to the regularity of large numbers and the central limit theorem of probability theory. The regularity of large numbers describes the stability of the sequence of random variables, while the central limit theorem describes the stability of the distribution function, and the stable distribution of alpha is the only distribution that satisfies the stability rate (Nikias & Shao, 1995). The normal distribution, which widely used in natural science and engineering technology, is stable distributed, and normal distribution is only a case in stable distribution, and the theory of stable distribution is an essential basis for the study of self-similar stochastic processes.

The Alpha-stable distribution model has the following two advantages over other models. First, the path fitting can be better. That is because the Alpha-stable distribution model can be adjusted and fitted according to its own needs and data characteristics, and different closed forms are selected. Moreover, the Alpha-stable distribution model has infinite separability, which is convenient for the verification and calibration of parameters after the model established. Second, most Alpha-stable distribution models have their feature functions, which can be used to perform more efficient analysis of data based on feature functions.

This chapter first gives four different definitions of alpha-stable distribution, then discusses the basic properties and fundamental theorems of alpha-stable distribution, compared with the traditional standard distribution statistics, the fractional low-order statistics briefly discussed. A normal distribution is a particular case of alpha-stable distribution, but the general alpha-stable distribution contains four basic parameters, which represent position, scale, peak, and deflection so these parameters can reflect the characteristic quantity in the income distribution of economic and financial time series more flexible. In natural phenomena, many random variables are subject to a normal distribution, traditional economic data analysis and modeling are regarded as general natural phenomena because their inherent Property and characteristics not considered, so often the random variables (such as earnings) are assumed to be compliant with the normal distribution. However, many experiments and studies have shown the inherent characteristics of the economic data game itself. For example, the pattern of heavy peak tail, the price fluctuation of global stock as an example, the logarithmic income of stock price is deviated from the normal distribution, because the stock price appears very small and the frequency of significant changes is much higher than the standard distribution value Mandelbrot (1963). Therefore, it would be better to use a stable model to fit the effect.

2.1 Definition of stable distribution

Probability space is a unique measure space to describe stochastic phenomena, and stable stochastic distribution refers to the probability density function of random variables attenuation at square rate. There are four different ways to define the stability of probability distribution (Samoradnitsky, 2017), of which the first two are related to the stability characteristic, the third is related to the function of stable distribution in the central limit theorem, and the fourth definition gives the characteristic function of the stochastic variable with stable distribution.

Definition 1: Random variable X has a stable distribution, for positive A,B,C and a real number D, such as:

$$\mathrm{AX}_1 + B\mathrm{X}_2 \approx CX + D$$

Was founded X_1, X_2 is a separate sample in X, and "≈" represents the same probability distribution. If D = 0, called X is strictly stable, if X is symmetrically distributed, X is called symmetric stability SaS.

Theorem 1: For any stable random variable X, there are several$\alpha\epsilon(0,2]$, And set the following formula:

$$C^{\alpha} = A^{\alpha} + B^{\alpha}$$

Number A is called the stability characteristic factor (Characteristic Exponent), and the stable random variable X with a factor is called a stable random variable. Like Definition 1, a second definition can be given.

Definition 2: Random variable X has a stable distribution, for$\mathrm{n} \geq 2$Positive$\mathrm{C_n}$ And a real number. $\mathrm{D_n}$ if:

$$\mathrm{X}_1 + \mathrm{X}_2 + 1 - \cdots \mathrm{X}_\mathrm{n} \approx \mathrm{C_n X} + D_n$$

$X_1, X_2, X_3, \cdots, X_n$ it's a separate sample from X, ≈ Represents the same probability distribution. At this point $c_n = n^{1/a}, a \in (0,2]$. The third definition, also known as Generalized Central limit theorem (Samoradnitsky, 2017) (Prokhorov, 1965).

Definition 3: Random variable X has a stable distribution, X exists an attraction domain, event that $Y_1, Y_2, \cdots, Y_n$ is a independent same distribution (independent and identically distributed) Sequence, for positive number sequences $\{d_n\}$ and a sequence of real numbers $\{a_n\}$ if:

$$\frac{Y_1 + Y_2 + \cdots + Y_n}{d_n} + a_n \Rightarrow X$$

The sample "⇒" means "distribution converges". As can be seen, when X is a random variable of the Gaussian distribution, andY_1 When a random variable with finite variance, formula (2.4) is the original expression of the central limit theorem, whend_n=$n^{1/a}$WhenY_1 It's called falling into the normal state of X. The lead fields. Generally speaking d_n=$n^{1/a}$h(n), for h (x), x > 0 is ultimately a slowly changing function: $\lim_{x \to \infty} h(ux) / h(x) = 1$, U>0, such as function h (x) = ln x, when$x \to \infty$, The function changes very slowly.

A feature function gives the fourth definition based on a stable random variable. It is worth noting that the stable distribution of PDFs except for a few exceptions, there is no closed expression, generally using feature functions to describe its distribution characteristics.

Definition 4: Random variable X has a stable distribution if there are parameters: $0 < \alpha \leq 2$, $\sigma \geq 0$, $-1 \leq \beta \leq 1$ and real numbers μ, and has the following feature function form:

$$\mathrm{E} \exp i\theta X = \begin{cases} \exp\{-\sigma^a |\theta|^a \left(1 - i\beta (\operatorname{sign}\theta) \tan\frac{\pi a}{2}\right) + i\mu\theta\}, & a \neq 1 \\ \exp\{-\sigma|\theta| \left(1 + i\beta \frac{2}{\pi} (\sin\theta) \ln|\theta|\right) + i\mu\theta\}, & a = 1 \end{cases}$$

Which $\operatorname{sign}\theta = \begin{cases} 1, & \theta > 0 \\ 2, & \theta = 0 \\ -1, & \theta < 0 \end{cases}$ is a symbolic function. In accordance with the four parameters expressed by such a feature function, we call it a standard parameter system.

2.2 Basic properties of alpha-stable distribution

alpha-stable Distribution $S_a(\sigma, \beta, \mu)$ (Samoradnitsky, 2017) has four parameters: Feature factor α, Scale parameters (Scale Parameter) σ, Skew factor (Skewness Parameter) β, Center position offset parameter (Shift Parameter) μ. A determines the trailing degree of the distribution, σ Represents the scale of the distribution (that is, the degree of dispersion that describes the distribution), β Represents the degree of asymmetry (that is, the degree of deflection) of the distribution relative to its center point, μ Represents the location of the distribution. In general, alpha-stable distribution has the following essential characteristics:

- Distribution attenuation speed and several α about α the smaller it is, the more distributed the heavier the trailing. As shown in the Figure 2.1~2.2;
- Whena ≤ 1, The distribution has infinite mean value and variance;
- Distribution can have asymmetric distribution, parameters β as its skew factor;
- Range of values for parameters: $0 < a \leq 2, \sigma \geq 0, -1 \leq \beta \leq 1$, and μ can be any real number.

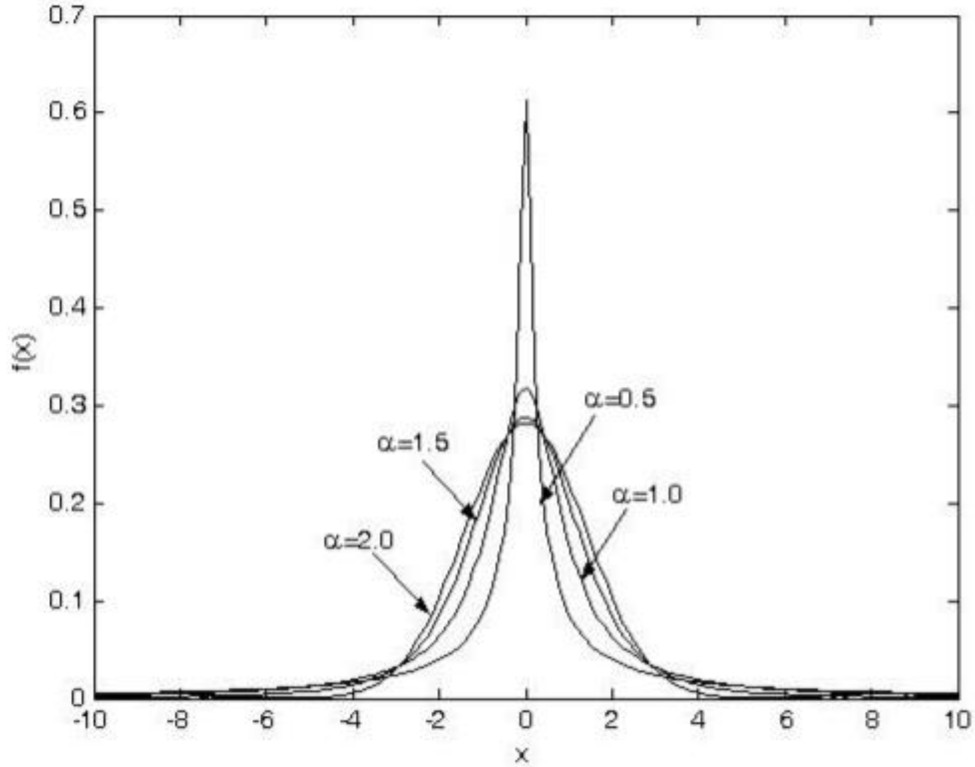

Figure 2.1 Different α only $S_a(1,0,0)$ PDF (Samoradnitsky, 2017)

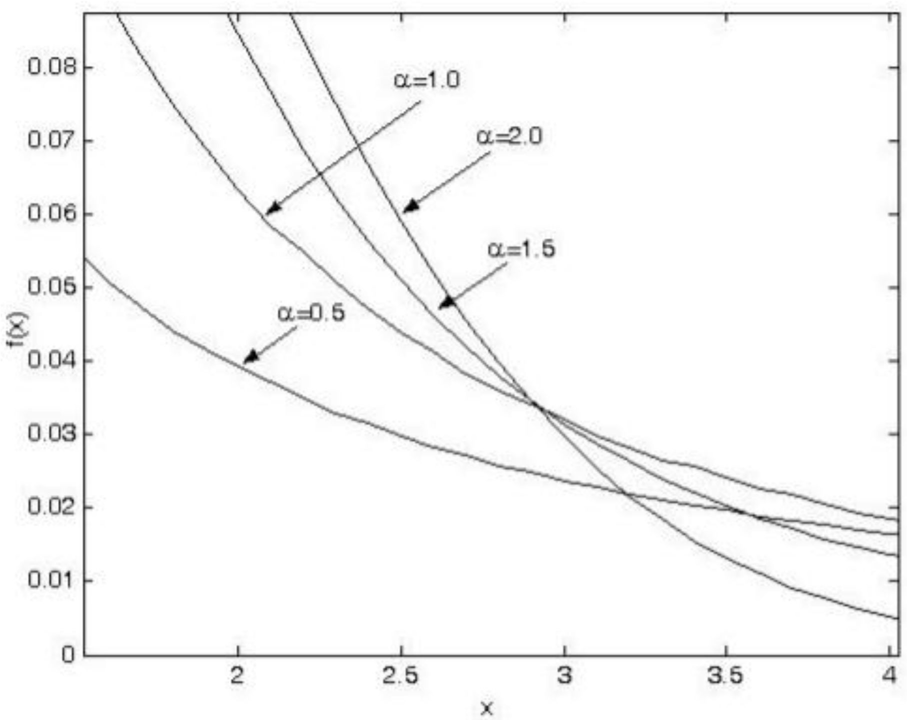

Figure 2.2 PDF trailing degree of different the value of $S_a(1,0,0)$(Samoradnitsky, 2017)

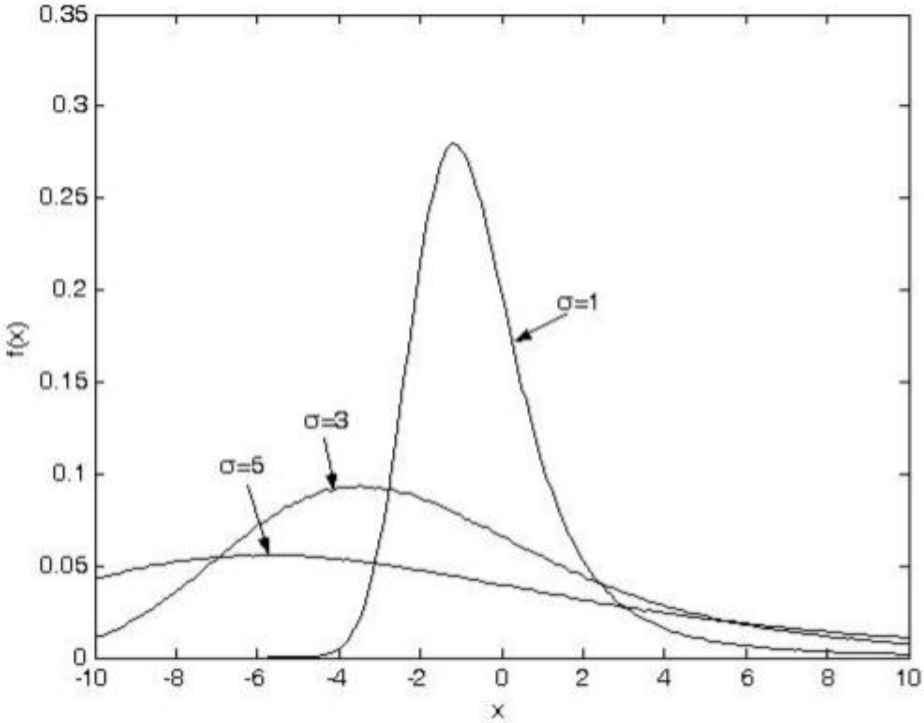

Figure 2.3 Different value of σ and the PDF of $S_{1.5}(\sigma, 1, 0)$(Samoradnitsky, 2017)

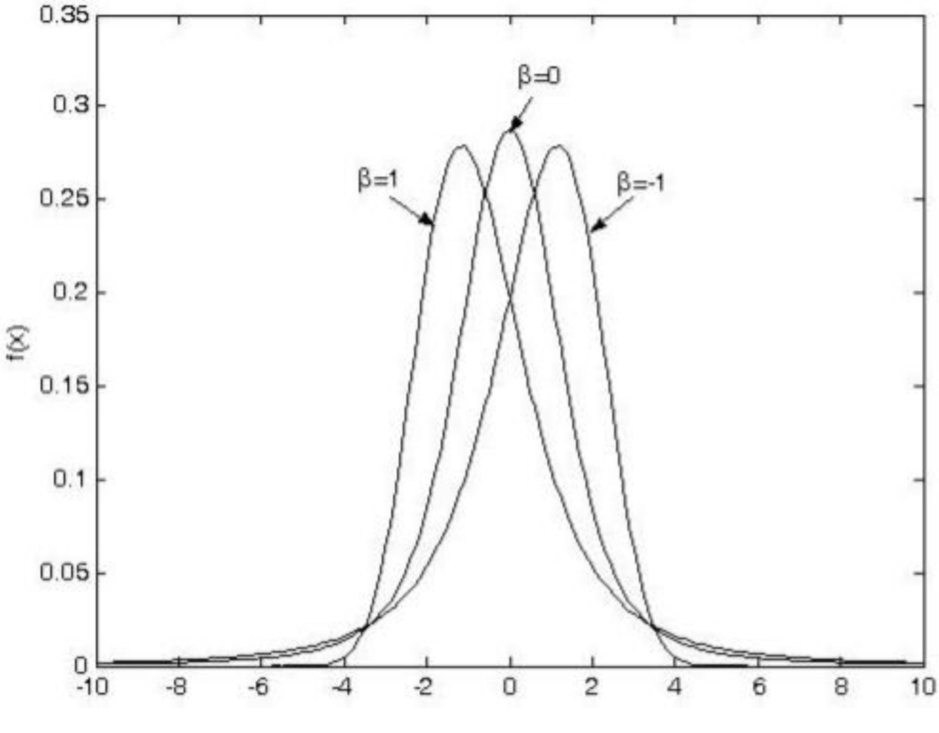

Figure 2.4 Different value of β and the PDF of $S_{1.5}(1, \beta, 0)$(Samoradnitsky, 2017)

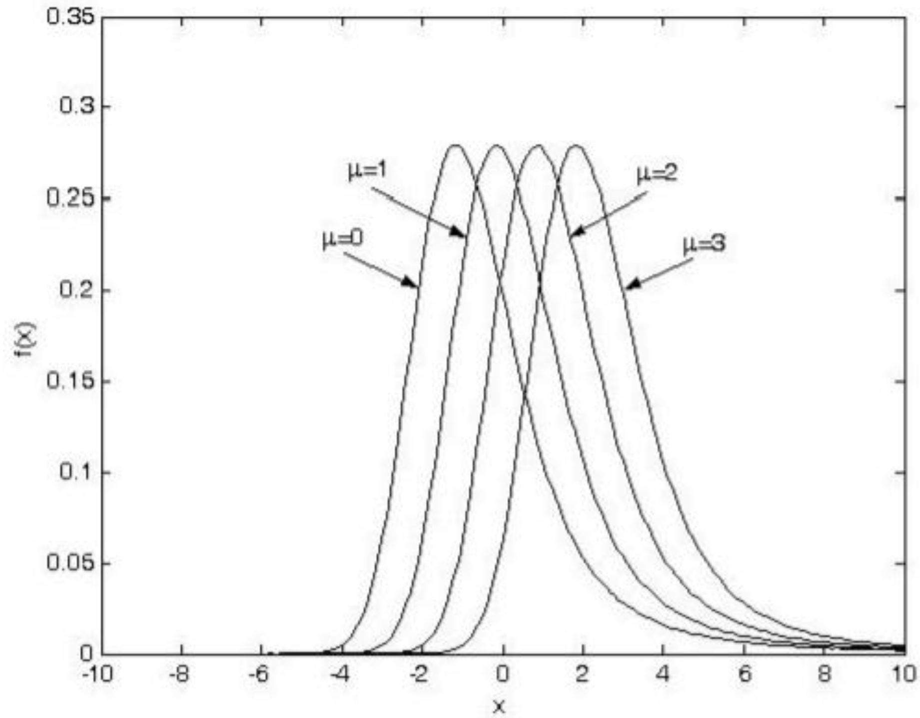

Figure 2.5 Different value of μ and the PDF of $S_{1.5}(1,1,\mu)$(Samoradnitsky, 2017)

The Figure 2.3, the Figure 2.4 and the Figure 2.5 shows a PDF of alpha-stable distribution corresponding to different parameter values, indicating the parameters σ、β and the effect of μ on the stable distribution of Alpha: $\beta = 0$ The distribution is symmetrical, $\beta = -1$ For the left negative deflection distribution, $\beta = 1$ For forward skew distribution (Positive alpha-stable, Ps) (Tsakalides, Trinic & Nikias, 2000),σThe larger the distribution, the greater the degree of dispersion,μControl the location of the distribution. It is worth noting that alpha-stable distribution of PDF and cumulative probability density functions (Cumulative Density Function, CDF) does not exist in closed mathematical expressions, only its feature functions meet the Figure 2.5, Stable distribution $s_a(\sigma, \beta, \mu)$ Has the following basic properties (Samoradnitsky, 2017):

Property 1: If X_1 and X_2 are obedient $x_i \sim S_a(\sigma_i, \beta_i, \mu_i), i = 1,2$ of independent random variables, the

$$X_1 + X_2 \sim s_a(\sigma, \beta, \mu), \text{ And: } \sigma = (\sigma_1^a + \sigma_2^a)^{1/a}, \beta = \frac{\beta_1\sigma_1^a + \beta_2\sigma_2^a}{\sigma_1^a + \sigma_2^a}, \mu = \mu_1 + \mu_2.$$

The Property 1 provides the conditions for the distribution transformation, and the specific transformation will be given in chapter fourth. The properties 2~3 indicate the displacement properties and scale properties of the distribution:

Property 2: If $X \sim s_a(\sigma, \beta, u)$, alpha is a real constant,

then $X + a \sim s_a(\sigma, \beta, u + a)$.

