FREE-BODY AND FLEXURAL MOTION OF A FLOATING ELASTIC PLATE UNDER WAVE MAKER FORCING

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Talk Abstract

A boundary integral equation (BIE) method is presented for the calculation of the motion of an elastic plate in a two-dimensional setting. A wave maker is placed next to the plate and the model is therefore representative of recent wave tank experiments. The model includes the plate's draught and surge motion, the latter of which is restrained by a spring and damping system. However, the corners created by the non-zero draught produce first-order singularities in the normal derivative of the Green's function, in addition to the logarithmic singularities in the Green's function itself. Both of these singularities are separated using Kummer transforms, and the associated boundary integrals are evaluated analytically. The numerical computation of the modified BIEs explicitly decomposes the plate's displacement into its heave, pitch and flexural motions.

Introduction

The model outlined in this paper is motivated by a series of wave-tank experiments that were recently conducted at Laboratoire de Méchanique des Fluides, École Centrale de Nantes, and which are described in [1]. In these unique experiments the wave induced motions of a floating elastic plate were recorded under controlled conditions. The experiments highlighted the presence of appreciable surge motion, which is not included in most theoretical models. The reason for such omission is that surge is only possible when a non-zero draught is accommodated, and this is often ignored to simplify the solution procedure.

Surge motion needs to be taken into account when the size of the plate is comparable to the wavelength. This is also the case for most off-shore structures, such as pontoon-type very large floating structures.

This paper considers a two-dimensional model of a wave tank, containing a floating elastic plate. Motion of the plate is forced by a wave maker, which is located at a vertical boundary. The model can also accommodate an incident wave from the far field or a forced motion of the plate. Spring and damping constraints on the surge motion, which could result from a vertical rod or a mooring system, are included.

A system of boundary integral equations (BIEs) are constructed for this problem using a Green's function and Green's theorem. The integrals are posed on the wetted surface of the plate, which has corners. In this situation the the normal derivative of the Green's function is not bounded when the source and field points approach the same corner from different limits. Our innovation is to derive numerically stable BIEs. To achieve this, the singularities are separated using the Kummer transform, and the boundary integrals containing those singular parts are evaluated analytically. Consequently the numerical computation involves only bounded functions.

The modified BIEs are solved using the Galerkin technique, using trial and test functions based on the eigenfunctions of an elastic beam and the orthogonal set of Gegenbauer polynomials. The leading terms of the displacement capture the heave and pitch of the plate and the remaining terms are the flexure.

The following sections provide a brief derivation of the BIEs and the series expansions over orthogonal polynomials. Some preliminary numerical results are given in the final section. A more complete set of simulations will be shown during the conference presentation.

Governing Equations

The two-dimensional geometry is defined by the horizontal and vertical Cartesian coordinates x and z respectively. In the absence of the plate, the fluid occupies the domain $\Omega = \{x, z : x > 0, -h < z < 0\}$, so that $z = 0$ defines the equilibrium position of the fluid surface, $z = -h$ is the floor of the tank and $x = 0$ is the location of the wave maker. A schematic of the geometry is shown in figure 1.

Under the regular assumptions of linear motions, the fluid's velocity field is sought as the gradient of a velocity potential, denoted $\Phi = \Phi(x, z, t)$. Imposing time-harmonic conditions, the velocity potential may be written as $\Phi(x, z, t) = \Re\mathrm{e}\{(g/\mathrm{i}\omega)\phi(x, z)\mathrm{e}^{-\mathrm{i}\omega t}\}$, where $g \approx 9.81\,\mathrm{m\,s^{-2}}$ is acceleration due to gravity, ω is a prescribed angular momentum and ϕ is a complex-valued (reduced) potential that must be calculated.

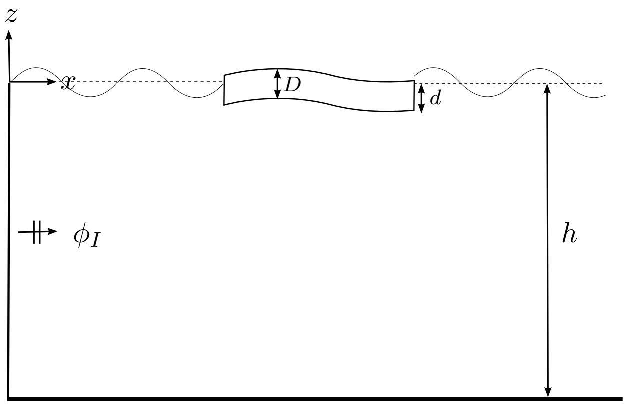

Figure 1: A schematic of the geometry.

