A Closed-form Exact Solution for Pricing Variance Swaps with Stochastic Volatility

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Abstract

In this paper, we present a highly efficient approach to price variance swaps with discrete sampling times. We have found a closed-form exact solution for the partial differential equation (PDE) system based on the Heston (1993) two-factor stochastic volatility model embedded in the framework proposed by Little and Pant (2001). In comparison with all the previous approximation models based on the assumption of continuous sampling time, the current research of working out a closed-form exact solution for variance swaps with discrete sampling times at least serves for two major purposes: (i) to verify the degree of validity of using a continuous-sampling-time approximation for variance swaps of relatively short sampling period; (ii) to demonstrate that significant errors can result from still adopting such an assumption for a variance swap with small sampling frequencies or long tenor. Other key features of our new solution approach include: (a) with the newly found analytic solution, all the hedging ratios of a variance swap can also be analytically derived; (b) numerical values can be very efficiently computed from the newly found analytic formula.

Key Words: variance swaps, Heston model, closed-form exact solution, explicit formula, stochastic volatility

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1 Introduction

In today's financial markets, trading volatility risk increasingly becomes important to market practitioners ranging from individuals to financial institutes and pension funds. As illustrated by Demeterfi et al. (1999), there are at least three reasons for trading volatility. Firstly, one may want to take a long or short position simply due to a personal directional view of the future volatility level. Secondly, speculators may want to trade the spread between the realized volatility and the implied volatility. These two reasons involve direct speculation on the future trend of stock or index volatility. Thirdly, one may need to hedge against volatility risk of his portfolios. This is the more important reason for trading volatility since bad estimation or inefficient hedging of volatility risk may result in financial disasters, such as the Asian financial crisis, the crash of Barings Bank and the collapse of Long Term Capital Management (LTCM). In practice, derivative products related to volatility and variance have been experiencing sharp increases in trading volume recently. Jung (2006) showed that there was still growing interest in volatility products, such as conditional and corridor variance swaps, among hedge funds and proprietary desks.

Effectively providing volatility exposure, volatility and variance swaps are among the most popular trading products. There is no cost to enter these contracts as they are essentially forward contracts. The payoff at expiry for the long position of a volatility or variance swap is equal to the realized volatility or variance over a pre-specified period minus a pre-set delivery price of the contract multiplied by a notional amount of the swap in dollars per annualized volatility point. Generally, there are two types of volatility or variance swap products (see Dupire 2005). One is historical volatility- or variance-based products, the payoff function of which is the realized volatility or variance discretely sampled at some pre-specified sampling points. Most products of this type are over-the-counter (OTC) contracts. There are some listed products of this kind as well, such as futures on realized variance. NYSE Euronext started to offer cleared-only, on-exchange solution for variance futures in

2006, which are in essence "exchange-listed" version of OTC variance swaps. There are variance futures traded in CBOE as well. The second type of volatility or variance swaps is implied-volatility based products, such as VIX futures in CBOE. Due to the square root relationship between volatility and variance, it turns out to be easier to price and hedge variance swaps than volatility swaps. Also, due to such a square root relationship between the two, the price of a volatility swap should be closely correlated to that of a variance swap anyway. Therefore, we shall primarily focus our attention on variance swaps in this paper. In particular, we shall concentrate on the variance swaps based on discretely-sampled realized variance.

Since the sharp increase in the trading volume of variance swaps recently, it has drawn considerable research interests to develop appropriate valuation approaches for variance swaps. In the literature, there have been two types of valuation approaches, numerical methods and analytical methods.

Of all the analytical methods, there are two subcategories. The most influential ones were proposed by Carr and Madan (1998) and Demeterfi et al. (1999). They have shown how to theoretically replicate a variance swap by a portfolio of standard options. Without requiring to specify the function of volatility process, their models and analytical formulae are indeed very attractive. However, as pointed out by Carr and Corso (2001), the replication strategy has a drawback that the sampling time of a variance swap is assumed to be continuous rather than discrete; such an assumption implies that the results obtained from a continuous model can only be viewed as an approximation for the actual cases in financial practice, in which all contacts are written with the realized variance being evaluated on a set of discrete sampling points. Another drawback is that this strategy also requires options with a continuum of exercise prices, which is not actually available in marketplace. The second kind of analytical methods is the stochastic volatility models. Grunbichler and Longstaff (1996) first developed a pricing model for volatility futures based on mean-reverting squared-root volatility process. Heston (2000) derived an analytical solution for both variance and volatility swaps based on the GARCH volatility process. Javaheri et al. (2004) also discussed the valuation and calibration for variance

swaps based on the GARCH(1,1) stochastic volatility model. They used the flexible PDE approach to determine the first two moments of the realized variance in the context of continuous as well as discrete sampling, and then obtained a closed-form approximate solution after the so-called convexity correction was made. Howison et al. (2004) also considered the valuation of variance swaps and volatility swaps under a variety of diffusion and jump-diffusion models. In their work, approximate solutions of the PDE for pricing volatility-related products are derived. Swishchuk (2004) used an alternative probabilistic approach to value variance and volatility swaps under the Heston (1993) stochastic volatility model. More recently, Elliott et al. (2007) proposed a model to evaluate variance swaps and volatility swaps under a continuous-time Markov-modulated version of the stochastic volatility with regime switching, with both probabilistic and PDE approaches being discussed. All these stochastic volatility models, however, are based on the assumption that the realized variance is approximated with a continuously-sampled one, which will result in a systematic bias for the price of a variance swap. As will be shown later, while the approximation methods provide fairly reasonable estimates for the value of variance swaps with high sampling frequencies, they may lead to large relative errors for variance swaps with small sampling frequencies or long tenors.

Various numerical methods, as an alternative to analytical methods, were also intensively developed recently. A typical article in this category belongs to Little and Pant (2001). In their article, it is shown how to price a variance swap using the finite-difference method in an extended Black-Scholes framework, in which the local volatility is assumed to be a known function of time and spot price of the underlying asset. By exploring a dimension reduction technique, their numerical approach achieves high efficiency and accuracy for discretely-sampled variance swaps. Windcliff et al. (2006) also explored a numerical algorithm to evaluate discretely-sampled volatility derivatives using numerical partial-integro differential equation approach. Under this framework, they investigated a variety of modeling assumptions including local volatility models, jump-diffusion models and models with transaction cost being taken into consideration. Although these two numerical methods evaluate vari-

ance swaps based on discretely-sampled realized variance and achieve high accuracy, the major limitation is that their models do not incorporate stochastic volatilities that are the most commonly used to model the dynamics of equity indices. To remedy this drawback, Little and Pant (2001) and Windcliff et al. (2006) pointed out, respectively, in the conclusions of their papers that for better pricing and hedging general variance swaps one needs to adopt an appropriate model that incorporates the stochastic volatility characteristics observed in financial markets.

Very recently, Brodie and Jain (2008) published a paper, in which they have presented a closed-form solution for volatility as well as variance swaps with discrete sampling. In that paper, they have examined the effects of jumps and stochastic volatility on the price of volatility and variance swaps by comparing calculated prices under various models such as the Black-Scholes model, the Heston stochastic volatility model, the Merton (1973) jump diffusion model and the Bates (1996) and Scott (1997) stochastic volatility and jump model. However, their solution approach is primarily based on integrating the underlying stochastic processes directly and such an approach cannot be adopted for the payoff function we focus on in this paper.

In this paper, we price discretely-sampled variance swaps based on the Heston (1993) two-factor stochastic volatility model embedded in the framework proposed by Little and Pant (2001). In this way, the nature of stochastic volatility is included in the model and most importantly, a closed-form exact solution is worked out, even when the sampling times are discrete. Furthermore, it is shown that our solution degenerates to continuous sampling model when sampling frequency approaches infinity, as expected. Our explicit pricing formula for variance swaps presented here should be valuable in both theoretical and practical senses. Theoretically, although there are many existing models, as mentioned above, to price variance swaps, the closed-form exact solution for discretely-sampled variance swaps with the realized variance defined as the sum of the percentage increment of the underlying asset price is presented for the first time in the stochastic volatility framework. Secondly, our discrete model can be used to verify the validity of the corresponding continuous

models for the specific payoff discussed here and thus would fill a gap that has been in the field of variance swaps pricing. Thirdly, the Fourier inverse transform in our model has been analytically worked out, which is a significant step forward in the literature of Heston model. Practically, the final form of our solution is simple enough in a closed form and thus can be easily used by market practitioners. Furthermore, our explicit solution shows substantial advantage, in terms of both accuracy and efficiency, over previous numerical or approximate approaches, and thus it can satisfy the increasing demand of trading variance swaps in financial markets.

