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INFINITE DIMENSIONAL MIXED ECONOMIES WITH ASYMMETRIC INFORMATION

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Abstract. In this paper, we study asymmetric information economies consisting of both non-negligible and negligible agents and having ordered Banach spaces as their commodity spaces. In answering a question of Hervés-Beloso and Moreno-García in [17], we establish a characterization of Walrasian expectations allocations by the veto power of the grand coalition. It is also shown that when an economy contains only negligible agents a Vind's type theorem on the private core with the exact feasibility can be restored. This solves a problem of Pesce in [20].

1. Introduction

In their seminal papers [3] and [19], Arrow, Debreu and McKenzie considered an economic model consisting of finitely many agents. Since only finitely many coalitions can be formed in such an economy, the characterization of Walrasian allocations by the veto mechanism is asymptotic [7]. Later, Aumann [4] considered an economic model consisting of a continuum of agents by taking [0,1] with Lebesgue measure as the space of agents and established a characterization of Walrasian allocations in terms of the core. The main advantage of Aumann's model is that perfect competition prevails, that is, the influence of any individual agent on the economy is negligible. However, the competition in many real economies is imperfect, for instance, in an economy which has some individual agents who own large portions of initial endowments of some commodities. This is the main motivation to consider mixed economies or oligopolistic markets, refer to [8], [12], [20], and [24]. In Chapter 7 of [6], uncertainty was introduced in the Arrow-Debreu-McKenzie model by allowing finitely many states of nature and viewing the commodities as differentiated by state. In this model, each agent possesses the same full information and makes a contract contingent on the realized state of nature. However, such a model does not capture the idea of contracts under asymmetric information. This analysis was extended by Radner in [21], where each agent is characterized by a private information set, a state-dependent utility function, a random initial endowment and a prior belief. The trade of an agent is measurable with respect to his information so that he cannot act differently on states that he cannot distinguish and an agent makes a contract for trading commodities before he obtains any information about

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the realized state of nature. Radner also extended the notion of a Walrasian equilibrium in the Arrow-Debreu-McKenzie model to that of a Walrasian expectations equilibrium in his model so that better informed agents are generally better off.

In this paper, we consider a mixed economy with asymmetric information and infinitely many commodities. In Section 2, we provide a general description on our model. Section 3 is devoted to study a special case of our model, where the space of agents is an atomless measure space. Two results on the private blocking power of a coalition are established, and measures of blocking coalitions when agents are asymmetrically informed are studied. Schmeidler [23] first improved Aumann's equivalence result by only considering the blocking power of small coalitions in a complete information and atomless economy with finitely many commodities. Schmeidler's result was further generalized in Grodal [13]. Finally, Vind [26] showed that if some coalition blocks an allocation then there is a blocking coalition with any measure less than the measure of the grand coalition. Although Hervés-Beloso et al. [14] pointed out that analogous results of Vind's theorem are generally false for an atomless economy with the space of real bounded sequences as the commodity space, extensions of Vind's theorem for special economies with asymmetric information and the free disposal condition can be found in [5], [15] and [16]. Recently, Hervés-Beloso et al. [18] established a Vind's type theorem for the process of information shared by coalitions in an asymmetric information economy having a finite dimensional commodity space and the free disposal assumption. Considering an ordered Banach space whose positive cone admitting an interior point as the commodity space and a complete finite positive atomless measure space of agents, Evren and Hüsseinov [11] established a Vind's type result on the private core of an economy under the free disposal condition and other additional assumptions. However, as mentioned in [20], whether there is a version of Vind's theorem on the private core of an economy with the exact feasibility for finite dimensional economies is still an open problem. Here, we investigate this question for an asymmetric information economy with an ordered Banach space whose positive cone has an interior point as the commodity space and give a full solution. As a result, the equivalence theorem for finite dimensional economies in [2] is further generalized. The corresponding problems on the (strong) fine core of an economy are also considered.

Concerning a complete information economy, Hervés-Beloso and Mareno-García [17] provided a characterization of Walrasian allocations by robustly efficient allocations when the economy has a continuum of agents and finitely many commodities. More precisely, if f is a Walrasian allocation then it is non-dominated in not only the initial economy but also all economies obtained by modifying the initial endowments of any coalition in the direction of f. In the same paper, they also showed that such a result holds for economies with asymmetric information and the space of real bounded sequences as the commodity space. In Section 4, a similar result is established in an asymmetric information economy whose space of agents is a complete finite positive measure space and commodity space is an ordered separable Banach space whose positive cone has an interior point. Other results in Section 4 concern the relationships among different types of cores. Einy et al. [9] showed that the fine core is a subset of the ex-post core for an asymmetric information economy with an atomless measure space of agents and a finite dimensional commodity space. One year later, they established a characterization of the weak fine core by the private core in a complete information economy in [10], where it was assumed that the grand coalition is a finite union of pairwise disjoint measurable subsets having positive measure and any two agents in the same measurable subset have the same information. Here, these results are extend to mixed economies with asymmetric information and ordered separable Banach spaces whose positive cones contain interior points as commodity spaces. Furthermore, in our framework there may exist an information type associated with a null measurable subset of the grand coalition.

2. The model

Let $\mathcal{E}$ be an exchange economy with asymmetric information as in [21] and [22]. Suppose that $(\Omega, \mathcal{F})$ is a measurable space, where Ω is a finite set denoting all possible states of nature and the σ-algebra $\mathcal{F}$ denotes all events. Following from the well-known mixed market model, the space of agents is a measure space (T, Σ, μ) with a complete, finite and positive measure μ, where T is the set of agents, Σ is the σ-algebra of measurable subsets of T whose economic weights on the market are given by μ. Following from a classical result in measure theory, T can be decomposed into two parts: one is atomelss and the other contains countably many atoms. That is, $T = T_0 \cup T_1$, where T_0 is the atomless part and T_1 is the countable union of μ-atoms. Since each μ-atom is treated as an agent, $A \in T_1$ is used instead of $A \subseteq T_1$ if A is a μ-atom. Agents in T_0 are called "*small agents*" and those in T_1 are called "*large agents*". In each state, infinitely many commodities are assumed. Throughout, the commodity space of $\mathcal{E}$ is an ordered Banach space Y whose positive cone has an interior point. The order on Y is denoted by $\leq$, and $Y_+ = \{x \in Y : x \geq 0\}$ denotes the positive cone of Y. The symbol $x \gg 0$ (resp. $x > 0$) denotes a strictly positive (resp. non-zero positive) element x of Y_+. The economy extends over two time periods $\tau = 0, 1$. Consumption takes place at $\tau = 1$. At $\tau = 0$, there is uncertainty over the states and agents make contracts that are contingent on the realized state at $\tau = 1$. Thus, $\mathcal{E}$ can be defined by

$$\mathcal{E} = \{(\Omega, \mathcal{F});\ (T, \Sigma, \mu);\ Y_+;\ (\mathcal{F}_t, U_t, a(t, \cdot), q_t)_{t \in T}\}.$$

Here, Y_+ is the *consumption set* in every state $\omega \in \Omega$ for every agent $t \in T$; $\mathcal{F}_t$ the σ-algebra generated by a partition Π_t of Ω representing the *private information* of agent t; $U_t : \Omega \times Y_+ \to \mathbb{R}$ is the *state-dependent utility function* of agent t; $a(t, \cdot) : \Omega \to Y_+$ is the *random initial endowment* of agent t, assumed to be constant on elements of Π_t; and q_t is a probability measure on Ω giving the *prior* of agent t. It is assumed that q_t is positive on all elements of Ω. The quadruple $(\mathcal{F}_t, U_t, a(t, \cdot), q_t)$ is called the *characteristics* of the agent $t \in T$. A function $x : \Omega \to Y_+$ is interpreted as a random consumption bundle in $\mathcal{E}$. The *ex ante expected utility* of an agent t for a given random consumption bundle x is defined by $V_t(x) = \sum_{\omega \in \Omega} U_t(\omega, x) q_t(\omega)$.

Any set $S \in \Sigma$ with $\mu(S) > 0$ is called a *coalition* of $\mathcal{E}$. If S and S' are two coalitions of $\mathcal{E}$ with $S' \subseteq S$, then S' is called a *sub-coalition* of S. For a coalition S in $\mathcal{E}$, an S-*assignment* in $\mathcal{E}$ is a function $f : S \times \Omega \to Y_+$ such that $f(\cdot, \omega) \in L_1^S(\mu, Y_+)$ for all $\omega \in \Omega$, where $L_1^S(\mu, Y_+)$ is the set of all Bochner integrable functions from S into Y_+. It is assumed that $a(\cdot, \omega) \in L_1^T(\mu, Y_+)$ for each $\omega \in \Omega$. Put $L_t = \{x \in (Y_+)^\Omega : x \text{ is } \mathcal{F}_t\text{-measurable}\}$. An S-assignment f in $\mathcal{E}$ is called an S-*allocation* if $f(t, \cdot) \in L_t$ for almost all $t \in S$, and it is said to be S-*feasible* if $\int_S f(\cdot, \omega) d\mu \leq \int_S a(\cdot, \omega) d\mu$ for all $\omega \in \Omega$. T-assignments, T-allocations and T-feasible allocations are simply called *assignments*, *allocations* and *feasible allocations*. A coalition S

privately blocks an allocation f in $\mathcal{E}$ if there is an S-feasible allocation g such that $V_t(g(t,\cdot)) > V_t(f(t,\cdot))$ for almost all $t \in S$. The *private core* of $\mathcal{E}$ is the set of all feasible allocations which are not privately blocked by any coalition. A *price system* is an $\mathcal{F}$-measurable, non-zero function $\pi : \Omega \to Y_+^*$, where Y_+^* is the positive cone of the norm-dual space Y^* of Y. The *budget set* of agent t can be defined by

$$B_t(\pi) = \left\{ x \in L_t : \sum_{\omega \in \Omega} \langle \pi(\omega), x(\omega) \rangle \leq \sum_{\omega \in \Omega} \langle \pi(\omega), a(t,\omega) \rangle \right\}.$$

A *Walrasian expectations equilibrium* of $\mathcal{E}$ in the sense of Radner is a pair (f, π), where f is a feasible allocation and π is a price system such that for almost all $t \in T$, $f(t,\cdot) \in B_t(\pi)$ and $f(t,\cdot)$ maximizes V_t on $B_t(\pi)$, and

$$\sum_{\omega \in \Omega} \left\langle \pi(\omega), \int_T f(\cdot,\omega) d\mu \right\rangle = \sum_{\omega \in \Omega} \left\langle \pi(\omega), \int_T a(\cdot,\omega) d\mu \right\rangle.$$

Two agents are said to be the *same type* if they have the same characteristics. The family of partitions of Ω is denoted by $\mathfrak{P}$. For any $\mathcal{Q} \in \mathfrak{P}$, let $T_{\mathcal{Q}} = \{t \in T : \Pi_t = \mathcal{Q}\}$. For any coalition S, put $\mathfrak{P}_S = \{\mathcal{Q} \in \mathfrak{P} : S \cap T_{\mathcal{Q}} \neq \emptyset\}$ and $\mathfrak{P}(S) = \{\mathcal{Q} \in \mathfrak{P}_S : \mu(S \cap T_{\mathcal{Q}}) > 0\}$. Then, $\bigcup_{\mathcal{Q} \in \mathfrak{P}_T} T_{\mathcal{Q}} = T$ and $L_t = L_{t'}$ if and only if $t, t' \in T_{\mathcal{Q}}$ for some $\mathcal{Q} \in \mathfrak{P}_T$. For any $S \in \Sigma$, $\bigvee \mathfrak{Q}$ denotes the σ-algebra generated by the smallest refinement of all members of $\mathfrak{Q} \subseteq \mathfrak{P}_S$.

Assumptions:

- (A_1) *Measurability*: The functions $t \mapsto q_t$ and $t \mapsto \mathcal{F}_t$ are measurable. This means that $\{t \in T : q_t \in A\} \in \Sigma$ for any Borel subset A of $|\Omega| - 1$ dimensional unit simplex, and $T_{\mathcal{Q}} \in \Sigma$ for all $\mathcal{Q} \in \mathfrak{P}$.
- (A_2) *Carathéodory*: For each $\omega \in \Omega$, $(t,x) \mapsto U_t(\omega, x)$ is a Carathéodory function on $T \times Y_+$. This means that $U_{(\cdot)}(\omega, x)$ is measurable for all $(\omega, x) \in \Omega \times Y_+$, and $U_t(\omega, \cdot)$ is norm-continuous for all $(t, \omega) \in T \times \Omega$.
- (A_3) *Monotonicity*: For each $(t,\omega) \in T \times \Omega$, if $x, y \in Y_+$ with $y \gg 0$, then $U_t(\omega, x+y) > U_t(\omega, x)$.
- (A_3') *Strong monotonicity*: For each $(t,\omega) \in T \times \Omega$, if $x, y \in Y_+$ with $y > 0$, then $U_t(\omega, x+y) > U_t(\omega, x)$.
- (A_4) *Partial concavity*: For each $(t_0, \omega_0) \in T_1 \times \Omega$ and S-feasible assignment f with $\mu(S \cap T_1) > 0$ in $\mathcal{E}$, $U_{t_0}(\omega_0, \hat{f}(\omega_0)) \geq \frac{1}{\mu(S \cap T_1)} \int_{S \cap T_1} U_{t_0}(\omega_0, f(\cdot, \omega_0)) d\mu$, where $\hat{f}(\omega_0) = \frac{1}{\mu(S \cap T_1)} \int_{S \cap T_1} f(\cdot, \omega_0) d\mu$.
- (A_4') *Concavity*: For each $(t,\omega) \in T \times \Omega$, $U_t(\omega, \cdot)$ is concave.
- (A_5) *Strict positivity*: For each $(t,\omega) \in T \times \Omega$, $a(t,\omega) \gg 0$.
- (A_6) *Similarity*: All large agents are the same type.
- (A_7) *Minimality*: T_1 contains at least two elements.
- (A_8) *Informativeness*: $\bigvee \mathfrak{P}_T = \mathcal{F}$.
- (A_9) *$\mathcal{F}$-measurability*: For almost all $t \in T$ and $x \in Y_+$, $U_t(\cdot, x)$ is $\mathcal{F}$-measurable.

