Bornologies and Hyperspaces

Jiling Cao

School of Computing and Mathematical Sciences
Auckland University of Technology
Private Bag 92006
Auckland 1142
New Zealand

Email: jiling.cao@aut.ac.nz

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What is a bornology?

Let X be a nonempty set.

A *bornology* on X is a family of subsets $\mathfrak{B}$ of X satisfying the following three conditions:

1. $\mathfrak{B}$ is closed under taking finite unions;
2. $\mathfrak{B}$ is closed under taking subsets;
3. $\mathfrak{B}$ forms a cover of X.

Examples

Example

1. The power set of X, $\mathfrak{P}(X)$, is a bornology on X. This is the largest bornology on X.
2. The family of all finite subsets of X, $\mathfrak{F}(X)$, is a bornology on X. This is the smallest bornology on X.
3. Let (X, d) be a metric space. The family of d-bounded subsets of X, $\mathfrak{B}_d(X)$, is a bornology on (X, d).
4. Let X be a topological space. The family of subsets of X with compact closure, $\mathfrak{K}_c(X)$, is a bornology on X.

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Vector bornologies

Let E be a vector space over $\mathbb{K}$. A bornology $\mathfrak{B}$ is said to a *vector bornology* on E, if $\mathfrak{B}$ is closed under vector addition, scalar multiplication, and the formation of circled hulls, or in other words, the sets $A + B$, λA, $\bigcup_{|\alpha|\leq 1} \alpha A$ belong to $\mathfrak{B}$ whenever A and B belong to $\mathfrak{B}$ and $\lambda \in \mathbb{K}$.

Example (Vector bornologies)

1. Let p be a semi-norm, the subsets of E which are bounded for p form a vector bornology on E.
2. If E is a topological vector space, the *von Neumann bornology* is the family of subsets that are absorbed by every neighbourhood of zero in E.

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Hyperspaces via bornologies

For a metric space (X, d), let $CL(X)$ be the family of nonempty closed subsets of (X, d). If we equip $CL(X)$ with a metric, a uniformity or a topology, the resulting object is called a *hyperspace*.

For each $A \in CL(X)$, the distance functional $d(A, \cdot) : X \to \mathbb{R}$ defined by

$$d(A, x) = \inf\{d(a, x) : a \in A\}, \qquad \forall x \in X$$

is a continuous real-valued function. Thus, if A is identified with $d(A, \cdot)$, then $CL(X) \hookrightarrow C(X)$.

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Structure of the hyperspace

Let $\mathfrak{B}$ be a bornology on a metric space (X, d). For any $S \in \mathfrak{B}$ and $\varepsilon > 0$, consider

$$U_{S,\varepsilon} = \left\{(A, B) : \sup_{x \in S} |d(A, x) - d(B, x)| < \varepsilon\right\}.$$

Now, $\{U_{S,\varepsilon} : S \in \mathfrak{B}, \varepsilon > 0\}$ is a base for some uniformity $\mathfrak{U}_{\mathfrak{B},d}$ on $CL(X)$, which induces a topology $\tau_{\mathfrak{B},d}$ on $CL(X)$. In this way, a hyperspace $(CL(X), \mathfrak{U}_{\mathfrak{B},d})$ or $(CL(X), \tau_{\mathfrak{B},d})$ is generated.

General issue

Using properties of (X, d) and $\mathfrak{B}$, determine properties of the hyperspace of $(CL(X), \mathfrak{U}_{\mathfrak{B},d})$ or $(CL(X), \tau_{\mathfrak{B},d})$.

Sample results

1. $\tau_{\mathfrak{B},d} = \tau_{\mathfrak{C},d}$ iff $\mathfrak{B}$ and $\mathfrak{C}$ determine the same family of totally bounded sets in X .
2. If $\mathfrak{B}$ has a countable base, then $(CL(X), \mathfrak{U}_{\mathfrak{B},d})$ is metrizable.
3. $(CL(X), \tau_{\mathfrak{B},d})$ is compact iff (X, d) is compact.

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Special cases

1. $\tau_{\mathfrak{P}(X),d}$ is the well-known *Hausdorff metric topology*, which is usually denoted by $\tau_{H(d)}$ [F. Hausdorff, 1914]
2. $\tau_{\mathfrak{F}(X),d}$ is called the *Wijsman topology*, which is usually denoted by $\tau_{W(d)}$ [R. A. Wijsman, 1966].
3. $\tau_{\mathfrak{B}_d(X),d}$ is the *Attouch-Wets topology*, which is usually denoted by $\tau_{AW(d)}$ [H. Attouch and R. J. B. Wets, 1983].
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Completeness

Note that $\{X\}$ is a countable base for $\mathfrak{P}(X)$. So $(CL(X), \tau_{H(d)})$ is metrizable with the compatible metric

$$H(d)(A,B) = \sup_{x \in X} |d(x,A) - d(x,B)|.$$

Then $(CL(X), \tau_{H(d)})$ is complete iff (X,d) is complete.

For any fix any point $x_0 \in X$, note that $\{S(x_0, n) : n \in \mathbb{N}\}$ is a countable base for $\mathfrak{B}_d(X)$. Thus, $(CL(X), \tau_{AW(d)})$ is metrizable.

Theorem (Attouch, Lucchetti and Wets, 1991)

If (X,d) is complete, then $(CL(X), \tau_{AW(d)})$ is completely metrizable.

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Theorem (Attouch, Lucchetti and Wets, 1991)

If (X,d) is complete, then $\big(CL(X), \tau_{AW(d)}\big)$ is completely metrizable.

Theorem (Costantini, 1995)

If X is Polish, then $\big(CL(X), \tau_{W(d)}\big)$ is Polish for any compatible metric d on X.

Example

1. There is a 3-valued metric d on $\mathbb{R}$ such that $\big(CL(\mathbb{R}), \tau_{W(d)}\big)$ is not Čech-complete [Costantini, 1998].
2. There is metric d on $(\omega_1^\omega)_0$ such that $((\omega_1^\omega)_0, d)$ is of the first category, but $\big(CL((\omega_1^\omega)_0), \tau_{W(d)}\big)$ is countably base compact [C and Junnila, 2010].

A space X is countably base compact if there is a base $\mathfrak{B}$ such that each countable centered family $\mathfrak{F} \subseteq \mathfrak{B}$ has a cluster point.

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The Baire property

A space T is *Baire* if for any sequence $\{G_n : n \in \mathbb{N}\}$ of dense open sets in T, $\bigcap_{n\in\mathbb{N}} G_n$ is still dense in T. In addition, if every nonempty closed subspace of T is Baire, then T is called hereditarily Baire.

Theorem (Zsilinszky, 1996)

If (X, d) is separable and Baire, then $(CL(X), \tau_{W(d)})$ is Baire.

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Continuous selections

A function $f : (CL(X), \tau_{\mathfrak{B},d}) \to (X, d)$ is called a *selection* on $(CL(X), \tau_{\mathfrak{B},d})$ if $f(A) \in A$ for all $A \in CL(X)$.

Theorem (Bertacchi and Costantini, 1998)

Let (X, d) be a separable complete metric space, where d is non-Archimedean. Then $(CL(X), \tau_{W(d)})$ admits a continuous selection iff $(CL(X), \tau_{W(d)})$ is totally disconnected.

An observation

Let (X, d) be a separable complete metric space, where d is non-Archimedean. Then $(CL(X), \tau_{AW(d)})$ admits a continuous selection.

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