This paper is organized into four sections. In Section 2, a detailed description of variance swaps is first provided, followed by our analytical formula for the variance swaps. In Section 3, some numerical examples are given, demonstrating the correctness of our solution from various aspects. Comparison with continuous sampling models and discussion for other properties of the variance swaps are also carried out. In Section 4, a brief summary is provided.

2 Our Model

In this section, we use the Heston (1993) stochastic volatility model to describe the dynamics of the underlying asset. To evaluate the discretely-sampled realized variance swaps, we employ the dimension reduction technique proposed by Little and Pant (2001) to analytically solve the associated PDE.

2.1 The Heston Model

It is a well-known fact by now that the Black and Scholes (1973) model may fail to reflect certain features of the reality of financial markets due to some unrealistic assumptions, such as the constant volatility assumption; numerous phenomena such as smile effect (Wilmott 1998), skewness and kurtosis effects (Voit 2005) have been observed and reported, suggesting necessary improvements of the Black-Scholes model.

In the hope of remedying some apparent drawback of the Black-Scholes model, many models have been proposed to incorporate stochastic volatility, stochastic volatility with jump, stochastic volatility and stochastic interest rate (c.f., Stein and Stein 1991; Heston 1993; Scott 1997; Schöbel and Zhu 1999). In order to assess the performance of these models, Bakshi et al. (1997) systematically analyzed the performance of incorporating stochastic volatility, jump diffusion, and stochastic interest rate, and concluded that the most important improvement over the Black-Scholes model was achieved by introducing stochastic volatility into option pricing models. Once this is done, introducing jumps and stochastic interest rate leads to only marginal improvement in option pricing. For this reason, we shall focus on the stochastic volatility model in this paper, leaving stochastic interest rate model and jump diffusion model for further research. Among all the stochastic volatility models in the literature, model proposed by Heston (1993) has received the most attention since it can give a satisfactory description of the underlying asset dynamics (Daniel et al. 2005; Silva et al. 2004). In the Heston (1993) model, the underlying asset S_t is modeled by the following diffusion process with a stochastic instantaneous variance v_t.

$$\begin{cases} dS_t = \mu S_t dt + \sqrt{v_t} S_t dB_t^S \\ dv_t = \kappa(\theta - v_t)dt + \sigma_V \sqrt{v_t} dB_t^V \end{cases} \tag{2.1}$$

where μ is the expected return of the underlying asset, θ is the long-term mean of variance, κ is a mean-reverting speed parameter of the variance, σ_V is the so-called volatility of volatility. The two Wiener processes dB_t^S and dB_t^V describe the random noise in asset and variance respectively. They are assumed to be correlated with a constant correlation coefficient ρ, that is $(dB_t^S, dB_t^V) = \rho dt$. The stochastic volatility process is the familiar squared-root process. To ensure the variance is always positive, it is required that $2\kappa\theta \geq \sigma^2$ (see Cox et al. 1985; Heston 1993; Zhang and Zhu 2006).

According to the existence theorem of equivalent martingale measure, we are

able to change the real probability measure to a risk-neutral probability measure and describe the processes as:

$$\begin{cases} dS_t = rS_t dt + \sqrt{v_t} S_t d\widetilde{B}_t^S \\ dv_t = \kappa^*(\theta^* - v_t)dt + \sigma_V \sqrt{v_t} d\widetilde{B}_t^V \end{cases} \tag{2.2}$$

where $\kappa^* = \kappa + \lambda$ and $\theta^* = \frac{\kappa\theta}{\kappa+\lambda}$ are the risk-neutral parameters, the new parameter λ is the premium of volatility risk (Heston 1993). As illustrated in Heston's paper, applying Breeden (1979)'s consumption-based model yields a volatility risk premium of the form $\lambda(t, S_t, v_t) = \lambda v$ for the CIR square-root process. For the rest of this paper, our analysis will be based on the risk-neutral probability measure. The conditional expectation at time t is denoted by $E_t^Q = E^Q[\cdot \mid \mathscr{F}_t]$, where $\mathscr{F}_t$ is the filtration up to time t.

2.2 Variance Swaps

Variance swaps are forward contracts on the future realized variance of the returns of the specified underlying asset. The long position of a variance swap pays a fixed delivery price at expiry and receives the floating amounts of annualized realized variance, whereas the short position is just the opposite. Thus it can be easily used for investors to gain exposure to volatility risk.

Usually, the value of a variance swap at expiry can be written as $V_T = (\sigma_R^2 - K_{var}) \times L$, where the σ_R^2 is the annualized realized variance over the contract life $[0, T]$, K_{var} is the annualized delivery price for the variance swap, and L is the notional amount of the swap in dollars per annualized volatility point squared. The T is the life time of the contract.

At the beginning of a contract, it is clearly specified the details of how the realized variance should be calculated. Important factors contributing to the calculation of the realized variance include the underlying asset(or assets), the observation frequency of the price of the underlying asset(s), the annualization factor, the contract lifetime, the method of calculating the variance. A typical formula for the measure

of realized variance is

$$\sigma_R^2 = \frac{AF}{N}\sum_{i=1}^{N}(\frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}})^2 \times 100^2 \tag{2.3}$$

where S_{t_i} is the closing price of the underlying asset at the i-th observation time t_i, and there are altogether N observations. AF is the annualized factor converting this expression to an annualized variance. If the sampling frequency is every trading day, then $AF = 252$, assuming that there are 252 trading days in one year, if every week then $AF = 52$, if every month then $AF = 12$ and so on. We assume equally-spaced discrete observations in this paper so that the annualized factor is of a simple expression $AF = \frac{1}{\Delta t} = \frac{N}{T}$.

In the risk-neutral world, the value of a variance swap at time t is the expected present value of the future payoff, $V_t = E_t^Q[e^{-r(T-t)}(\sigma_R^2 - K_{var})L]$. This should be zero at the beginning of the contract since there is no cost to enter into a swap. Therefore, the fair variance delivery price can be easily defined as $K_{var} = E_0^Q[\sigma_R^2]$, after setting the value of $V_t = 0$ initially. The variance swap valuation problem is therefore reduced to calculating the expectation value of the future realized variance in the risk-neutral world.

2.3 Our Approach to Price Variance Swaps

In this subsection, we discuss our approach to produce an analytical solution for the fair delivery price of a variance swap. As we shall see later, the associated PDE is analytically solved and an explicit closed-form solution is obtained. While we focus on calculating the expected value of realized variance σ_R^2 defined in (2.3) in this paper, our approach could be easily extended to handle other definitions of realized variances.

As illustrated in (2.3), the expected value of realized variance in the risk neutral world is defined as:

$$E_0^Q[\sigma_R^2] = E_0^Q[\frac{1}{N\Delta t}\sum_{i=1}^{N}(\frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}})^2] \times 100^2 = \frac{100^2}{N\Delta t}\sum_{i=1}^{N}E_0^Q[(\frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}})^2] \tag{2.4}$$

So the problem of pricing variance swap is reduced to calculating the N expectations in the form of:

$$E_0^Q[(\frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}})^2] \tag{2.5}$$

for some fixed equal time period Δt and N different tenors $t_i = i\Delta t$ $(i = 1, \cdots, N)$. In the rest of this section, we will focus our main attention on calculating the expectation of this expression. As shall be shown later, we need to consider two cases, $i = 1$ and $i > 1$, due to the difference in the calculation procedures. In the process of calculating of this expectation, i, unless otherwise stated, is regarded as a constant. And hence both t_i and t_{i-1} are regarded as known constants.