Under (A_1) and (A_2), $V_{(\cdot)}(\cdot) : T \times (Y_+)^\Omega \to \mathbb{R}$ is a Carathéodory function. Under (A_3) (resp. (A_3')), V_t is monotone (resp. strongly monotone) in the sense that if $x, y \in (Y_+)^\Omega$ with $y(\omega) \gg 0$ (resp. $y(\omega) > 0$) for some $\omega \in \Omega$, then $V_t(x+y) > V_t(x)$. Clearly, (A_4) implies that V_{t_0} is partially concave for all $t_0 \in T_1$, that is, $V_{t_0}(\hat{f}(\cdot)) \geq \frac{1}{\mu(S \cap T_1)} \int_{S \cap T_1} V_{t_0}(f(\cdot,\cdot)) d\mu$ for all $t_0 \in T_1$ and S-feasible assignment f in $\mathcal{E}$ with $\mu(S \cap T_1) > 0$, where $\hat{f}$ is defined in (A_4). Similarly, (A_4') implies that

V_t is concave for all $t \in T$. By (A_6), all agents in T_1 have the same characteristics, so we use $(\mathcal{F}_{T_1}, U_{T_1}, a(T_1,\cdot), q_{T_1})$ to denote their common characteristics. Similarly, V_{T_1} denotes the common ex ante expected utility of agents in T_1. Note that (A_8) is similar to $(A.4)$ in [9], and (A_1)-(A_3), (A_5) are the same as those in [11]. For undefined mathematical concepts and terminologies in this paper, refer to [1].

3. Privately blocking and exact feasibility in atomless economies

In this section, we study privately blocking and exactly feasible allocations in an atomless economy. Thus, we assume $T = T_0$ in this section. Two lemmas are established in Subsection 3.1, which are used in Section 4. Similar to that in [26], we also investigate the blocking power of a coalition for the (strong) fine core and the private core when the exact feasibility is imposed on allocations.

3.1. **Privately blocking coalitions.** The following result is similar to Lemma 1 in [11]. In order to obtain a slightly different conclusion, we provide a proof here.

Lemma 3.1. *Let an allocation f in $\mathcal{E}$ be privately blocked by a coalition S and $\alpha \in (0,1)$. Under (A_1), (A_2) and (A_5), there exist an S-allocation g and a subcoalition S' of S such that*

(i) *$g(t,\omega) \gg 0$ for all $(t,\omega) \in S' \times \Omega$, and $V_t(g(t,\cdot)) > V_t(f(t,\cdot))$ for almost all $t \in S$*
(ii) *$\int_S (a(\cdot,\omega) - g(\cdot,\omega))d\mu \gg 0$ for all $\omega \in \Omega$,*
(iii) *$\mu(S' \cap T_{\mathcal{Q}}) > \alpha\mu(S \cap T_{\mathcal{Q}})$ for all $\mathcal{Q} \in \mathfrak{P}(S)$.*

Proof. Since f is privately blocked by the coalition S, there exists an S-feasible allocation h such that $V_t(h(t,\cdot)) > V_t(f(t,\cdot))$ for almost all $t \in S$. Define a correspondence $P_f : S \rightrightarrows (Y_+)^\Omega$ by $P_f(t) = \{y \in L_t : V_t(y) > V_t(f(t,\cdot))\}$ for each $t \in S$. Then $h(t,\cdot) \in P_f(t)$ for almost all $t \in S$. By ignoring a μ-null subset of S, one can choose a separable, closed linear subspace Z of Y^Ω such that $f(S,\cdot) \cup h(S,\cdot) \cup a(S,\cdot) \subseteq Z$. Consider a correspondence $\widetilde{P}_f : S \rightrightarrows Z$ defined by $\widetilde{P}_f(t) = Z \cap P_f(t)$. By Remark 6 in [11], $\mathrm{Gr}_{\widetilde{P}_f} \in \Sigma_S \otimes \mathfrak{B}(Z)$, where $\Sigma_S = \{A \in \Sigma : A \subseteq S\}$, $\mathrm{Gr}_{\widetilde{P}_f}$ denotes the graph of $\widetilde{P}_f$ and $\mathfrak{B}(Z)$ is the family of Borel subsets of Z. For any $\epsilon > 0$, define a correspondence $N_\epsilon : S \rightrightarrows Z$ by $N_\epsilon(t) = \{y \in Z : \|y - h(t,\cdot)\| < \epsilon\}$. Then, $\mathrm{Gr}_{N_\epsilon} \in \Sigma_S \otimes \mathfrak{B}(Z)$. Furthermore, $\mathrm{Gr}_{\tilde{L}} \in \Sigma_S \otimes \mathfrak{B}(Z)$, where $\tilde{L} : S \rightrightarrows Z$ is defined by $\tilde{L}(t) = Z \cap L_t$. For all $t \in S$, choose ϵ_t such that $\epsilon_t = \sup\{\epsilon > 0 : y \in \widetilde{P}_f(t)$ whenever $y \in \tilde{L}(t) \cap N_\epsilon(t)\}$. Continuity of V_t implies $\epsilon_t > 0$ for almost all $t \in S$. Let $\beta > 0$. Then,

$$\{t \in S : \epsilon_t < \beta\} = \bigcup_{r \in \mathbb{Q} \cap (0,\beta)} \{t \in S : N_r(t) \cap \tilde{L}(t) \cap (Z \setminus \widetilde{P}_f(t)) \neq \emptyset\},$$

which is the projection of the set

$$\bigcup_{r \in \mathbb{Q} \cap (0,\beta)} \left(\mathrm{Gr}_{N_r} \cap \mathrm{Gr}_{\tilde{L}} \cap \left(S \times Z \setminus \mathrm{Gr}_{\widetilde{P}_f}\right)\right) \in \Sigma_S \otimes \mathfrak{B}(Z)$$

on S. By the projection theorem [1, p.608], the set $\{t \in S : \epsilon_t < \beta\} \in \Sigma$, which means that the function $t \mapsto \epsilon_t$ is measurable. Choose a sequence $\{c_m\} \subset (0,1)$ such that $c_m \to 0$ as $m \to \infty$. For each $m \geq 1$, define $h_m : S \times \Omega \to Y_+$ such that $h_m(t,\omega) = (1-c_m)h(t,\omega) + \frac{c_m}{2}a(t,\omega)$. Then, h_m is an S-allocation, and $h_m(t,\omega) \gg$

0 for all $(t,\omega) \in S \times \Omega$. For each $m \geq 1$, put $S_m = \{t \in S : \|h_m(t,\cdot) - h(t,\cdot)\| < \epsilon_t\}$. Clearly, $S_m \in \Sigma_S$ and $S_m \subseteq S_{m+1}$ for all $m \geq 1$. Moreover, $\bigcup_m S_m \sim S$ and hence, $\lim_{m\to\infty} \mu(S \setminus S_m) = 0$. By the definition of ϵ_t, $h_m(t,\cdot) \in P_f(t)$ for almost all $t \in S_m$. For each $m \geq 1$, we now define a function $g_m : S \times \Omega \to Y_+$ by

$$g_m(t,\omega) = \begin{cases} h(t,\omega), & \text{if } t \in (S \setminus S_m) \times \Omega; \\ h_m(t,\omega), & \text{if } (t,\omega) \in S_m \times \Omega. \end{cases}$$

Then g_m is an S-allocation, $V_t(g_m(t,\cdot)) > V_t(f(t,\cdot))$ for almost all $t \in S$, and $g_m(t,\omega) \gg 0$ for all $(t,\omega) \in S_m \times \Omega$. Now, for each $\omega \in \Omega$,

$$\begin{aligned} \int_S g_m(\cdot,\omega)d\mu &= \int_{S\setminus S_m} h(\cdot,\omega)d\mu + \int_{S_m} h_m(\cdot,\omega)d\mu \\ &= \int_{S\setminus S_m} (h(\cdot,\omega) - h_m(\cdot,\omega))d\mu + \int_S h_m(\cdot,\omega)d\mu. \end{aligned}$$

In addition, $\int_S h_m(\cdot,\omega)d\mu \leq \left(1 - \frac{c_m}{2}\right)\int_S a(\cdot,\omega)d\mu$. Consequently, we obtain

$$\int_S g_m(\cdot,\omega)d\mu \leq \int_{S\setminus S_m} c_m \left(h(\cdot,\omega) - \frac{1}{2}a(\cdot,\omega)\right) d\mu + \left(1 - \frac{c_m}{2}\right)\int_S a(\cdot,\omega)d\mu,$$

which is equivalent to

$$\int_S (a(\cdot,\omega) - g_m(\cdot,\omega))d\mu \geq c_m \left(\frac{1}{2}\int_S a(\cdot,\omega)d\mu - z_m(\omega)\right),$$

where $z_m(\omega) = \int_{S\setminus S_m}\left(h(\cdot,\omega) - \frac{1}{2}a(\cdot,\omega)\right)d\mu$. Since $\int_S a(\cdot,\omega)d\mu \gg 0$, by absolute continuity of the Bochner integral, $\frac{1}{2}\int_S a(\cdot,\omega)d\mu - z_m(\omega) \gg 0$ for all $\omega \in \Omega$ when m is sufficiently large. Pick a $\mathcal{Q}_0 \in \mathfrak{P}(S)$ satisfying $\mu(S \cap T_{\mathcal{Q}_0}) \leq \mu(S \cap T_{\mathcal{Q}})$ for all $\mathcal{Q} \in \mathfrak{P}(S)$ and select a $1 < \delta < \frac{1}{\alpha}$. Then for all $\mathcal{Q} \in \mathfrak{P}(S)$,

$$(1 - \alpha\delta)\mu(S \cap T_{\mathcal{Q}_0}) < (1 - \alpha)\mu(S \cap T_{\mathcal{Q}}). \tag{3.1}$$

Choose some integer m such that $\mu(S_m) > \alpha\delta\mu(S \cap T_{\mathcal{Q}_0}) + \mu(S \setminus T_{\mathcal{Q}_0})$. Obviously, $\mu(S_m \cap T_{\mathcal{Q}_0}) > \alpha\mu(S \cap T_{\mathcal{Q}_0})$. It is claimed that $\mu(S_m \cap T_{\mathcal{Q}}) \leq \alpha\mu(S \cap T_{\mathcal{Q}})$ implies $(1-\alpha\delta)\mu(S\cap T_{\mathcal{Q}_0}) \geq (1-\alpha)\mu(S\cap T_{\mathcal{Q}})$ for any $\mathcal{Q} \in \mathfrak{P}(S)\setminus\{\mathcal{Q}_0\}$. If not, there is some $\mathcal{Q}' \in \mathfrak{P}(S) \setminus \{\mathcal{Q}_0\}$ such that $\mu(S_m \cap T_{\mathcal{Q}'}) \leq \alpha\mu(S \cap T_{\mathcal{Q}'})$ but $(1 - \alpha\delta)\mu(S \cap T_{\mathcal{Q}_0}) < (1 - \alpha)\mu(S \cap T_{\mathcal{Q}'})$. It follows that

$$\begin{aligned} \mu(S_m) &= \mu(S_m \cap T_{\mathcal{Q}'}) + \sum_{\mathcal{Q}\in\mathfrak{P}(S)\setminus\{\mathcal{Q}'\}} \mu(S_m \cap T_{\mathcal{Q}}) \\ &\leq \alpha\mu(S \cap T_{\mathcal{Q}'}) + \mu(S \cap T_{\mathcal{Q}_0}) + \sum_{\mathcal{Q}\in\mathfrak{P}(S)\setminus\{\mathcal{Q}_0,\mathcal{Q}'\}} \mu(S \cap T_{\mathcal{Q}}) \\ &< \alpha\delta\mu(S \cap T_{\mathcal{Q}_0}) + \mu(S \setminus T_{\mathcal{Q}_0}), \end{aligned}$$

which contradicts with the choice of S_m. This verifies the claim. By (3.1) and the claim, we conclude that $\mu(S_m \cap T_{\mathcal{Q}}) > \alpha\mu(S \cap T_{\mathcal{Q}})$ for all $\mathcal{Q} \in \mathfrak{P}(S)$. The proof is completed by letting $g = g_m$ and $S' = S_m$. □

Remark 3.2. The conclusion of Lemma 3.1 is also true if the atomless measure space is replaced by a complete finite positive measure space.

Lemma 3.3. [25] *Suppose that (X,Σ,μ) is an atomless measure space and E is a Banach space. If $f \in L_1^X(\mu,E)$, then the set $H = \mathrm{cl}\{\int_B f : B \in \Sigma\}$ is convex.*

The following result is an extension of the result used in the main theorem of [26] to an asymmetric information economy whose commodity space is an ordered Banach space having an interior point in its positive cone.

Lemma 3.4. *Let f be an allocation in $\mathcal{E}$. Suppose there exist a coalition S, a sub-coalition S' of S and an S-allocation g such that $g(t,\omega) \gg 0$ for all $(t,\omega) \in S' \times \Omega$, $\mathfrak{P}(S) = \mathfrak{P}(S')$ and $V_t(g(t,\cdot)) > V_t(f(t,\cdot))$ for almost all $t \in S$. Under* (A$_1$)-(A$_3$), *for each $r \in (0,1)$, there exists an S-allocation h such that $V_t(h(t,\cdot)) > V_t(f(t,\cdot))$ for almost all $t \in S$, and $\int_S h(\cdot,\omega)d\mu = \int_S (rg(\cdot,\omega)+(1-r)f(\cdot,\omega))d\mu$ for all $\omega \in \Omega$.*

Proof. For each $m \geq 1$, let $g_m : S\times\Omega \to Y_+$ be defined by $g_m(t,\omega) = (1-c_m)g(t,\omega)$. Then g_m is an S-allocation and $g_m(t,\omega) \gg 0$ for all $(t,\omega) \in S' \times \Omega$. Pick an $r \in (0,1)$ and a $\mathcal{Q} \in \mathfrak{P}(S)$. Let $\{c_m\}$ be a sequence in $(0,1)$ such that $c_m \to 0$ as $m \to \infty$. Applying an argument similar to that in Lemma 3.1, it can be shown that there is an increasing sequence $\{S^{\mathcal{Q}}_m\} \subseteq \Sigma_{S\cap T_{\mathcal{Q}}}$ such that $\bigcup_m S^{\mathcal{Q}}_m \sim S \cap T_{\mathcal{Q}}$, $\lim_{m\to\infty} \mu((S\cap T_{\mathcal{Q}}) \setminus S^{\mathcal{Q}}_m) = 0$, and $V_t(g_m(t,\cdot)) > V_t(f(t,\cdot))$ for almost all $t \in S^{\mathcal{Q}}_m$. Choose an $m_{\mathcal{Q}}$ such that $\mu(S' \cap T_{\mathcal{Q}} \cap S^{\mathcal{Q}}_{m_{\mathcal{Q}}}) > 0$. Consider the function $y^{\mathcal{Q}} : (S\cap T_{\mathcal{Q}}) \times \Omega \to Y_+$ defined by

$$y^{\mathcal{Q}}(t,\omega) = \begin{cases} g_{m_{\mathcal{Q}}}(t,\omega), & \text{if } (t,\omega) \in S^{\mathcal{Q}}_{m_{\mathcal{Q}}} \times \Omega; \\ g(t,\omega), & \text{otherwise.} \end{cases}$$