Property 3: If $X \sim s_a(\sigma, \beta, u)$, alpha is a non-zero real constants, then:

$$aX \sim \begin{cases} S_{\alpha(|a|\,\sigma,\, \mathrm{sign}(a)\beta, aU)}, & a \neq 1 \\ S_1\left(|a|\,\sigma, \mathrm{sign}(a)\beta\,, aU - \frac{2}{\pi}a\left(\ln|a|\sigma^{\beta}\right)\right), & a = 1 \end{cases}$$

Property 4 and 5 for the symmetrical properties of the distribution:

Properties 4: For any 0 < a < 2,$X \sim S_a(\sigma, \beta, 0) \Leftrightarrow -X \sim S_a(\sigma, -\beta, 0)$.

Property 5: when and only if$\beta = 0$ And $\mu = 0$ when $X \sim S_a(\sigma, \beta, \mu)$ about 0 symmetrical distribution; when and only when $\beta = 0$, and when X about μ symmetric distribution.

Property 6 and 7 involves skew factor β.

Property 6: If $X \sim S_a(\sigma, \beta, 0)$ and $\alpha < 2$, there are two obedience $S_a(\sigma, 1,0)$ Distributed IID random variable Y1 And2 Meet:

$$x \approx \left(\frac{1+\beta}{2}\right)^{1/a} Y_1 - \left(\frac{1-\beta}{2}\right)^{1/a} Y_2, \text{ when } \alpha \neq 1$$

$$x \approx \left(\frac{1+\beta}{2}\right)^{\frac{1}{a}} Y_1 - \left(\frac{1-\beta}{2}\right)^{\frac{1}{a}} Y_2 + \sigma\left(\frac{1+\beta}{\pi}\ln\frac{1+\beta}{2} - \frac{1-\beta}{\pi}\ln\frac{1-\beta}{2}\right),$$
When $\alpha = 1$

Property 7: When α< 1 and σ take the fixed value $X_\beta \sim S_a(\sigma, \beta. 0)For\beta_1 \leq \beta_2$ and all the X has relationship

$P\{x_{\beta_1} \geq x\} \leq P\{x_{p_2} \geq x\}$ established, and this conclusion for all real values$-1 < \beta < 1$ are established.

Property 8: If $X \sim S_a(\sigma, \beta, \mu)$, 0 < α < 2, there is

$$\begin{cases} \lim_{\lambda \to \infty} \lambda^\alpha P\{x > \lambda\} = C_\alpha \frac{1+\beta}{2} \sigma^\alpha \\ \lim_{\lambda \to \infty} \lambda^\alpha P\{x > -\lambda\} = C_\alpha \frac{1+\beta}{2} \sigma^\alpha \end{cases} \text{ was founded}$$

Which $c_a = \left(\int_0^\infty x^{-a} \sin x dx\right)^{-1} = \begin{cases} \frac{1-a}{r(2-\alpha)\cos(\pi\alpha/2)} & \alpha \neq 1 \\ 2/\pi & \alpha = 1 \end{cases}$

When $X_2 \sim SaS$ and when $P(x > \lambda) \sim \sigma^a \frac{C_a}{2} \lambda^{-a}$

Property 9 to 12 The theoretical basis of fractional low-order statistics is established:

Property 9: If $x \sim S_a(\sigma, \beta, 0)$, And 0 < 2, there are:

For arbitrary$0 < P < a, E|x|^P < \infty$, For arbitrary$p \geq a, , E|x|^P = \infty$.

Property 10: If $x \sim S_a(\sigma, \beta, 0)$ Whena<0< 2 and when $\alpha = 1$ When $\beta = 0$, There are:

For arbitrary$0 < P < a$, There is a constant $C_{\alpha\beta}(P)$ Makes$(E|x|^p)^{1/p} = \sigma C_{\alpha\beta}(P)$, And when

$X_0 \sim S_a(1, \beta, \mu)$When$C_{\alpha\beta}(P) = (E|x|^p)^{1/p}$.

Property 11: if $x \sim S_a(\sigma, \beta, \mu)$ It is

$\min_{r \uparrow \alpha}(\alpha - r)E|x|^r = \alpha C_\alpha \sigma^\alpha$ and $\min_{r \uparrow \alpha}(\alpha - r)EX^{<r>} = \alpha C_\alpha \sigma^\alpha$,

Which $a^{<b>} = |a|^b \text{sign}(a)$.

Property 12 provides an estimated distribution position parameter μ.

Property 12: when $1 < \alpha \leq 2$ is position offset parameter μ Is equal to the median value of the distribution.

For a detailed proof of the above properties, see (Samoradnitsky, 2017). In the proof, the following two propositions are needed. The proposition gives the relationship between the stable distribution and the Poisson distribution and the Laplace transformation equation of the random variable:

Proposition 1 fixed $0 < \alpha < 2$, $\delta > 0$, and N_δ Is an obedience Poisson the mean value of a distributed random variable, which is $EN_\delta = \delta^{-\alpha}$. $Y_{\delta,k}$, $k = 1,2,\cdots$, Is a positive random variable of IID and is independent of the N_δ , Its distribution serviceFrom:

$P(Y_{\delta,k} > \lambda) = \begin{cases} \delta^\alpha \lambda^2, \lambda > \delta \\ 1, \quad \lambda \leq \delta \end{cases}$, The synthetic Poisson Random variables

$X_\delta = \sum_{k=1}^{N_\delta} Y_{\delta,k}$, The distribution when $\delta \rightarrow 0$, converge on $x \sim S_a(\sigma, 1,0)$ And$\sigma^\alpha = \Gamma(1-\alpha)\cos(\pi\alpha/2)$. And X's Laplace transformation.$Ee^{-\gamma^X} = e^{-a^a \gamma^\alpha}$, $\gamma > 0$, when $\alpha > 0$, and

$$a^\alpha = \Gamma(1-\alpha) = \sigma^\alpha/\cos(\pi\alpha/2) \quad .$$

Proposition 2: random variables $X \sim S_\alpha(\sigma, \beta, \mu)$, $0 < \alpha \leq 2$, $\sigma > 0$ of the Laplace transforms $Ee^{-\gamma^X}$, $\gamma \geq 0$ To meet the following relationships:

$$Ee^{-\gamma^X} = \exp\left\{-\frac{\sigma}{\cos\frac{\pi\alpha}{2}} y^\alpha\right\}, \text{when } \alpha \neq 1$$

$$Ee^{-\gamma^X} = \exp\left\{\sigma \frac{2}{\pi} \gamma \ln \gamma\right\}, \text{when } \alpha = 1$$

In addition to the special exceptions of Gaussian distribution, Cauchy distribution and Levi distribution, the alpha-stable distribution is difficult for theoretical analysis and practical application due to the lack of closed expression of PDF and CDF. The true PDF can only be replaced by a progressive approximation expression. The two theorems below provide a way to represent the asymptotic representation.

Theorem 2: A PDF with a standard alpha-stable distribution $(\gamma = 1, \mu = 0)$, when

$x > 0$, can be progressively represented by a convergent series of stages as (Shao & Nikias, 1993):

$$
f(x;\alpha,\beta)=\begin{cases}\frac{1}{\pi x}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k!}\Gamma\,(\alpha k+1)^{-\alpha K}\sin\left[\frac{k\pi}{2}(\alpha+\xi)\right], for\ 0<\alpha<1\\ \frac{1}{\pi x}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k!}\Gamma\left(\frac{k}{\alpha}+1\right)^{-\alpha K}\sin\left[\frac{k\pi}{2\alpha}(\alpha+\xi)\right], for\ 1<\alpha\leq 2\end{cases}
$$

the $\eta=\beta\tan(\pi\alpha/2)$,$r=(1+\eta^2)^{-1/(2\alpha)}$, $\zeta=-(2/\pi)\arctan\eta$,

$\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}\,dt$ is gamma function.

Theorem 3: If the alpha-stable distribution is the standard symmetric distribution of SaS, then the PDF is

$$
f(x)=\begin{cases}\frac{1}{\pi x}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k!}\pi\alpha k+1, x^{-\alpha k}\sin\left[\frac{k\alpha\pi}{2}\right], & for\ 0<\alpha<1\\ \frac{1}{\pi(x^2+1)}, & for\ \alpha=1\\ \frac{1}{\pi\alpha}\sum_{k=0}^{\infty}\frac{(-1)^{k}}{2k!}\Gamma\left(\frac{2k+1}{a}\right)x^{2k}, & for\ 1<\alpha<2\\ \frac{1}{2\sqrt{\pi}}\exp[-x^2/4], & for\ \alpha=2\end{cases}
$$

SaS stable distribution progressive approximation

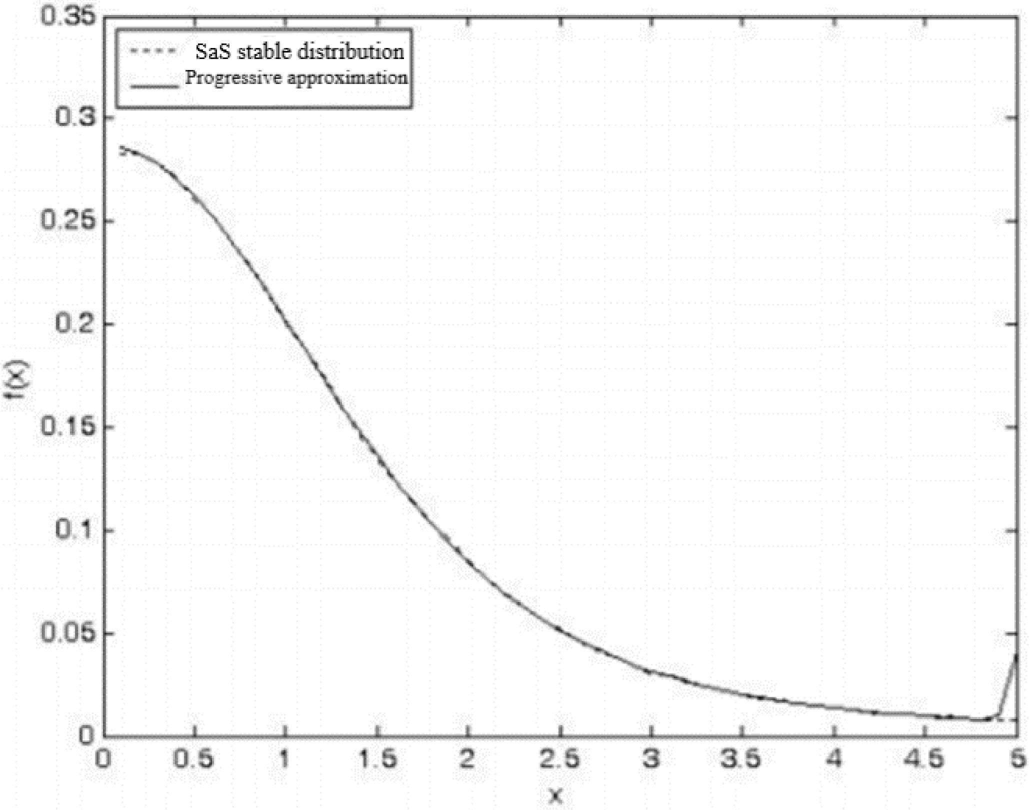

Figure 2.6 progressive estimation of infinite series$S_{1.5}(1,0,0)$(Samoradnitsky, 2017)

However, the above method of series approximation in practical application series the order of the number is impossible to Take infinity, if the finite item been taken, the value of the order is difficult to determine, usually determined by experiments, so it has considerable randomness. The method of progressive approximation of infinite series for smaller X Values can Satisfactory approximation is obtained, while the tail deviation of the distribution is significant (the series is no longer convergent), as shown in the Figure 2.6, the simulation results of the progressive estimation $S_{1.5}(1,0,0)$ are more and more skewed when x > 4.8.

2.3 Three special cases of alpha-stable distribution

The stable distribution $S_\alpha(\sigma, \beta, \mu)$ can be divided into two major categories: symmetric and asymmetric. When the skewness factor $\beta = 0$, the distribution of $S_\alpha(\sigma, \beta, \mu)$ is symmetric with respect to μ for SaS. Here are some specific probability distributions:

2.3.1 Gaussian distribution

$S_{1/2}(\sigma, 0, \mu) = N(u, 2\sigma^2)$, Density function $f(x) = \frac{1}{2\sigma\sqrt{\pi}} e - (x - \mu)^2 \,/\, 4\sigma^2$, so the Gaussian distribution can take specific parameters through stable distribution$(a = 2, \beta = 0)$And obtained, the Gaussian distribution belongs to SaS .

2.3.2 Gauchy distribution

$S_1(\sigma, 0, \mu)$, Density function $f(x) = \frac{\sigma}{\pi((x-\mu)^2+\sigma^2)}$ if $X \sim S_1(\sigma, 0,0)$, for any x>0 have $P(X \leq x) = \frac{1}{2} + \frac{1}{\pi} \text{Arc} \tan \frac{x}{\sigma}$

2.3.3 Levy distribution

$S_{1/2}(\sigma, 1, \mu)$, Density function

$f(x) = \left(\frac{\sigma}{2\pi}\right)^{1/2} \frac{1}{(x-\mu)^{3/2}} \exp\left\{-\frac{\sigma}{2(x-\mu)}\right\}$ in (u, ∞) on the convergence, if $X \sim S_{1/2}(\sigma, 1,0)$, for any $x > 0$ have

$P(x \leq x) = 2\left(1 - \phi\left(\sqrt{\frac{\sigma}{\pi}}\right)\right)$. Which Φ for the standard normal distribution of the probability distribution function CDF.

2.4 Fractional low-order statistic

The classical statistical signal processing theory hardly shakes off the Gaussian model and the second-order statistical theory. On the premise that the error or noise obeys the stable distribution, the stable signal processing based on the fractional low-order statistic FLOS is very different from the Gaussian signal processing method based on the integer moment. The latter uses the minimum mean square error (Minimum Mean Square Error, MMSE) is the criterion, while the former based on the minimum dispersion MD (Shao & Nikias, 1993). Stable signal processing is suitable for processing noise with sharp spikes and used in some areas for applications superior to Gaussian signal processing. Commonly used fractional low-order statistics have fractional low-order, covariant, and alpha spectra.

2.4.1 Fractional lower order

Although $0 < \alpha < 2$ of SaS of the second-order moment of a random variable does not exist, but all the fractional moments below a are present., it's called Fractional Low order Moment FLOM (Shao & Nikias, 1993), FLOM can be made from deviation (Dispersion) $\gamma = \sigma^{\alpha}$ And characteristic factors α Characterization.

Theorem 4 if X ~ SaS Is

$$E|x|^{P} = C(P,\alpha)\sigma^{P} = C(P,\alpha)\gamma^{\frac{p}{\alpha}}$$

Which $C(P,\alpha) = \dfrac{2^{P+1}\Gamma\left(\frac{P+1}{2}\right)\Gamma\left(-\frac{p}{2}\right)}{\alpha\sqrt{\pi}\Gamma\left(-\frac{p}{2}\right)}$

If X~SaS, Then the norm of X is defined as

$$\|x\|_{a} = \begin{cases} \sigma, & 1 \le \alpha \le 2 \\ \sigma^{\alpha}, & 0 < \alpha < 1 \end{cases} = \begin{cases} \gamma^{1/\alpha}, & 1 \le \alpha \le 2 \\ \gamma, & 0 < \alpha < 1 \end{cases}$$

Therefore, the norm $\|x\|_{\alpha}$ is distance difference of γ, a measure of scale. For two random variables X, Y, if all are the SaS Random variable, the distance measure defined by the norm can be obtained $d_{\alpha}(X,Y) = \|X - Y\|_{\alpha}$ by two formulas in theorem 4It is not difficult to see:

$$d_a(X,Y) = \begin{cases} \left(E|x-y|^p/C(p,\alpha)\right)^{1/p}, for\ 0 < P < \alpha, 1 \le \alpha \le 2 \\ \left(E|x-y|^p/C(p,\alpha)\right)^{\alpha/p}, for\ 0 < P < \alpha, 0 < \alpha < 1 \end{cases}$$

2.4.2 Covariation and common difference

Since the limited variance does not exist, it SaS the covariance of a random variable also does not exist. For two random variables X, Y, if all are SaS random variables, when $1 < a \le 2$, If the deviation is γ_x And γ_y, The total change of covariant definition (Samoradnitsky, 2017) For:

$$[X,Y]_{\alpha} = \frac{E(XY^{<p-1>})}{E(|Y|^P)} y_X$$

For real number Z,$z^{<a>} = |z|^{\alpha} \text{sign}(z),\ \alpha \ge 0$. Obviously$[X,Y]_{\alpha} = \gamma_X = \|X\|_{\alpha}^{\alpha}$, $[Y,Y]_{\alpha} = \gamma_y = \|Y\|_{\alpha}^{\alpha}$, The co-variable coefficient can therefore be defined as:

$$\lambda_{x,y} = \frac{[x,y]_{\alpha}}{[x,y]_{\alpha}} = \frac{E(XY^{<p-1>})}{L = (|Y|^P)}, 1 \le P < a$$

If X, Y independent, there is $[X,Y]_{\alpha} = 0$, But the opposite is not true.

Like common change, the difference (Samoradnitsky, 2017) Provides an alternative two stable measure of the relationship of a random variable, for two random variables X, Y, if all are SaS Random variables, when $0 < a \le 2$ WhenCommon-difference definitionFor:

$$\tau_{xy} = \|X\|_{\alpha}^{\alpha} + \|Y\|_{\alpha}^{\alpha} - \|X - Y\|_{\alpha}^{\alpha}$$

The co-difference has the Property of symmetry, that is, $\tau_{Xy} = \tau_{Y,X}$, and when $\alpha = 2$, Co-difference with the covariance is the same, $\tau_{Xy} = \text{Cov}(X,Y)$. If X, Y independent, there is, $\tau_{xy} = 0$ Conversely, if, $\tau_{xy} = 0$ and $0 < a < 1$, X, Y is independent.

2.4.3 Alpha spectrum

Consider a stable noise sequence x_n Output of the excited FIR filter Y_n, The general form of its co-transformation is $[Y_n, Y_n]_\alpha$, Which

$$W_n = \sum_{i=-q}^{q} a_i Y_{a-i}$$

The $\{\alpha_i\}$ is an arbitrary constant, the Y_n and W_n of the co-transformation is:

$$[Y_n, w_n]_a = \gamma_x \sum_{k=\vartheta}^{9} h_k \left(\sum_{i=0}^{q} a_{k-i} h_i \right)^{<\alpha-1>}$$

h_k For the impact response of the Finite Impulse Response (FIR) filter, the α Spectrum (Tsihrintzis & Nikias, 1995) is defined as:

$$S_a(z) = [Y_n, W_n(z)]_\alpha = \gamma_x H\left(\left(\frac{1}{z} \right)^{<\alpha-b>} \right) \left(H(Z) \right)^{<\alpha-b>}$$

Which $w_n(2) = \sum_{i=-q}^{q} Y_{n-i} z^i$ output for the filterY_nThe window Z transform, $H(z)$ Is the transfer function of the filter. So, by the α amplitude response and phase ringing of the filter can be obtained by spectrum (Ma & Nikias, 1995).

2.5 Sub-Gaussian random variable

The following theorems illustrate any SaS the random variables can be represented as the composite of Gaussian random variables, in which the variance of Gaussian random variables obeys the stable distribution of A/2 with forward skew. This type of random variable is also known as the sub-Gaussian random variable (Sub-Gaussian Random Variables) (Samoradnitsky, 2017) (Ma & Nikias, 1995).