Firstly we consider the case $i > 1$. In this case the time $t_{i-1} > 0$ and thus $S_{t_{i-1}}$ is also an unknown at the current time $t = 0$. Therefore, the payoff function depends on two unknown variables $S_{t_{i-1}}$ and S_{t_i} which are the underlying price in the future. This two-dimensional payoff function makes the problem extremely difficult to deal with. We will however show that the problem could be solved by firstly introducing a new variable I_t and then decomposing the original problem into two one-dimensional problems which could be relatively easier to be solved analytically. This technique was firstly proposed by Little and Pant (2001).

Let us first introduce a new variable I_t

$$I_t = \int_0^t \delta(t_{i-1} - \tau) S_\tau d\tau \tag{2.6}$$

where the $\delta(\cdot)$ is the Dirac delta function. Note that $I_t = 0$ for $t < t_{i-1}$ and $I_t = S_{t_{i-1}}$ for $t \geq t_{i-1}$.

We now consider a contingent claim $U_i = U_i(S_t, v_t, I_t, t)$ whose payoff at expiry t_i is $(\frac{S_{t_i}}{I_{t_i}} - 1)^2$. Following the general asset valuation theory by Garman (1977), or the standard analysis of Asian options with stochastic volatility (Fouque et al. 2000; Wilmott 1998), we obtain the PDE for U_i (Subscripts have been omitted in the PDE

without ambiguity).

$$\begin{aligned}(2.7)\qquad &\frac{\partial U_i}{\partial t}+\frac{1}{2}vS^2\frac{\partial U_i^2}{\partial S^2}+\rho\sigma_V vS\frac{\partial U_i^2}{\partial S\partial v}+\frac{1}{2}\sigma_V^2 v\frac{\partial U_i^2}{\partial v^2}+rS\frac{\partial U_i}{\partial S}\\ &+[\kappa^*(\theta^*-v)]\frac{\partial U_i}{\partial v}-rU_i+\delta(t_{i-1}-t)\frac{\partial U_i}{\partial I}=0\end{aligned}$$

The terminal condition is

$$(2.8)\qquad U_i(S,v,I,t_i)=(\frac{S}{I}-1)^2$$

Howison et al. (2004) also derived a similar PDE based on their model, however, they didn't solve the PDE directly.

The Feynman-Kac theorem (Karatzas et al. 1991) states that the solution of the PDE system satisfies:

$$(2.9)\qquad E_0^Q[(\frac{S_{t_i}}{I_{t_i}}-1)^2]=e^{rt_i}U_i(S_0,v_0,I_0,0)$$

Thus it is sufficient to solve the PDE (2.7) with terminal condition (2.8) to obtain the expectation (2.5) we require. To solve this PDE system, we need to utilize the properties of variable I_t and the Dirac delta function in the equation.

The property of Dirac delta function indicates that any time away from t_{i-1} the PDE (2.7) could be reduced as

$$(2.10)\qquad \frac{\partial U_i}{\partial t}+\frac{1}{2}vS^2\frac{\partial U_i^2}{\partial S^2}+\rho\sigma_V vS\frac{\partial U_i^2}{\partial S\partial v}+\frac{1}{2}\sigma_V^2 v\frac{\partial U_i^2}{\partial v^2}+rS\frac{\partial U_i}{\partial S}+[\kappa^*(\theta^*-v)]\frac{\partial U_i}{\partial v}-rU_i=0$$

This means that we have managed to get rid of variable I_t in the equation except at the time t_{i-1}. However, we cannot declare that we have succeeded in getting rid of one spatial dimension due to the presence of I_t in the terminal condition (2.8). To handle the I_t in the terminal condition, we turn to the so-called jump condition.

As mentioned previously, $I_t=0$, $t<t_{i-1}$ and $I_t=S_{t_{i-1}}$, $t\geq t_{i-1}$. The variable I_t therefore experiences a jump in value across time t_{i-1}. The no-arbitrary pricing theory however requires the claim's value should remain continuous. This leads to

an additional jump condition at time t_{i-1} (refer to Wilmott et al. 1993 for a further discussion of jump conditions),

$$\lim_{t\uparrow t_{i-1}} U_i(S,v,I,t) = \lim_{t\downarrow t_{i-1}} U_i(S,v,I,t) \tag{2.11}$$

From this viewpoint, we can equivalently solve the PDE (2.10) with terminal condition (2.8) and jump condition (2.11) in order to obtain the expectation we are interested in. Furthermore, inspired by the property of variable I_t, we consider dividing the time domain $[0,t_i]$ into two parts $[0,t_{i-1}]$ and $[t_{i-1},t_i]$ since during each of the two time sub-domains, I_t could be regarded as constant. Hence, it is a clever idea to solve the PDE system by two stages, the first stage in $[t_{i-1},t_i]$ and the second stage in $[0,t_{i-1}]$. During each of the two stages the PDE systems have one dimension less than the original PDE system. The obtained solution of the first stage will provide the terminal condition for PDE system in second stage through the jump condition (2.11). We need to remark that this is one of the key features of this paper. Little and Pant (2001) were the first to use the dimension reduction approach which provides many computational benefits in their instantaneous local volatility model. In this paper, the approach is applied to the stochastic volatility model and provides us with a closed-form solution.

Now, the PDE system (2.7) could be equivalently expressed by two PDE systems as

$$\begin{cases} \dfrac{\partial U_i}{\partial t} + \dfrac{1}{2}vS^2\dfrac{\partial U_i^2}{\partial S^2} + \rho\sigma_V vS\dfrac{\partial U_i^2}{\partial S\partial v} + \dfrac{1}{2}\sigma_V^2 v\dfrac{\partial U_i^2}{\partial v^2} + rS\dfrac{\partial U_i}{\partial S} + [\kappa^*(\theta^* - v)]\dfrac{\partial U_i}{\partial v} - rU_i = 0 \\ U_i(S,v,I,t_i) = (\dfrac{S}{I} - 1)^2 \qquad t_{i-1} \le t \le t_i \end{cases} \tag{2.12}$$

and

$$\begin{cases} \dfrac{\partial U_i}{\partial t} + \dfrac{1}{2}vS^2\dfrac{\partial U_i^2}{\partial S^2} + \rho\sigma_V vS\dfrac{\partial U_i^2}{\partial S\partial v} + \dfrac{1}{2}\sigma_V^2 v\dfrac{\partial U_i^2}{\partial v^2} + rS\dfrac{\partial U_i}{\partial S} + [\kappa^*(\theta^* - v)]\dfrac{\partial U_i}{\partial v} - rU_i = 0 \\ \lim\limits_{t\uparrow t_{i-1}} U_i(S,v,I,t) = \lim\limits_{t\downarrow t_{i-1}} U_i(S,v,I,t) \qquad 0 \le t \le t_{i-1} \end{cases} \tag{2.13}$$

Note that I_t is a fixed number $I_t = S_{t_{i-1}}$ in the domain $t_{i-1} \le t \le t_i$ and $I_t = 0$

in $0 \leq t < t_{i-1}$. We firstly analytically solve the PDE system (2.12) using the generalized Fourier transform method (see Lewis 2000; Poularikas 2000).