Obviously, $y^{\mathcal{Q}}$ is an $(S\cap T_{\mathcal{Q}})$-allocation, and $V_t(y^{\mathcal{Q}}(t,\cdot)) > V_t(f(t,\cdot))$ for almost all $t \in S\cap T_{\mathcal{Q}}$. Furthermore, for all $\omega \in \Omega$,

$$\int_{S\cap T_{\mathcal{Q}}} y^{\mathcal{Q}}(\cdot,\omega)d\mu = \int_{S\cap T_{\mathcal{Q}}} g(\cdot,\omega)d\mu - c_{m_{\mathcal{Q}}} \int_{S^{\mathcal{Q}}_{m_{\mathcal{Q}}}} g(\cdot,\omega)d\mu.$$

Let $x^{\mathcal{Q}} \gg 0$ be chosen such that $x^{\mathcal{Q}} \leq \frac{c_{m_{\mathcal{Q}}}}{2} \int_{S^{\mathcal{Q}}_{m_{\mathcal{Q}}}} g(\cdot,\omega)d\mu$ for all $\omega \in \Omega$. Let $U(r,\mathcal{Q})$ be an open neighborhood of 0 such that $rx^{\mathcal{Q}} - U(r,\mathcal{Q}) \subseteq \mathrm{int}Y_+$. By Lemma 3.3,

$$H_{\mathcal{Q}} = \mathrm{cl}\left\{\left(\mu(E^{\mathcal{Q}}), \int_{E^{\mathcal{Q}}} (y^{\mathcal{Q}} - f)d\mu\right) \in \mathbb{R}\times Y^{\Omega} : E^{\mathcal{Q}} \in \Sigma_{S\cap T_{\mathcal{Q}}}\right\}$$

is a convex set. So, there is a sequence $\{E^{\mathcal{Q}}_n\} \subseteq \Sigma_{S\cap T_{\mathcal{Q}}}$ such that for each $\omega \in \Omega$,

$$\lim_{n\to\infty} \left(\mu(E^{\mathcal{Q}}_n), \int_{E^{\mathcal{Q}}_n} (y^{\mathcal{Q}}(\cdot,\omega) - f(\cdot,\omega))d\mu\right) = r\left(\mu(S\cap T_{\mathcal{Q}}),\ z^{\mathcal{Q}}(\omega)\right),$$

where $z^{\mathcal{Q}}(\omega) = \int_{S\cap T_{\mathcal{Q}}} (y^{\mathcal{Q}}(\cdot,\omega) - f(\cdot,\omega))d\mu$. Define a function $b^{\mathcal{Q}}_n : \Omega \to Y$ such that $b^{\mathcal{Q}}_n(\omega) = \int_{E^{\mathcal{Q}}_n} (y^{\mathcal{Q}}(\cdot,\omega) - f(\cdot,\omega))d\mu - rz^{\mathcal{Q}}(\omega)$. Since $\|b^{\mathcal{Q}}_n(\omega)\| \to 0$ as $n \to \infty$, there is an $n_{\mathcal{Q}}$ such that $b^{\mathcal{Q}}_{n_{\mathcal{Q}}}(\omega) \in U(r,\mathcal{Q})$ for all $\omega \in \Omega$ and $\mu(E^{\mathcal{Q}}_{n_{\mathcal{Q}}}) < \mu(S\cap T_{\mathcal{Q}})$. Consider the function $g^{\mathcal{Q}} : (S\cap T_{\mathcal{Q}}) \times \Omega \to Y_+$ defined by

$$g^{\mathcal{Q}}(t,\omega) = \begin{cases} y^{\mathcal{Q}}(t,\omega), & \text{if } (t,\omega) \in E^{\mathcal{Q}}_{n_{\mathcal{Q}}} \times \Omega; \\ f(t,\omega) + \frac{rx^{\mathcal{Q}}}{\mu\left((S\cap T_{\mathcal{Q}})\setminus E^{\mathcal{Q}}_{n_{\mathcal{Q}}}\right)}, & \text{if } (t,\omega) \in \left((S\cap T_{\mathcal{Q}}) \setminus E^{\mathcal{Q}}_{n_{\mathcal{Q}}}\right) \times \Omega. \end{cases}$$

By (A$_3$), we have $V_t(g^{\mathcal{Q}}(t,\cdot)) > V_t(f(t,\cdot))$ for almost all $t \in S\cap T_{\mathcal{Q}}$ and $g^{\mathcal{Q}}$ is an $(S\cap T_{\mathcal{Q}})$-allocation. Thus, we have

$$\int_{S\cap T_{\mathcal{Q}}} g^{\mathcal{Q}}(\cdot,\omega)d\mu \quad = \quad \int_{E^{\mathcal{Q}}_{n_{\mathcal{Q}}}} (y^{\mathcal{Q}}(\cdot,\omega) - f(\cdot,\omega))d\mu + \int_{S\cap T_{\mathcal{Q}}} f(\cdot,\omega)d\mu + rx^{\mathcal{Q}}.$$

Furthermore, for all $\omega \in \Omega$,

$$\int_{E^{\mathcal{Q}}_{n_{\mathcal{Q}}}} (y^{\mathcal{Q}}(\cdot,\omega) - f(\cdot,\omega))d\mu - b^{\mathcal{Q}}_{n_{\mathcal{Q}}}(\omega) = r\int_{S\cap T_{\mathcal{Q}}} (y^{\mathcal{Q}}(\cdot,\omega) - f(\cdot,\omega))d\mu.$$

Consequently, we obtain

$$\int_{S\cap T_{\mathcal{Q}}} g^{\mathcal{Q}}(\cdot,\omega)d\mu \ll \int_{S\cap T_{\mathcal{Q}}} (ry^{\mathcal{Q}}(\cdot,\omega) + (1-r)f(\cdot,\omega))d\mu + 2rx^{\mathcal{Q}}$$

for each $\omega \in \Omega$, which implies that for each $\omega \in \Omega$,

$$\int_{S\cap T_{\mathcal{Q}}} g^{\mathcal{Q}}(\cdot,\omega)d\mu \ll \int_{S\cap T_{\mathcal{Q}}} (rg(\cdot,\omega) + (1-r)f(\cdot,\omega))d\mu.$$

We now define a $\mathcal{Q}$-measurable $d^{\mathcal{Q}} : \Omega \to Y_+$ such that for each $\omega \in \Omega$,

$$d^{\mathcal{Q}}(\omega) = \frac{1}{\mu(S\cap T_{\mathcal{Q}})}\left[\int_{S\cap T_{\mathcal{Q}}} (rg(\cdot,\omega) + (1-r)f(\cdot,\omega))d\mu - \int_{S\cap T_{\mathcal{Q}}} g^{\mathcal{Q}}(\cdot,\omega)d\mu\right].$$

Clearly, $d^{\mathcal{Q}}(\omega) \gg 0$ for each $\omega \in \Omega$. Define an $(S\cap T_{\mathcal{Q}})$-allocation by $h^{\mathcal{Q}}(t,\omega) = g^{\mathcal{Q}}(t,\omega) + d^{\mathcal{Q}}(\omega)$ for all $(t,\omega) \in (S\cap T_{\mathcal{Q}}) \times \Omega$. Then, $V_t(h^{\mathcal{Q}}(t,\cdot)) > V_t(f(t,\cdot))$ for almost all $t \in S\cap T_{\mathcal{Q}}$ and $\int_{S\cap T_{\mathcal{Q}}} h^{\mathcal{Q}}(\cdot,\omega)d\mu = \int_{S\cap T_{\mathcal{Q}}}(rg(\cdot,\omega) + (1-r)f(\cdot,\omega))d\mu$ for all $\omega \in \Omega$. Let $h : S\times\Omega \to Y_+$ be defined by

$$h(t,\omega) = \begin{cases} h^{\mathcal{Q}}(t,\omega), & \text{if } (t,\omega) \in (S\cap T_{\mathcal{Q}})\times\Omega \text{ and } \mathcal{Q} \in \mathfrak{P}(S)\ ; \\ g(t,\omega), & \text{otherwise.} \end{cases}$$

It can be readily checked that h is the desired S-allocation. □

3.2. **Allocations with the exact feasibility.** In this subsection, we provide a characterization of exactly feasible allocations of $\mathcal{E}$ that are not in various types of cores. Given a coalition S of $\mathcal{E}$, an S-assignment f in $\mathcal{E}$ is called *S-exactly feasible* if $\int_S f(\cdot,\omega)d\mu = \int_S a(\cdot,\omega)d\mu$ for all $\omega \in \Omega$. For simplicity, T-exactly feasible assignment is just termed as exactly feasible assignment. An allocation f in $\mathcal{E}$ is *NY-strongly fine*[1] *blocked by a coalition S* [28] if there exist a sub-coalition S_0 and an S-exactly feasible assignment g such that $g(t,\cdot)$ is $\bigvee\mathfrak{P}_S$-measurable and $V_t(g(t,\cdot)) \geq V_t(f(t,\cdot))$ for almost all $t \in S$, and $V_t(g(t,\cdot)) > V_t(f(t,\cdot))$ for almost all $t \in S_0$. The *NY-strong fine core* [28] of $\mathcal{E}$ is the set of exactly feasible allocations which are not NY-strongly fine blocked by any coalition of $\mathcal{E}$.

Lemma 3.5. *Let an allocation f in $\mathcal{E}$ be NY-strongly fine blocked by a coalition S of $\mathcal{E}$. Under* (A$_1$)-(A$_2$), (A$'_3$) *and* (A$_5$)*, there exist a sub-coalition S' of S and an S-assignment g such that*

(i) *$g(t,\cdot)$ is $\bigvee\mathfrak{P}_S$-measurable and $V_t(g(t,\cdot)) > V_t(f(t,\cdot))$ for almost all $t \in S$,*
(ii) *$g(t,\omega) \gg 0$ for all $(t,\omega) \in S'\times\Omega$,*
(iii) *$\int_S(a(\cdot,\omega) - g(\cdot,\omega))d\mu \gg 0$ for all $\omega \in \Omega$.*

Proof. Since f is NY-strongly fine blocked by S, there are a sub-coalition S_0 of S and an S-exactly feasible assignment y such that $y(t,\cdot)$ is $\bigvee\mathfrak{P}_S$-measurable and $V_t(y(t,\cdot)) \geq V_t(f(t,\cdot))$ for almost all $t \in S$, and $V_t(y(t,\cdot)) > V_t(f(t,\cdot))$ for almost all $t \in S_0$. Without loss of generality, we may assume that $\mu(S_0) < \mu(S)$. Otherwise, the argument will be similar to that in Lemma 3.1. By (A$'_3$) and the fact that

[1]NY is the abbreviation of Nicholas Yannelis. Here, we follow some idea of his definition in [28], to distinguish it from the concept of Wilson in [27].

$V_t(y(t,\cdot)) > V_t(f(t,\cdot))$ for almost all $t \in S_0$, there exist an atom A of $\bigvee \mathfrak{P}_S$ and a sub-coalition S_1 of S_0 such that $y(t,\omega) > 0$ for all $\omega \in A$ and almost all $t \in S_1$. Let $\{c_m\}$ be a sequence in $(0,1)$ converging to 0. For each $m \geq 1$, we define a function $y_m : S_1 \times \Omega \to Y_+$ such that $y_m(t,\omega) = (1 - c_m)y(t,\omega)$. Then $y_m(t,\cdot)$ is $\bigvee \mathfrak{P}_S$-measurable for almost all $t \in S_1$. By an argument similar to that in the proof of Lemma 3.1, it can be shown that there is a sub-coalition S_m of S_1 such that $V_t(y_m(t,\cdot)) > V_t(f(t,\cdot))$ for almost all $t \in S_m$. Note that the function $b : A \to Y_+$, defined by $b(\omega) = \frac{c_m}{2} \int_{S_m} y(\cdot,\omega)d\mu$, is $\bigvee \mathfrak{P}_S$-measurable. Define a function $\hat{y} : (S \setminus S_0) \times \Omega \to Y_+$ by

$$\hat{y}(t,\omega) = \begin{cases} y(t,\omega) + \frac{b(\omega)}{\mu(S \setminus S_0)}, & \text{if } (t,\omega) \in (S \setminus S_0) \times A; \\ y(t,\omega), & \text{otherwise.} \end{cases}$$

Furthermore define another function $h : S \times \Omega \to Y_+$ by

$$h(t,\omega) = \begin{cases} \hat{y}(t,\omega), & \text{if } (t,\omega) \in (S \setminus S_0) \times \Omega; \\ y(t,\omega), & \text{if } (t,\omega) \in (S_0 \setminus S_m) \times \Omega; \\ y_m(t,\omega), & \text{if } (t,\omega) \in S_m \times \Omega. \end{cases}$$

Then, $\hat{y}(t,\cdot)$ is $\bigvee \mathfrak{P}_S$-measurable and by (A_3'), $V_t(\hat{y}(t,\cdot)) > V_t(f(t,\cdot))$ for almost all $t \in S \setminus S_0$. It follows that $h(t,\cdot)$ is $\bigvee \mathfrak{P}_S$-measurable and $V_t(h(t,\cdot)) > V_t(f(t,\cdot))$ for almost all $t \in S$, and $\int_S h(\cdot,\omega)d\mu \leq \int_S a(\cdot,\omega)d\mu$ for each $\omega \in \Omega$. Next, for each $m \geq 1$, define a function $h_m : S \times \Omega \to Y_+$ by $h_m(t,\omega) = (1-c_m)h(t,\omega) + \frac{c_m}{2} a(t,\omega)$. Clearly, $h_m(t,\cdot)$ is $\bigvee \mathfrak{P}_S$-measurable for almost all $t \in S$, and $h_m(t,\omega) \gg 0$ for all $(t,\omega) \in S \times \Omega$. Applying an argument similar to that in the proof of Lemma 3.1, one can find an increasing sequence $\{R_m\} \subseteq \Sigma_S$ such that $\bigcup_m R_m \sim S$ and $V_t(h_m(t,\cdot)) > V_t(f(t,\cdot))$ for almost all $t \in R_m$. Finally, for each $m \geq 1$, consider the function $g_m : S \times \Omega \to Y_+$ defined by

$$g_m(t,\omega) = \begin{cases} h(t,\omega), & \text{if } (t,\omega) \in (S \setminus R_m) \times \Omega; \\ h_m(t,\omega), & \text{if } (t,\omega) \in R_m \times \Omega. \end{cases}$$

Following from the steps at the end of the proof of Lemma 3.1, it can be verified that the conclusion of this lemma is true when m is sufficiently large. Hence, the proof is completed by selecting such an m and setting $S' = R_m$ and $g = g_m$. □

Theorem 3.6. *Let an exactly feasible allocation f be not in the NY-strong fine core of $\mathcal{E}$. Under* (A_1)-(A_2), (A_3')-(A_4') *and* (A_5), *for any $0 < \epsilon < \mu(T)$, there is a coalition S with $\mu(S) = \epsilon$ which NY-strongly fine blocks f.*