The potential ϕ satisfies Laplace's equation throughout the fluid domain, that is $\nabla^2\phi = 0$ for $(x, z) \in \Omega$. On the floor of the tank, $z = -h$, the no-flow condition $\partial_z\phi = 0$ is imposed. At the linearized free-surface, $z = 0$, the condition $\partial_z\phi = \sigma\phi$ holds, where $\sigma = \omega^2/g$ is a frequency parameter. Waves are generated through a prescribed horizontal velocity at the linearized position of the wave maker, $x = 0$. In this study, forcing is provided by an incident plane wave, so that $\partial_x\phi = \partial_x\phi_I$ at $x = 0$, where the incident wave is

$$\phi_I(x, z) = \mathrm{e}^{\mathrm{i}k_0x}\cosh\{k_0(z + h)\}/\cosh(k_0h),$$

with k_0 the propagating wavenumber (to be defined shortly). The potential must also describe outgoing waves in the far field $x \to \infty$.

A thin-elastic plate occupies the interval $a < x < b$, where $0 < a < b$ and $b - a \equiv l$ is the length of the plate. In equilibrium the lower surface of the plate is located at $z = -d = -\rho_pD/\rho_w$, where D is the thickness of the plate, $\rho_p =$ is its density, and $\rho_w =$ is the fluid density. The linearized fluid domain in the presence of the plate is therefore $\Omega_w \equiv \Omega\backslash\{x, z : a < x < b, -d < z < 0\}$.

Fluid motion causes the plate to oscillate, and the position of its lower surface at time t is denoted $z = -d + \Re\mathrm{e}\{\xi(x)\mathrm{e}^{-\mathrm{i}\omega t}\}$. The displacement function ξ is related to the potential ϕ through the linearized equations

$$\phi = F\xi'''' + (1 - \sigma d)\xi, \quad \partial_z\phi = \sigma\xi, \tag{1a}$$

where $F = F_0/(\rho_wg)$, and $F_0 \propto D^3$ is the flexural rigidity of the plate.

The plate is also permitted to surge to and fro, although these motions are restrained by a mooring system. The horizontal position of the plate, $\Re\mathrm{e}\{u\mathrm{e}^{-\mathrm{i}\omega t}\}$ say, where u is a constant (complex-valued) amplitude, is coupled to the potential ϕ by the linearized equation of motion

$$(S - \sigma M - \mathrm{i}A)u = \rho_w\int_{-d}^{0}\{\phi(b, z) - \phi(a, z)\}\,\mathrm{d}z, \tag{1b}$$

where $M = \rho_pDl$ is the mass of the plate, S in the spring constant, and $A = \omega A_0/g$, in which A_0 is the damping constant. The amplitude u and the potential ϕ are also coupled by the kinematic conditions

$$\partial_x\phi(a, z) = \partial_x\phi(b, z) = \sigma u \quad (-d < z < 0). \tag{1c}$$

Solution Method

Consider the Green's function $G = G(x_0, z_0|x, z)$, satisfying

$$\nabla^2G = \delta(x - x_0)\delta(z - z_0) \quad (x, z) \in \Omega,$$

$\partial_zG = 0$ on $z = -h$, $\partial_zG = \sigma G$ on $z = 0$, $\partial_xG = 0$ on $x = 0$, and G represents outgoing waves as $x \to \infty$. The function may be represented in a series form as

$$G = \frac{1}{2\mathrm{i}}\sum_{n=0}^{\infty}\frac{\mathrm{e}^{\mathrm{i}k_n(x+x_0)} + \mathrm{e}^{\mathrm{i}k_n|x-x_0|}}{k_nc_n}w_n(z)w_n(z_0),$$

where k_n ($n \in \mathbb{N}$) are the roots k of the dispersion relation $k\tanh(kh) = \sigma$, the vertical functions $w_n(z) = \cosh\{k_n(z + h)\}$, and the constants $c_n = ||w_n||^2$.

Applying Green's theorem in the plane to ϕ and G over Ω_w produces the integral expression

$$\epsilon\phi = \phi_I - \int_\Gamma\{(\partial_{n_0}G)\phi_0 - G(\partial_{n_0}\phi_0)\}\,\mathrm{d}s_0, \tag{2}$$

where a subscript 0 indicates that a function is evaluated at the source point (x_0, z_0) rather than at the field point (x, z). The integral is around the wetted surface of the plate, Γ say, with tangential coordinate s and (outward) normal n. The quantity ϵ is defined as $\hat{\epsilon}/2\pi$, where $\hat{\epsilon}$ is the angle around the point (x, z) in Ω_w.

A system of BIEs are formed from (2) by allowing the field point (x, z) to tend to the three continuous components of Γ. For what follows, these components will be denoted $\Gamma_a = \{x, z : x = a, -d < z < 0\}$, $\Gamma_b = \{x, z : x = b, -d < z < 0\}$ and $\Gamma_d = \{x, z : a < x < b, z = -d\}$.