Proposition 2.1 *If the underlying asset follows the dynamic process (2.2) and a European-style derivative written on this underlying asset has a payoff function $U(S, v, T) = H(S)$ at expiry T, then the solution of the associated PDE system of the derivative value*

(2.14)
$$\begin{cases} \dfrac{\partial U}{\partial t} + \dfrac{1}{2} v S^2 \dfrac{\partial U^2}{\partial S^2} + \rho \sigma_V v S \dfrac{\partial U^2}{\partial S \partial v} + \dfrac{1}{2} \sigma_V^2 v \dfrac{\partial U^2}{\partial v^2} + rS \dfrac{\partial U}{\partial S} + [\kappa^*(\theta^* - v)] \dfrac{\partial U}{\partial v} - rU = 0 \\ U(S, v, T) = H(S) \end{cases}$$

can be expressed in closed form as:

$$U(x, v, t) = \mathcal{F}^{-1}[e^{C(\omega, T-t) + D(\omega, T-t)v} \mathcal{F}[H(e^x)]] \tag{2.15}$$

using generalized Fourier transform method (see Lewis 2000; Poularikas 2000), where $x = \ln S$, $j = \sqrt{-1}$ and ω is the Fourier transform variable, and

$$\begin{cases} C(\omega, \tau) = r(\omega j - 1)\tau + \dfrac{\kappa^* \theta^*}{\sigma_V^2}[(a+b)\tau - 2\ln(\dfrac{1 - ge^{b\tau}}{1-g})] \\ D(\omega, \tau) = \dfrac{a+b}{\sigma_V^2} \dfrac{1 - e^{b\tau}}{1 - ge^{b\tau}} \\ a = \kappa^* - \rho \sigma_V \omega j, \quad b = \sqrt{a^2 + \sigma_V^2(\omega^2 + \omega j)}, \quad g = \dfrac{a+b}{a-b} \end{cases} \tag{2.16}$$

The proof of this proposition is left in Appendix A.

It should be noted that Equation (2.15) has been deliberately left in a rather general form. This is because the payoff function $H(S)$ hasn't been specified yet. In this most general form, Proposition 1 is applicable to most derivatives whose payoffs depend on spot price S of underlying asset in the framework of the Heston stochastic volatility. The original result of Heston (1993) is actually a special case covered by this proposition.

However, for some payoffs, the Fourier transform in Proposition 1 has to be interpreted as the generalized Fourier transform, which is a useful tool for pricing derivatives. For most popularly used financial derivatives, such as vanilla call op-

tions with $H(S) = \max(S - K, 0)$, performing the generalized Fourier transform is straightforward. The main difficulty with this approach, however, is associated with the Fourier inverse transform needed to be performed, if one wishes to reduce the computational time substantially. For our specific case, $H(S) = (\frac{S}{I} - 1)^2$, the Fourier inverse transform could be explicitly worked out and hence the solution can be written in a much simple and elegant form.

Based on the generalized Fourier transform, we can perform the transformation as

$$\mathcal{F}[e^{j\alpha t}] = 2\pi\delta_\alpha(\omega) \tag{2.17}$$

where $j = \sqrt{-1}$, α is any complex number and $\delta_\alpha(\omega)$ is the generalized delta function satisfying

$$\int_{-\infty}^{\infty} \delta_\alpha(t)\Phi(t)dt = \Phi(\alpha) \tag{2.18}$$

In our specific case PDE (2.12), $H(S) = (\frac{S}{I} - 1)^2$. By setting $x = \ln S$ and noting I a constant, we perform the generalized Fourier transform to the payoff function $H(e^x)$ with regards to x.

$$\mathcal{F}[(\frac{e^x}{I} - 1)^2] = 2\pi[\frac{\delta_{-2j}(\omega)}{I^2} - 2\frac{\delta_{-j}(\omega)}{I} + \delta_0(\omega)] \tag{2.19}$$

Using the Proposition 1, the solution of PDE (2.12) is given by

$$\begin{aligned}
&U_i(S, v, I, t) = \mathcal{F}^{-1}[e^{C(\omega,t_i-t)+D(\omega,t_i-t)v} 2\pi[\frac{\delta_{-2j}(\omega)}{I^2} - 2\frac{\delta_{-j}(\omega)}{I} + \delta_0(\omega)]] \\
&= \int_{-\infty}^{\infty} e^{C(\omega,t_i-t)+D(\omega,t_i-t)v}[\frac{\delta_{-2j}(\omega)}{I^2} - 2\frac{\delta_{-j}(\omega)}{I} + \delta_0(\omega)]e^{x\omega j}d\omega \\
&= \tfrac{1}{I^2} e^{C(\omega,t_i-t)+D(\omega,t_i-t)v+x\omega j}\big|_{\omega=-2j} - \tfrac{2}{I} e^{C(\omega,t_i-t)+D(\omega,t_i-t)v+x\omega j}\big|_{\omega=-j} \\
&\quad + e^{C(\omega,t_i-t)+D(\omega,t_i-t)v+x\omega j}\big|_{\omega=0} \\
&= \tfrac{e^{2x}}{I^2} e^{\widetilde{C}(t_i-t)+\widetilde{D}(t_i-t)v} - \tfrac{2e^x}{I} + e^{-r(t_i-t)}
\end{aligned} \tag{2.20}$$

where $x = \ln S$ and $t_{i-1} \le t \le t_i$, and $\widetilde{C}(\tau)$ and $\widetilde{D}(\tau)$ are equal to $C(-2j, \tau)$,

$D(-2j,\tau)$ respectively, and have simple forms as

$$(2.21)\quad \begin{cases} \widetilde{C}(\tau) = r\tau + \dfrac{\kappa^*\theta^*}{\sigma_V^2}[(\widetilde{a}+\widetilde{b})\tau - 2\ln(\dfrac{1-\widetilde{g}e^{\widetilde{b}\tau}}{1-\widetilde{g}})] \\ \widetilde{D}(\tau) = \dfrac{\widetilde{a}+\widetilde{b}}{\sigma_V^2}(\dfrac{1-e^{\widetilde{b}\tau}}{1-\widetilde{g}e^{\widetilde{b}\tau}}) \\ \widetilde{a} = \kappa^* - 2\rho\sigma_V, \quad \widetilde{b} = \sqrt{\widetilde{a}^2 - 2\sigma_V^2}, \quad \widetilde{g} = (\dfrac{\widetilde{a}}{\sigma_V})^2 - 1 + (\dfrac{\widetilde{a}}{\sigma_V})\sqrt{(\dfrac{\widetilde{a}}{\sigma_V})^2 - 2} \end{cases}$$

Now, we have succeeded in obtaining the solution for the PDE system (2.12), which is the first stage in calculating $E_0^Q[(\dfrac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}})^2]$. It should be remarked that we have actually solved an option pricing problem based on the Heston stochastic volatility model. The very reason that we have explicitly worked out the Fourier inverse transform so that our final solution (2.20) of the first stage can be written in such a simple and closed form, whereas the Fourier inverse transform could not be worked out by Heston (1993), is because of the very special form of the payoff function (2.8). One may argue that Heston's solution for a simple European call is still in closed form, because there is only an explicit integral left to be calculated, the same as the calculation of the cumulative distribution function required in using the Black-Scholes formula. But, a sharp difference between the two is that the integrand of the latter is a well-defined and smooth real function whereas the integrand of the former (i.e., Heston's original solution as well as the solutions presented in many other follow-up papers based on the Heston model, such as Bakshi et al. 1997; Bates 1996; Pan 2002), is a complex-value function, as a result of the Fourier inverse transform not being analytically performed. The main disadvantage of a solution being left in terms of complex-valued integrals is that the numerical calculation of these integrals has to be handled very carefully as they are multi-valued complex functions, which may cause some problems when one needs to decide which root is the correct one to take. There have been examples reported in the literature (e.g., Kahl and Jackel 2005) for the wrong numerical integration that those complex-valued integrand may result in. In comparison with those complicated integral calculations, the advantage of our compact solution (2.20) is obvious. Although our success in

analytically performing Fourier inverse transform under the Heston model may be limited for a special form of payoff function, it made us to believe that there might be other payoff functions, with which the Fourier inverse transform can be worked out analytically as well. This belief has not been clearly articulated in the relevant literature before; all the papers following Heston's work stopped at the same point where Heston did, i.e., did not bother to analytically perform the Fourier inverse transform at all.

To finish off the calculation of $E_0^Q[(\frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}})^2]$, we need to move to the second stage, i.e. solving the PDE system (2.13), after the imposition of the jump condition (2.11). As we shall show later, the simple form of solution (2.20) has paved an easy way of obtaining an analytical solution in the second stage.