Proof. Suppose that f is NY-strongly fine blocked by a coalition S. By Lemma 3.5, there are a sub-coalition S' of S and an S-assignment g such that (i)-(iii) of Lemma 3.5 hold. Define a function $z : \Omega \to Y_+$ such that for all $\omega \in \Omega$,

$$z(\omega) = \int_S (a(\cdot,\omega) - g(\cdot,\omega))d\mu. \tag{3.2}$$

Then $z(\omega) \gg 0$ for all $\omega \in \Omega$. For any fixed $\mathcal{Q} \in \mathfrak{P}(S)$, by Lemma 3.3,

$$H_{\mathcal{Q}} = \mathrm{cl}\left\{ \left(\mu(E^{\mathcal{Q}}), \int_{E^{\mathcal{Q}}} (a-g)d\mu \right) \in \mathbb{R} \times Y^{\Omega} : E^{\mathcal{Q}} \in \Sigma_{S \cap T_{\mathcal{Q}}} \right\}$$

is convex. For any given $\delta \in (0,1)$, there is a sequence $\{E_n^{\mathcal{Q}}\} \subseteq \Sigma_{S\cap T_{\mathcal{Q}}}$ such that

$$\lim_{n\to\infty}\left(\mu(E_n^{\mathcal{Q}}),\int_{E_n^{\mathcal{Q}}}(a(\cdot,\omega)-g(\cdot,\omega))d\mu\right)=\delta(\mu(S\cap T_{\mathcal{Q}}),\ z^{\mathcal{Q}}(\omega))$$

for all $\omega\in\Omega$, where $z^{\mathcal{Q}}(\omega)=\int_{S\cap T_{\mathcal{Q}}}(a(\cdot,\omega)-g(\cdot,\omega))d\mu$. Since μ is atomless, we can select a sequence $\{F_n^{\mathcal{Q}}\}\subseteq\Sigma_{S\cap T_{\mathcal{Q}}}$ such that $\mu(F_n^{\mathcal{Q}})=\delta\mu(S\cap T_{\mathcal{Q}})$ and $\mu(F_n^{\mathcal{Q}}\Delta E_n^{\mathcal{Q}})=|\delta\mu(S\cap T_{\mathcal{Q}})-\mu(E_n^{\mathcal{Q}})|$. Indeed, if $\mu(E_n^{\mathcal{Q}})\geq\delta\mu(S\cap T_{\mathcal{Q}})$, we select any $F_n^{\mathcal{Q}}\subseteq E_n^{\mathcal{Q}}$ with $\mu(F_n^{\mathcal{Q}})=\delta\mu(S\cap T_{\mathcal{Q}})$; Otherwise, we first select $C_n^{\mathcal{Q}}\subseteq(S\cap T_{\mathcal{Q}})\setminus E_n^{\mathcal{Q}}$ with $\mu(C_n^{\mathcal{Q}})=\delta\mu(S\cap T_{\mathcal{Q}})-\mu(E_n^{\mathcal{Q}})$ and put $F_n^{\mathcal{Q}}=E_n^{\mathcal{Q}}\cup C_n^{\mathcal{Q}}$. As a result, $\lim_{n\to\infty}\mu(F_n^{\mathcal{Q}}\Delta E_n^{\mathcal{Q}})=0$, which implies that $\lim_{n\to\infty}\int_{F_n^{\mathcal{Q}}}(a(\cdot,\omega)-g(\cdot,\omega))d\mu=\delta z^{\mathcal{Q}}(\omega)$ for all $\omega\in\Omega$. Let

$$F_n=\left(\bigcup_{\mathcal{Q}\in\mathfrak{P}(S)}F_n^{\mathcal{Q}}\right)\bigcup\left(\bigcup_{\mathcal{Q}\in\mathfrak{P}_S\setminus\mathfrak{P}(S)}(S\cap T_{\mathcal{Q}})\right)$$

for all $n\in\mathbb{N}$. Then $\mu(F_n)=\delta\mu(S)$ and $\lim_{n\to\infty}\int_{F_n}(a(\cdot,\omega)-g(\cdot,\omega))d\mu=\delta z(\omega)$ for all $\omega\in\Omega$. Hence there is an n_0 such that $\int_{F_{n_0}}(a(\cdot,\omega)-g(\cdot,\omega))d\mu\gg 0$ for all $\omega\in\Omega$. Since $\bigvee\mathfrak{P}_{F_{n_0}}=\bigvee\mathfrak{P}_S$, the function $z_{n_0}:\Omega\to Y_+$, defined by $z_{n_0}(\omega)=\int_{F_{n_0}}(a(\cdot,\omega)-g(\cdot,\omega))d\mu$, is $\bigvee\mathfrak{P}_{F_{n_0}}$-measurable. Define a function $\hat{g}:F_{n_0}\times\Omega\to Y_+$ such that $\hat{g}(t,\omega)=g(t,\omega)+\frac{z_{n_0}(\omega)}{\delta\mu(S)}$. By (A$_3$), f is NY-strongly fine blocked by F_{n_0} via $\hat{g}$, which proves the theorem for $\epsilon\leq\mu(S)$. If $\mu(S)=\mu(T)$, the proof has been completed. Otherwise, $\mu(T\setminus S)>0$. Let $R=T\setminus S$. Again by Lemma 3.3,

$$G_{\mathcal{Q}}=\mathrm{cl}\left\{\left(\mu(B^{\mathcal{Q}}),\int_{B^{\mathcal{Q}}}(a-f)d\mu\right)\in\mathbb{R}\times Y^{\Omega}:B^{\mathcal{Q}}\in\Sigma_{R\cap T_{\mathcal{Q}}}\right\}$$

is convex for all $\mathcal{Q}\in\mathfrak{P}(R)$. Given any $\alpha\in(0,1)$ and $\mathcal{Q}\in\mathfrak{P}(R)$, applying an argument similar to the previous one, one can find a sequence $\{B_n^{\mathcal{Q}}\}\subseteq\Sigma_{R\cap T_{\mathcal{Q}}}$ such that $\mu(B_n^{\mathcal{Q}})=(1-\alpha)\mu(R\cap T_Q)$ and for all $\omega\in\Omega$,

$$\lim_{n\to\infty}\int_{B_n^{\mathcal{Q}}}(a(\cdot,\omega)-f(\cdot,\omega))d\mu=(1-\alpha)\kappa^{\mathcal{Q}}(\omega),$$

where $\kappa^{\mathcal{Q}}(\omega)=\int_{R\cap T_{\mathcal{Q}}}(a(\cdot,\omega)-f(\cdot,\omega))d\mu$. Let

$$B_n=\left(\bigcup_{\mathcal{Q}\in\mathfrak{P}(R)}B_n^{\mathcal{Q}}\right)\bigcup\left(\bigcup_{\mathcal{Q}\in\mathfrak{P}_R\setminus\mathfrak{P}(R)}(R\cap T_{\mathcal{Q}})\right)$$

for all $n\in\mathbb{N}$ and $\kappa(\omega)=\int_R(a(\cdot,\omega)-f(\cdot,\omega))d\mu$ for all $\omega\in\Omega$. For all $n\geq 1$, define a function $b_n:\Omega\to Y_+$ such that

$$b_n(\omega)=(1-\alpha)\kappa(\omega)-\int_{B_n}(a(\cdot,\omega)-f(\cdot,\omega))d\mu. \tag{3.3}$$

Then b_n is $\bigvee\mathfrak{P}_{B_n}$-measurable for all $n\geq 1$, and $\|b_n(\omega)\|\to 0$ as $n\to\infty$ for all $\omega\in\Omega$. Choose an n_1 satisfying $\alpha z(\omega)-b_{n_1}(\omega)\gg 0$ for all $\omega\in\Omega$, define $g_\alpha:S\times\Omega\to Y_+$ such that

$$g_\alpha(t,\omega)=\alpha g(t,\omega)+(1-\alpha)f(t,\omega)+\frac{1}{\mu(S)}(\alpha z(\omega)-b_{n_1}(\omega)),$$

and take $\widetilde{S}=S\cup B_{n_1}$. Note that $\mu(\widetilde{S})=\mu(S)+(1-\alpha)\mu(T\setminus S)$ and g_α is $\bigvee\mathfrak{P}_{\widetilde{S}}$-measurable for almost all $t\in S$. By (A$_3'$) and (A$_4'$), $V_t(g_\alpha(t,\cdot))>V_t(f(t,\cdot))$ for almost all $t \in S$. It remains to verify that f is NY-strongly fine blocked by $\widetilde{S}$. To this end, define $y_\alpha : \widetilde{S} \times \Omega \to Y_+$ by

$$y_\alpha(t,\omega) = \begin{cases} g_\alpha(t,\omega), & \text{if } (t,\omega) \in S \times \Omega; \\ f(t,\omega), & \text{if } (t,\omega) \in B_{n_1} \times \Omega. \end{cases}$$

Then $y_\alpha(t,\cdot)$ is $\bigvee \mathfrak{P}_{\widetilde{S}}$-measurable and $V_t(y_\alpha(t,\cdot)) \geq V_t(f(t,\cdot))$ for almost all $t \in \widetilde{S}$, and $V_t(y_\alpha(t,\cdot)) > V_t(f(t,\cdot))$ for almost all $t \in S$. Using (3.2) and (3.3), one has

$$\int_{\widetilde{S}} (a(\cdot,\omega) - y_\alpha(\cdot,\omega)) d\mu = (1-\alpha) \int_T (a(\cdot,\omega) - f(\cdot,\omega)) d\mu = 0$$

for all $\omega \in \Omega$. This completes the proof. □

An allocation f in $\mathcal{E}$ is *NY-fine blocked by a coalition* S [28] if there is an S-exactly feasible assignment g such that $g(t,\cdot)$ is $\bigvee \mathfrak{P}_S$-measurable and $V_t(g(t,\cdot)) > V_t(f(t,\cdot))$ for almost all $t \in S$. The *NY-fine core* [28] of $\mathcal{E}$ is the set of exactly feasible allocations which are not NY-fine blocked by any coalition of $\mathcal{E}$.

Remark 3.7. Under (A_1)-(A_3), (A'_3)-(A'_4) and (A_5), an analogous result can be derived for allocations not in the NY-fine core of $\mathcal{E}$ by modifying the functions g_α and y_ϵ in the following way:

$$g_\alpha(t,\omega) = \alpha g(t,\omega) + (1-\alpha) f(t,\omega) + \frac{1}{\mu(S)} (\alpha z(\omega) - b_{n_1}(\omega) - x),$$

and

$$y_\alpha(t,\omega) = \begin{cases} g_\alpha(t,\omega), & \text{if } (t,\omega) \in S \times \Omega; \\ f(t,\omega) + \frac{x}{\mu(B_{n_1})}, & \text{if } (t,\omega) \in B_{n_1} \times \Omega, \end{cases}$$

where $x \gg 0$ such that $\alpha z(\omega) - b_{n_1}(\omega) - x \gg 0$.

Definition 3.8. An allocation f in $\mathcal{E}$ is *NY-privately blocked by a coalition* S [28] if there exists an S-exactly feasible allocation g such that $V_t(g(t,\cdot)) > V_t(f(t,\cdot))$ for almost all $t \in S$. The *NY-private core* [28] of $\mathcal{E}$ is the set of exactly feasible allocations which are not NY-privately blocked by any coalition of $\mathcal{E}$.

Now, we are ready to present one of the main results of this paper, which completely answers a question of Pesce in [20, Remark 1].

Theorem 3.9. *Assume that f is an exactly feasible allocation in $\mathcal{E}$ which is not in the NY-private core and $0 < \epsilon < \mu(T)$. Under* (A_1)-(A_3), (A'_4) *and* (A_5), *f is NY-privately blocked by some coalition S with $\mu(S) = \epsilon$.*

Proof. Since f is not in the NY-private core of $\mathcal{E}$, there exist a coalition S and an S-exactly feasible allocation g such that $V_t(g(t,\cdot)) > V_t(f(t,\cdot))$ for almost all $t \in S$. For all $\omega \in \Omega$ and $\mathcal{Q} \in \mathfrak{P}(S)$, let

$$e_{\mathcal{Q}}(\omega) = \frac{1}{\mu(S \cap T_{\mathcal{Q}})} \int_{S \cap T_{\mathcal{Q}}} a(\cdot,\omega) d\mu.$$

Choose an $e \gg 0$ such that $e \leq \frac{e_{\mathcal{Q}}(\omega)}{3}$ for all $\omega \in \Omega$ and $\mathcal{Q} \in \mathfrak{P}(S)$, an open ball U with center 0 and radius $\epsilon > 0$ such that $e - U \subseteq \text{int}Y_+$ and a $\lambda \in (0,1)$. Let $\{c_m\}$ be a sequence in $(0,1)$ such that $c_m \to 0$ as $m \to \infty$. Pick an arbitrary element $\mathcal{Q} \in \mathfrak{P}(S)$, and define a function $g_m^{\mathcal{Q}} : (S \cap T_{\mathcal{Q}}) \times \Omega \to Y_+$ such that $g_m^{\mathcal{Q}}(t,\omega) = (1-c_m) g(t,\omega) + c_m(e_{\mathcal{Q}}(\omega) - 2e)$. By an argument similar to that in Lemma 3.1, one can find an increasing sequence $\{S_m^{\mathcal{Q}}\} \subseteq \Sigma_{S\cap T_{\mathcal{Q}}}$ such that $\bigcup_m S_m^{\mathcal{Q}} \sim S \cap T_{\mathcal{Q}}$, $\lim_{n\to\infty}((S\cap T_{\mathcal{Q}}) \setminus S_m^{\mathcal{Q}}) = 0$ and $V_t(g_m^{\mathcal{Q}}(t,\cdot)) > V_t(f(t,\cdot))$ for almost all $t \in S_m^{\mathcal{Q}}$. By absolute continuity of the Bochner integral, there is some $\delta > 0$ such that

$$\frac{2}{\mu(S\cap T_{\mathcal{Q}})} \int_{R_{\mathcal{Q}}} (g(\cdot,\omega) - e_{\mathcal{Q}}(\omega))\, d\mu \in U$$

for all $R_{\mathcal{Q}} \in \Sigma_{S\cap T_{\mathcal{Q}}}$ with $\mu(R_{\mathcal{Q}}) < \delta$ and $\mathcal{Q} \in \mathfrak{P}(S)$. For each $\mathcal{Q} \in \mathfrak{P}(S)$, choose an $m_{\mathcal{Q}}$ such that

$$\mu\left(S_{m_{\mathcal{Q}}}^{\mathcal{Q}}\right) > \left(1 - \frac{\lambda}{2}\right) \mu(S\cap T_{\mathcal{Q}})$$

and $\mu((S\cap T_{\mathcal{Q}}) \setminus S_{m_{\mathcal{Q}}}^{\mathcal{Q}}) < \delta$. Let $m_0 = \max\{m_{\mathcal{Q}} : \mathcal{Q} \in \mathfrak{P}(S)\}$. It follows that

$$\frac{1}{\mu(S_{m_0}^{\mathcal{Q}})} \int_{(S\cap T_{\mathcal{Q}})\setminus S_{m_0}^{\mathcal{Q}}} (g(\cdot,\omega) - e_{\mathcal{Q}}(\omega))\, d\mu \in U$$

for all $\mathcal{Q} \in \mathfrak{P}(S)$. For each $\mathcal{Q} \in \mathfrak{P}(S)$ and $(t,\omega) \in S_{m_0}^{\mathcal{Q}} \times \Omega$, set

$$x(t,\omega) = e_{\mathcal{Q}}(\omega) - \frac{1}{\mu(S_{m_0}^{\mathcal{Q}})} \int_{(S\cap T_{\mathcal{Q}})\setminus S_{m_0}^{\mathcal{Q}}} (g(\cdot,\omega) - e_{\mathcal{Q}}(\omega))\, d\mu.$$