Theorem 4: a random variable that obeys SaS $Z \sim S_\alpha(\sigma, 0, 0)$, $\alpha < 2$ Can be expressed as $Z = \sigma G S^{1/2}$, Which $G \sim N(0, \sigma^2)$, $S \sim S_{\alpha/2}\left(\left(\cos\frac{\pi\alpha}{4}\right)^{2/\alpha}, 1, 0\right)$, S obey the positive deflection of the$\alpha/2$Stable distribution.

The above theorem also provides a progressive representation of a stable noise PDF, Kuruoglu A similar hybrid high is derived from Theorem 4. Approximate form of (Kay, 1998) (Kuruoglu, Fitzgerald & Rayner, 1998):

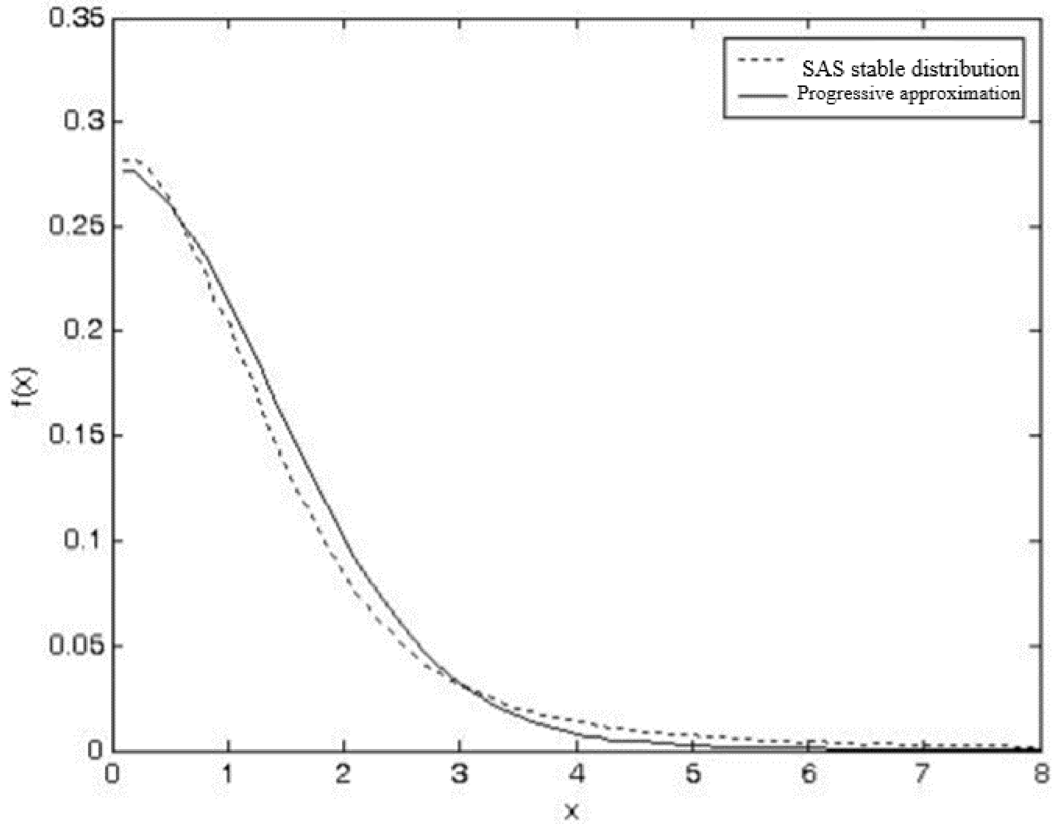

Figure 2.7 Asymptotic estimation of multi-Gaussian mixture $S_{1.5}(1,0,0)$(Samoradnitsky, 2017)

Theorem 5 If $Z \sim S\alpha S$, It is

$$f_z(z; \alpha, \gamma, \mu) = \frac{1}{\sqrt{2}\pi} \int_{-\infty}^{\infty} \exp\left[-\frac{(2-\mu)^2}{2yv^2}\right] f_v(v) v^{-1}\, dv$$

The discrete forms are:

$$\mathrm{fz}(\mathrm{z};\alpha,\gamma,\mu)\frac{\sum_{i=1}^{N}\frac{1}{\sqrt{2\pi v_i}}\exp\left[-\frac{(z-\mu)^2}{2yv_i^2}\right]f_v(v_i)}{\sum_{i=1}^{N}f_v(v_i)}$$

Which $V=Y^{1/2}$, $Y\sim S_{a/2}\left(\left(\cos\frac{\pi\alpha}{4}\right)^{2/\alpha},1,0\right)$ obey α / 2 Steady distribution of forward skew $p\frac{\alpha}{2}S$. However, the accuracy of the approximation of the method is limited, and there is still the problem of approximation option selection. The Figure 2.7 shows the progressive estimation of the method $S_{1.5}(1,0,0)$Simulation results.

If the stochastic process $\{x(t),t\in T\}$ The finite dimension distribution obeys the stable distribution, the stochastic process is called the stabilization process. Therefore, if the finite dimension distribution of the stochastic process is subject to the sub-Gaussian distribution, the stochastic process is called the secondary Gaussian process (Sub-Gaussian Processes) (Samoradnitsky, 2017). There are three basic types of stabilization processes: the sub-Gaussian process, the linear stabilization process, and the harmonic (Harmonizable) stabilization process. The Gaussian process is both harmonic and linear, but the stability process does not have such a property, in which the sub-Gaussian process is more special, it is neither harmonic nor nonlinear (Shao & Nikias, 1993).

Table 1: Comparison of stable distribution and Gaussian distribution statistics

Statistic Amount	**SaS Random variables**	**Gaussian random variables**
Degree of dispersion	Distance difference $\gamma=\sigma^\alpha$	Variance σ^2
Moment measure	Scores Low order Moment	Integer Order Moment
Related metrics	Co-transformation-aberration	Covariance
Spectral metrics	α Spectrum	Power spectrum, high order spectrum

2.6 Summary

This chapter first discusses the four definition forms of alpha-stable distribution, explain what the stability of the distribution and the generalized central limit theorem is, and point out that the central limit theorem is a particular case of the generalized central limit Theorem when $\alpha=1$. This chapter also discusses the basic properties and fundamental theorems of alpha-stable distribution. It pointed out that Gaussian distribution, Cauchy distribution. Levy distribution are three exceptional cases of alpha-stable distribution, and the statistics describing stable distribution are very different from those of Gaussian distribution, so four kinds of fractional low-order statistics discussed, that is, fractional low order moment, co-change, Co-aberration, and Alpha Spectrum; Finally, the basic concepts of sub-Gaussian stochastic variables and stabilization process briefly introduced.

Alpha's stable distribution of PDFs lacks closed expression, so regular approximation expressions can only replace the real PDF. There are two approaches to the gradual approximation of PDF: The method of infinite series approximation and multi-gaussian mixing, and people can study the performance of the two methods through simulation. Although the approximation degree of infinite series approximation is better than that of multi-Gaussian mixing in the low-value region, a satisfactory approximation cannot obtain in the high-value region. In practical application, there approximate both. Order is challenging to Identify the problem.

The alpha-stable distribution contains. Gaussian distribution, the statistics of the two are analogical, when $\alpha=2$, the statistical amount of stable distribution is equal to the statistics of Gaussian distribution. In order to facilitate the understanding of fractional low-order statistics, compared with the traditional Gaussian system, and make a summary comparison, as shown in table 1. The content of this chapter establishes the theoretical basis for the study of subsequent chapters.

The above is to describe in detail alpha-stable distribution, and only after understanding the distribution can a useful analysis be carried out in subsequent experiments. The results of the experiment need to be combined, alpha-stable the definition property of distribution and the meaning of parameters can be analyzed in more depth.

3 Case studies in New Zealand

The deployment of network base stations always updates because of the market demand for base stations. With the arrival of the 5G period, the number of 5G base stations much larger than the 4G base stations. Hence, how to deploy 5G base stations more accurately becomes essential research. Now 4G networks technology is matured, and the number of 4G base stations can fit the requirements of mobile users around the world. Although the current 4G network is still not perfect in some isolated areas, the 4G signal in the urban environment can fully meet the requirements of users. Therefore, people can use the 4G cellular network base station deployment to study the deployment of 5G cellular network base station.

The deployment of cellular base stations does not follow a theoretical model such as the Figure 3.1 due to the influence of random events, the shape of the cellular base station after deployment is irregular in the actual situation, and the reason for that may be due to the terrain or the number of users. Those reasons are often randomly generated and complex. It is theoretically impossible to understand the causes of this situation entirely. However, people still need a model to predict the space deployment of network base stations. Therefore, a mathematical model with extremely high convergence is necessary. The traditional Poisson distribution model needs to assume that the deployment of base stations is entirely random, but actual base station deployments are rarely entirely random. Based on Poisson distribution, Zhou et al. (2014) pointed out that the alpha-stable distribution model is a better choice for network base stations deployment problem.

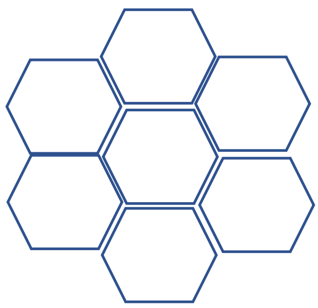

Figure 3.1 the shape of cell towers radiation range

4G and 3G are the most popular cellular network technology in New Zealand. The cellular network base station distribution is affected by different communication companies. Some communication companies such as Vodafone have strong signals in urban areas, while in some non-urban areas, signals are weak or completely no services. This situation is not conducive to the development of New Zealand network communication technology. The cause of this situation is the difference between the number of users in a different area. It is easier to get income from urban areas, but hard to get income non-urban areas. That makes base station deployment difficult in non-urban areas. Therefore, a reasonable profit from the base station cannot recover the cost. That cause a decline in the quality of service of some New Zealand companies. The advent of the 5G period is not only the innovation of network communication technology but also the turning point of other industries. In the future, many services and equipment will depend on 5G technology. If the New Zealand government do not pay attention to 5G technology development, the other industries will be affected in the future.

3.1 Architecture evolution of network base station

The mobile communication system has gradually developed from the first-generation mobile communication system (1G) and has now developed to the fourth-generation mobile communication system (4G). The fifth-generation mobile communication system (5G) also be standardized. It is expected to invest in 2020. Therefore, understanding the evolution of a network base station is necessary for the network base stations distribution analyzing. The reason for the current situation of the distribution of network base stations is technological innovation. This section explores the advantages and disadvantages of different generations of network base stations. This section aims to help researchers better understand the changes of network base station distribution.

3.1.1 2G network

he 2G communication system adopts a 3-level network architecture, namely: BTS-base stationC-core network. The 2G core network includes both a CS domain (Circuit Switched) and a PS domain (Packet Switch).

The 2G communication system initially used an integrated base station architecture. The integrated base station architecture displays in the Figure below (the Figure 3.2). The antenna of the base station located on the tower and the rest is in the equipment room next to the base station. The antenna connects to the indoor computer room through a feeder.

The integrated base station architecture needs to establish a machine room under each tower, which has a significant construction cost and a long period and is not convenient for the expansion of the network architecture.

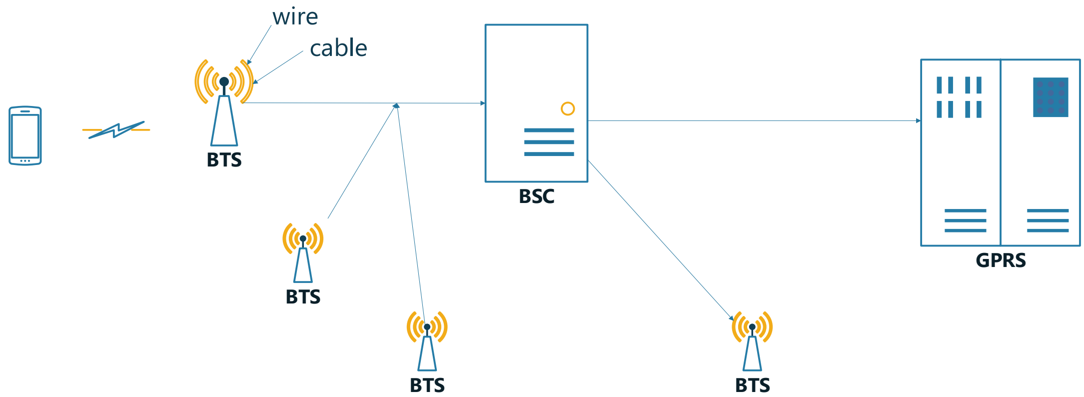

Figure 3.2. The structure of the 2G network

The 2G network developed into a distributed base station architecture. The distributed base station architecture divides the BTS into RRUs and BBUs. The RRU is mainly responsible for radio frequency related modules, including four modules: IF module, transceiver module, power amplifier, and filtering module. The BBU is mainly responsible for baseband processing and protocol stack processing. The RRU located on the tower and the BBU is in the indoor equipment room. Each BBU can connect multiple (3-4) RRUs. A fiber connection used between the BBU and the RRU.

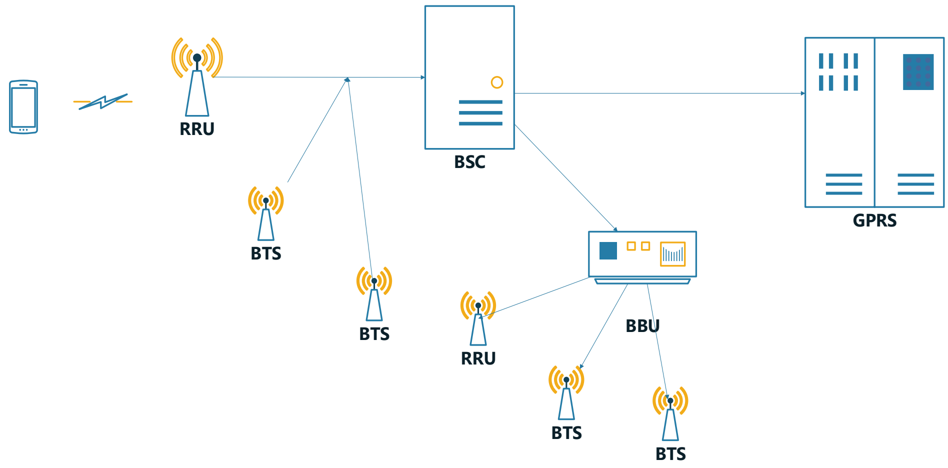

Figure 3.3 The structure of 2G network second vision

3.1.2 3G network

When developing 3G networks, in order to save network construction costs, the 3G network architecture is consistent with 2G.

The 3G communication system also uses a 3-level network architecture, namely Node B – RNC - Core Network. The 3G core network includes both the CS domain and the PS domain.

The 3G era mainly uses a distributed base station architecture. Similarly, the distributed base station architecture divides the Node B into two parts, a BBU, and an RRU.

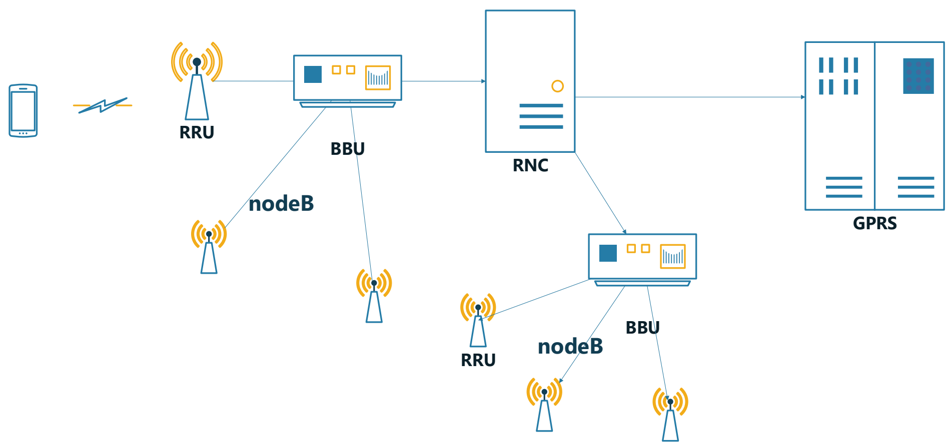

Figure 3.4 The structure of the 3G network

3.1.3 4G network

When the 4G era arrived, the base station architecture changed a lot. To reduce end-to-end latency, 4G uses a flat network architecture. The original 3-level network architecture is "flattened" to level 2: eNode B-core network. Part of the RNC's functionality can split into eNode Bs and part of it is moved to the core network. The 4G core network only contains the PS domain.

The 4G base station adopts the architecture of the distributed base station. At the same time, the C-RAN architecture proposed and promoted by China Mobile has gradually been promoted. The C-RAN architecture further centralizes, cloud, and virtualizes the functions of the BBU. Each BBU can connect 10 to 100 RRUs, further reducing the network deployment cycle and cost.

Unlike traditional distributed base stations, C-RAN breaks the fixed connection between the remote radio unit and the baseband processing unit. Each remote radio unit does not belong to any of the baseband processing unit entities. The processing of transmitting and receiving signals on each remote radio unit performs at a virtual baseband base station, and the processing capability of the virtual base station is composed of a part of processors in the baseband pool allocated by the real-time virtual technology.

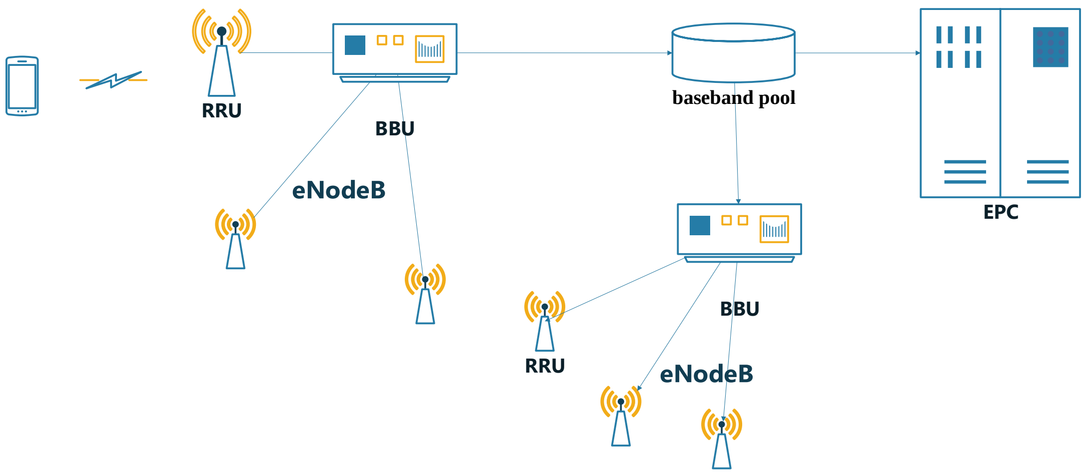

Figure 3.5 The structure of the 4G network

3.1.4 5G network

In order to further improve the flexibility of the 5G mobile communication system, 5G adopts a level 3 network architecture, a level DU-CU-core network (5GC). The DU and the CU together form a next-generation NodeB (gNB), and each CU can connect to one or more DUs. There are multiple functional splitting schemes between the CU and the DU, which can adapt to different communication scenarios and different communication requirements.