By noting the fact that $\lim_{t \downarrow t_{i-1}} \ln S_t = \ln I$ due to the definition of I, we have,

$$\lim_{t \downarrow t_{i-1}} U_i(S, v, I, t) = e^{\widetilde{C}(\Delta t) + \widetilde{D}(\Delta t)v} + e^{-r\Delta t} - 2 \tag{2.22}$$

For the simplicity of notation, the right hand side of above equation is denoted as $f(v)$, i.e.,

$$f(v) = e^{\widetilde{C}(\Delta t) + \widetilde{D}(\Delta t)v} + e^{-r\Delta t} - 2 \tag{2.23}$$

which is now the terminal condition for the PDE system (2.13) in the period $0 \leq t \leq t_{i-1}$, according to the jump condition (2.11).

It should be noticed that the terminal condition (2.23) for the PDE system (2.13) in the period $0 \leq t \leq t_{i-1}$ happens to contain one independent variable, v, only. One can thus take the advantage of this fact and solve the problem neatly with the following proposition.

Proposition 2.2 *If the underlying asset follows the dynamic process (2.2), the derivative written on some stochastic aggregated property of this underlying asset with payoff function depending on the v_T only, i.e., $U(S, v, T) = G(v_T)$ at expiry T*

will satisfy the PDE

(2.24)
$$\begin{cases} \dfrac{\partial U}{\partial t} + \dfrac{1}{2}vS^2\dfrac{\partial U^2}{\partial S^2} + \rho\sigma_V vS\dfrac{\partial U^2}{\partial S\partial v} + \dfrac{1}{2}\sigma_V^2 v\dfrac{\partial U^2}{\partial v^2} + rS\dfrac{\partial U}{\partial S} + [\kappa^*(\theta^* - v)]\dfrac{\partial U}{\partial v} - rU = 0 \\ U(S, v, T) = G(v) \end{cases}$$

The solution of this PDE can be obtained analytically in the form of

$$U(S, v, t) = \int_0^{+\infty} e^{-r(T-t)} G(v_T) p(v_T|v_t) dv_T \tag{2.25}$$

where

$$p(v_T|v_t) = ce^{-W-V}(\frac{V}{W})^{q/2} I_q(2\sqrt{WV})$$

(2.26)
$$c = \frac{2\kappa^*}{\sigma_V^2(1 - e^{-\kappa^*(T-t)})}, \qquad W = cv_t e^{-\kappa^*(T-t)}, \qquad V = cv_T, \qquad q = \frac{2\kappa^*\theta^*}{\sigma_V^2} - 1$$

and $I_q(\cdot)$ is the modified Bessel function of the first kind of order q.

The proof of Proposition 2 is trivial, as it is actually implied by the Feynman-Kac formula, which states that the solution of PDE (2.24) can be derived from the conditional expectation of the payoff function under the risk-neutral probability measure. Hence, the solution can be expressed in the form of

$$U(S, v, t) = E_t^Q[e^{-r(T-t)} G(v_T)] \tag{2.27}$$

where the associated two processes S_t and v_t follow the stochastic processes in (2.2), respectively. The expectation is actually not related to the process S since the payoff function is independent of S. The process v_t is the well-known CIR squared-root process (Cox et al. 1985) and the distribution is the noncentral chi-square, $\chi^2(2V; 2q + 2, 2W)$, with $2q + 2$ degrees of freedom and parameter of non-centrality $2W$ proportional to the current variance, v_t. Once we realized that the needed transition probability density function $p(v_T|v_t)$ has been given in Cox et al. (1985), as shown in Equation (2.26), the proof naturally follows.

Using the Proposition 2, we can express the solution of PDE system (2.13) as

$$U_i(S,v,I,t) = \int_0^\infty e^{-r(t_{i-1}-t)} f(v_{t_{i-1}}) p(v_{t_{i-1}}|v_t) dv_{t_{i-1}} \tag{2.28}$$

where $f(v)$ and $p(v_{t_{i-1}}|v_t)$ are given in Equation (2.23) and Equation (2.26) respectively, and $0 \le t < t_{i-1}$. This means for each $i > 1$ the expectation (2.5) has been found by solving the PDE systems (2.12) and (2.13) in two stages,

$$\begin{aligned} & E_0^Q[(\frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}})^2] = e^{rt_i} U_i(S_0, v_0, I_0, 0) \\ & = \int_0^\infty e^{r\Delta t} f(v_{t_{i-1}}) p(v_{t_{i-1}}|v_0) dv_{t_{i-1}} \end{aligned} \tag{2.29}$$

As Zhang and Zhu (2006) commented in their paper, the integration in the above equation usually cannot be explicitly carried out; we had initially decided to leave our final solution in this integral form too. However, after a careful examination of the properties of the integrand, we realized that the elegant form of $f(v)$, which is the solution of the first stage, could be explored again. Utilizing the characteristic function of noncentral chi-squared distribution (Johnson et al. 1970), we have successfully carried out the above integral analytically and obtain a fully closed-form solution as our final solution for the price of a variance swap with the realized variance defined by (2.3). This has made our solution in a remarkably simple form as

$$E_0^Q[(\frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}})^2] = e^{r\Delta t} f_i(v_0) \tag{2.30}$$

where

$$\begin{aligned} f_i(v_0) &= \int_0^\infty f(v_{t_{i-1}}) p(v_{t_{i-1}}|v_0) dv_{t_{i-1}} \\ &= e^{\widetilde{C}(\Delta t) + \frac{c_i e^{-\kappa^* t_{i-1}}}{c_i - \widetilde{D}(\Delta t)} \widetilde{D}(\Delta t) v_0} (\frac{c_i}{c_i - \widetilde{D}(\Delta t)})^{\frac{2\kappa^*\theta^*}{\sigma_V^2}} + e^{-r\Delta t} - 2 \end{aligned} \tag{2.31}$$

and $c_i = \frac{2\kappa^*}{\sigma_V^2(1-e^{-\kappa^* t_{i-1}})}$. To a certain extent, it is even simpler than that of the classic Black-Scholes formula, because the latter still involves the calculation of the

cumulative distribution function, which is an integral of a smooth real-value function, whereas there is no need to calculate any integral at all in our final solution! The details of analytically carrying out the integration in Equation (2.31) are left in Appendix B.

Utilizing (2.30), the summation in (2.4) can now be carried out all the way except for the very first period with $i = 1$. We need to treat the case $i = 1$, separately, simply because in this case we have $t_{i-1} = 0$ and $S_{t_{i-1}} = S_0$, which is the current underlying asset price and is a known value, instead of an unknown value of $S_{t_{i-1}}$ for any other cases with $i > 1$. So the expectation that needs to be calculated in this special case is reduced to

$$E_0^Q[(\frac{S_{t_i}}{S_0} - 1)^2] \tag{2.32}$$

which can be easily derived by invoking Proposition 1 directly,

$$E_0^Q[(\frac{S_{t_1}}{S_0} - 1)^2] = e^{r\Delta t} f(v_0) \tag{2.33}$$

Summarizing the calculation procedure discussed above, we finally obtain the fair strike price for the variance swap as:

$$K_{var} = E_0^Q[\sigma_R^2] = \frac{e^{r\Delta t}}{T}[f(v_0) + \sum_{i=2}^{N} f_i(v_0)] \times 100^2 \tag{2.34}$$

where N is a finite number denoting the total sampling times of the swap contract. This formula is obtained by solving the associated PDEs in two stages. Since we have managed to express the solution of the associated PDEs, in both stages, in terms of simple and elementary functions, we are able to write the fair strike price of a variance swap with discretely-sampled realized variance defined in its payoff in a simple and closed form.

One may wonder why not use the Feynman-Kac formula to calculate the expectation of the payoff function directly instead of painfully detouring around to solve a PDE (2.12) in Stage 1 first and then using the Feynman-Kac formula in Stage 2.

This is actually due to the dimensionality of the payoff function $(\frac{S_{t_i}-S_{t_{i-1}}}{S_{t_{i-1}}})^2$, that involves two stochastic variables, S_{t_i} and $S_{t_{i-1}}$. To use the Feynman-Kac formula for this two dimensional payoff function, one needs to find the joint transition probability function of the two stochastic variables, which is a very difficult task, and even if it could be successfully found, there are still difficulties involved in the numerical computation of the resulted two-dimensional integral. This is why we chose to use this two-stage approach to reduce the dimensionality of solving the original problem with the Feynman-Kac formula directly. The great benefit of using this analytic formula for the price of a variance swap with the realized variance being defined in (2.3) is illustrated in the next section through some examples.