Consider a function $y^{\mathcal{Q}} : (S\cap T_{\mathcal{Q}}) \times \Omega \to Y_+$ defined by

$$y^{\mathcal{Q}}(t,\omega) = \begin{cases} (1-c_{m_0})g(t,\omega) + c_{m_0}x(t,\omega), & \text{if } (t,\omega) \in S_{m_0}^{\mathcal{Q}} \times \Omega; \\ g(t,\omega), & \text{otherwise.} \end{cases}$$

Since $y^{\mathcal{Q}}(t,\omega) \gg g_{m_0}^{\mathcal{Q}}(t,\omega) + c_{m_0}e$ for all $(t,\omega) \in S_{m_0}^{\mathcal{Q}} \times \Omega$, by (A$_3$), $V_t(y^{\mathcal{Q}}(t,\cdot)) > V_t(f(t,\cdot))$ for almost all $t \in S\cap T_{\mathcal{Q}}$ and $y^{\mathcal{Q}}$ is an $(S\cap T_{\mathcal{Q}})$-allocation. Moreover,

$$\int_{S\cap T_{\mathcal{Q}}} y^{\mathcal{Q}}(\cdot,\omega)d\mu = \int_{S\cap T_{\mathcal{Q}}} ((1-c_{m_0})g(\cdot,\omega) + c_{m_0}a(\cdot,\omega))\, d\mu \tag{3.4}$$

for all $\omega \in \Omega$. By Lemma 3.3, the set

$$H_{\mathcal{Q}} = \mathrm{cl}\left\{\left(\mu(E^{\mathcal{Q}}), \int_{E^{\mathcal{Q}}} \left(y^{\mathcal{Q}} - a\right) d\mu\right) \in \mathbb{R} \times Y^{\Omega} : E^{\mathcal{Q}} \in \Sigma_{S\cap T_{\mathcal{Q}}}\right\}$$

is convex. Using an argument similar to that in the proof of Theorem 3.6, one can find a sequence $\{F_n^{\mathcal{Q}}\} \subseteq \Sigma_{S\cap T_{\mathcal{Q}}}$ such that $\mu(F_n^{\mathcal{Q}}) = \lambda\mu(S\cap T_{\mathcal{Q}})$ and for all $\omega \in \Omega$,

$$\lim_{n\to\infty} \int_{F_n^{\mathcal{Q}}} (y^{\mathcal{Q}}(\cdot,\omega) - a(\cdot,\omega))d\mu = \lambda z^{\mathcal{Q}}(\omega),$$

where

$$z^{\mathcal{Q}}(\omega) = \int_{S\cap T_{\mathcal{Q}}} \left(y^{\mathcal{Q}}(\cdot,\omega) - a(\cdot,\omega)\right) d\mu. \tag{3.5}$$

The function $b_n^{\mathcal{Q}} : \Omega \to Y_+$, defined by

$$b_n^{\mathcal{Q}}(\omega) = \lambda z^{\mathcal{Q}}(\omega) - \int_{F_n^{\mathcal{Q}}} \left(y^{\mathcal{Q}}(\cdot,\omega) - a(\cdot,\omega)\right) d\mu,$$

is $\mathcal{Q}$-measurable for all $n \geq 1$ and $\|b_n^{\mathcal{Q}}(\omega)\| \to 0$ as $n \to \infty$ for all $\omega \in \Omega$. Note that $\min\left\{\mu(F_n^{\mathcal{Q}} \cap S_{m_0}^{\mathcal{Q}}) : n \geq 1\right\} \geq \frac{\lambda}{2}\mu(S\cap T_{\mathcal{Q}}) > 0$. Choose an $n_{\mathcal{Q}}$ such that

$\frac{2b^{\mathcal{Q}}_{n_{\mathcal{Q}}}(\omega)}{\lambda\mu(S\cap T_{\mathcal{Q}})} \in c_{m_0}U$ for all $\omega \in \Omega$. Then $c_{m_0}e + \frac{b^{\mathcal{Q}}_{n_{\mathcal{Q}}}(\omega)}{\mu\left(F^{\mathcal{Q}}_{n_{\mathcal{Q}}}\cap S^{\mathcal{Q}}_{m_0}\right)} \gg 0$ for all $\omega \in \Omega$. Define a function $g^{\mathcal{Q}} : F^{\mathcal{Q}}_{n_{\mathcal{Q}}} \times \Omega \to Y_+$ such that

$$g^{\mathcal{Q}}(t,\omega) = \begin{cases} y^{\mathcal{Q}}(t,\omega) + \frac{b^{\mathcal{Q}}_{n_{\mathcal{Q}}}(\omega)}{\mu\left(F^{\mathcal{Q}}_{n_{\mathcal{Q}}}\cap S^{\mathcal{Q}}_{m_0}\right)}, & \text{if } (t,\omega) \in \left(F^{\mathcal{Q}}_{n_{\mathcal{Q}}} \cap S^{\mathcal{Q}}_{m_0}\right) \times \Omega; \\ y^{\mathcal{Q}}(t,\omega), & \text{otherwise.} \end{cases}$$

By (A_3) and the fact that $V_t\left(g^{\mathcal{Q}}_{m_0}(t,\cdot)\right) > V_t(f(t,\cdot))$ for almost all $t \in F^{\mathcal{Q}}_{n_{\mathcal{Q}}} \cap S^{\mathcal{Q}}_{m_0}$, we have $V_t(g^{\mathcal{Q}}(t,\cdot)) > V_t(f(t,\cdot))$ for almost all $t \in F^{\mathcal{Q}}_{n_{\mathcal{Q}}} \cap S^{\mathcal{Q}}_{m_0}$. So, $g^{\mathcal{Q}}$ is $F^{\mathcal{Q}}_{n_{\mathcal{Q}}}$-allocation and $V_t(g^{\mathcal{Q}}(t,\cdot)) > V_t(f(t,\cdot))$ for almost all $t \in F^{\mathcal{Q}}_{n_{\mathcal{Q}}}$. Furthermore,

$$\int_{F^{\mathcal{Q}}_{n_{\mathcal{Q}}}} (g^{\mathcal{Q}}(\cdot,\omega) - a(\cdot,\omega))d\mu = \lambda z^{\mathcal{Q}}(\omega) \tag{3.6}$$

for all $\omega \in \Omega$. Let $F = \bigcup\{F^{\mathcal{Q}}_{n_{\mathcal{Q}}} : \mathcal{Q} \in \mathfrak{P}(S)\}$. So $\mu(F) = \lambda\mu(S)$. Define a function $h : F \times \Omega \to Y_+$ such that $h(t,\omega) = g^{\mathcal{Q}}(t,\omega)$ if $(t,\omega) \in F^{\mathcal{Q}}_{n_{\mathcal{Q}}} \times \Omega$. Then h is an F-allocation and $V_t(h(t,\cdot)) > V_t(f(t,\cdot))$ for almost all $t \in F$. By (3.4)-(3.6), we have $\int_F (h(\cdot,\omega) - a(\cdot,\omega))d\mu = 0$ for all $\omega \in \Omega$. Thus, f is NY-privately blocked by F via h. This proves the theorem for $\epsilon \le \mu(S)$. If $\mu(S) = \mu(T)$, the proof has been completed. Otherwise, $\mu(T \setminus S) > 0$. Let $S' = \bigcup\{S^{\mathcal{Q}}_{m_0} : \mathcal{Q} \in \mathfrak{P}(S)\}$. Let $A = T \setminus S$ and $u = \frac{\lambda c_{m_0}\mu(S')e}{2(1-\lambda)\mu(A)}$. Again pick an arbitrary element $\mathcal{Q} \in \mathfrak{P}(A)$. By Lemma 3.3,

$$G_{\mathcal{Q}} = \mathrm{cl}\left\{\left(\mu(B^{\mathcal{Q}}), \int_{B^{\mathcal{Q}}} (a - f - u)\, d\mu\right) \in \mathbb{R} \times Y^{\Omega} : B^{\mathcal{Q}} \in \Sigma_{A\cap T_{\mathcal{Q}}}\right\}$$

is convex. Hence, there exists a sequence $\{B^{\mathcal{Q}}_k\} \subseteq \Sigma_{A\cap T_{\mathcal{Q}}}$ such that $\mu(B^{\mathcal{Q}}_k) = (1-\lambda)\mu(A \cap T_{\mathcal{Q}})$ and for all $\omega \in \Omega$,

$$\lim_{k\to\infty} \int_{B^{\mathcal{Q}}_k} (a(\cdot,\omega) - f(\cdot,\omega) - u)d\mu = (1-\lambda)v^{\mathcal{Q}}(\omega),$$

where

$$v^{\mathcal{Q}}(\omega) = \int_{A\cap T_{\mathcal{Q}}} (a(\cdot,\omega) - f(\cdot,\omega) - u)d\mu. \tag{3.7}$$

The function $d^{\mathcal{Q}}_k : \Omega \to Y_+$, defined by

$$d^{\mathcal{Q}}_k(\omega) = (1-\lambda)v^{\mathcal{Q}}(\omega) - \int_{B^{\mathcal{Q}}_k} (a(\cdot,\omega) - f(\cdot,\omega) - u)\, d\mu,$$

is $\mathcal{Q}$-measurable for all $k \ge 1$ and $\|d^{\mathcal{Q}}_k(\omega)\| \to 0$ as $k \to \infty$ for all $\omega \in \Omega$. Choose a $k_{\mathcal{Q}}$ such that $u - \frac{d^{\mathcal{Q}}_{k_{\mathcal{Q}}}(\omega)}{(1-\lambda)\mu(A\cap T_{\mathcal{Q}})} \gg 0$ for each $\omega \in \Omega$. It is obvious that the function $f^{\mathcal{Q}} : B^{\mathcal{Q}}_{k_{\mathcal{Q}}} \times \Omega \to Y_+$, defined by

$$f^{\mathcal{Q}}(t,\omega) = f(t,\omega) + u - \frac{d^{\mathcal{Q}}_{k_{\mathcal{Q}}}(\omega)}{(1-\lambda)\mu(A\cap T_{\mathcal{Q}})},$$

is an $B^{\mathcal{Q}}_{k_{\mathcal{Q}}}$-allocation. By ($\mathrm{A}_3$), $V_t(f^{\mathcal{Q}}(t,\cdot)) > V_t(f(t,\cdot))$ for almost all $t \in B^{\mathcal{Q}}_{k_{\mathcal{Q}}}$. Furthermore, for each $\omega \in \Omega$,

$$\int_{B^{\mathcal{Q}}_{k_{\mathcal{Q}}}} (a(\cdot,\omega) - f^{\mathcal{Q}}(\cdot,\omega))d\mu = (1-\lambda)v^{\mathcal{Q}}(\omega). \tag{3.8}$$

Let $B = \bigcup\{B^{\mathcal{Q}}_{k_{\mathcal{Q}}} : \mathcal{Q} \in \mathfrak{P}(A)\}$. Then, $\mu(B) = (1-\lambda)\mu(A)$. Now, define a function $f_\lambda : B \times \Omega \to Y_+$ such that $f_\lambda(t,\omega) = f^{\mathcal{Q}}(t,\omega)$ if $(t,\omega) \in B^{\mathcal{Q}}_{k_{\mathcal{Q}}} \times \Omega$, and for any $\mathcal{Q} \in \mathfrak{P}(S)$, consider the function $\hat{y}^{\mathcal{Q}} : S \cap T_{\mathcal{Q}} \to Y_+$ defined by

$$\hat{y}^{\mathcal{Q}}(t,\omega) = \begin{cases} y^{\mathcal{Q}}(t,\omega) - \frac{c_{m_0}}{2}e, & \text{if } (t,\omega) \in S^{\mathcal{Q}}_{m_0} \times \Omega; \\ y^{\mathcal{Q}}(t,\omega), & \text{otherwise.} \end{cases}$$

Since $\hat{y}^{\mathcal{Q}}(t,\omega) \gg g^{\mathcal{Q}}_{m_0}(t,\omega) + \frac{c_{m_0}}{2}e$ for all $(t,\omega) \in S^{\mathcal{Q}}_{m_0} \times \Omega$, by ($A_3$), $V_t(\hat{y}^{\mathcal{Q}}(t,\cdot)) > V_t(f(t,\cdot))$ for almost all $t \in S \cap T_{\mathcal{Q}}$. Note that $\hat{y}^{\mathcal{Q}}$ is an $(S \cap T_{\mathcal{Q}})$-allocation. Take $\widehat{S} = \bigcup\{S \cap T_{\mathcal{Q}} : \mathcal{Q} \in \mathfrak{P}(S)\}$. Then, $\mu(\widehat{S}) = \mu(S)$. Define $y_\lambda : \widehat{S} \times \Omega \to Y_+$ by $y_\lambda(t,\omega) = \hat{y}^{\mathcal{Q}}(t,\omega)$ if $(t,\omega) \in (S \cap T_{\mathcal{Q}}) \times \Omega$. It can be checked that for each $\omega \in \Omega$,

$$\int_{\widehat{S}} a(\cdot,\omega)d\mu - \int_{\widehat{S}} y_\lambda(\cdot,\omega)d\mu = \frac{c_{m_0}\mu(S')}{2}e. \tag{3.9}$$

Consider $h_\lambda : \widehat{S} \times \Omega \to Y_+$ defined by $h_\lambda(t,\omega) = \lambda y_\lambda(t,\omega) + (1-\lambda)f(t,\omega)$. By ($A'_4$), $V_t(h_\lambda(t,\cdot)) > V_t(f(t,\cdot))$ for almost all $t \in \widehat{S}$, and further h_λ is an $\widehat{S}$-allocation. Let $\widetilde{S} = \widehat{S} \cup B$. Since $\mu(\widetilde{S}) = \mu(S) + (1-\lambda)\mu(T \setminus S)$, it remains to verify that f is NY-privately blocked by $\widetilde{S}$. To show this, consider $g_\lambda : \widetilde{S} \times \Omega \to Y_+$ defined by

$$g_\lambda(t,\omega) = \begin{cases} h_\lambda(t,\omega), & \text{if } (t,\omega) \in \widehat{S} \times \Omega; \\ f_\lambda(t,\omega), & \text{if } (t,\omega) \in B \times \Omega. \end{cases}$$

Obviously, g_λ is an $\widetilde{S}$-allocation and $V_t(g_\lambda(t,\cdot)) > V_t(f(t,\cdot))$ for almost all $t \in \widetilde{S}$. Furthermore, using (3.7)- (3.9), it can be simply verified that

$$\int_{\widetilde{S}} (a(\cdot,\omega) - g_\lambda(\cdot,\omega))d\mu = (1-\lambda)\int_T (a(\cdot,\omega) - f(\cdot,\omega))d\mu = 0$$

holds for all $\omega \in \Omega$. This completes the proof. □

Remark 3.10. If Y is separable, then without (A'_4) the conclusions of Theorem 3.6, Remark 3.7 and Theorem 3.9 hold. Indeed, to restore the conclusions in Theorem 3.6 and Remark 3.7, note that $\int_S g d\mu, \int_S f d\mu$ are in the convex set $\mathrm{cl}\int_S P_f d\mu$. So, $\int_S (\alpha g + (1-\alpha)f)d\mu \in \mathrm{cl}\int_S P_f d\mu$ and by (A'_3), $\int g_\alpha d\mu \in \int_S P_f d\mu$. Similarly, to restore the conclusion of Theorem 3.9, note that $\int_S g^{\mathcal{Q}}_{m_0} d\mu$ and $\int_S f d\mu$ are elements of the convex set $\mathrm{cl}\int_S P_f d\mu$. Thus, $\int_S (\lambda g^{\mathcal{Q}}_{m_0} + (1-\lambda)f)d\mu \in \mathrm{cl}\int_S P_f d\mu$ and by (A_3), $\int h_\lambda d\mu \in \int_S P_f d\mu$.