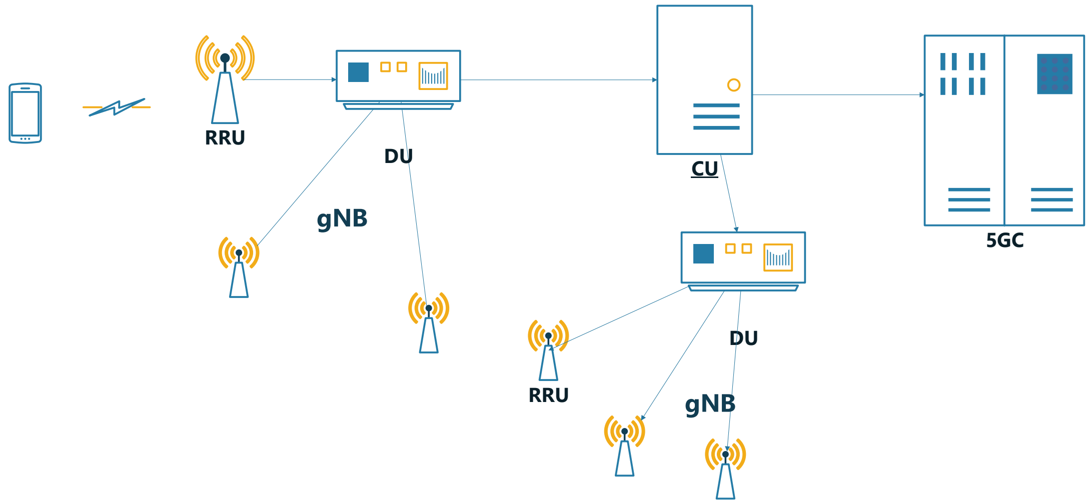

Figure 3.6 The structure of the 5G network

The update of different generation network base stations has caused users to change the network requirements. These transformations will affect the distribution of network base stations. Considering the difference between the number of urban users and non-urban users, the cost of equipment deploying is different. The services will prioritize the deployment of more advanced technologies and equipment in urban areas. That will lead to slow development or use the previous generation technology and equipment in the non-urban areas, plus advanced technologies and equipment development in the urban area. Therefore, the total number of network base stations in urban areas will be higher than the total number of network base stations in non-urban areas. Researchers can use the alpha-stable distribution model to validate the opinion further.

3.2 Data discretion

This project focused on the distribution of network base stations in New Zealand. The mobile network of New Zealand mainly owned by two different cellular operators (Spark, Vodafone). As shown in table 2

Table 2: The user number of the network operator

Operator	User number of operators
Spark	2,390,000
Vodafone	2,490,000

They are considering that the total number of deployed base stations in the entire operator set exceeds 80,000 base stations. This project report will focus on the distribution of base stations of the whole New Zealand and different parts of New Zealand. Then, by studying the base station distribution in different regions, it can verify whether the alpha-stable distribution model conforms to the base station distribution in New Zealand. This project divided the total number of base stations in New Zealand into two main parts by region. The first part is the base station distribution of North Island, and the second part is the base station distribution of South Island. Then this project studied the base station distribution in major New Zealand cities. For example, researchers chose Auckland and Wellington. Those cities located on North Island. Then chose Christchurch as an example of South Island. The alpha-stable distribution model is analyzed by studying large areas and small areas.

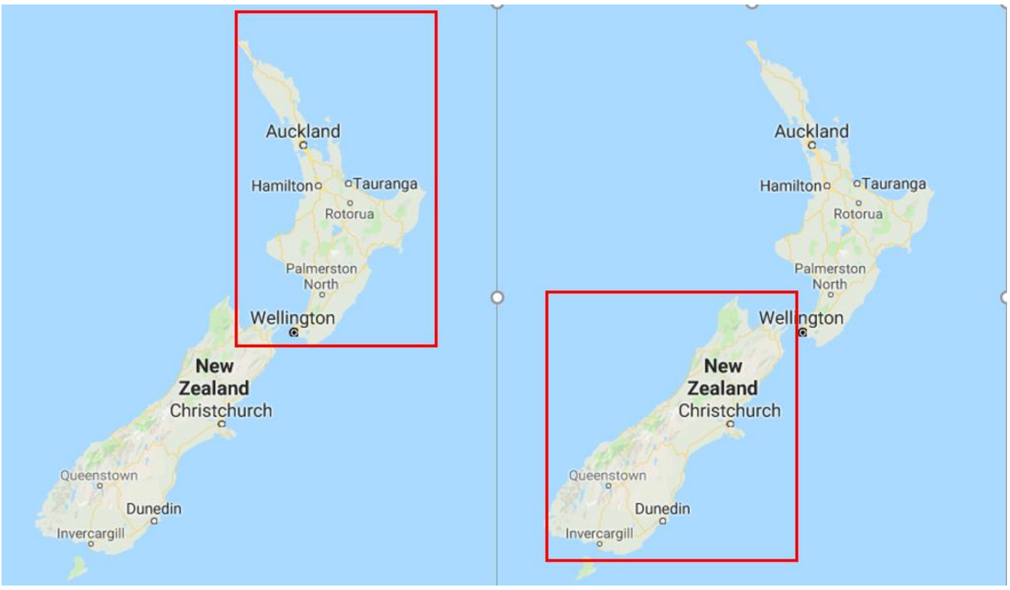

Figure 3.7 New Zealand dataset: Base station location (in a rectangular box). Left: Distribution of base stations in the North Island of New Zealand. Right: Base station distribution in South Island, New Zealand

This work interested in the spatial distribution of the base station in different fields. This project first selected the North Island region, as displayed on the left side of the Figure 3.7, and there are over 1,000 square kilometers. In this area, approximately 20,000 network base stations are deployed, which is equivalent to about 25% of the total number of all New Zealand base station number. The resulting average base station density is equal to 0.05/km^2. Besides, researchers also picked a set of data from Auckland for analyzing. The South Island of New Zealand is mainly composed of rural and small towns, as shown in the Figure 3.7 right. The GPS coordinates of the base station are available for each region of New Zealand. In order to obtain the spatial position of the base station, this project involved the Universal Transverse Mercator (UTM) conformal projection (Grafarend, 1995). In this way, GPS coordinates mapped to the surface.

3.3 Candidate distributions

The project now focuses on the theoretical statistical distribution and simulating the spatial distribution of base stations in the area. The distribution data was downloaded from OpenCelliD (an open source data base), and the dataset of New

Zealand has no missing data and data imbalance problem. Therefore, data classify is the only action before the data analysis.

Many of the phenomena in the telecommunications industry can be described by heavy-tailed distribution, including remote traffic statistics (Resnick, 1997) and Internet topologies (Faloutsos, Faloutsos & Faloutsos, 1999). At the same time, there are also many statistical distributions that prove to be heavy tails. Among them, the Generalized Pareto (GP) distribution, the Weibull distribution, and the Log-normal distribution belong to a closed distribution with a Probability Density Function (PDF) (see Table 2). Another well-known heavy-tailed distribution is the alpha-stable distribution, which reflected the independent distribution of random variables with the ability to characterize normalized and distributed distributions (Samoradnitsky, 2017). However, the alpha-stable distribution with a few exceptions lacks a final form of PDF, which is usually specified by its feature function.

Table 3: List of candidate distributions.

Distribution	**Probability Density Function**
generalized Pareto (GP)	ax^{-b}
Weibull	$abx^{b-1}e^{-ax^b}$
Log-normal	$\frac{1}{\sqrt{2\pi b^*}}e^{-\frac{(\ln x-a)^2}{2b^2}}$
alpha-stable	Closed form not always exists. Shown in part 3
Poisson	$\frac{\lambda^k}{k!}e^{-\lambda}$

According to the above mentioned, if there are parameters $0 < \alpha \leq 2$, $\sigma \geq 0$, $-1 \leq \beta \leq 1$ and $\mu \in R$ such that their characteristic functions are as follows, the random variable X is considered to be formed by the α stable distribution:

$$\begin{aligned}\phi(w) &= E(\exp j\omega x) \\ &= \exp\{-\sigma^{\alpha}|w|^{a}(1 - j\beta(\operatorname{sgn}(w))\phi) + j\mu\omega\}\end{aligned}$$

The ϕ is given by:

$$\phi = \begin{cases} \tan\frac{\pi\alpha}{2}, & \alpha \neq 1 \\ -\frac{2}{\pi}\ln|\omega|, & \alpha = 1 \end{cases}$$

in this case, the function E represents the desired operation concerning a random variable. α is called a characteristic index and is expressed as an index of stability, and β is a skewness parameter. Together, α and β determine the specific shape of the model. Also, σ and μ are scale and shift parameters, respectively. Specifically, if α = 2, the α stable distribution is shown as a Gaussian distribution.

3.4 Case study result

Through analyzing the data set, the network base station distribution in New Zealand is in line with the alpha-stable distribution. The analysis experiment not only used New Zealand's national network base station coordinate data for analysis but also analyzed the location coordinates network base station (North and South Island). Secondly, this analysis also used three major New Zealand cities (Auckland, the network base station coordinate data of Wellington and Christchurch were analyzed. Two of the three cities located on the North Island of New Zealand, one city located on the South Island of New Zealand). This project expects to verify that the alpha-stable distribution model fits all regions of New Zealand network base station coordinate data and fit the different regions of New Zealand. However, since the experimental network base station data comes from the location users upload, the data set is lack accuracy. Therefore, in some cases, no valid experimental results.

The Figure below (the Figure 3.8) is the result obtained by analyzing the network base station coordinate data of New Zealand. The experimental results show that the network base station coordinates distribution following the alpha-stable distribution model. According to the following Figure 3.8, we can also see that in addition to the Poisson distribution model, the distribution of network base stations in New Zealand is also in line with the other candidate distribution models. Therefore, the Root Mean Square Error (RMSE) has been conducted to further measure the accuracy of each model.

The sample area range used in the Figure 3.8 is 4x4 km^2. That is, if the land area divided into small areas of 16 square kilometers, the overall distribution of network base stations in the country is in line with the alpha-stable distribution.

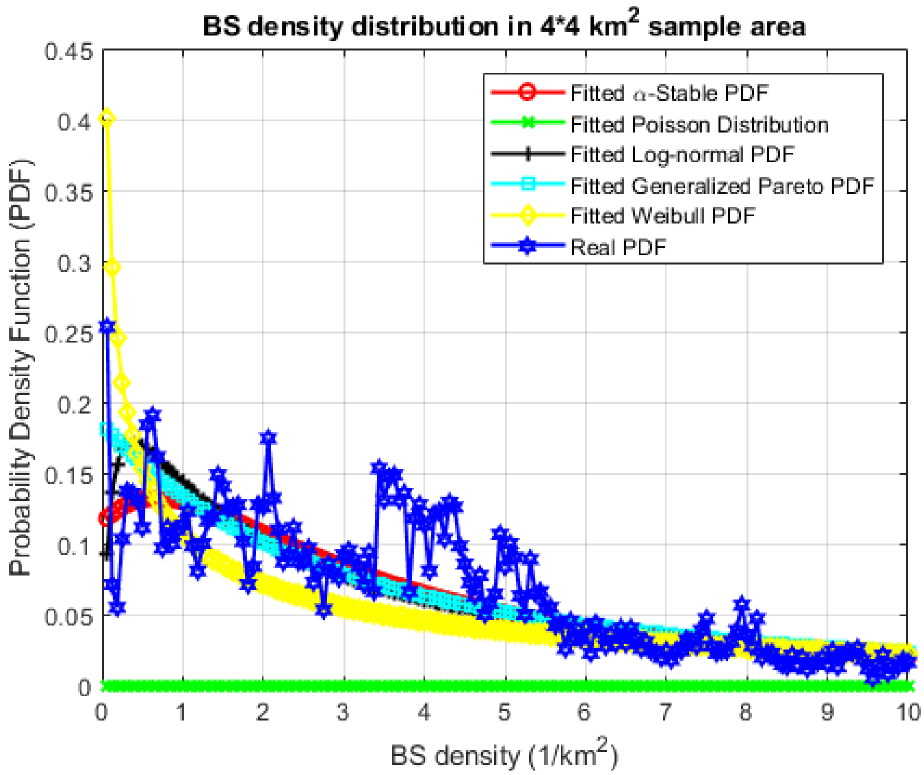

Figure 3.8. 4x4 km^2 of all New Zealand base station distribution analysis result

The Figure 3.9 shows the distribution of network base stations in the country after the small area divided into 4x4 km^2 by using log-log plot. The result shows that the distribution of network base stations in New Zealand in addition to the Poisson distribution is consistent with the alpha-stable distribution model and other candidate distribution models. That is to say, and the alpha-stable distribution model can be used to study the distribution of network base stations throughout New Zealand.

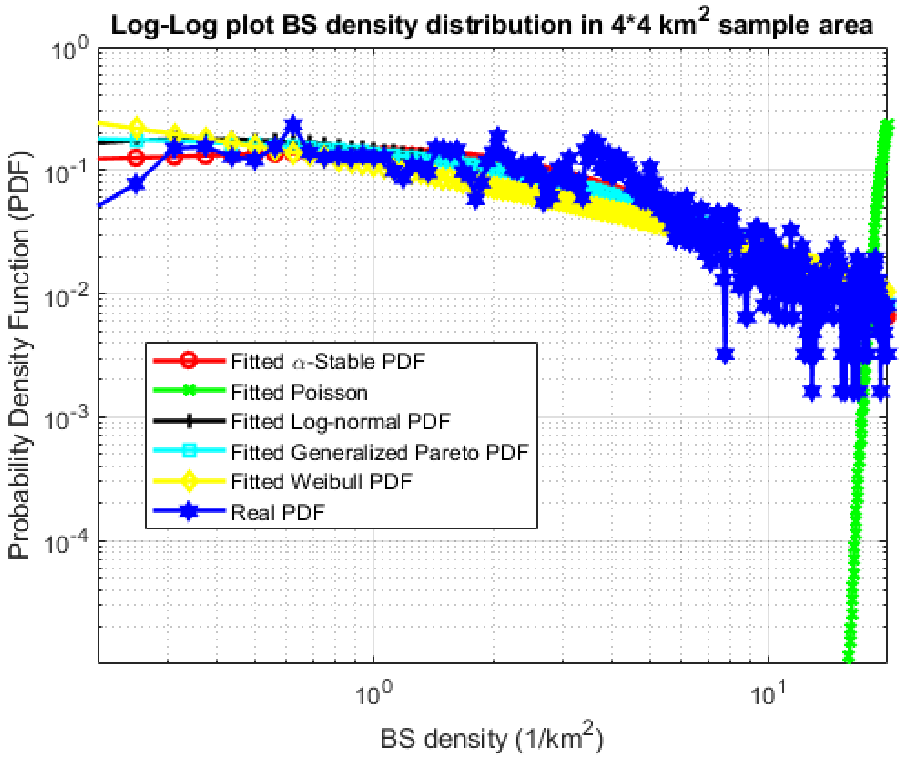

Figure 3.9. log-log plot of all New Zealand base station distribution analysis result

From the Figure 3.8 and 3.9, the most suitable model is unclear, while according to table 4 of RMSE analysis, the most suitable model is alpha-stable which has lowest value, the least error to the real PDF.

Table 4: the RMSE analysis on New Zealand national case

Distribution model	**RMSE value**
generalized Pareto (GP)	0.0042
Weibull	0.0058
Log-normal	0.0045
alpha-stable	0.0041
Poisson	0.0224

The Figure 3.10 and the Figure 3.11 show the experimental results of the network base station coordinate data of the North Island of New Zealand. The result in the Figure 3.10 would be the experimental result if the South Island of New Zealand divided into small areas of 4x4 km^2. The experimental results show that the distribution of the coordinates of the network base stations in New Zealand is roughly consistent with all candidate models except Poisson distribution. However, the experimental results of the distribution of real data fluctuate significantly, so it does not indicate that the distribution of network base stations in South Island is consistent. Other candidate distribution models also fit. That may due to the small size of the land area of territory or due to the lack of accuracy of the location coordinates of the network base station.

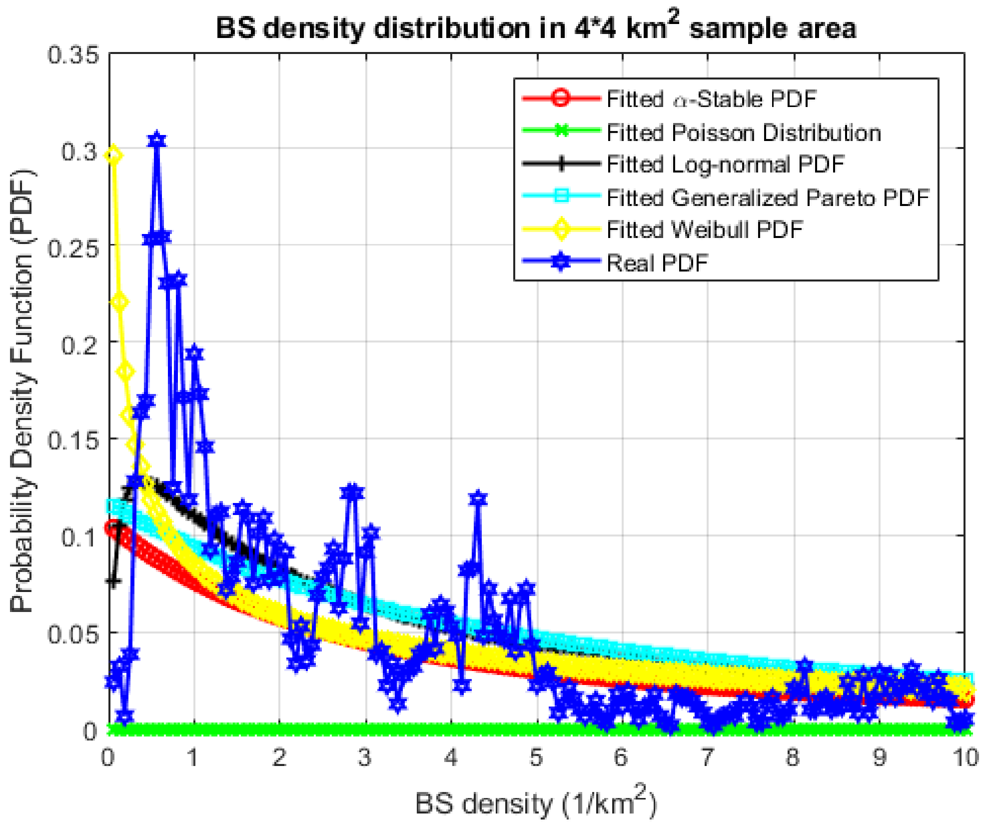

Figure 3.10 4x4 km^2 of New Zealand South Island base station distribution analysis result

The Figure 3.11 shows that the real distribution of the network base station in the North Island, it conforms to all other distribution models except the Poisson distribution, but it can give accurate experimental conclusions due to the experimental results.

According to the Figure 3.8 and the Figure 3.10, it can be the distribution curves presented by the experimental results and the reallocation of the network base station are distributed in 4x4 km^2 for the range of the distribution curve shape is relatively flat, the real shape of network base station distribution is generally relatively stable. There is no visible crest from the distribution curve. That means a relatively fragmented distribution of network base stations across New Zealand. According to the Figure 3.10 can be seen in the 4x4 km^2 the network base station distribution for the range of New Zealand North is relatively fragmented, but according to the Figure 3.9 and the Figure 3.11 In the log-log plot for the experimental results of the range. Both the distribution curve and the real network base station distribution have visible peaks, which indicates that in the Figure 3.9 and the Figure 3.11 in the area studied, there was a situation in which the network base station was extremely dense. According to the experimental results, it can be the network base station of North in New Zealand is mainly deployed in a minimal number of large cities. Therefore, other network base stations scattered in other areas, and the distribution density is shallow. That reflects from the side that the development of New Zealand mainly concentrated in very few large cities such as Auckland. The Figure 3.22 is the use of Auckland network Base station data in the log-log plot. The experimental results distributed for the range of network base stations. The experimental results show that there is a significant peak in the distribution curve of the network base station and the distribution of the actual network base station in Auckland. That indicates that the network base station of Auckland deployed in the sort of outward radiation from the center. Combination the Figure 3.16 and the Figure 3.18 the results of the experiment show that the development of New Zealand is hugely imbalanced, the development of significant individual cities is much higher than that of other parts of New Zealand. This imbalance in development reflected in the uneven allocation of resources for infrastructure construction.