Singularities and Corners

A Green's function in a two-dimensional plane contains a logarithmic singularity at the point at which the field and source points coincide, i.e. $(x, z) = (x_0, z_0)$. In the above definition of G, the singularity is manifest as the non-convergence of the series at this point. The presence of the singularity will impede the numerical evaluation of the integrals in equation (2).

A standard method for dealing with this issue is to apply a Kummer transformation. This involves a manipulation of the logarithmic singularity into a more convenient form. The Green's function may then be expressed as

$$G = \widetilde{G} + \frac{1}{2\pi}\Big(\log(\mathcal{R}_-) + \log(\mathcal{R}_+)\Big),$$

where $\widetilde{G}$ is a bounded function and

$$(x_0 - x) + \mathrm{i}(z_0 \pm z) = \mathcal{R}_\pm \mathrm{e}^{\mathrm{i}\Theta_\pm}.$$

The singularity is in the term $\log(\mathcal{R}_-)$, and the term $\log(\mathcal{R}_+)$ is also separated due to its near-singular behaviour for small values of the draught d.

The method for dealing with integrals involving G is demonstrated with the following example. Consider

$$\int_a^b G(\partial_{n_0}\phi_0)\,\mathrm{d}x_0 = \int_a^b \widetilde{G}(\partial_{n_0}\phi_0)\,\mathrm{d}x_0 + \frac{1}{2\pi}\sum_{i=\pm}\int_a^b \log(\mathcal{R}_i)(\partial_{n_0}\phi_0)\,\mathrm{d}x_0.$$

The integrals involving the logarithmic functions are then written

$$\int_a^b \log(\mathcal{R}_\pm)(\partial_{n_0}\phi_0)\,\mathrm{d}x_0 = \partial_n\phi\int_a^b \log(\mathcal{R}_\pm)\,\mathrm{d}x_0 + \int_a^b \log(\mathcal{R}_\pm)(\partial_{n_0}\phi_0 - \partial_n\phi)\,\mathrm{d}x_0.$$

The first integral on the right hand side of the above equation involves only a known function and can be calculated explicitly. The function beneath the second integral tends to zero as $x \to x_0$ and can be evaluated numerically at a low cost.

In the present case, the normal derivative of the Green's function is not bounded due to the corners in Γ. Specifically, a first-order singularity occurs when the field and source points tend to the same corner from opposing limits. An approach is described here for circumventing this issue.

The approach is demonstrated through another example. Consider

$$\int_a^b (\partial_{n_0}G)\phi_0\,\mathrm{d}x_0 = \int_a^b (\partial_{n_0}\widetilde{G})\phi_0\,\mathrm{d}x_0 + \frac{1}{2\pi}\sum_{i=\pm}\int_a^b \big(\partial_{n_0}\log(\mathcal{R}_i)\big)\phi_0\,\mathrm{d}x_0.$$

The singular terms are then extricated by writing

$$\int_a^b (\partial_{n_0}\log(\mathcal{R}_\pm))\phi_0\,\mathrm{d}x_0 = \phi\int_a^b (\partial_{n_0}\log(\mathcal{R}_\pm))\,\mathrm{d}x_0 + \int_a^b (\partial_{n_0}\log(\mathcal{R}_\pm))(\phi_0 - \phi)\,\mathrm{d}x_0.$$

The function beneath the final integral is bounded as $x \to x_0$ and can therefore be evaluated numerically. The integrals of the normal derivative of the logarithmic function alone can be treated analytically by noting the following identity

$$\partial_{n_0}\log(\mathcal{R}_\pm) = \partial_{s_0}\Theta_\pm,$$

(see [2]). Accounting for the jump in Θ_- at the point $x = x_0$, it can be shown that

$$\int_a^b \big(\partial_{n_0}\log(\mathcal{R}_-)\big)\,\mathrm{d}x_0 = 0,$$

whereas Θ_+ is continuous along the interval and therefore

$$\int_a^b \big(\partial_{n_0}\log(\mathcal{R}_+)\big)\,\mathrm{d}x_0 = -\Big[\Theta_+\Big]_{x_0=a}^{b} \quad (z = z_0 = -d).$$

Expansions

The system of BIEs can now be solved using a form of the Galerkin technique, in which the unknown functions are approximated as a linear combination of a finite set of chosen trial functions. Inner-products of the BIEs are then taken in turn with the members of a corresponding set of test functions.

First though, the number of unknowns present in the BIEs is reduced by implementing the boundary conditions (1a,c). This leaves the system of BIEs to be solved for the constant u and the functions $\phi_i = \phi_i(z) \equiv \phi(i, z)$ $(-d < z < 0)$ for $i = a, b$, and $\xi = \xi(x)$.