4. Robust efficiency and different types of cores of mixed market economies

In this section, we study cores and Walrasian expectations allocations in mixed economies. We characterize Walrasian expectations allocations in terms of robust efficiency, and establish relationships among various types of cores. To achieve these goals, we associate the mixed economy $\mathcal{E}$ in Section 2 with an atomless economy $\mathcal{E}^*$, and then apply results established in Section 3. The space of agents of $\mathcal{E}^*$ is denoted by (T^*, Σ^*, μ^*), where $T^* = T_0 \cup T_1^*$ and T_1^* is an atomless measure space such that $\mu^*(T_1^*) = \mu(T_1)$ and $T_0 \cap T_1^* = \emptyset$. We assume that (T^*, Σ^*, μ^*) is obtained by the direct sum of $(T_0, \Sigma_{T_0}, \mu_{T_0})$ and the measure space T_1^*, where μ_{T_0} is the restriction of μ to T_0. It is also assumed that each agent $A \in T_1$ one-to-one corresponds to a measurable subset A^* of T_1^* with $\mu^*(A^*) = \mu(A)$. Each agent

$t \in A^*$ is characterized by the private information set $\mathcal{F}_t = \mathcal{F}_A$; the consumption set Y_+ in each state $\omega \in \Omega$; the initial endowment $a(t, \cdot) = a(A, \cdot)$; the utility function $U_t = U_A$; and the prior $q_t = q_A$. Therefore, the ex ante expected utility function of every agent $t \in A^*$ is $V_t = V_A$.

4.1. **Robust efficiency.** In this subsection, we characterize a Walrasian expectations equilibrium of a mixed economy by the private blocking power of the grand coalition. For any coalition S, allocation f in $\mathcal{E}$ and any $0 \leq r \leq 1$, we introduce an asymmetric information economy $\mathcal{E}(S, f, r)$ which coincides with $\mathcal{E}$ except for the initial endowment allocation that is given by

$$a(S, f, r)(t, \cdot) = \begin{cases} a(t, \cdot), & \text{if } t \in T \setminus S; \\ (1 - r)a(t, \cdot) + rf(t, \cdot), & \text{if } t \in S. \end{cases}$$

A feasible allocation f in $\mathcal{E}$ is said to be *robustly efficient* [17] if f is not privately blocked by the grand coalition in every economy $\mathcal{E}(S, f, r)$.

Lemma 4.1. *Assume that an allocation f^* in $\mathcal{E}^*$ is privately blocked by a coalition S^* with $\mu^*(S^* \cap T_1^*) > 0$. Under* (A$_1$)-(A$_2$) *and* (A$_5$)*, for any $0 < \epsilon \leq \mu^*(S^* \cap T_1^*)$, there exist a coalition $R^* \subseteq \bigcup_{\mathcal{Q} \in \mathfrak{P}(T^*)} (S^* \cap T_{\mathcal{Q}}^*)$, a sub-coalition R_1^* of R^* and an R^*-allocation g^* such that*

(i) *$\int_{R^*} (a(\cdot, \omega) - g^*(\cdot, \omega)) d\mu^* \gg 0$ for all $\omega \in \Omega$ and $V_t(g^*(t, \cdot)) > V_t(f^*(t, \cdot))$ for almost all $t \in R^*$,*

(ii) *$g^*(t, \omega) \gg 0$ for all $(t, \omega) \in R_1^* \times \Omega$ and $\mathfrak{P}(R_1^*) = \mathfrak{P}(R^*)$,*

(iii) *$\mu^*(R^* \cap T_1^*) = \epsilon$ and $\mu^*(R^* \cap T_{\mathcal{Q}}^*) = \frac{\epsilon \mu^*(S^* \cap T_{\mathcal{Q}}^*)}{\mu^*(S^* \cap T_1^*)}$ for all $\mathcal{Q} \in \mathfrak{P}(S^*)$.*

Proof. If $\epsilon = \mu^*(S^* \cap T_1^*)$, the conclusion directly follows from Lemma 3.1. Assume $0 < \epsilon < \mu^*(S^* \cap T_1^*)$. Let $\delta = \frac{\epsilon}{\mu^*(S^* \cap T_1^*)}$ and $\alpha = 1 - \frac{\delta}{2}$. Applying Lemma 3.1, one has a sub-coalition S_1^* of S^* and an S^*-allocation g^* satisfying (i)-(iii) of Lemma 3.1. For each $\mathcal{Q} \in \mathfrak{P}(S^*)$, by Lemma 3.3, the set

$$H_{\mathcal{Q}} = \mathrm{cl} \left\{ \left(\mu^*(E^{\mathcal{Q}}), \mu^*(E^{\mathcal{Q}} \cap T_1^*), \int_{E^{\mathcal{Q}}} (a - g^*) d\mu^* \right) \in \mathbb{R}^2 \times Y^{\Omega} : E^{\mathcal{Q}} \in \Sigma^*_{S^* \cap T_{\mathcal{Q}}^*} \right\}$$

is convex. Similar to the proof of Theorem 3.6, for each $\mathcal{Q} \in \mathfrak{P}(S^*)$, there exists a sequence $\{E_n^{\mathcal{Q}}\} \subseteq \Sigma^*_{S^* \cap T_{\mathcal{Q}}^*}$ such that $\mu^*(E_n^{\mathcal{Q}}) = \delta \mu^*(S^* \cap T_{\mathcal{Q}}^*)$, $\mu^*(E_n^{\mathcal{Q}} \cap T_1^*) = \delta \mu^*(S^* \cap T_{\mathcal{Q}}^* \cap T_1^*)$ and

$$\lim_{n \to \infty} \int_{E_n^{\mathcal{Q}}} (a - g^*) d\mu^* = \delta \int_{S^* \cap T_{\mathcal{Q}}^*} (a - g^*) d\mu^*.$$

Since $\mu^*(S_1^* \cap T_{\mathcal{Q}}^*) > \alpha \mu^*(S^* \cap T_{\mathcal{Q}}^*)$ for all $\mathcal{Q} \in \mathfrak{P}(S^*)$, then $\mu^*(S_1^* \cap E_n^{\mathcal{Q}}) > 0$ for all $n \geq 1$ and all $\mathcal{Q} \in \mathfrak{P}(S^*)$. Let $E_n = \bigcup_{\mathcal{Q} \in \mathfrak{P}(S^*)} E_n^{\mathcal{Q}}$ for all $n \geq 1$. Then

$$\lim_{n \to \infty} \int_{E_n} (a - g^*) d\mu^* = \delta \int_{S^*} (a - g^*) d\mu^*.$$

Pick an n_0 such that $\int_{E_{n_0}} (a - g^*) d\mu^* \gg 0$, and put $R^* = E_{n_0}$, $R_1^* = R^* \cap S_1^*$. $\square$

Lemma 4.2. [11] *Assume Y is separable. Under* (A$_1$)-(A$_3$) *and* (A$_5$)*, f^* is a Walrasian expectations allocation of $\mathcal{E}^*$ if and only if it is in the private core of $\mathcal{E}^*$.*

Lemma 4.3. [11] *Assume that $\mathcal{E}$ satisfies* (A_1)-(A_3) *and* (A_5). *Let f^* be a feasible allocation of $\mathcal{E}^*$ and $0 < \epsilon < \mu^*(T^*)$. If f^* is not in the private core of $\mathcal{E}^*$, then there is a coalition S with $\mu^*(S) = \epsilon$ privately blocking f^*.*

The following lemma is similar to Theorem 3.5 in [8].

Lemma 4.4. *Assume that f is a robustly efficient allocation of $\mathcal{E}$. Under* (A_1)-(A_7), *there is an allocation $\hat{f}$ in $\mathcal{E}$ such that $\hat{f}|_{T_0\times\Omega} = f$, $\hat{f}(\cdot,\omega)$ is constant on T_1 for each $\omega \in \Omega$, $V_{T_1}(\hat{f}(t,\cdot)) = V_{T_1}(f(t,\cdot))$ for all $t \in T$ and $\int_T f d\mu = \int_T \hat{f} d\mu$.*

Proof. Consider the allocation $\hat{f} : T \times \Omega \to Y_+$ defined by

$$\hat{f}(t,\omega) = \begin{cases} f(t,\omega), & \text{if } (t,\omega) \in T_0 \times \Omega; \\ \frac{1}{\mu(T_1)} \int_{T_1} f(\cdot,\omega) d\mu, & \text{if } (t,\omega) \in T_1 \times \Omega. \end{cases}$$

To complete the proof, one only needs to verify $V_{T_1}(\hat{f}(t,\cdot)) = V_{T_1}(f(t,\cdot))$ holds for all $t \in T_1$. Suppose that there exists a coalition $D \subseteq T_1$ such that $V_{T_1}(\hat{f}(t,\cdot)) > V_{T_1}(f(t,\cdot))$ for all $t \in D$. Then applying an argument similar to that in Lemma 3.1, one can find some $r_1 \in (0,1)$ and a sub-coalition $C \subseteq D$ such that $V_{T_1}(r_1\hat{f}(t,\cdot)) > V_{T_1}(f(t,\cdot))$ for all $t \in C$. Let $r_2 = \frac{\mu(C)}{\mu(T_1)}$ and $r_3 = r_1 + \eta$ for some $\eta > 0$ such that $r_3 \in (0,1)$. Then $r_2 \in (0,1]$. Suppose that for each $\omega \in \Omega$,

$$\alpha(\omega) = r_2 r_3 \left(\int_T f(\cdot,\omega) d\mu - \int_T a(\cdot,\omega) d\mu \right) - r_2(1-r_3) \int_{T_1} a(\cdot,\omega) d\mu.$$

Note that $\alpha(\omega) \in -\mathrm{int}Y_+$ for each $\omega \in \Omega$. Choose an $\epsilon > 0$ such that for each $\omega \in \Omega$, $\alpha(\omega) + B(0,2\epsilon) \subseteq -\mathrm{int}Y_+$. By Lemma 3.3, $H = \mathrm{cl}\left\{ \int_E (f-a) \in Y^\Omega : E \in \Sigma_{T_0} \right\}$ is convex. So there is an $E_0 \in \Sigma_{T_0}$ such that $\| \int_{E_0}(f-a) - r_2r_3 \int_{T_0}(f-a) \| < \epsilon$. Pick an $u \in B(0,\epsilon) \cap \mathrm{int}Y_+$ and put $S = E_0 \cup C$. Then, $\mu(S) < \mu(T)$. Note that the function $g : S \times \Omega \to Y_+$, defined by

$$g(t,\omega) = \begin{cases} \hat{f}(t,\omega) + \frac{u}{2\mu(E_0)}, & \text{if } (t,\omega) \in E_0 \times \Omega; \\ r_3\hat{f}(t,\omega) + \frac{u}{2\mu(C)}, & \text{if } (t,\omega) \in C \times \Omega, \end{cases}$$

is an S-allocation and $V_t(g(t,\cdot)) > V_t(f(t,\cdot))$ for almost all $t \in S$. Further, $g(t,\omega) \gg 0$ for all $(t,\omega) \in S \times \Omega$ and $\int_S g(\cdot,\omega)d\mu = \int_{E_0} f(\cdot,\omega)d\mu + r_2r_3 \int_{T_1} f(\cdot,\omega)d\mu + u$ for all $\omega \in \Omega$. By (A_6), $\int_C a(\cdot,\omega)d\mu = r_2 \int_{T_1} a(\cdot,\omega)d\mu$ for all $\omega \in \Omega$. Then it can be easily verified that for all $\omega \in \Omega$,

$$-\alpha(\omega) + \int_S (g(\cdot,\omega) - a(\cdot,\omega))d\mu = \int_{E_0} (f-a) - r_2r_3 \int_{T_0} (f-a) + u \in B(0,2\epsilon).$$

It follows that $\int_S g(\cdot,\omega)d\mu - \int_S a(\cdot,\omega)d\mu \ll 0$ for all $\omega \in \Omega$. Select an $z \gg 0$ such that $\int_S a(\cdot,\omega)d\mu - \int_S g(\cdot,\omega)d\mu \gg z$ for each $\omega \in \Omega$ and pick an $r \in (0,1)$ such that $r_1\hat{f}(t,\omega) \le rg(t,\omega)$ for all $(t,\omega) \in C \times \Omega$. Note that the function $h_1 : C \times \Omega \to Y_+$, defined by $h_1(t,\omega) = r_1\hat{f}(t,\omega)$, is a C-allocation and $V_{T_1}(h_1(t,\cdot)) > V_{T_1}(f(t,\cdot))$ for all $t \in C$. By Lemma 3.4, there is an E_0-allocation $h_2 : E_0 \times \Omega \to Y_+$ such that $V_t(h_2(t,\cdot)) > V_t(f(t,\cdot))$ for almost all $t \in E_0$, and

$$\int_{E_0} h_2(\cdot,\omega)d\mu = \int_{E_0} \left(rg(\cdot,\omega) + (1-r)f(\cdot,\omega) \right) d\mu$$

for all $\omega \in \Omega$. Now, $h : S \times \Omega \to Y_+$, defined by

$$h(t,\omega) = \begin{cases} h_2(t,\omega), & \text{if } (t,\omega) \in E_0 \times \Omega; \\ h_1(t,\omega), & \text{if } (t,\omega) \in C \times \Omega, \end{cases}$$

is an S-allocation, $V_t(h(t,\cdot)) > V_t(f(t,\cdot))$ for almost all $t \in S$, and

$$\int_S h(\cdot,\omega)d\mu \le \int_S (rg(\cdot,\omega) + (1-r)f(\cdot,\omega))d\mu \text{ for all } \omega \in \Omega. \tag{4.1}$$