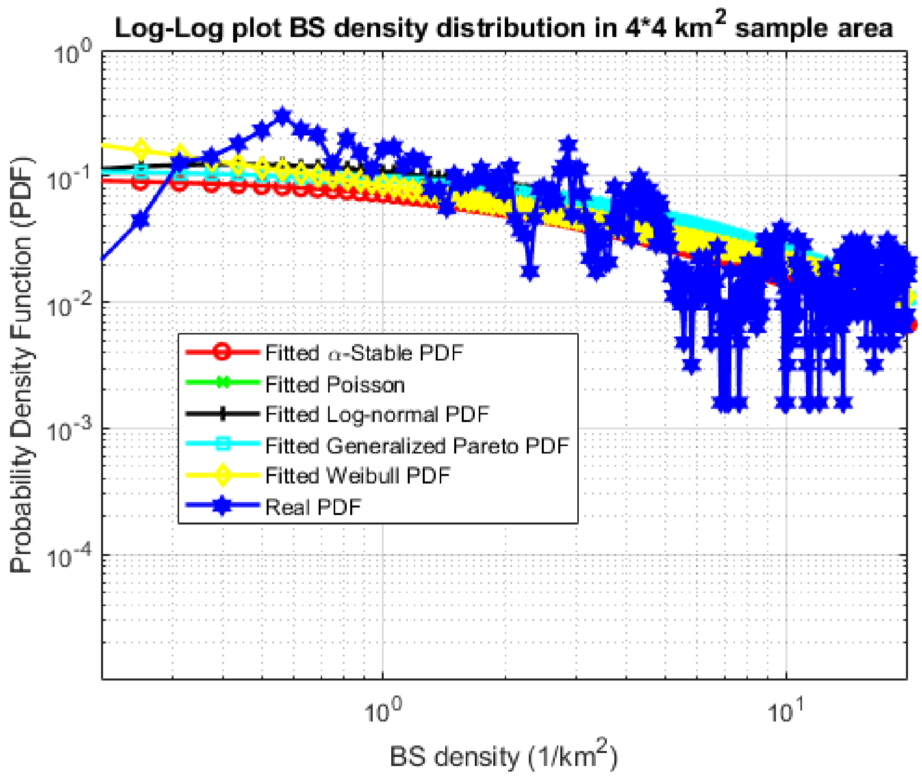

Figure 3.11 log-log plot of New Zealand South Island base station distribution analysis result

From the Figure 3.10 and 3.11, the most suitable model is unclear, while according to table 5 of RMSE analysis, the most suitable model is Log-normal which has lowest value, the least error to the real PDF.

Table 5: the RMSE analysis on South Island case

Distribution model	**RMSE value**
generalized Pareto (GP)	0.0054
Weibull	0.0065
Log-normal	0.0049
alpha-stable	0.0057
Poisson	0.0207

The Figure 3.12 shows that the data set divided the area of New Zealand's South Island into 4x4 km^2 small areas. The real network base station coordinate distribution is following the alpha-stable distribution model. However, due to the large fluctuations in the distribution of real network base stations, researchers cannot blindly judge the accuracy before researchers can obtain more accurate experimental results. The cause of this problem could be a lack of accurate data.

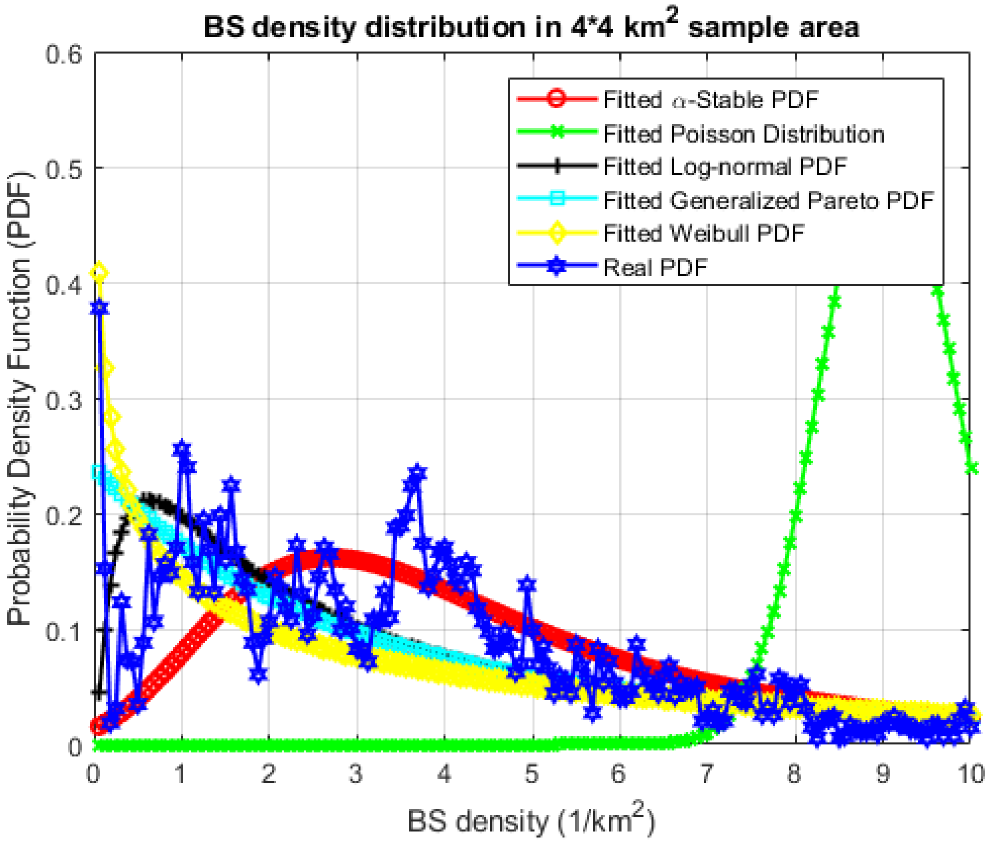

Figure 3.12 4x4 km^2 of New Zealand North Island base station distribution analysis result

Contrast the Figure 3.12 and the Figure 3.10. Through analyze the result, the Figure 3.12 There is a more apparent trend of peaks. That illustrates the distribution of network base stations in South Island, which is slightly stronger than the North Island network base station. The Figure 3.13 shows the log-log plot of the network base station distribution of North Island analysis results. It means that the distribution of network base stations in the South Island of New Zealand is relatively dense (considering the geographical environment of the South Island) rather than in the case of the North Island, where there is an extremely dense problem. However, that does not mean that the development of base station of South Island is growing better than North Island.

The analysis results that shown in the Figure 3.14 ~ 3.17 and Figure 3.19 have slight error (there is a severe error in the actual distribution of the network base station coordinate data). The reason for this problem should be roughly as same as the reason that guessed above. Therefore, a more accurate network base station coordinate position data is necessary. Moreover, due to the small size of the land area of New Zealand, the number of network base station coordinate positions planned in the unit area is limited.

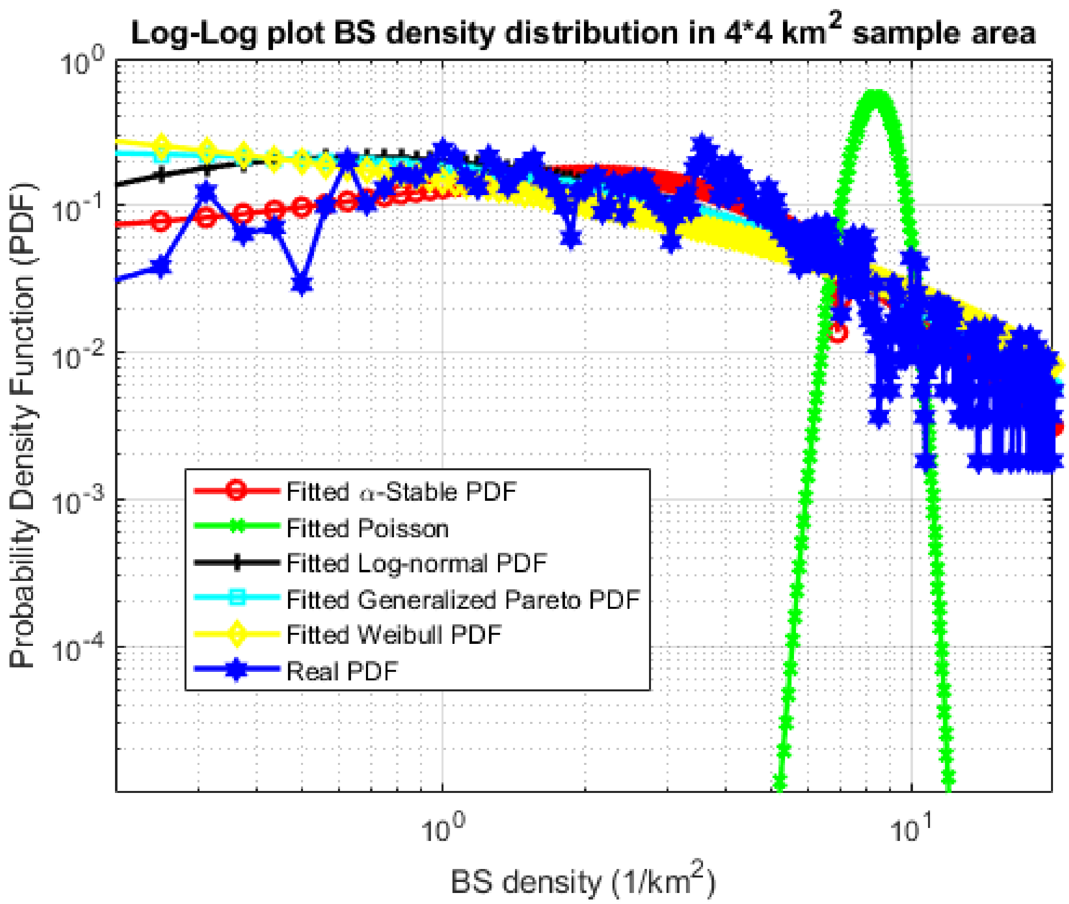

Figure 3.13 log-log plot of New Zealand North Island base station distribution analysis result

From the Figure 3.12 and 3.13, the most suitable model is unclear, while according to table 6 of RMSE analysis, the most suitable model is alpha-stable which has lowest value, the least error to the real PDF.

Table 6: the RMSE analysis on North Island case

Distribution model	RMSE value
generalized Pareto (GP)	0.0110
Weibull	0.0126
Log-normal	0.0114
alpha-stable	0.0100
Poisson	0.0445

This experiment not only uses data from network base stations in New Zealand, South Island, and North Island but also uses network base station distribution of three other major cities such as Auckland, Wellington, and Christchurch. The three cities of Auckland and Wellington located on the North Island, and Christchurch located on the South Island. Those three cities are all major cities in New Zealand, and they are developing rapidly. Using those network base station distribution data for research can provide a deeper understanding of whether network base station distribution of New Zealand is consistent with the alpha-stable model or other distribution models.

In this project of the distribution of network base stations in these major cities, the base station splits into small areas of 16 square kilometers. Users also need to upload the distribution data of these network base stations so the experimental results may be inaccurate.

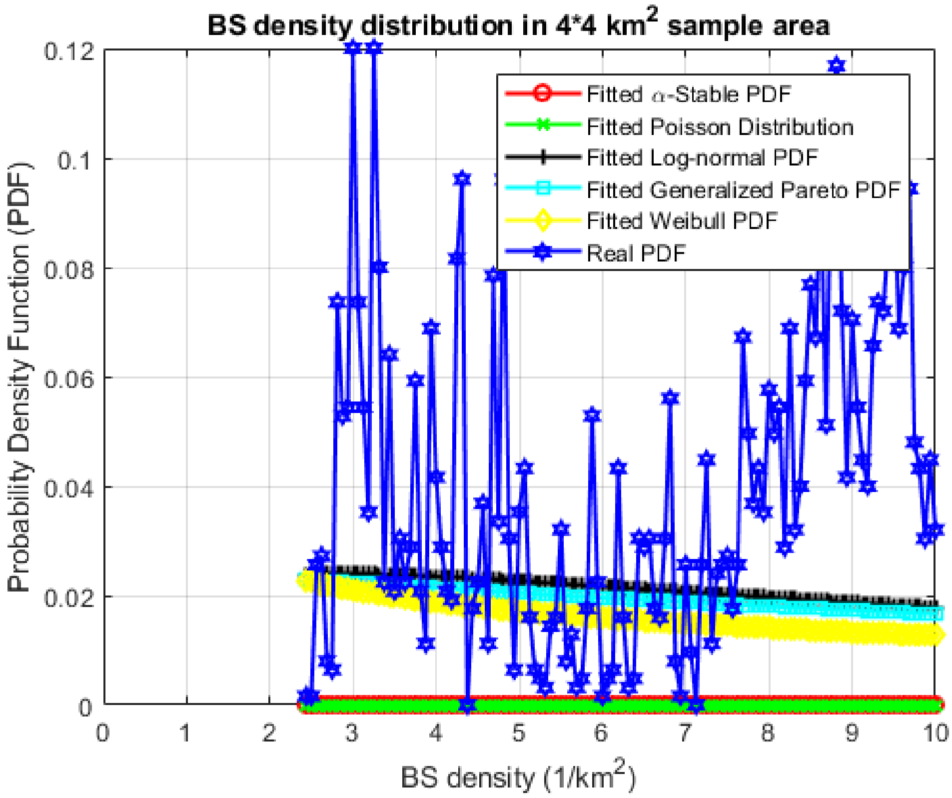

Figure 3.14 4x4 km^2 of Auckland base station distribution analysis result

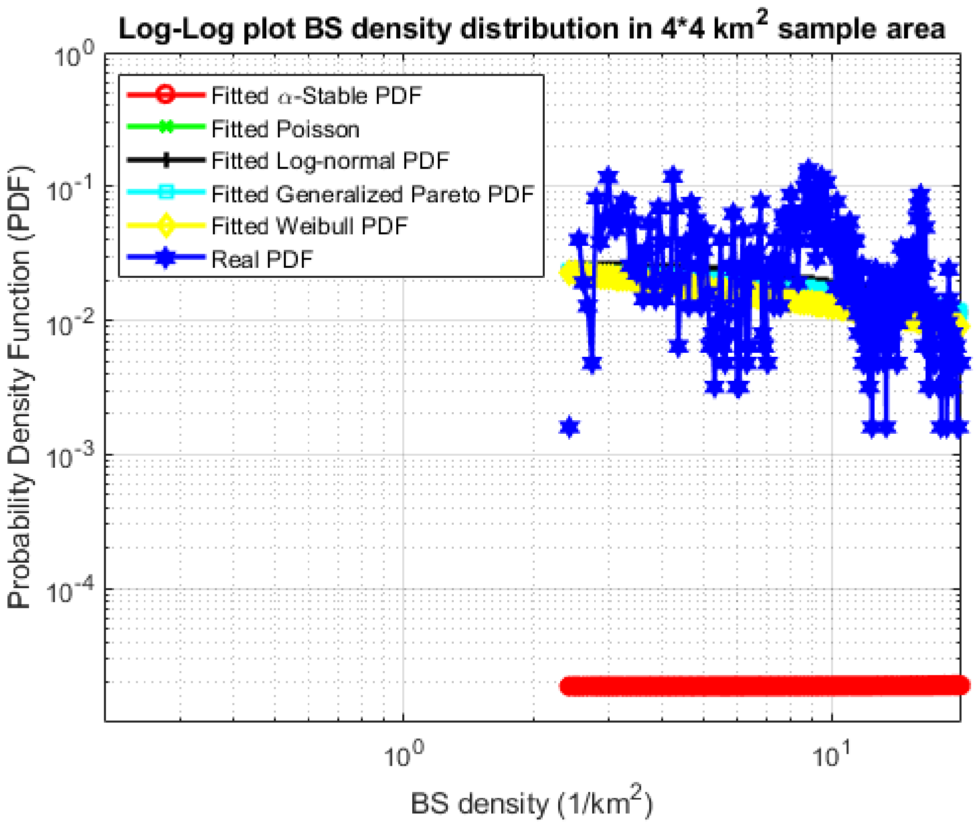

Figure 3.15 log-log plot of Auckland base station distribution analysis result

The Figure 3.14 and the Figure 3.15, that the results of the experimental data in Auckland are relatively large. That could be due to the limited area of the Auckland region. Although the population of New Zealand is relatively small compared to other developed countries, and Auckland is one of the major cities in New Zealand, its population art is relatively small for the world's developed cities. That causes the mobile Internet market to develop more slowly than the major cities of other developed countries. They are resulting in a relatively small number of network base stations in Auckland. Therefore, the network base station distribution in Auckland does not apply to the alpha-stable distribution model because the alpha-stable distribution model requires a specific amount of data to get useful analysis results.

From the Figure 3.14 and 3.15, the most suitable model is unclear, while according to table 7 of RMSE analysis, the most suitable model is Log-normal which has lowest value, the least error to the real PDF.