3 Numerical Examples and Discussions

In this section, we show some numerical examples for illustration purposes. Although theoretically there would be no need to discuss the accuracy of a closed-form exact solution and present numerical results, some comparisons with the Monte Carlo (MC) simulations may give readers a sense of verification for the newly found solution. This is particularly so for some market practitioners who are very used to MC simulations and would not trust analytical solutions that may contain algebraic errors unless they have seen numerical evidence of such a comparison. In addition, comparisons with the previous continuous sampling model will also help readers understand the improvement in accuracy with our exact solution. Furthermore, we shall discuss some essential properties of variance swaps as well, utilizing the newly found analytical solution.

To achieve these purposes, we use the following parameters (unless otherwise stated): $v_0 = 0.04$, $\theta^* = 0.022$, $\kappa^* = 11.35$, $\rho = -0.64$, $\sigma_V = 0.618$, $r = 0.1$, $T = 1$ in this section. This set of parameters for the square root process was also adopted by Dragulescu and Yakovenko (2002). As for the MC simulations, we took asset price $S_0 = 1$ and the number of the paths $N = 200,000$ for all the simulation results presented here. All the numerical values of variance swaps presented in this section

are quoted in variance points (the square of volatility points).

3.1 Monte Carlo Simulations

Our MC simulations are based on a simple simulation of the CIR variance process, which is anything but straightforward. Glasserman (2003) proposed a method to simulate the square-root process by sampling the transition density function. Broadie and Kaya (2006) developed an approach for exact simulation of the Heston dynamical process. Andreasen (2006) also suggested a method using log-normal approximation for the transition density of the variance with matched first two moments. Higham and Mao (2005) proved that the Euler-Maruyama discretization is an attractive approach, providing qualitatively correct approximations. Since our aim is primarily to some obtain benchmark values for our solution Equation (2.34), we will not focus our attention on the use of other variance reduction techniques that could further enhance the computational efficiency. In our MC simulations, we have employed the simple Euler-Maruyama discretization for the Heston model

$$\text{(3.1)} \qquad \begin{cases} S_t = S_{t-1} + rS_{t-1}\Delta t + \sqrt{|v_{t-1}|}S_{t-1}\sqrt{\Delta t}W_t^1 \\ v_t = v_{t-1} + \kappa^*(\theta^* - v_{t-1})\Delta t + \sigma\sqrt{|v_{t-1}|}\sqrt{\Delta t}(\rho W_t^1 + \sqrt{1-\rho^2}W_t^2) \end{cases}$$

where W_t^1 and W_t^2 are two independent standard normal random variables. Shown in Figure 3.1, as well as in Table 3.1, are three sets of data, for the strike price of variance swaps obtained with the numerical implementation of Equation (2.34), those from MC simulations (3.1) and the numerical results obtained from the continuously-sampled realized variance Equation (3.3). One can clearly observe that the results from our exact solution perfectly match the results from the MC simulations. To make sure that readers have some quantitative concept of how large the difference between the results from our exact solution and those from the MC simulations, we have also tabulated the relative difference of the two as a function of the number of paths, using our exact solution as the reference in the calculation, in Table 3.2. Clearly, when the number of paths reaches 200,000 in MC simulations, the relative

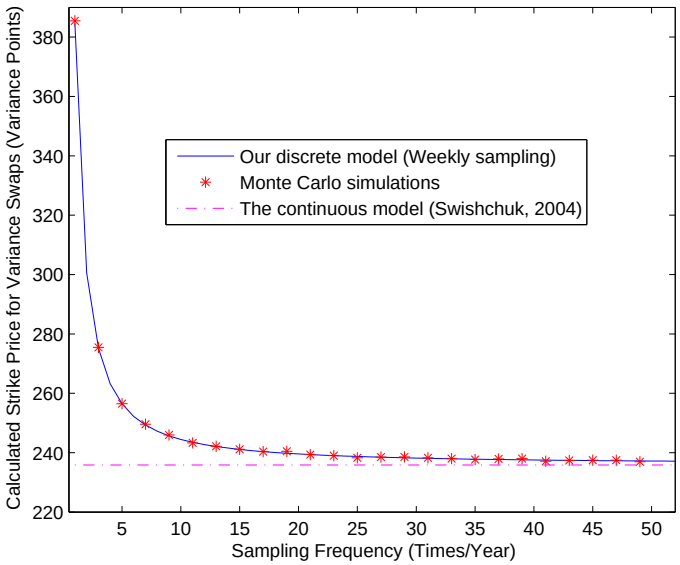

Figure 3.1: A comparison of fair strike values based on the discrete model, continuous model and the MC simulations

Table 3.1: The numerical results of discrete model, continuous model and MC simulations

Sampling Frequency	Discrete Model	Continuous Model	MC Simulations
Quarterly(N=4)	267.6	235.9	267.3
Monthly(N=12)	242.7	235.9	243.2
Fortnightly(N=26)	238.6	235.9	238.1
Weekly(N=52)	237.1	235.9	237.4
Daily(N=252)	236.1	235.9	

difference of the two is less than 0.1% already. Such a relative difference is further reduced when the number of paths is increased; demonstrating the convergence of the MC simulations towards our exact solution. On the other hand, in terms of computational time, the MC simulations take a much longer time than our analytical solution does. To illustrate it clearly, we compare the computational times of implementing Equation (2.34) and the MC simulations with sampling frequency for the realized variance equalling to 5 times per year. Table 3.2 shows the computational times for different path numbers in the MC simulations. In contrast to a formidable computational time of 2,184.239 seconds using the MC simulations with 500,000 paths, implementing Equation (2.34) just consumed 0.011 seconds; a roughly 200 thousands folds of reduction in computational time for one data point.

Table 3.2: Relative errors and computational time of MC simulations

Path numbers of the MC	Relative Error %	Computational time(s)
10,000	0.233	5.126
100,000	0.191	89.549
200,000	0.074	360.268
500,000	0.012	2,184.239

The difference is even more significant when the sampling frequency is increased; we had to abandon the calculation when the sampling frequency became daily as it just simply took too long to finish off the calculation on our PC (as a result, one cell in Table 3.1 is left empty). This is not surprising at all since time-consuming is a well-known drawback of MC simulations.

3.2 The continuous model

In the literature, many researchers, such as Swishchuk (2004), Zhang and Zhu (2006), have proposed continuous sampling models for variance swaps based on the Heston model. In their papers, the realized variance (2.3) is approximated by

$$\sigma_R^2 = \frac{1}{T}\int_0^T v_t dt \times 100^2 \tag{3.2}$$

for the convenience of calculation. This is because Swishchuk (2004) has shown that once the realized variance is defined in terms of an integral, the expectation of this continuous integral can be easily obtained, utilizing the second stochastic process defined in (2.2). The resulting fair delivery price for the variance swap is thus written as

$$E_0^Q[\sigma_R^2] = [v_0\frac{1-e^{-\kappa^* T}}{\kappa^* T} + \theta^*(1-\frac{1-e^{-\kappa^* T}}{\kappa^* T})] \times 100^2 \tag{3.3}$$

which can be interpreted as a weighted average of the spot variance v_0 and the long-term mean of variance θ^*. Indeed, this formula is very simple and can be easily implemented in calculating the numerical value of $E_0^Q[\sigma_R^2]$. For the convenience

of referencing, this formula will be referred to as the Swishchuk formula hereafter, although many others also derived this formula.