Define a function $y : T \times \Omega \to Y_+$ such that

$$y(t,\omega) = \begin{cases} h(t,\omega), & \text{if } (t,\omega) \in S \times \Omega; \\ f(t,\omega) + \frac{rz}{\mu(T\setminus S)}, & \text{if } (t,\omega) \in (T \setminus S) \times \Omega. \end{cases}$$

By (A_3), $V_t(y(t,\cdot)) > V_t(f(t,\cdot))$ for almost all $t \in T \setminus S$. Thus, y is an allocation and $V_t(y(t,\cdot)) > V_t(f(t,\cdot))$ for almost all $t \in T$. Furthermore, using (4.1) and $\int_S (a(\cdot,\omega) - g(\cdot,\omega))d\mu \gg z$, one can simply verify that for each $\omega \in \Omega$,

$$\int_T (y(\cdot,\omega) - a(T \setminus S, f, r)(\cdot,\omega))d\mu \le (1-r)\int_T (f(\cdot,\omega) - a(\cdot,\omega))d\mu \le 0.$$

This means that f is privately blocked by the grand coalition in $\mathcal{E}(T\setminus S, f, r)$, which contradicts with the fact that f is robustly efficient. So $V_{T_1}(f(t,\cdot)) \ge V_{T_1}(\hat{f}(t,\cdot))$ for all $t \in T_1$. Suppose that there is a coalition $W \subseteq T_1$ such that $V_{T_1}(f(t,\cdot)) > V_{T_1}(\hat{f}(t,\cdot))$ for all $t \in W$. By (A_4), one can easily derive

$$V_{T_1}(\hat{f}(t,\cdot)) > \frac{1}{\mu(T_1)} \int_{T_1} V_{T_1}(\hat{f}(t,\cdot))d\mu = V_{T_1}(\hat{f}(t,\cdot)),$$

which is a contradiction. Thus, $V_{T_1}(f(t,\cdot)) = V_{T_1}(\hat{f}(t,\cdot))$ for all $t \in T_1$. □

Next, in answering a question mentioned Hervés-Beloso and Moreno-García in [17, p.705], we provide a characterization of Walrasian expectations equilibria by the veto power of the grand coalition in a mixed economy with asymmetric information and an ordered separable Banach space whose positive cone has an interior point as the commodity space.

Theorem 4.5. *Assume that Y is separable. Under* (A_1)-(A_7)*, f is a Walrasian expectations allocation of $\mathcal{E}$ if and only if it is a robustly efficient allocation of $\mathcal{E}$.*

Proof. Suppose that f is a Walrasian expectations allocation of $\mathcal{E}$. Applying an argument similar to that in [17], one can show that it is robustly efficient.

Conversely, let f be a robustly efficient allocation of $\mathcal{E}$. By Lemma 4.4, there is an allocation $\hat{f}$ in $\mathcal{E}$ such that $\hat{f}|_{T_0\times\Omega} = f$, $\hat{f}(\cdot,\omega)$ is a constant $\mathbf{c}(\omega)$ on T_1 for each $\omega \in \Omega$, $V_{T_1}(\hat{f}(t,\cdot)) = V_{T_1}(f(t,\cdot))$ for all $t \in T_1$ and $\int_T f d\mu = \int_T \hat{f} d\mu$. Suppose that f is not a Walrasian expectations allocation of $\mathcal{E}$. Then $\hat{f}$ is not a Walrasian expectations allocation for $\mathcal{E}$. To see this, let $(\hat{f}, \pi)$ be a Walrasian expectations equilibrium for $\mathcal{E}$, $d \in \mathrm{int}Y_+$ and $\alpha > 0$. By (A_3), one has $V_t(f(t,\cdot) + \alpha d) > V_t(f(t,\cdot)) = V_t(\hat{f}(t,\cdot))$ for all $t \in T$. It follows that for almost all $t \in T$,

$$\sum_{\omega\in\Omega} \langle \pi(\omega), f(t,\omega) + \alpha d \rangle > \sum_{\omega\in\Omega} \langle \pi(\omega), \hat{f}(t,\omega) \rangle.$$

Letting $\alpha \to 0$, one has $\sum_{\omega\in\Omega}\langle \pi(\omega), f(t,\omega)\rangle \geq \sum_{\omega\in\Omega}\langle \pi(\omega), \hat{f}(t,\omega)\rangle$. So,

$$\sum_{\omega\in\Omega}\langle \pi(\omega), f(t,\omega)\rangle = \sum_{\omega\in\Omega}\langle \pi(\omega), \hat{f}(t,\omega)\rangle \leq \sum_{\omega\in\Omega}\langle \pi(\omega), a(t,\omega)\rangle$$

holds for almost all $t \in T$, and one has a contradiction. Therefore, the allocation $\hat{f}^* : T^* \times \Omega \to Y_+$ defined by

$$\hat{f}^*(t,\omega) = \begin{cases} \hat{f}(t,\omega), & \text{if } (t,\omega) \in T_0 \times \Omega; \\ \mathbf{c}(\omega), & \text{if } (t,\omega) \in T_1^* \times \Omega, \end{cases}$$

is not a Walrasian expectations allocation of $\mathcal{E}^*$. By Lemma 4.2, $\hat{f}^*$ is not in the private core of $\mathcal{E}^*$. Pick any $A_0 \in T_1$ with $\mu(A_0) = \epsilon > 0$. According to Lemma 4.3, $\hat{f}^*$ is privately blocked by a coalition S^* of $\mathcal{E}^*$ with $\mu^*(S^*) = \mu^*(T_0) + \epsilon$, which yields $\mu^*(S^* \cap T_1^*) \geq \epsilon$. By Lemma 4.1, there exists a coalition $R^* \subseteq S^*$, a sub-coalition R_1^* of R^* and an R^*-allocation g^* such that (i)-(iii) of Lemma 4.1 hold. Take a coalition E of $\mathcal{E}$ such that $E = (R^* \cap T_0) \cup A_0$, and define a function $\tilde{g} : E \times \Omega \to Y_+$ by

$$\tilde{g}(t,\omega) = \begin{cases} g^*(t,\omega), & \text{if } (t,\omega) \in (R^* \cap T_0) \times \Omega; \\ \frac{1}{\epsilon}\int_{R^* \cap T_1^*} g^*(\cdot,\omega)d\mu^*, & \text{otherwise.} \end{cases}$$

Further, define another function $\tilde{g}^* : E^* \times \Omega \to Y_+$ such that

$$\tilde{g}^*(t,\omega) = \begin{cases} \tilde{g}(t,\omega), & \text{if } (t,\omega) \in (R^* \cap T_0) \times \Omega; \\ \tilde{g}(A_0,\omega), & \text{if } (t,\omega) \in A_0^* \times \Omega. \end{cases}$$

By (A_4), one concludes that $\tilde{g}^*$ is an E^*-allocation such that $V_t(\tilde{g}^*(t,\cdot)) > V_t(\hat{f}^*(t,\cdot))$ for almost all $t \in E^*$ and $\int_{E^*} \tilde{g}^*(\cdot,\omega)d\mu^* \ll \int_{E^*} a(\cdot,\omega)d\mu^*$ for all $\omega \in \Omega$. Select some $b \gg 0$ such that $\int_{E^*}(a(\cdot,\omega)d\mu^* - \tilde{g}^*(\cdot,\omega))d\mu^* \gg b$ for all $\omega \in \Omega$, and consider the function $g_b^* : E^* \times \Omega \to Y_+$ defined by $g_b^*(t,\omega) = \tilde{g}^*(t,\omega) + \frac{b}{2\mu^*(E^*)}$. By ($A_3$), $V_t(g_b^*(t,\cdot)) > V_t(\hat{f}^*(t,\cdot))$ for almost all $t \in E^*$. Note that the function $g_b : E \times \Omega \to Y_+$, defined by

$$g_b(t,\omega) = \begin{cases} g_b^*(t,\omega), & \text{if } (t,\omega) \in (E \cap T_0) \times \Omega; \\ \frac{1}{\epsilon}\int_{A_0^*} g_b^*(\cdot,\omega)d\mu^*, & \text{otherwise,} \end{cases}$$

is an E-allocation such that $V_t(g_b(t,\cdot)) > V_t(f(t,\cdot))$ for almost all $t \in E$. Choose an $r \in (0,1)$ satisfying $\tilde{g}(A_0,\omega) \leq r g_b(A_0,\omega)$ for each $\omega \in \Omega$. By Lemma 3.4, there exists an $(E \cap T_0)$-allocation h_b such that $V_t(h_b(t,\cdot)) > V_t(f(t,\cdot))$ for almost all $t \in E \cap T_0$ and $\int_{E\cap T_0} h_b(\cdot,\omega)d\mu = \int_{E\cap T_0}(r g_b(\cdot,\omega) + (1-r)f(\cdot,\omega))d\mu$. Finally, consider the function $h : E \times \Omega \to Y_+$ defined by

$$h(t,\omega) = \begin{cases} h_b(t,\omega), & \text{if } (t,\omega) \in (E \cap T_0) \times \Omega; \\ \tilde{g}(t,\omega), & \text{otherwise.} \end{cases}$$

Note that h is an E-allocation. Applying an argument similar to the final part of Lemma 4.4, one can show that f is not robustly efficient. This is a contradiction and so f is a Walrasin expectations allocation. □

Remark 4.6. It is clear that the conclusion of Theorem 4.5 is valid in atomless economies whenever assumptions (A_1)-(A_3) and (A_5) hold. By Lemma 4.3, the coalition S of $\mathcal{E}(S, f, r)$ in Theorem 4.5 can be chosen arbitrarily small in an atomless economy. Thus, perturbation of small coalition is enough to characterize the Walrasian expectations allocations.

4.2. **The RW-fine core and the ex-post core.** In this subsection, we establish a relationship between the RW-fine core and the ex-post core of $\mathcal{E}$. An *information structure* for a coalition S is a family $\{\mathcal{G}_t : t \in S\}$ of σ-algebras such that $\mathcal{G}_t \subseteq \mathcal{F}$ for all $t \in S$ and $\{t \in S : \mathcal{G}_t = \mathcal{H}\} \in \Sigma$ for every σ-algebra $\mathcal{H} \subseteq \mathcal{F}$. Since Ω is finite, the family $\{\mathcal{G} \subseteq \mathcal{F} : \mathcal{G}$ is a σ-algebra$\}$ is finite. Thus, it is possible that for an information structure $\{\mathcal{G}_t : t \in S\}$ of S and two distinct agents t and t' of S, $\mathcal{G}_t = \mathcal{G}_{t'}$. A *communication system* for a coalition S is an information structure $\{\mathcal{G}_t : t \in S\}$ for S such that $\mathcal{F}_t \subseteq \mathcal{G}_t \subseteq \bigvee \mathfrak{P}_S$ for almost all $t \in S$, and it is called a *full communication system* if $\mathcal{G}_t = \bigvee \mathfrak{P}_S$ for almost all $t \in S$. Further, for any σ-algebra $\mathcal{H}$ with $\mathcal{H} \subseteq \mathcal{F}$, $\mathcal{F}$-measurable function $f : \Omega \to Y_+$ and $t \in T$, let $\mathbb{E}_t[f|\mathcal{H}]$ be the conditional expectation of f given $\mathcal{H}$ with respect to q_t. For any coalition S, we now assume that an S-allocation (including initial endowment) is a function $f : S \times \Omega \to Y_+$ such that $f(\cdot, \omega) \in L_1^S(\mu, Y_+)$ for each $\omega \in \Omega$ and $f(t, \cdot)$ is $\mathcal{F}$-measurable for almost all $t \in S$. As mentioned previously, T-allocations are simply called *allocations*.

Definition 4.7. [27] An allocation f in $\mathcal{E}$ is *RW-fine*[2] *blocked* by a coalition S if there are an S-allocation g, a communication system $\{\mathcal{G}_t\}_{t \in S}$ for S, and a nonempty event $A \in \bigcap_{t \in S} \mathcal{G}_t$ such that $\int_S g(\cdot, \omega) d\mu = \int_S a(\cdot, \omega) d\mu$ for all $\omega \in A$, and

$$\mathbb{E}_t[U_t(\cdot, g(t, \cdot))|\mathcal{G}_t](\omega) > \mathbb{E}_t[U_t(\cdot, f(t, \cdot))|\mathcal{G}_t](\omega)$$

for all $\omega \in A$ and almost all $t \in S$. The *RW-fine core* of $\mathcal{E}$ is the set of all feasible allocations that cannot be RW-fine blocked by any coalition.

Definition 4.8. [9] An allocation f in $\mathcal{E}$ is *ex-postly blocked* by a coalition S if there exist an S-allocation g and a state $\omega_0 \in \Omega$ such that $\int_S g(\cdot, \omega_0) d\mu = \int_S a(\cdot, \omega_0) d\mu$, and $U_t(\omega_0, g(t, \omega_0)) > U_t(\omega_0, f(t, \omega_0))$ for almost all $t \in S$. The *ex-post core* of $\mathcal{E}$ is the set of all feasible allocations that cannot be ex-postly blocked by any coalition.