Table 7: the RMSE analysis on Auckland case

Distribution model	**RMSE value**
generalized Pareto (GP)	0.0052
Weibull	0.0054
Log-normal	0.0051
alpha-stable	0.0065
Poisson	0.0157

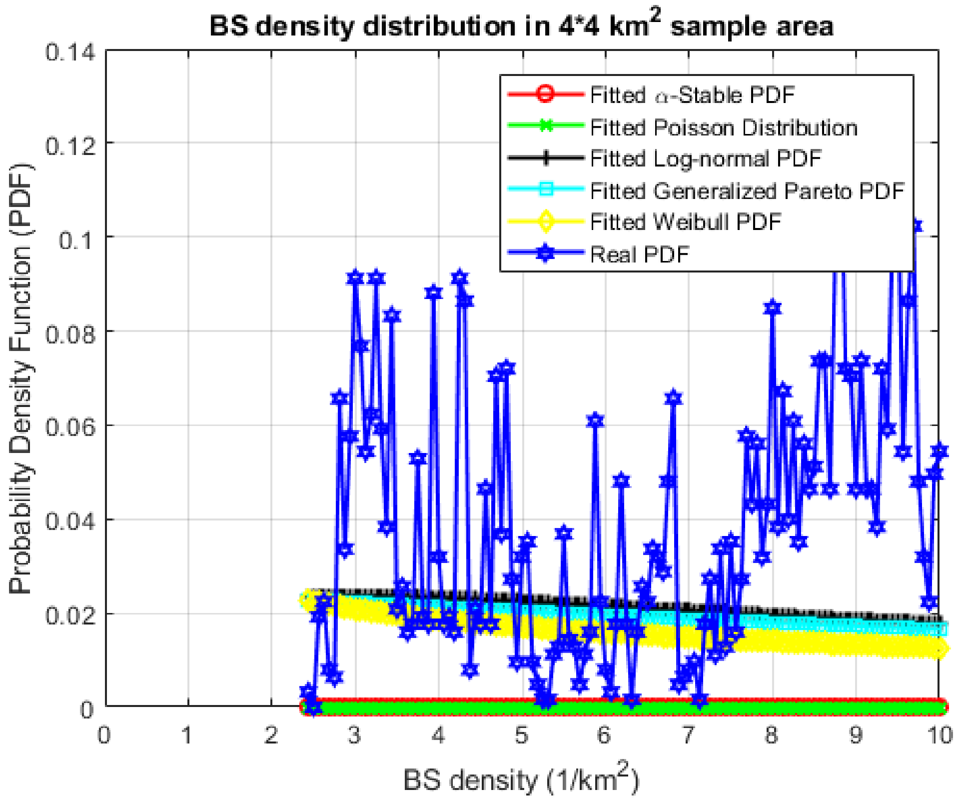

Figure 3.16. 4x4 km^2 of Wellington base station distribution analysis result

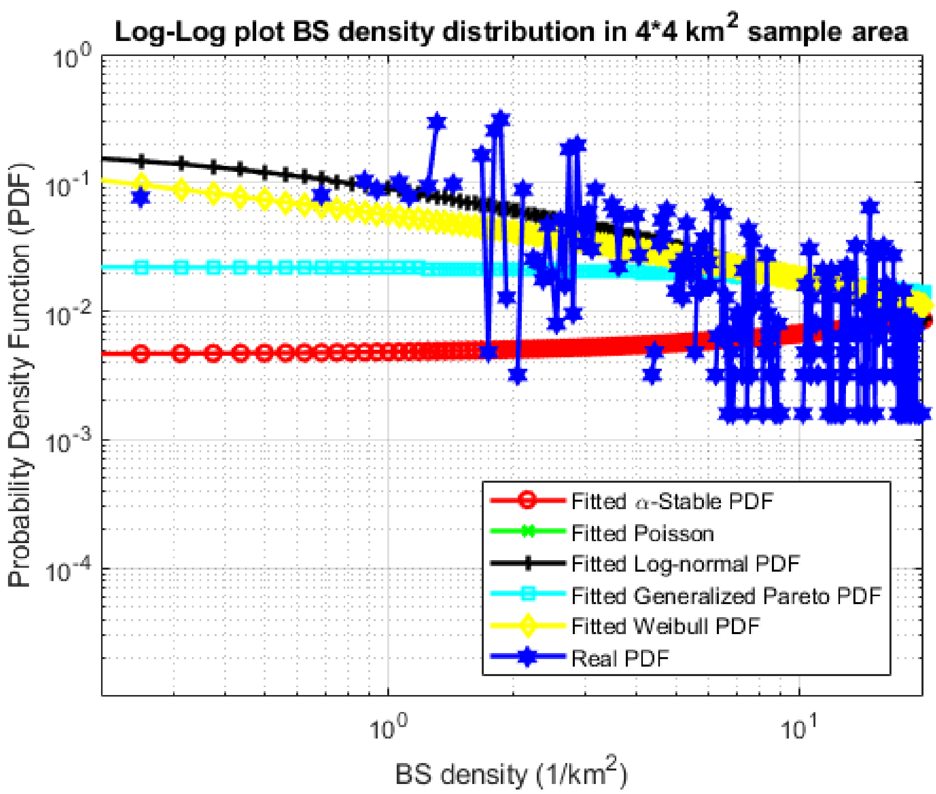

Figure 3.17 log-log plot of Wellington base station distribution analysis result

The Figure 3. 16 and the Figure 3. 17 shows the analysis distribution result of the network base station of Wellington. The experimental results are like those in the Figure 13. 14 and the Figure 13.15. The real PDF of Wellington to the online battle is as volatile as the real PDF of Auckland. Although the two cities are in different locations on the North Island of New Zealand and separated by a long distance, the results of the two sets of experiments are similar. Analysis result of network base station distribution of Wellington and Auckland cannot see which real PDF is closest to which distribution curve, neither can judge whether the real PDF of the two cities meet all candidate distribution models. The reasons for this analysis result should be related to the area of Wellington, plus network mobile Internet market is like the analysis results of Auckland.

From the Figure 3.16 and 3.17, the most suitable model is unclear, while according to table 8 of RMSE analysis, the most suitable model is Weibull which has lowest value, the least error to the real PDF.

Table 8: the RMSE analysis on Wellington case

Distribution model	**RMSE value**
generalized Pareto (GP)	0.0293
Weibull	0.0264
Log-normal	0.0267
alpha-stable	0.0300
Poisson	0.0435

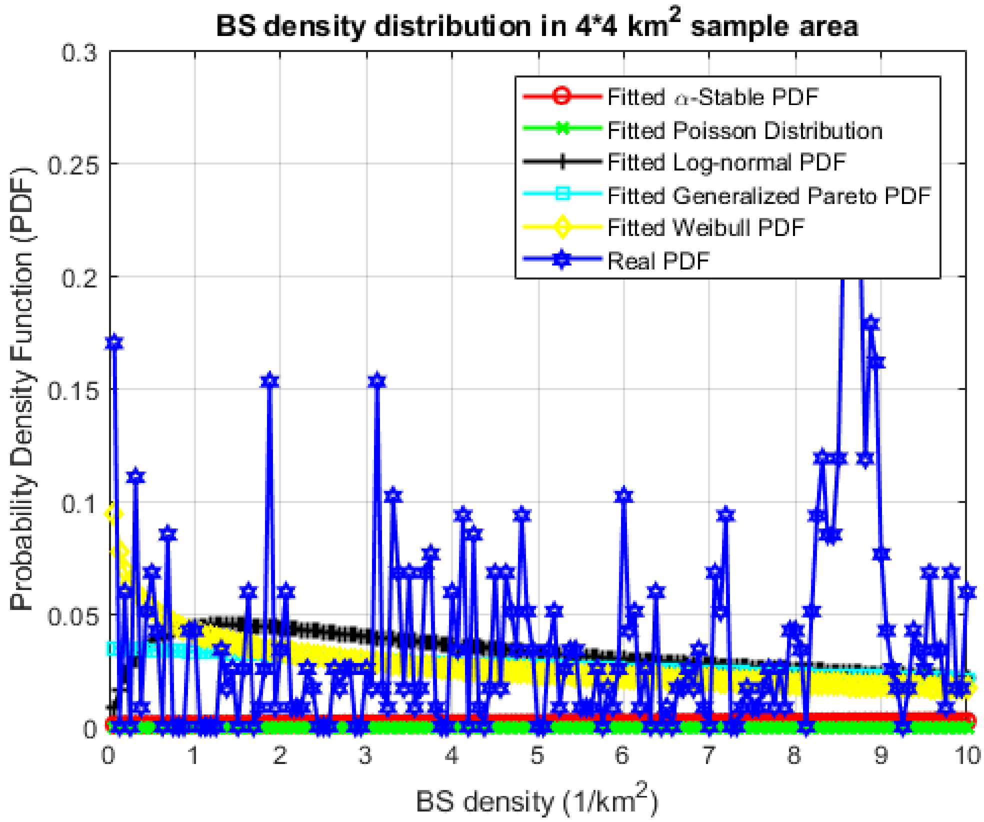

Figure 3.18. 4x4 km^2 of Christchurch base station distribution analysis result

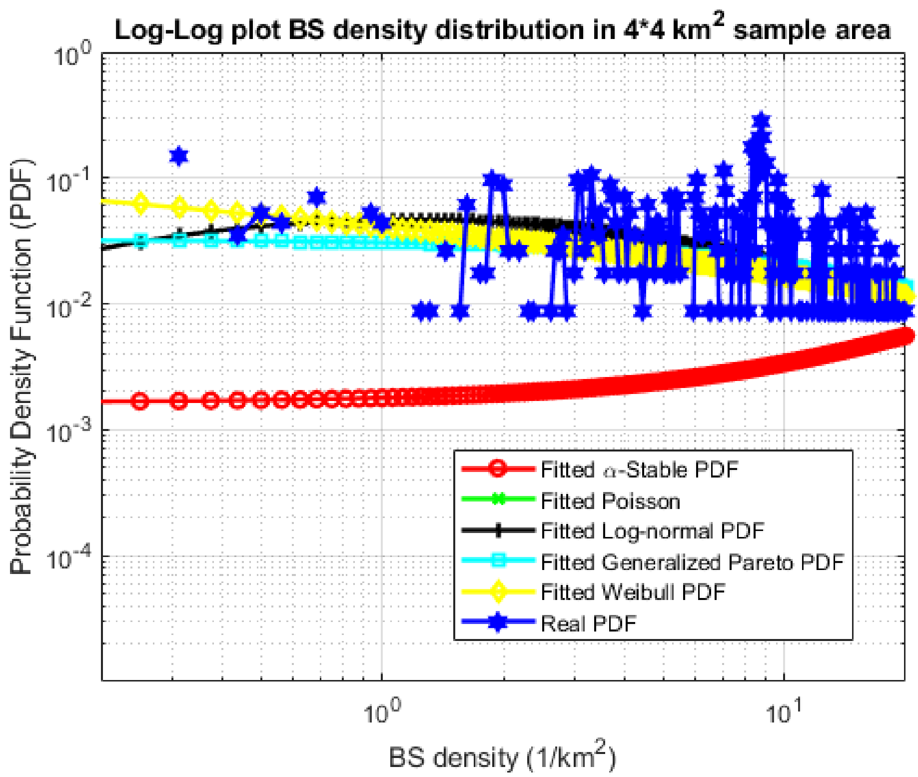

Figure 3.19. log-log plot of Christchurch base station distribution analysis result

The Figure 3. 18 and the Figure3. 19 is the result of base station analysis data on the location of the network base station in Christchurch. Christchurch located on the South Island. The Figure 3.18 displays the real PDF (the blue line) of Christchurch is fluctuating, and it is not possible to judge, which distribution model of network base station of Christchurch is consistent. However, according to the Figure 3.19, the network base station distribution of Christchurch is not consistent with alpha-stable distribution model. The reason for this result might be due to the distribution of network base stations in the South Island, that conform with the distribution of network base stations in non-urban areas.

Moreover, according to the Figure 3.19, there is a precise breakpoint in the real PDF. The reason for that could be some areas of Christchurch are too empty, and there are no mobile Internet users in that area. The active uploading of the location coordinate data of the base station has resulted in the lack of experimental result data, which reflects the small number of mobile Internet users in Christchurch, and reflects the development of the network base station in Christchurch, is under the development of network base stations in Auckland and Wellington.

From the Figure 3.18 and 3.19, the most suitable model is unclear, while according to table 9 of RMSE analysis, the most suitable model is generalized Pareto (GP) which has lowest value, the least error to the real PDF.

Table 9: the RMSE analysis on Christchurch case

Distribution model	**RMSE value**
generalized Pareto (GP)	0.0122
Weibull	0.0123
Log-normal	0.0123
alpha-stable	0.0138
Poisson	0.0294

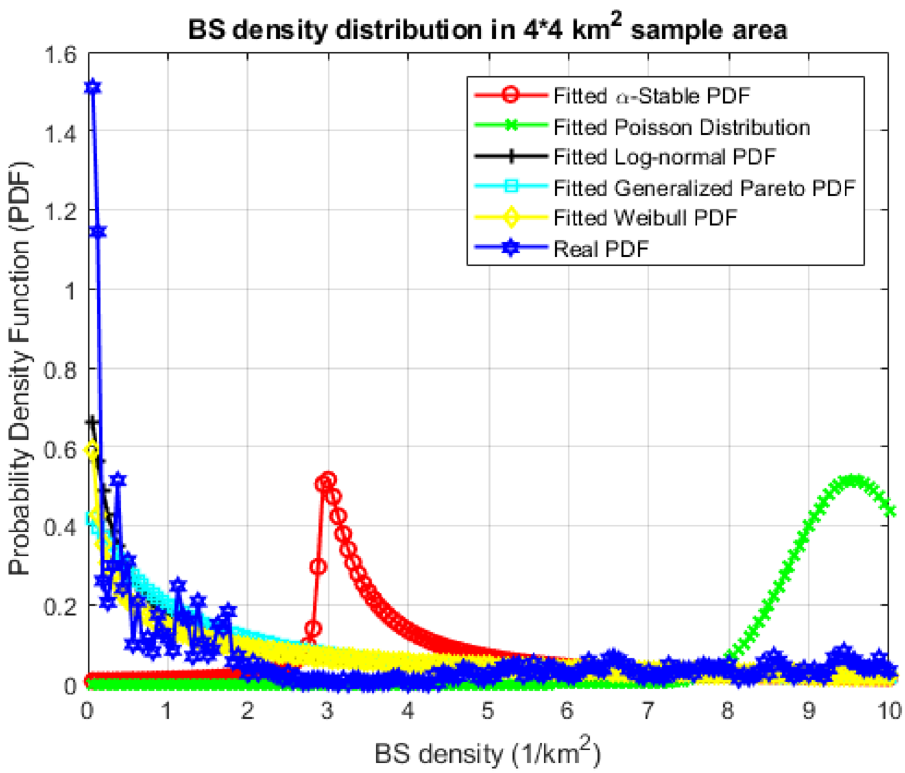

Figure 3.20 4x4 km^2 of Bay of Plenty base station distribution analysis result

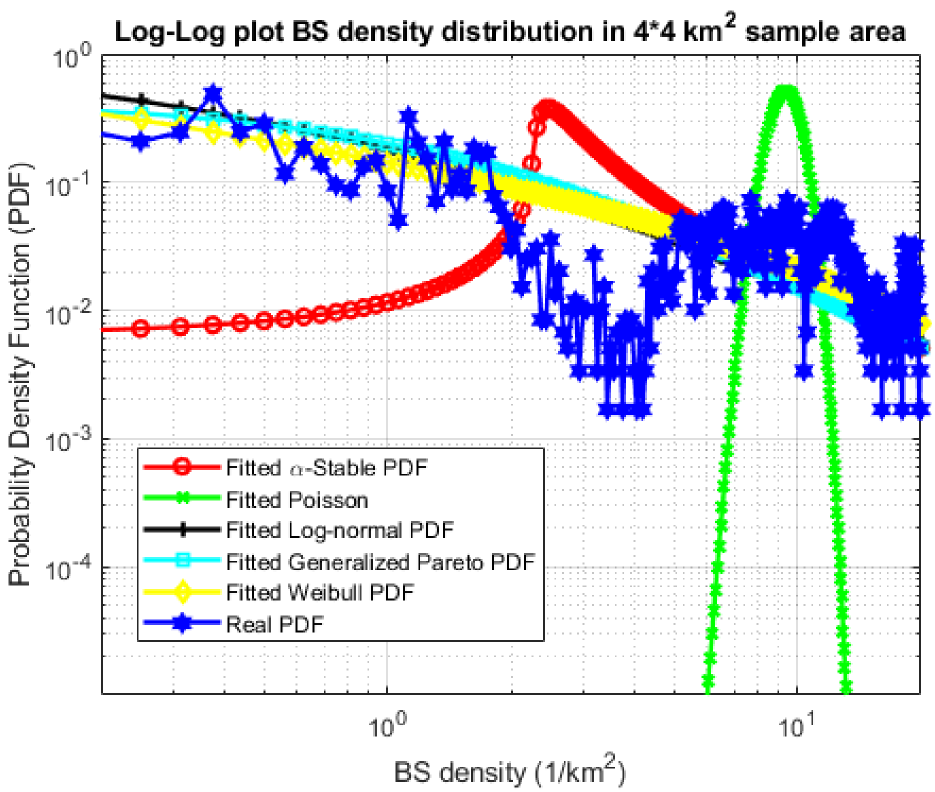

Figure 3.21 log-log plot of Bay of Plenty base station distribution analysis result

New Zealand's Bay of Plenty is a slower-growing region that is not a city. The Figure 3. 20 and the Figure 3. 21 experimental results of network base station distribution in the Bay of Plenty area. The results of these two data sets of experiments show that the distribution of network base stations in the Bay of Plenty does not conform to the alpha-stable distribution model. That is in line with the experimental result of Italy Chiaraviglio, et al. (2016). The alpha-stable model applies to the distribution of network base stations in urban areas but not fit for the network base station distribution in non-urban areas.

From the Figure 3.20 and 3.21, the most suitable model is unclear, while according to table 10 of RMSE analysis, the most suitable model is Log-normal which has lowest value, the least error to the real PDF.

Table 10: the RMSE analysis on Bay of Plenty case

Distribution model	**RMSE value**
generalized Pareto (GP)	0.0416
Weibull	0.0359
Log-normal	0.0341
alpha-stable	0.0663
Poisson	0.0843

The Figure 3. 22, the Figure 3. 23, the Figure 3. 24, and the Figure 3. 25 is the result of all Root Mean Square Error (RMSE) for the smallest region from 4x4 km^2 to 8x8 km^2. According to the Figure 3.22 that all RMSEs of alpha-stable is the smallest, that is the alpha-stable distribution model best matches the actual distribution of network base stations in New Zealand.

The experimental data of the Figure 3.23, the Figure 3.24, and the Figure 3.25 show that the alpha-stable distribution model is not the best distribution model. According to the Figure 3. 23 and the Figure 3. 24, the full RMSE value of North Island and Auckland has shown. The analysis results show that the network base stations in those two regions are most conform to the Log-normal, Generalized Pareto, and Weibull distribution models.

The Figure 3.25 shows all the RMSE values of the Bay of Plenty. The analysis result shows the network base stations distribution of Bay of Plenty conforms to the Log-normal and Weibull distribution models. The experimental result in Italy is like the case Bay of Plenty (in the experimental results in Italy, the Weibull distribution model applies to rural areas in non-urban areas).