Let the displacement function be approximated by

$$\xi(x) \approx \frac{2}{l}\sum_{m=0}^{M}\xi_m X_m(\hat{x}); \quad \hat{x} = \frac{2}{l}(x - a) - 1,$$

for some chosen constant M. The orthonormal set $\{X_m\}$ are the eigenfunctions of the spectral problem

$$X_m'''' - \alpha_m^4 X_m = 0 \quad (-1 < \hat{x} < 1),$$

with boundary conditions $X'' = 0$ and $= X''' = 0$ at $\hat{x} = \pm 1$, and corresponding eigenvalues α_m. Note that the boundary conditions are the natural conditions for a plate with free ends. The above expansion is convenient for analysis of the motion of the plate because the first mode, X_0, supports its heave motion and the second mode, X_1, supports its pitch motion. The subsequent modes X_m $(m \geq 2)$ describe the flexural motion of the plate.

The potential functions at the ends of the plate are similarly approximated, with

$$\phi_i(z) \approx \frac{2}{d}c_i + \frac{2}{d}\sum_{n=0}^{N}\phi_{i,n}\mathcal{C}_{2n}(\hat{z}); \quad \hat{z} = \frac{z}{d}, \qquad (3)$$

for $i = a, b$ and for some chosen constant N. The functions $\mathcal{C}_{2n}$ are a set of weighted even Gegenbauer polynomials. The weighting is chosen so that the expected singular behaviour of the fluid velocity at the submerged corners of the plate are captured in the approximation (see [3]).

Inner-products of the integral equations on the ends of the plate Γ_a and Γ_b are taken with the set of unweighted even Gegenbauer polynomials, and the set $\{X_m\}$ is used as the test functions for the integral equation posed on Γ_d. Extra equations must also be added in order to close the system. Firstly, the condition (1b) is applied in conjunction with the approximations (3). Continuity of the potential is also ensured at the submerged corners of the plate, which sets the values of the constants c_i $(i = a, b)$.

Preliminary Numerical Results

A set of preliminary results is given in figure 2 for parameters resembling the wave-tank experiments. The plate is of length $l = 2\,\text{m}$ and thickness $D = 10\,\text{mm}$ and is placed on the interval $(a, b) = (5, 7)\,\text{m}$. The relative plate and fluid densities mean that the draught is $d = 5\,\text{mm}$.

Figure 2 shows the decomposed motions of the plate, as functions of wavelength $2\pi/k_0$. The top panel contains the moduli of the amplitude of heave motion, $\tilde{\xi}_0 = \sqrt{2}\,\xi_0$ (black curve), and the scaled coefficient of the pitch motion, $\tilde{\xi}_1 = \sqrt{2/3}\,\xi_1/k_0$ (grey). The middle panel contains the moduli of the coefficients of the primary symmetric (black) and antisymmetric (grey) terms, ξ_2 and ξ_3, respectively. The bottom panel contains the modulus of the surge amplitude, u, and in this case no restraints are placed on the surge motion.

Each of the motions here displays fine structure when the wavelength is smaller than the length of the plate. As the wavelength increases to approximately the length of the plate and beyond, the rigid body motions generally grow, whereas the flexural motions decay. Throughout the range of wavelengths considered, the symmetric component of the plate's flexure is consistently greater than the antisymmetric component. Note that for these particular parameters the moduli of the heave and surge motions are almost identical, and that they vanish at a wavelength slightly shorter than the length of the plate.

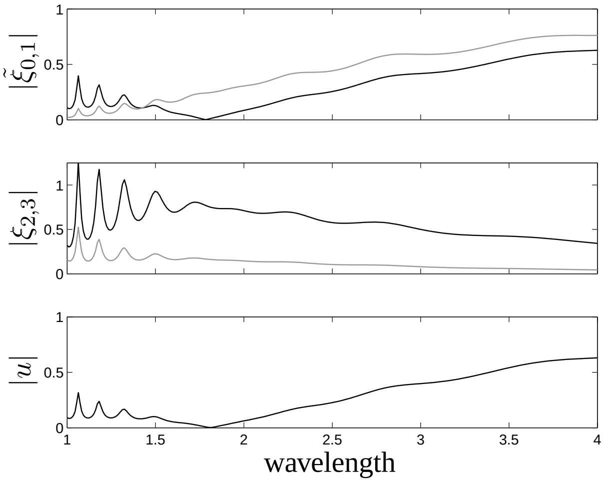

Figure 2: Results for a plate of length $l = 2\,\text{m}$ and thickness $D = 10\,\text{mm}$, as described in the text.

Further results will be presented at the conference.

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