Lemma 4.9. *Assume that f is in the RW-fine core of $\mathcal{E}$. Under* (A_1)-(A_9)*, there exists an allocation $\hat{f}$ in $\mathcal{E}$ such that $\hat{f}|_{T_0 \times \Omega} = f$, $\hat{f}(\cdot, \omega)$ is constant on T_1 for each $\omega \in \Omega$, $U_{T_1}(\omega, \hat{f}(t, \omega)) = U_{T_1}(\omega, f(t, \omega))$ for all $(t, \omega) \in T_1 \times \Omega$ and $\int_T f d\mu = \int_T \hat{f} d\mu$.*

Proof. Consider the allocation $\hat{f} : T \times \Omega \to Y_+$ defined by

$$\hat{f}(t, \omega) = \begin{cases} f(t, \omega), & \text{if } (t, \omega) \in T_0 \times \Omega; \\ \frac{1}{\mu(T_1)} \int_{T_1} f(\cdot, \omega) d\mu, & \text{if } (t, \omega) \in T_1 \times \Omega. \end{cases}$$

One needs to verify $U_{T_1}(\omega, \hat{f}(t, \omega)) = U_{T_1}(\omega, f(t, \omega))$ for all $(t, \omega) \in T_1 \times \Omega$. Suppose that there exist a coalition $D \subseteq T_1$ and a state $\omega_0 \in \Omega$ such that $U_{T_1}(\omega_0, \hat{f}(t, \omega_0)) > U_{T_1}(\omega_0, f(t, \omega_0))$ for all $t \in D$. Then, a contradiction can be derived by a proof similar to that of Lemma 4.4 except for the fact that the coalition E_0 can be chosen as $\bigcup_{\mathcal{Q} \in \mathfrak{P}(T_0)} E_0^{\mathcal{Q}}$, where each $E_0^{\mathcal{Q}}$ satisfies the condition

$$\left\| \int_{E_0^{\mathcal{Q}}} (f(\cdot, \omega_0) - a(\cdot, \omega_0)) d\mu - r_2 r_3 \int_{T_0 \cap T_{\mathcal{Q}}} (f(\cdot, \omega_0) - a(\cdot, \omega_0)) d\mu \right\| < \frac{\epsilon}{|\mathfrak{P}(T_0)|},$$

[2] RW is the abbreviation of Robert Wilson.

the blocking coalition is of the form $R = S \bigcup \left(T_0 \setminus \bigcup_{\mathcal{Q}\in\mathfrak{P}(T_0)} T_{\mathcal{Q}}\right)$, where S is defined in Lemma 4.4, and the function $g : R \to Y_+$ is defined by

$$g(t) = \begin{cases} \hat{f}(t,\omega_0) + \frac{u}{2\mu(E_0)}, & \text{if } t \in E_0; \\ r_3\hat{f}(t,\omega_0) + \frac{u}{2\mu(C)}, & \text{if } t \in C; \\ f(t,\omega_0), & \text{otherwise.} \end{cases}$$

Note that $\bigvee \mathfrak{P}_R = \bigvee \mathfrak{P}_T = \mathcal{F}$ and $\int_R g d\mu \leq \int_R a(\cdot,\omega_0)d\mu$. Let $b = \int_R a(\cdot,\omega_0) - \int_R g d\mu$. Consider a function $h : R \to Y_+$ defined by $h(t) = g(t) + \frac{b}{\mu(R)}$. By ($A_3$), $U_t(\omega_0, h(t)) > U_t(\omega_0, f(t,\omega_0))$ for almost all $t \in R$. Let $A(\omega_0)$ denote the atom of $\mathcal{F}$ containing ω_0. Define a function $y : R \times \Omega \to Y_+$ by

$$y(t,\omega) = \begin{cases} h(t), & \text{if } (t,\omega) \in R \times A(\omega_0); \\ a(t,\omega), & \text{otherwise.} \end{cases}$$

Then, y is an R-allocation. Since $a(t,\cdot)$ is $\mathcal{F}$-measurable, $a(t,\omega) = a(t,\omega')$ for almost all $t \in R$ and all $\omega,\omega' \in A(\omega_0)$. Hence, $\int_R y(\cdot,\omega)d\mu = \int_R a(\cdot,\omega)d\mu$ for all $\omega \in A(\omega_0)$. By (A_8), $\bigvee \mathfrak{P}_R = \mathcal{F}$. Thus using ($A_9$), one has $\mathbb{E}_t[U_t(\cdot, f(t,\cdot))| \bigvee \mathfrak{P}_R] = U_t(\cdot, f(t,\cdot))$ and $\mathbb{E}_t[U_t(\cdot, y(t,\cdot))| \bigvee \mathfrak{P}_R] = U_t(\cdot, y(t,\cdot))$. Further, for all $\omega \in A(\omega_0)$ and almost all $t \in R$, one has that

$$\begin{aligned} \mathbb{E}_t\left[U_t(\cdot, y(t,\cdot))\middle| \bigvee \mathfrak{P}_R\right](\omega) &= U_t(\omega, y(t,\omega)) = U_t(\omega_0, \hat{h}(t)) \\ &> U_t(\omega_0, \hat{f}(t,\omega_0)) \\ &= \mathbb{E}_t\left[U_t(\cdot, f(t,\cdot))\middle| \bigvee \mathfrak{P}_R\right](\omega), \end{aligned}$$

which implies that f is RW-fine blocked by R via y. This contradicts with the assumption. Hence, $U_{T_1}(\omega, f(t,\omega)) \geq U_{T_1}(\omega, \hat{f}(t,\omega))$ for all $(t,\omega) \in T_1 \times \Omega$. By an argument similar to that in Lemma 4.4, one can further show $U_{T_1}(\omega, \hat{f}(t,\omega)) = U_{T_1}(\omega, f(t,\omega))$ for all $(t,\omega) \in T_1 \times \Omega$. □

The following theorem is an extension of Theorem 3.1 in [9] to mixed economies with infinitely many commodities and the exact feasibility. In addition, the assumption $\mathfrak{P}_T = \mathfrak{P}(T)$ used by Einy et al. is not assumed in our result. To this end, we assume that for each $\omega \in \Omega$, $\mathcal{E}(\omega)$ denotes the symmetric information economy whose space of agents are T, and whose the consumption set, the utility function and the initial endowment of agent t are Y_+, $U_t(\omega,\cdot)$ and $a(t,\omega)$ respectively.

Theorem 4.10. *Assume that $\mathcal{E}$ satisfies* (A_1)-(A_9). *If f is in the RW-fine core of $\mathcal{E}$, then it is also in the ex-post core of $\mathcal{E}$.*

Proof. Suppose that f is not in the ex-post core of $\mathcal{E}$. The allocation $\hat{f}$ defined in Lemma 4.9 is not in the ex-post core of $\mathcal{E}$ either. Then there is a state $\omega_0 \in \Omega$ such that $\hat{f}(\cdot,\omega_0)$ is a feasible allocation in the symmetric information economy $\mathcal{E}(\omega_0)$ and is not in the core of $\mathcal{E}(\omega_0)$. Consider an allocation $\hat{f}^* : T^* \times \Omega \to Y_+$ defined by $\hat{f}^*(t,\omega) = \hat{f}(t,\omega)$, if $(t,\omega) \in T_0 \times \Omega$; and $\hat{f}^*(t,\omega) = \hat{f}(T_1,\omega)$, if $(t,\omega) \in T_1^* \times \Omega$, where $\hat{f}(T_1,\omega)$ denotes the constant value of $\hat{f}(\cdot,\omega)$ on T_1 for each $\omega \in \Omega$. Then $\hat{f}^*(\cdot,\omega_0)$ is a feasible allocation in $\mathcal{E}^*(\omega_0)$ and $\hat{f}^*(\cdot,\omega_0)$ is not in the core of $\mathcal{E}^*(\omega_0)$. Choose an arbitrary $A_0 \in T_1$ and let $\mu(A_0) = \epsilon > 0$. Note that under (A_3), the conclusion of Lemma 4.3 also holds with the exact feasibility in a deterministic economy. Thus, $\hat{f}^*(\cdot,\omega_0)$ is blocked by a coalition S^* via $\hat{g}^*$ such that $\mu^*(S^*) = \mu^*(T_0)+\epsilon$, if

$$\mu^*(T_1^* \setminus A_0^*) < \min\{\mu^*(T_0 \cap T_{\mathcal{Q}}^*) : \mathcal{Q} \in \mathfrak{P}(T_0)\},$$

and otherwise, $\mu^*(S^*) > \mu^*(T_0) - \min\{\mu^*(T_0 \cap T_{\mathcal{Q}}^*) : \mathcal{Q} \in \mathfrak{P}(T_0)\} + \mu^*(T_1^*)$. Clearly, $\mu^*(S^* \cap T_1^*) \geq \epsilon$ and $\bigvee \mathfrak{P}(S^*) = \bigvee \mathfrak{P}(T)$. Let $\alpha = \frac{\epsilon}{\mu^*(S^* \cap T_1^*)}$. Applying ($A_3$) and an argument similar to that in Lemma 4.1, one can show that there exists a coalition $R^* \subseteq \bigcup_{\mathcal{Q}\in\mathfrak{P}(T^*)}(S^* \cap T_{\mathcal{Q}}^*)$ blocking $\hat{f}^*(\cdot,\omega_0)$ via $\hat{h}^* : R^* \to Y_+$ in $\mathcal{E}^*(\omega_0)$ such that $\mu^*(R^* \cap T_{\mathcal{Q}}^*) = \alpha\mu^*(S^* \cap T_{\mathcal{Q}}^*)$ for all $\mathcal{Q} \in \mathfrak{P}(S^*)$ and $\mu^*(R^* \cap T_1^*) = \epsilon$. Note that $\bigvee \mathfrak{P}(R^*) = \bigvee \mathfrak{P}(S^*)$. Consider a coalition R of $\mathcal{E}$ define by $R = (R^* \cap T_0) \cup A_0$. Then, $\bigvee \mathfrak{P}(R) = \bigvee \mathfrak{P}(T)$. We consider a function $\hat{h} : R \to Y_+$ defined by

$$\hat{h}(t) = \begin{cases} \hat{h}^*(t), & \text{if } t \in R^* \cap T_0; \\ \frac{1}{\epsilon}\int_{R^*\cap T_1^*} \hat{h}^* d\mu^*, & \text{otherwise.} \end{cases}$$

Obviously, $U_t(\omega_0, \hat{h}(t)) > U_t(\omega_0, \hat{f}(t,\omega_0))$ if $t \in R^* \cap T_0$. By (A_4), $U_{T_1}(\omega_0, \hat{h}(t)) > U_{T_1}(\omega_0, \hat{f}(t,\omega_0))$ if $t = A_0$. Moreover, $\int_R \hat{h} d\mu = \int_R a(\cdot,\omega_0) d\mu$. Define a coalition $E = R \cup \left(T_0 \setminus \bigcup_{\mathcal{Q}\in\mathfrak{P}(T_0)} T_{\mathcal{Q}}\right)$. Then $\bigvee \mathfrak{P}_T = \bigvee \mathfrak{P}_E$. Let $A(\omega_0)$ be the atom of $\bigvee \mathfrak{P}_T$ containing ω_0. Now, define a function $y : E \times \Omega \to Y_+$ such that

$$y(t,\omega) = \begin{cases} \hat{h}(t), & \text{if } (t,\omega) \in R \times A(\omega_0); \\ a(t,\omega), & \text{otherwise.} \end{cases}$$

Then, y is an E-allocation. Applying an argument similar to that in Lemma 4.9, one can show that f is RW-fine blocked by E via y. This contradicts with the assumption, which completes the proof. $\square$

Remark 4.11. It is obvious from the proof of Theorem 4.10 that a similar result holds for atomless economies under (A_1)-(A_3), (A_5) and (A_8)-(A_9) only.

4.3. **The weak fine core.** In this subsection, we extend Proposition 5.1 in [10] to mixed economies with infinitely many commodities and the exact feasibility. We also relax the assumption $\mathfrak{P}_T = \mathfrak{P}(T)$.

Definition 4.12. A feasible assignment f in $\mathcal{E}$ is said to be in the *weak fine core* of $\mathcal{E}$ if $f(t,\cdot)$ is $\bigvee \mathfrak{P}_T$-measurable for almost all $t \in T$, and f cannot be NY-fine blocked by any coalition.

In the sequel, the economy $\mathcal{E}^s$ is similar to $\mathcal{E}$ except for the information of every agent being $\bigvee \mathfrak{P}_T$. The proof of the next lemma is similar to that of Lemma 4.9.

Lemma 4.13. *Assume that f is in the weak fine core of $\mathcal{E}$. Under* (A_1)-(A_7), *there exists an allocation $\hat{f}$ such that $\hat{f}|_{T_0\times\Omega} = f$, $\hat{f}(\cdot,\omega)$ is constant on T_1 for each $\omega \in \Omega$, $V_{T_1}(\hat{f}(t,\cdot)) = V_{T_1}(f(t,\cdot))$ for almost all $t \in T_1$ and $\int_T f d\mu = \int_T \hat{f} d\mu$.*

Theorem 4.14. *Assume that $\mathcal{E}$ satisfies* (A_1)-(A_7). *Then f is in the weak fine core of $\mathcal{E}$ if and only if f is in the private core of $\mathcal{E}^s$.*

Proof. It is clear that if f is in the private core of $\mathcal{E}^s$, then f is in the weak fine core of $\mathcal{E}$. Now, assume that f is in the weak fine core of $\mathcal{E}$. Let $\bigvee \mathfrak{P}_T$ be generated by the partition $\{A_1, ..., A_k\}$ of Ω, and let X denote the set of all $\bigvee \mathfrak{P}_T$-measurable elements of $(Y_+)^\Omega$. Define a function $\gamma : X \to Y_+^k$ such that $\gamma(f) = f_s$, where

$f_s = (f(\omega_1), ..., f(\omega_k))$ if $\omega_j \in A_j$ for all $1 \leq j \leq k$. Now for all $t \in T$, consider a function $V_t^s : Y_+^k \to \mathbb{R}$ defined by $V_t^s(f_s) = V_t(\gamma^{-1}(f_s))$. Let $\widetilde{\mathcal{E}}^s$ be a symmetric information economy whose space of economic agents is (T, Σ, μ), and in which the consumption set of every agent is Y_+^k, the utility function and initial endowment of agent t are V_t^s and $a^s(t) = \gamma(a(t, \cdot))$ respectively. Suppose that f is not in the private core of $\mathcal{E}^s$. Then f_s is not in the private core of $\widetilde{\mathcal{E}}^s$. Thus $\hat{f}_s$ is is not in the private core of $\widetilde{\mathcal{E}}^s$. Applying an argument similar to that in the proof of Theorem 4.10, one can show that there is a coalition $R \subseteq \bigcup_{\mathcal{Q} \in \mathfrak{P}(T)} T_{\mathcal{Q}}$ blocking f^s via h^s such that $\bigvee \mathfrak{P}(R) = \bigvee \mathfrak{P}(T)$. Let $E = R \cup \left(T_0 \setminus \bigcup_{\mathcal{Q} \in \mathfrak{P}(T_0)} T_{\mathcal{Q}}\right)$. Obviously, $\bigvee \mathfrak{P}_T = \bigvee \mathfrak{P}_E$. Define a function $y^s : E \to Y_+^k$ by

$$y^s(t) := \begin{cases} h^s(t), & \text{if } t \in R; \\ a^s(t), & \text{otherwise.} \end{cases}$$

Note that $V_t^s(y^s(t)) > V_t^s(f^s(t))$ for almost all $t \in E$. Furthermore, $\int_E y^s d\mu = \int_E a^s d\mu$. Let $y(t, \cdot) = \gamma^{-1}(y^s(t))$ for all $t \in E$. Then, $y(t, \cdot)$ is $\bigvee \mathfrak{P}_E$-measurable and $V_t(y(t, \cdot)) > V_t(f(t, \cdot))$ for almost all $t \in E$. Moreover, $\int_E y(\cdot, \omega) d\mu = \int_E a(\cdot, \omega) d\mu$ for all $\omega \in \Omega$. Thus, f is also NY-fine blocked by E via y. This contradicts with the fact that f is in the weak fine core of $\mathcal{E}$. Consequently, f must be in the private core of $\mathcal{E}^s$. $\square$

Remark 4.15. A similar conclusion can be derived for atomless economies under (A_1)-(A_3) and (A_5) only.

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