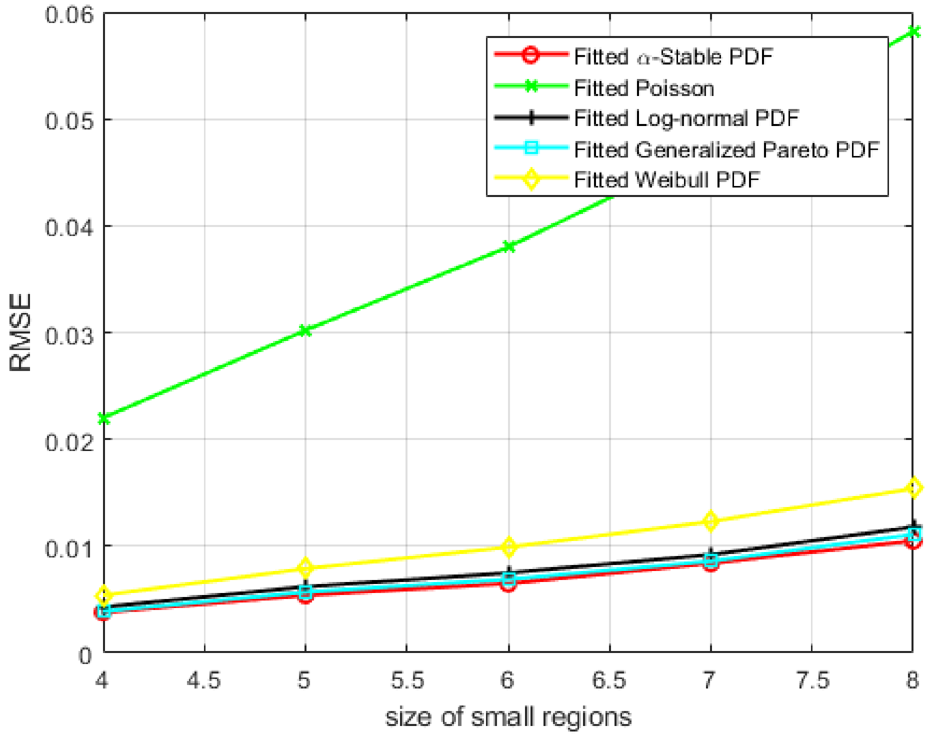

Figure 3.22 RMSE values of all New Zealand

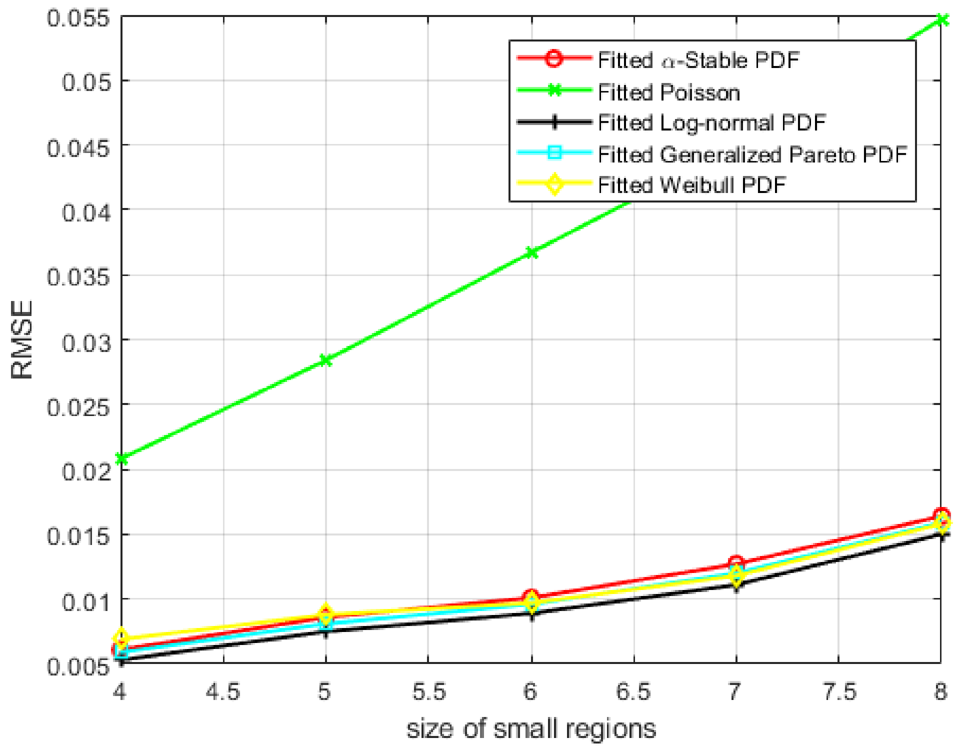

Figure 3.23 RMSE values of the South Island of New Zealand

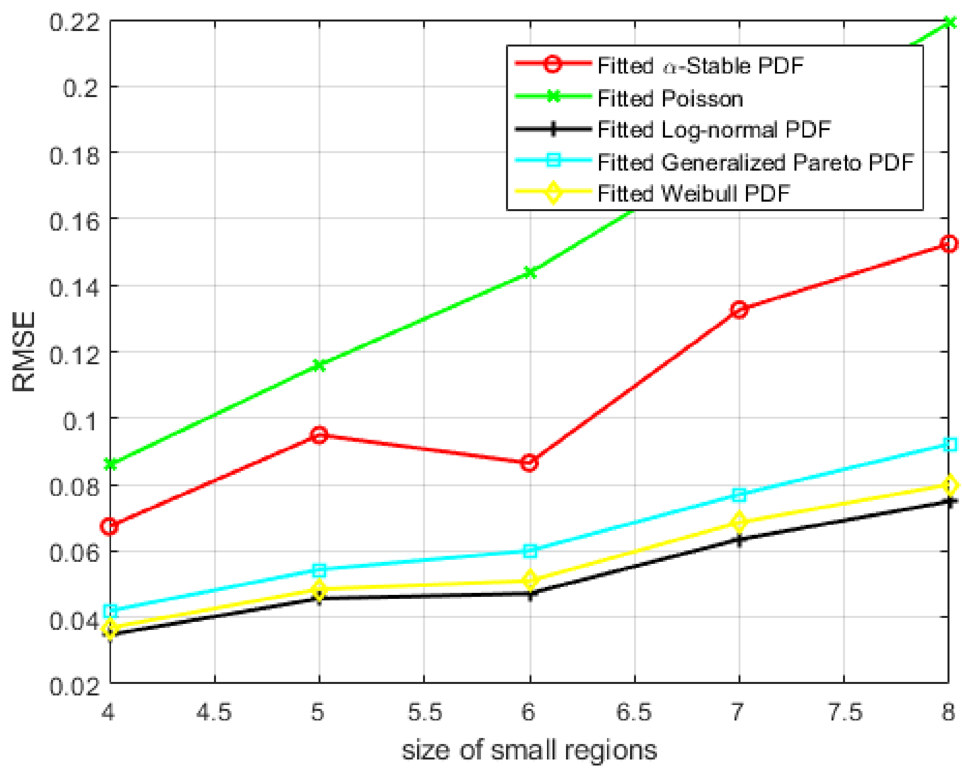

Figure 3.24 RMSE values of Auckland

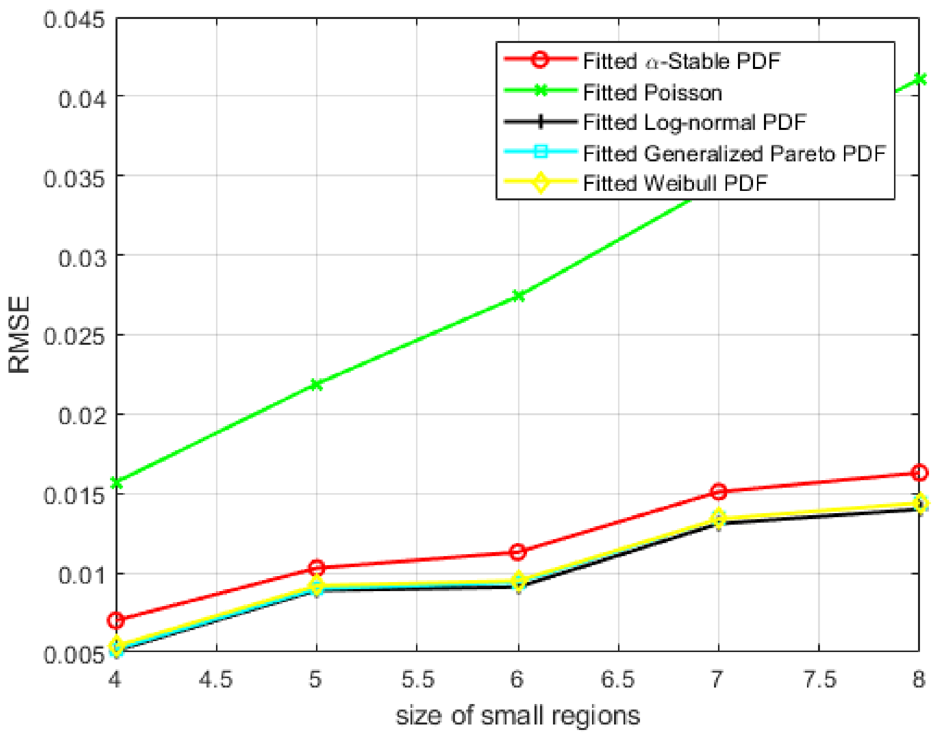

Figure 3.25 RMSE values of Bay of plenty

It is worth mentioning that a reliable result in Chiaraviglio, et al. (2016). The research on the distribution of network base stations in Italian urban areas is that only fit the alpha-stable distribution, and it is most suitable for real network base station distribution. Combined with the above experimental conclusions, the alpha-stable distribution model is suitable for analyzing the distribution of network base stations. However, New Zealand's results are experimental, but some problems have arisen. Those problems should be related to the conditions of New Zealand. The problem is that New Zealand itself does not have a balanced deployment of network base stations. The phenomenon reflects the fact that New Zealand does have shortcomings in other areas. The alpha-stable distribution model can be used to predict the distribution of base stations in the future for 5G network development.

3.5 New Zealand result compare with Italy result

The Figure 3.26 and the Figure 3.27 show the results of the Italian experiment. The line of alpha-stable distribution in the Figure 3.26 is the most consistent with the real PDF curve. Thus, the distribution of network base stations in urban areas of Italy can be analyzed using the alpha-stable model, and a very suitable result can achieve. However, the network base station distribution in Auckland is not suitable for the alpha-stable model (shown in the Figure 3.14). That may be due to the slower development of network base stations in New Zealand than the development of network base stations in Italy.

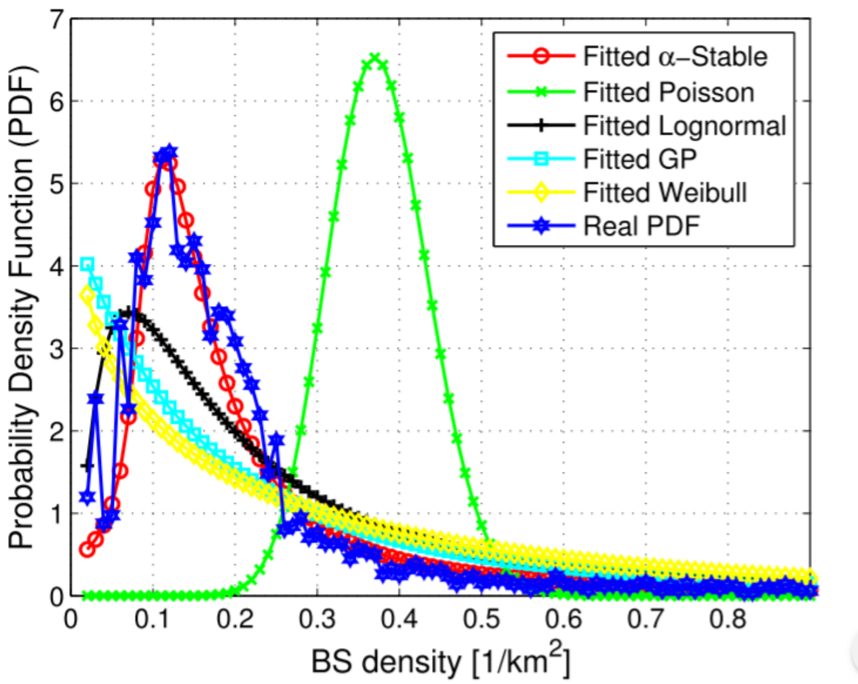

Figure 3.26 Italian urban scenario: probability density function of the base station density with all operators and sample squared area with size 10 km^2 Chiaraviglio, et al. (2016).

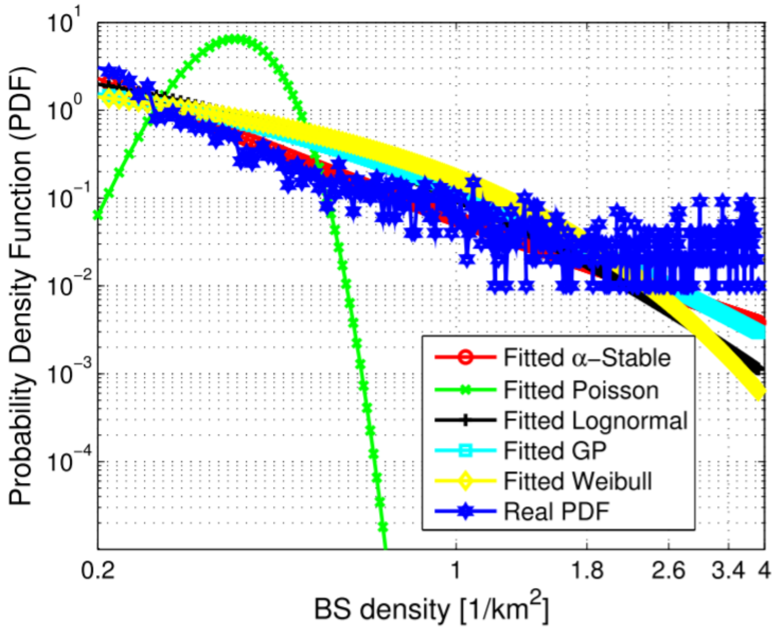

Figure 3.27 Italian urban scenario: Log-Log plot of the probability density function of the base station density with all operators and sample squared area with size 10 km^2 Chiaraviglio, et al. (2016).

The Figure 3.28 shows the RMSE values for all distribution models. The RMSE value of the alpha-stable model is minimal; that is to say, the alpha-stable distribution model has the smallest error relative to the real-to-network base station distribution model. That means the alpha-stable model is most suitable for analyzing the distribution of network base stations in Italian urban areas. The Figure 3.29 shows the RMSE value of a distribution model for the distribution of network base stations in rural Italy. We can see that the RMSE values of the Weibull model and the Log-normal model are similar, which is similar to our research on the distribution of network base stations in the Bay of Plenty, but the experimental results in Italy show that the RMSE value of the Weibull model has the smallest value (most fit). The Weibull model is most suitable for studying the distribution of network base stations in rural Italy. In New Zealand analysis case, the Log-normal model has similar RMSE with the Weibull model. However, the RMSE value of the Log-normal model is slightly smaller than the Weibull model. That means the Log-normal model is most suitable for analyzing non-urban cities in New Zealand regional network base station distribution.

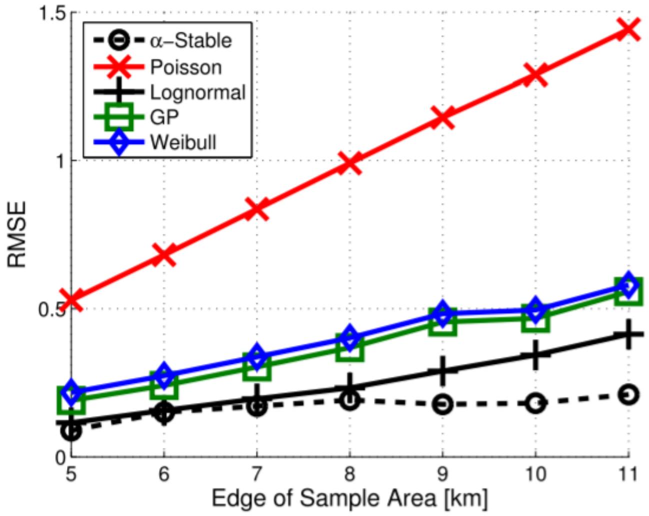

Figure 3.28 Italian urban scenario: RMSE vs. size of the sample squared area Chiaraviglio, et al. (2016).

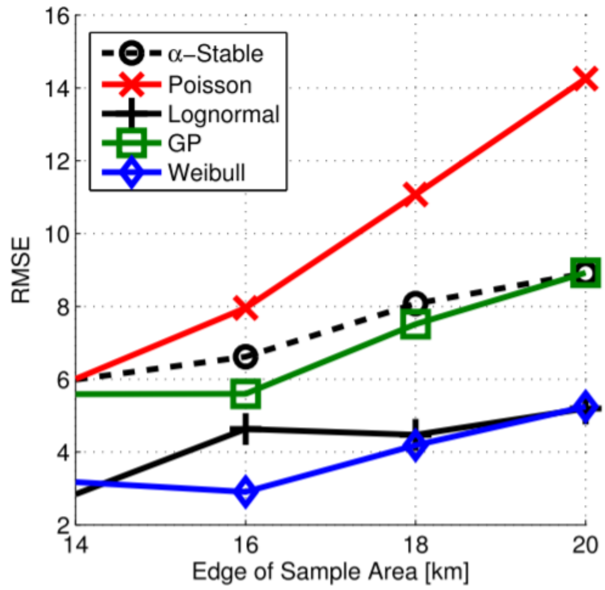

Figure 3.29 Italian rural scenario: RMSE vs. size of the sample squared area Chiaraviglio, et al. (2016).

3.6 Discussion of the analysis result

From the above analyses, the alpha-stable distribution is consistent with the distribution of network base stations in New Zealand. This section will focus on the underlying causes of this situation. First, according to the chapter in 3.1, the superposition of technology and equipment may cause uneven distribution of base stations, which can verify in this analysis. Second, we suspect that the main reason for the overlap in technology and equipment deployment is the demand of the market. For example, the population of North Island is much larger than the population of South Island. Therefore, the number of base stations deployed in the North Island is much larger than the number of base stations deployed in the South Island. Due to the large numbers of users in the North Island, the operators are more like the market of North Island. Therefore, there is a tendency to deploy base stations in the North Island. That also reflects imbalances economic and infrastructure development in New Zealand.

As the leading implementer of mobile communication network construction, operators will comprehensively grasp the demand factors and start from the internal and external levels to enhance the efficiency of network construction. The internal demand factors mainly refer to the three essential operator contents planning, actual user needs, and network resource allocation. The purpose of that is to improve the utilization of network resource allocation, efficiently use access and transmission networks, and minimize network construction. Operators will also pay attention to user needs during the construction process. They will comprehensively analyze external factors such as economic, transportation, and geographical conditions, fully understand the overall development of each region, and examine the user demand for network information. Connect the wired communication network with the wireless communication network is organically integrated, and the communication network structure layers such as the access network, the service network and the core network optimized according to the whole network planning principles and standards Tada, et al. (2016).

Furthermore, operators will also consider the construction needs of free networks. Based on existing network resources, they should adopt ways to establish network base stations and lease corresponding operators to improve the efficiency of network connections.

The external demand factors for the construction of mobile communication networks mainly refer to the construction standards of regional and government network planning. On the one hand, they must fit the basic requirements of operators' construction. On the other hand, they must meet national industry standards. Government agencies should take the lead in the adjustment of network construction planning, link the communication needs of different regions with the benefits of operators, ensure more coordinated implementation of network construction, and thus improve the feasibility and practicability of network construction. Another essential demand factor for the construction of mobile communication networks is business demand and competition. In the process of optimizing construction plans, operators must proceed from reality and combine their actual conditions to comprehensively analyze their level, strength, future development needs and planning positioning (Wikforss & Löfgren, 2007).

Operators are in the fierce market competition, but also pay attention to the improvement of their overall strong point, based on the needs of users. Conduct detailed market research, formulate a reasonable and scientific brand marketing strategy, divide the market into user groups, optimize communication, service quality pricing, and sales channel strategies to upgrade the mobile communications industry.

Population also can affect the base station distribution. Large cities in New Zealand concentrated in the North Island. Large cities such as Auckland and Wellington occupy most the population and network base stations. People perform live in large cities because larger cities have well-bred infrastructures. For Example, large cities have better public transport systems and better medical institutions. To some extent, those infrastructures fit the exact needs of people in their daily life. However, South Island dominated by the tourism industry. Visitors do not regard infrastructure as a necessary condition for a city, and the arrival of tourists will increase population growth. At the same time, the dividends brought by the tourism industry cannot support the long-term high-speed development of a city. Because of the limited population of New Zealand, international tourists have a vital role in the tourism industry. However, international visitors are not a strong reason to impress network operators to increase the number of base station deployments in the South Island.

The regularity of population distribution can help to study the distribution of network base stations. Population distribution has dynamic and static points. Static distribution is a spatial phenomenon in which the population process is relatively static during the historical period; dynamic distribution is the change of the population space phenomenon in the actual development process. The latter reflects changes in the distribution characteristics of the population in social development. These personnel changes can also help to analyze the distribution changes of network base stations. A comprehensive study of the two will help to understand the regional differences in population distribution and reveal the general pattern of population distribution in long-term history. Whether it is the static or dynamic distribution of the population, it has the following characteristics: 1) Regional and zonal distribution of the population. In a specific area with similar geographical and historical environments, population distribution has similar characteristics and vice versa. This feature is prominent in the zonal aspect of population distribution, with distinct zonal regularity in both the horizontal and vertical directions of the population. 2) imbalances in population distribution. Most of the land on the global continent is uninhabited. About half of the land is scarce, and the densely populated areas account for only a tiny part of the global land area. 3) The accumulation and spread of population distribution. It depends on the Property of the mode of production. Under different production methods, the characteristics of different populations distributed. For small-scale peasant economy, the population is mainly distributed in rural areas, forming many scattered residential areas. The commodity economy is dominant. The industry has become a specific production sector, with many rural populations flowing into cities, and the proportion of urban and rural populations has changed. Studying the regional differences and evolution process of population distribution, revealing the regularity of it, has a significant guiding role in formulating national and regional economic development plans.