Due to the lack of exact solution, in the past, for pricing a variance swap with discrete sampling, the Swishchuk formula was primarily used in pricing variance swaps, based on the assumption that the sampling period, such as daily sampling, is short enough so that the result obtained from the continuous model should be close to that without the continuum assumption of the sampling period. However, no one knew exactly how close the results were because there was no exact solution as a pricing formula for the case of discrete sampling times. Nor did any one know when the Swishchuk formula starts to yield large errors when the sampling time is large enough. In other words, there is a validity issue for the Swishchuk formula, since it is nevertheless an approximation in the trading practice where the sampling time, no matter how small, is always discrete. Our newly-derived formula can now be used not only as a pricing formula for any discrete sample period, but also as a validation tool for checking the accuracy level that the Swishchuk formula yields as a function of the sampling period.

In Figure 3.1, we illustrate the numerical results of the Swishchuk formula, Equation (3.3), which is obtained from the continuous approximation model. From this figure, one can clearly see that the values of our discrete model asymptotically approach the values of the continuous approximation model when the sampling frequency increases; the realized variance defined in (3.2) appears to be the limit of the realized variance defined in Equation (2.3) as $\Delta t \to 0$. Of course, one can theoretically prove that our solution (2.34) indeed approaches the Equation (3.3) when the discrete sampling period approaches zero, i.e.,

$$\lim_{\Delta t \to 0} \frac{e^{r\Delta t}}{T}[f(v_0) + \sum_{i=2}^{N} f_i(v_0)] = v_0(\frac{1 - e^{-\kappa^* T}}{\kappa^* T}) + \theta^*(1 - \frac{1 - e^{-\kappa^* T}}{\kappa^* T}) \tag{3.4}$$

With the proof of this limit, which is left in Appendix C, our solution is once again verified as the correct solution for the discrete sampling cases, taking the continuous sampling case as a special case with the sampling period shrinking down to zero.

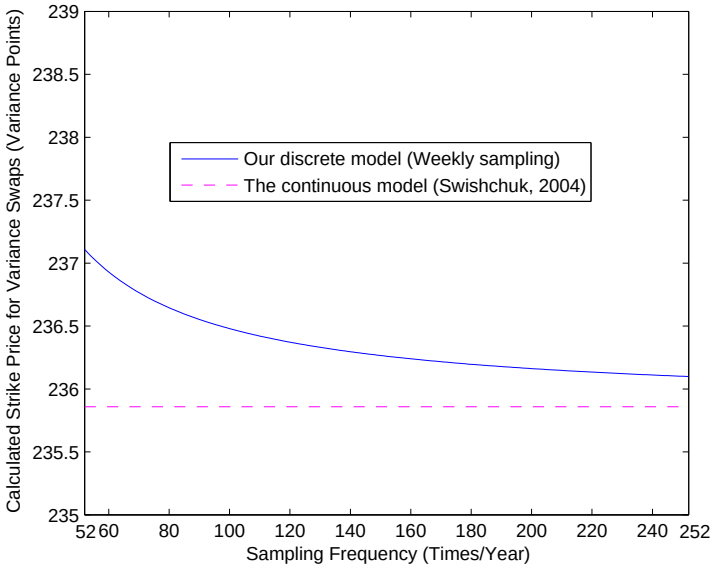

Figure 3.2: Calculated fair strike values as a function of sampling frequency

On the other hand, we now can use our discrete model to check the validity of the continuous model as an approximation. Shown in Figure 3.2 is a refined plot of Figure 3.1, in order to compare the degree of approximation between daily and weekly sampling. With the daily sampling, the relative difference between the results of our discrete model and the continuous model is 0.101%, whereas it has increased to 0.530% for weekly sampling. If the long-term mean variance is further reduced to $\theta^* = 0.01$ from $\theta^* = 0.022$ while the other parameters are held the same, the relative difference between the results of our discrete model for weekly sampling and the continuous model becomes more than doubled to reach 1.226%. With a relative difference of the order of one percent, adopting the continuous model as an approximation to price variance swaps with weekly sampling is clearly not justifiable. For example, when the error level reaches more than 0.5%, Little and Pant (2001) has already concluded, within the Black-Scholes framework, that such an error is "fairly large" so that adopting the continuous model might not be so justifiable any more. Our current findings not only confirm Little and Pant (2001)'s conclusion, but also show that, under the Heston model, the difference between the continuous model and the discrete model will exponentially grow, when the sampling frequency is reduced, as shown in Figure 3.1. Of course, contracts with sampling frequency

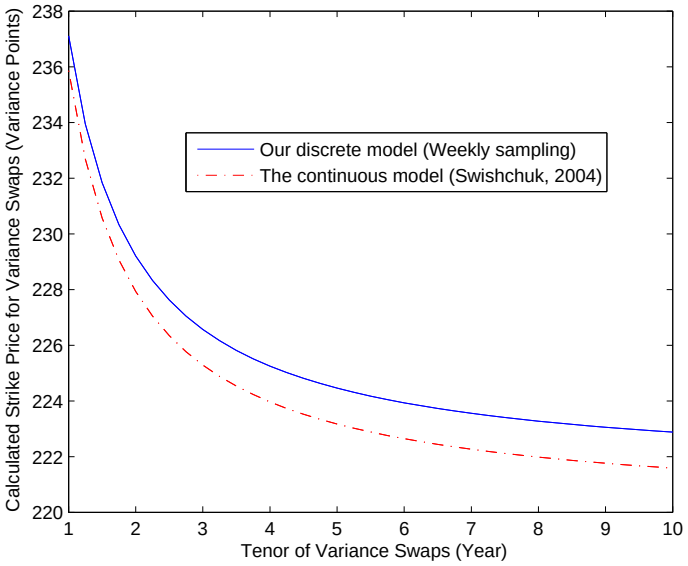

Figure 3.3: Calculated fair strike values as a function of tenor

higher than weekly are very rare in practice. However, specially designed over-the-counter (OTC) contacts of long tenor may still have sampling frequencies small enough to not warrant the realized variance being calculated with the continuous model.

The effect of contract lifetime has been demonstrated in Figure 3.3, in which the calculated fair strike price is plotted as a function of the tenor of a swap contract. Clearly, both models show that the fair strike price falls as tenor increases. However, the difference between the two becomes larger and larger as tenor increases, further demonstrating the need of using the correct formula presented in this paper for the discrete sampling case, rather than using the continuous model as an approximation.

A couple of more points should be remarked before leaving this section. Firstly, with the newly found analytic solution, all the hedging ratios of a variance swap can also be analytically derived by taking partial derivatives against various parameters in the model. With symbolic calculation packages, such Mathematica or Maple, widely available to researchers and market practitioners, these partial derivatives can be readily calculated and thus omitted here. However, to demonstrate how sensitive the strike price is to the change of the key parameters in the model, we performed some sensitivity tests for the example presented in this section. Shown in

Table 3.3: The sensitivity of strike price of variance swap (daily sampling)

Name	Value	Sensitivity
κ^*	11.35	-0.066%
θ^*	0.022	0.85%
σ_V	0.618	-0.0015%
v_0	0.04	0.15%

Table 3.3 are the results of the percentage change of the strike price when a model parameter is given a 1% change from its base value used in the example presented in this Section. Clearly, the strike price of a variance swap appears to be most sensible to the long-term mean variance θ^* for the case studied. On the other hand, the spot variance v_0 may also have significant influence in terms of the sensitivity of the strike price. Secondly, due to the notational amount factor L and the size of the contract traded per order, the 1% or 2% relative error may result in a considerable amount of absolute loss if the formula based on the continuous approximation is adopted. Combining these two points together, one may conclude that even with a relatively high sampling frequency, such as daily sampling, the approximation based on the continuous model could still lead to larger errors for a certain combination of parameter values. Thereby, having a closed-form formula for the case of discrete sampling would enable us to completely abandon the approximation formula based on the continuous model; whether the sampling period is small or not, the computational time of adopting our newly-derived formula, Equation (2.34), is virtually the same as that of adopting the traditional formula, Equation (3.3).

4 Conclusion

In this paper, we have applied the Heston stochastic volatility model to describe the underlying asset price and its volatility, and obtained a closed-form exact solution for discretely-sampled variance swaps with the realized variance defined as the sum of the percentage increment of the underlying asset price. This can be viewed as a substantial progress made in developing a more realistic pricing model for variance

swaps. Through numerical examples, we have shown that the our discrete model can improve the accuracy in pricing variance swaps. We have compared the results produced from our new solution with those produced by the MC simulations for the validation purposes and found that our results agree with those from the MC simulations perfectly.