Based on the above points, the population regularity is an internal, essential, and inevitable connection in the process of population development, and these properties may affect the distribution of network base stations. It reflects the fundamental characteristics of the demographic state under specific production methods. The population is an abundant population with many rules and relationships, so there is more than one regularity of population. For example, the regularity governing the natural changes of the population is the reproduction regularity, the change in the occupational composition of the population. However, in each mode of production, only one is the primary regularity that dominates.

The regularity of population can be the deep reason that affect base station distribution. It is an inevitable trend in the process of population development, the internal relations of various population phenomena and factors, and their changes and development. It is a population regularity system including population reproduction regularity, population migration regularity, population economic regularity. It objectively reflects and restricts the occurrence and development of the population process from different aspects. It has all the physiological functions such as its inheritance and mutation, which will affect the user base of the network and affect the distribution of the base station. The regularity of property genuinely restrict the population process; due to the essential difference between humans and other organisms, population regularity is a social regularity, and population is the main body of social and economic life, always in a particular political, economic, and cultural environment, with certain The level of social and economic development is compatible. The population regularity can divide into the common population regularity applicable to all social forms or several social forms and the different population regularity that only reflects the process of the population in a specific social form. The unique population regularity is a form of the existence, manifestation, and function of the standard population regularity in a specific social form. It occurs, disappears, and disappears with the emergence of social production methods. Fully understand the population regularity not only can correctly analyze the distribution of current and future base stations development but also can help the local development analysis.

The current status of base station distribution in New Zealand is not only caused by market demand, but the planning scheme of the network base station also affects the distribution of network base stations in New Zealand. The planning scheme of the network base station mainly divided into four points.

First, the main impact of base stations on the urban environment is the destruction of the urban landscape Montis, et al. (2016). With the rapid increase in the number of mobile communication users, the density of mobile communication base stations is increasing, and the form of antenna installation is various. For example, the antenna of the network base station placed on the roof of a high-rise building, or the network base station antenna erected. On the outer wall of the building, the base station also disguised as a tree. The erection of the network base station will affect the landscape of the city to a certain extent so the city planner will integrate the antenna of the base station into the urban landscape as much as possible, which affects the distribution of the network base station to some extent.

Second, the resistance of the masses to the base station also affects the distribution of network base stations over a certain length (Claassen, Dongen & Timmermans, 2017). With the people's awareness of electromagnetic radiation prevention and little understanding of the base station, it is easy to externally block during the construction of the base station. That will plus difficulty to the construction of the base station and is not conducive to timely optimization of the network.

Third, land-based base stations set up in urban construction areas constrain the intensive use of land (Swales, Beach, Edwards & McGeehan, 1990). The basic principle adopted by the existing base station layout is to build the base station on high buildings such as high-rise buildings, radio stations, post, and telecommunications buildings (other building groups do not easily shield signals), but at present, mobile base stations still have some forms of floor-standing towers for urban construction. Development (such as the establishment of new communities, demolition) has caused a certain degree of difficulty. If an angle steel tower used, it will hurt the urban areas where the land is tight due to its large ground area.

Fourth, the landing base stations set up in urban construction areas have affected the value of land development to some extent (Swales, Beach, Edwards & McGeehan, 1990). The floor tower base station set up in the city, and whether it is unbuilt or built, its existence will affect the value of land development.

Combining the above four points, the combination of base station location planning and urban development planning brings many benefits both for New Zealand's urban development and for the construction of New Zealand's communications industry. The location planning of the base station should ensure that the construction of the base station in the future will be followed by guidelines, which will promote the development of the urban communication industry. For each operator, the location planning of the base station is a prerequisite for communication development planning and network optimization. The development of the industry has provided an excellent guarantee. Careful selection of base stations plays a vital role in beautifying New Zealand's urban landscape and urban environment. Careful site selection for base stations is also conducive to the construction of base station networks, reducing the contradiction with urban planning: planning one than implementing one. Realizing a benign construction cycle for real investment savings, and building a long-term use at a time, which creates a network for a certain length. The superposition of base station deployments will affect the distribution of base stations in New Zealand from the other side.

3.7 Base station distribution forecasts for development in other industries

The arrival of the 5G period will mark a significant upgrade in the information industry. The impact of this upgrade will be no less than the invention of the telephone and the Internet. For the arrival of the 5G period, the first thing people think of is the development of driverless cars. The current network environment is not enough to support the speed of driverless information transmission, but the 5G network can meet the requirements of driverless network speed. At the same time, driverless can replace most of the workforce, which has the same concept as the phone mentioned above.

The distribution of network base stations not only used to study the speed of information transmission but also to study the development of industries such as driverless cars. It is well known that users who are closer to the physical location of the network base station have better information transfer efficiency than users who are far from the physical location of the base station. Then it is hard to find out the development trend of driverless cars will be like the development trend of 5G network base stations.

The future development of automated driving will also be uneven, and the development of automated driving in some areas with high population density will lead the area with smaller population density. Obviously, there are some exceptional cases in this kind of speculation. For example, the popularity of 5G technology may also bring about the popularity of fully automated factories. In some countries like New Zealand, where labor costs are high, fully automated factories will positively manufacture them. It also brings great benefits. However, since these automated factories require the support of 5G technology, the deployment of base stations in the base station will increase the number of base stations deployed in or around the factory, which will increase the density of base stations in the area. Besides, these factories do not need to invest a lot of human resources to maintain operations, so the population density in the region will not rise. So, there is a case about the above assumptions about the development of base station deployment for other industries. In general, the future distribution of 5G base stations can indicate the development of other industries.

4 Conclusion & Future work

This section mainly introduces the summary of the project report and the future research directions.

4.1 Conclusion

This report first introduces the application prospects of the alpha-stable distribution model. Then the example demonstrates that the case of alpha-stable distribution model can be used to analyze the coordinate data of the network base station, and the base station data of New Zealand from OpenCelliD has been used for experiment. This dataset has no data missing, so only the data classification is needed for analysis. The various parameters and properties of the alpha stable distribution model are further introduced. These parameters and properties of the alpha-stable distribution model are used to demonstrate the advantages of the alpha-stable distribution model in analyzing random data and this model can be used to analyze network base station coordinate data. This report demonstrates the advantages and limits of the alpha-stable distribution model by analyzing New Zealand's network base station distribution data. By analyzing the data, we find that the alpha-stable distribution model can well analyze the base stations distribution of whole New Zealand, but the alpha-stable distribution model is not the optimal distribution model for the network base stations distribution for small regions of New Zealand. Through the analysis, it is found that the heavy-tail phenomenon of the base station distribution in New Zealand, which shows that the base station distribution in New Zealand is relatively loose. Most of the base stations are concentrated in a few major cities, such as Auckland and Wellington. It has been found that the alpha-stable distribution model is the most suitable distribution model for national-wide, and other local areas are suitable for analysis using Weibull model and Log-normal model. When using this model to analyze network base station data across whole New Zealand, the alpha-stable distribution model has the smallest RMSE value. However, the alpha-stable distribution model does not perform well when analyzing the distribution of network base stations in regions of New Zealand. The RMSE values of alpha-stable model to analyze the data from the South Island, Auckland and the Bay of Plenty are not minimal. That shows that the alpha-stable distribution model cannot be used to analyze base station data of regions of New Zealand. The RMSE value of the Log-normal model is slightly smaller than the RMSE value of the Weibull model. This shows that the Log-normal model is the most suitable model for analyzing base station data in parts of New Zealand. In the experimental report in Italy, the alpha-stable distribution model is suitable for analyzing the distribution of network base stations in Italian urban areas but is not suitable for analyzing the distribution of network base stations in rural Italy. The distribution of network base stations in rural Italy is applicable to the Weibull distribution model. This shows that a certain amount of data is required when using the alpha-stable model. Obviously, the development of New Zealand is insufficient. Due to the disparity in the total population of Italy and the total population of New Zealand, the difference between the number of users in New Zealand and the number of users in Italy is wide. The shortage of users in New Zealand has caused a shortage of data for the base station (because the base station data requires user upload). So, in general, the alpha-stable distribution model is only suitable for analyzing network base stations in whole New Zealand.

4.2 Future work

The base station coordinate data used in this project was downloaded from an open database (OpenCelliD). The data in this database requires the user to upload base station coordinates, so there may be a problem of insufficient accuracy in those datasets. In future projects, more accurate base station coordinate data is desirable, and researchers could try to apply for the New Zealand government's official network base station data. Using more accurate data can help researchers get more useful results in future research. At the same time, in some areas with low population density, it may cause insufficient data. Because the number of users in areas with high population density is greater than the area with low population density, more users in high-density areas will upload base station coordinates. This situation can result in missing data and cannot be compensated by technical means. Therefore, the use of official data can effectively solve the above two problems and can get more useful results. Second, the project is to classify data in different regions of New Zealand. The purpose of this classification is to understand the distribution of network base stations in different parts of New Zealand, to analyze the development of other areas in different regions and to pave the way for the future development of 5G technologies and industries related to 5G technology. In future research, data can be classified by different operators. Analyze the development of each operator through the distribution of base stations of different operators. Finally, in the future research, the minimum amount of data that suitable for the alpha-stable distribution should also be studied, because in the case of New Zealand, the alpha-stable distribution model cannot effectively analyze the base station distribution in urban regions. because the size of New Zealand base station distribution data is too small. So, studying the minimum amount of data for an alpha-stable distribution can help researchers analyze base station distribution for countries like New Zealand. In order to ensure that the distribution of base stations in New Zealand is not a special case, the analysis of base station data from other regions and countries should also be used in future studies. For example, in the next analysis using Australian base station data. By using more data from countries near Oceania to improve the applicability of the alpha-stable distribution model to the study of base station distribution. At the same time, it should be studied whether the change of alpha-stable distribution parameters will affect the results of Base station distribution analysis in different countries.

Reference

Agarwal, V., & Naik, N. Y. (2004). Risks and portfolio decisions involving hedge funds. *The Review of Financial Studies*, *17*(1), 63-98.

Ameen, J. J. H., Ismail, W., & Idrus, S. M. (2011). Radio access network power management considering radio over fiber technique for 4G mobile system. *WSEAS Transactions on Communications*, (3).

Artzner, P., Delbaen, F., Eber, J. M., & Heath, D. (1999). Coherent measures of risk. *Mathematical finance*, *9*(3), 203-228.

Bodenschatz, J. S., & Nikias, C. L. (1999). Maximum-likelihood symmetric/spl alpha/-stable parameter estimation. *IEEE Transactions on Signal Processing*, *47*(5), 1382-1384.

Cashin, P. A., McDermott, C. J., & Scott, A. M. (1999). The myth of comoving commodity prices.

Chambers, J. M., Mallows, C. L., & Stuck, B. W. (1976). A method for simulating stable random variables. *Journal of the american statistical association*, *71*(354), 340-344.

Chang, K. H., & Mason, B. (2012, November). The IEEE 802.15. 4g standard for smart metering utility networks. In *2012 IEEE Third international conference on smart grid communications (SmartGridComm)* (pp. 476-480). IEEE.

Chiaraviglio, L., Cuomo, F., Gigli, A., Maisto, M., Zhou, Y., Zhao, Z., & Zhang, H. (2016, April). A reality check of base station spatial distribution in mobile networks. In *2016 IEEE Conference on Computer Communications Workshops (INFOCOM WKSHPS)* (pp. 1065-1066). IEEE.

Chiaraviglio, L., Cuomo, F., Maisto, M., Gigli, A., Lorincz, J., Zhou, Y., ... & Zhang, H. (2016). What is the best spatial distribution to model base station density? A deep dive into two European mobile networks. *IEEE Access*, *4*, 1434-1443.

Chiaraviglio, L., Cuomo, F., Maisto, M., Gigli, A., Lorincz, J., Zhou, Y., ... & Zhang, H. (2016). What is the best spatial distribution to model base station density? A deep dive into two European mobile networks. *IEEE Access*, *4*, 1434-1443.

Ding, Z., Fan, P., & Poor, H. V. (2016). Impact of user pairing on 5G nonorthogonal multiple-access downlink transmissions. *IEEE Transactions on Vehicular Technology*, *65*(8), 6010-6023.

DuMouchel, W. H. (1973). On the asymptotic normality of the maximum-likelihood estimate when sampling from a stable distribution. *The Annals of Statistics*, *1*(5), 948-957.

Faloutsos, M., Faloutsos, P., & Faloutsos, C. (1999, August). On power-regularity relationships of the internet topology. In *ACM SIGCOMM computer communication review* (Vol. 29, No. 4, pp. 251-262). ACM.

Frain, J. C. (2008). Value at Risk (VaR) and the alpha-stable distribution (No. tep0308). Trinity College Dublin, Department of Economics.

Gnedenko, B. V., & Kolmogorov, A. N. (1954). Limit Distributions for Sums of Independent Random Variables, Gostekhizdat, Moscow–Leningrad, (1949). *English translation: Addison Wesley, Reading, MA*.

Grafarend, E. (1995). The optimal universal transverse Mercator projection. In *Geodetic Theory Today* (pp. 51-51). Springer, Berlin, Heidelberg.

Gunay, S., & Khaki, A. (2018). Best Fitting Fat Tail Distribution for the Volatilities of Energy Futures: Gev, Gat and Stable Distributions in GARCH and APARCH Models. Journal of Risk and Financial Management, 11(2), 30.

Johansson, G., & Rumar, K. (1971). Drivers' brake reaction times. *Human factors*, *13*(1), 23-27.

Kabašinskas, A., Rachev, S. T., Sakalauskas, L., Sun, W., & Belovas, I. (2009). alpha-stable paradigm in financial markets. Journal of computational analysis and applications, 11(4), 641-668.

Kay, S. M. (1998). Statistical Signal Processing, Volume II, Detection Theory.

Komaty, A., Boudraa, A. O., Nolan, J. P., & Dare, D. (2015). On the Behavior of EMD and MEMD in Presence of Symmetric \alpha -Stable Noise. IEEE Signal Processing Letters, 22(7), 818-822.

Koutrouvelis, I. A. (1980). Regression-type estimation of the parameters of stable regularitys. *Journal of the American statistical association*, *75*(372), 918-928.

Koyon, S. M., & Williams, D. B. (1995, October). On the characterization of impulsive noise with/spl alpha/-stable distributions using Fourier techniques. In *Conference Record of the Twenty-Ninth Asilomar Conference on Signals, Systems and Computers* (Vol. 2, pp. 787-791). IEEE.

Kozubowski, T. J. (1999). Geometric stable regularitys: estimation and applications. *Mathematical and computer modelling*, *29*(10-12), 241-253.

Kuruoglu, E. E., Fitzgerald, W. J., & Rayner, P. J. (1998). Near optimal detection of signals in impulsive noise modeled with a symmetric/spl alpha/-stable distribution. IEEE Communications Letters, 2(10), 282-284.

Kuruoglu, E. E., Fitzgerald, W. J., & Rayner, P. J. (1998). Near optimal detection of signals in impulsive noise modeled with a symmetric/spl alpha/-stable distribution. *IEEE Communications Letters*, *2*(10), 282-284.

Lee, D., Zhou, S., Zhong, X., Niu, Z., Zhou, X., & Zhang, H. (2014). Spatial modeling of the traffic density in cellular networks. IEEE Wireless Communications, 21(1), 80-88.

Ma, X., & Nikias, C. L. (1995). Parameter estimation and blind channel identification in impulsive signal environments. *IEEE Transactions on signal processing*, *43*(12), 2884-2897.

Markowitz, H. (1952). Portfolio selection. *The journal of finance*, *7*(1), 77-91.

Marshall, C., & Siegel, M. (1996). Value-at-Risk: Implementing a risk measurement standard.

McCulloch, J. H. (1986). Simple consistent estimators of stable distribution parameters. *Communications in Statistics-Simulation and Computation*, *15*(4), 1109-1136.

Mears, T., Blick, G., Hampton, T., & Janssen, J. (2010). Fiscal institutions in New Zealand and the question of a spending cap. *Available at SSRN 1985208*.

Nikias, C. L., & Shao, M. (1995). *Signal processing with alpha-stable distributions and applications*. Wiley-Interscience.

Nolan, J. P. (1997). Numerical calculation of stable densities and distribution functions. *Communications in statistics. Stochastic models*, *13*(4), 759-774.

Nolan, J. P. (1997, November). Parameter estimation and data analysis for stable distributions. In *Conference Record of the Thirty-First Asilomar Conference on Signals, Systems and Computers (Cat. No. 97CB36136)* (Vol. 1, pp. 443-447). IEEE.

Nolan, J. P., & Rajput, B. (1995). Calculation of multidimensional stable densities. *Communications in Statistics-Simulation and Computation*, *24*(3), 551-566.

Prokhorov, Y. V. E. (1965). W. Feller, An Introduction to Probability Theory and Its Applications. *Teoriya Veroyatnostei i ee Primeneniya*, *10*(1), 204-206.

Resnick, S. I. (1997). Heavy tail modeling and teletraffic data: special invited paper. *The Annals of Statistics*, *25*(5), 1805-1869.

Rockafellar, R. T., & Uryasev, S. (2002). Conditional value-at-risk for general loss distributions. *Journal of banking & finance*, *26*(7), 1443-1471.

Rothenbücher, D., Li, J., Sirkin, D., Mok, B., & Ju, W. (2016, August). Ghost driver: A field study investigating the interaction between pedestrians and driverless vehicles. In *2016 25th IEEE international symposium on robot and human interactive communication (RO-MAN)* (pp. 795-802). IEEE.

Samoradnitsky, G. (2017). *Stable non-Gaussian random processes: stochastic models with infinite variance*. Routledge.

Shao, M., & Nikias, C. L. (1993). Signal processing with fractional lower order moments: stable processes and their applications. *Proceedings of the IEEE*, *81*(7), 986-1010.
Shao, M., & Nikias, C. L. (1993). Signal processing with fractional lower order moments: stable processes and their applications. *Proceedings of the IEEE*, *81*(7), 986-1010.
Tada, Y., Asakura, K., Shimizu, S., Konno, H., Murakami, M., & Kazutaka, J. O. E. (2016). U.S. Patent Application No. 15/064,991.

Tasche, D. (2002). Expected shortfall and beyond. *Journal of Banking & Finance*, *26*(7), 1519-1533.

Tsakalides, P., Trinic, F., & Nikias, C. L. (2000). Performance assessment of CFAR processors in Pearson-distributed clutter. *IEEE Transactions on Aerospace and electronic systems*, *36*(4), 1377-1386.

Tsihrintzis, G. A., & Nikias, C. L. (1995). Performance of optimum and suboptimum receivers in the presence of impulsive noise modeled as an alpha-stable process. *IEEE Transactions on communications*, *43*(2/3/4), 904-914.

Weron, R. (1996). On the Chambers-Mallows-Stuck method for simulating skewed stable random variables. *Statistics & probability letters*, *28*(2), 165-171.

Wikforss, Ö., & Löfgren, A. (2007). Rethinking communication in construction. *Journal of* Information Technology in Construction (ITcon), 12(23), 337-346.

Zhou, X., Zhao, Z., Li, R., Zhou, Y., Palicot, J., & Zhang, H. (2013). Human mobility patterns in cellular networks. *IEEE communications letters*, *17*(10), 1877-1880.

Zhou, Y., Li, R., Zhao, Z., Zhou, X., & Zhang, H. (2015). On the α-Stable Distribution of base stations in cellular networks. *IEEE Communications Letters*, *19*(10), 1750-1753.

Zhou, Y., Zhao, Z., Ying, Q., Li, R., Zhou, X., & Zhang, H. (2014, September). Two-tier spatial modeling of base stations in cellular networks. In *2014 IEEE 25th Annual International Symposium on Personal, Indoor, and Mobile Radio Communication (PIMRC)* (pp. 1570-1574). IEEE.

Zolotarev, V. M. (1989). One-dimensional stable distributions. *Bull. Amer. Math. Soc*, *20*, 270-277.