The significance of our work can be illustrated in two aspects. Theoretically, our discrete model can be used to verify the validity of the corresponding continuous models, and thus would fill a gap that has been in the field of variance swaps pricing. Fourier inverse transform in our model has been analytically worked out, which is a significant step forward in the literature of the Heston model. Practically, the computational efficiency is enormously enhanced in terms of assisting practitioners to price variance swaps, and thus it can be a very useful tool in trading practice when there is obviously increasing demand of trading variance swaps in financial markets.

Appendix A

We now present a brief proof of Proposition 1.

The PDE system is

(A.1)
$$\begin{cases} \dfrac{\partial U}{\partial t} + \dfrac{1}{2}vS^2\dfrac{\partial U^2}{\partial S^2} + \rho\sigma_V vS\dfrac{\partial U^2}{\partial S\partial v} + \dfrac{1}{2}\sigma_V^2 v\dfrac{\partial U^2}{\partial v^2} + rS\dfrac{\partial U}{\partial S} + [\kappa^*(\theta^* - v)]\dfrac{\partial U}{\partial v} - rU = 0 \\ U(S, v, T) = H(S) \end{cases}$$

Firstly, we do the transform by letting

(A.2)
$$\begin{cases} \tau = T - t \\ x = \ln S \end{cases}$$

After the transformation, the PDE system is converted to

(A.3)
$$\begin{cases} \dfrac{\partial U}{\partial \tau} = \dfrac{1}{2}v\dfrac{\partial U^2}{\partial x^2} + \rho\sigma_V v\dfrac{\partial U^2}{\partial x\partial v} + \dfrac{1}{2}\sigma_V^2 v\dfrac{\partial U^2}{\partial v^2} + (r - \dfrac{1}{2}v)\dfrac{\partial U}{\partial x} + [\kappa^*(\theta^* - v)]\dfrac{\partial U}{\partial v} - rU = 0 \\ U(x, v, 0) = H(e^x) \end{cases}$$

Solution for this PDE system can be obtained through generalized Fourier transform with respect to x. One can refer to Lewis (2000) and Poularikas (2000) for more details about the generalized Fourier transform. Based on the generalized Fourier transform, we can do the transformation

$$\mathcal{F}[e^{j\alpha t}] = 2\pi\delta_\alpha(\omega) \tag{A.4}$$

where $j = \sqrt{-1}$ and $\delta_\alpha(\omega)$ is the generalized delta function satisfying

$$\int_{-\infty}^{\infty} \delta_\alpha(t)\Phi(t)dt = \Phi(\alpha) \tag{A.5}$$

with α being any complex number.

Applying the transform to the PDE with respect to the variable x, we obtain the following problem for $\widetilde{U}(\omega, v, \tau) = \mathcal{F}[U(x, v, \tau)]$

(A.6)
$$\begin{cases} \dfrac{\partial \widetilde{U}}{\partial \tau} = \dfrac{1}{2}\sigma_V^2 v \dfrac{\partial \widetilde{U}^2}{\partial v^2} + [\kappa^*\theta^* + (\rho\sigma_V\omega j - \kappa^*)v]\dfrac{\partial \widetilde{U}}{\partial v} + [(r\omega j - r) - \dfrac{1}{2}(\omega j + \omega^2)v]\widetilde{U} \\ \widetilde{U}(\omega, v, 0) = \mathcal{F}[H(e^x)] \end{cases}$$

Following Heston's (1993) solution procedure, the solution of the above PDE system can be assumed of the form:

$$\widetilde{U}(\omega, v, \tau) = e^{C(\omega,\tau)+D(\omega,\tau)v}\widetilde{U}(\omega, v, 0) \tag{A.7}$$

One can then substitute this function into the PDE to reduce it to two ordinary differential equations,

$$\begin{cases} \dfrac{dD}{d\tau} = \dfrac{1}{2}\sigma_V^2 D^2 + (\rho\omega\sigma_V j - \kappa^*)D - \dfrac{1}{2}(\omega^2 + \omega j) \\ \dfrac{dC}{d\tau} = \kappa^*\theta^* D + r(\omega j - 1) \end{cases} \tag{A.8}$$

with the initial conditions

$$C(\omega, 0) = 0, \quad D(\omega, 0) = 0 \tag{A.9}$$

The solutions of these equations can be easily found as

$$\text{(A.10)} \qquad \begin{cases} C(\omega,\tau) = r(\omega j - 1)\tau + \dfrac{\kappa^*\theta^*}{\sigma_V^2}[(a+b)\tau - 2\ln(\dfrac{1-ge^{b\tau}}{1-g})] \\ D(\omega,\tau) = \dfrac{a+b}{\sigma_V^2}\dfrac{1-e^{b\tau}}{1-ge^{b\tau}} \end{cases}$$

where

$$\text{(A.11)} \qquad a = \kappa - \rho\sigma_V\omega j, \quad b = \sqrt{a^2 + \sigma_V^2(\omega^2 + \omega j)}, \quad g = \frac{a+b}{a-b}$$

One should note that the Fourier transform variable ω appears as a parameter in function C and D.

Therefore, the solution of the original PDE can be obtained after the inverse Fourier transform in form as

$$\text{(A.12)} \qquad \begin{aligned} U(x,v,\tau) &= \mathcal{F}^{-1}[\widetilde{U}(\omega,v,\tau)] \\ &= \mathcal{F}^{-1}[e^{C(\omega,T-t)+D(\omega,T-t)v}\mathcal{F}[H(e^x)]] \end{aligned}$$

Appendix B

If setting stochastic variable $\chi_t^2 = 2cv_t$, then χ_t^2 is subject to noncentral chi-squared distribution, $\chi^2(2V; 2q+2, 2W)$, with probability density function denoted by $p_{\chi_t^2}(x)$. We can easily verify that $p(v_T|v_t) = 2cp_{\chi^2_{T-t}}(2cv_T)$. c, W, q and $p(v_T|v_t)$ are given in Equation (2.26) and Equation (2.23). Hence,

$$\text{(B.1)} \qquad \begin{aligned} &E_0^Q[(\frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}})^2] = \int_0^\infty e^{r\Delta t} f(v_{t_{i-1}}) p(v_{t_{i-1}}|v_0) dv_{t_{i-1}} \\ &= e^{r\Delta t} E_0^Q[e^{\widetilde{C}(\Delta t)+\widetilde{D}(\Delta t)v_{t_{i-1}}} + e^{-r\Delta t} - 2] \\ &= e^{r\Delta t}(e^{\widetilde{C}(\Delta t)} E_0^Q[e^{\widetilde{D}(\Delta t)v_{t_{i-1}}}] + e^{-r\Delta t} - 2) \\ &= e^{r\Delta t}(e^{\widetilde{C}(\Delta t)} E_0^Q[e^{\frac{\widetilde{D}(\Delta t)}{2c}\chi^2_{t_{i-1}}}] + e^{-r\Delta t} - 2) \\ &= e^{r\Delta t}(e^{\widetilde{C}(\Delta t)}(1-2\Phi)^{-(q+1)} e^{\frac{2W\Phi}{1-2\Phi}}|_{\Phi=\frac{\widetilde{D}(\Delta t)}{2c}} + e^{-r\Delta t} - 2) \\ &= e^{r\Delta t}(e^{\widetilde{C}(\Delta t)+\frac{W\widetilde{D}(\Delta t)}{c-\widetilde{D}(\Delta t)}}(\frac{c}{c-\widetilde{D}(\Delta t)})^{\frac{2\kappa^*\theta^*}{\sigma_V^2}} + e^{-r\Delta t} - 2) \end{aligned}$$

It should be noted the parameters c, W are determined by the time t_{i-1} in Equation

(2.26) with $T = t_{i-1}$ and $t = 0$.

$$f_i(v_0) = e^{\widetilde{C}(\Delta t)+\frac{c_i e^{-\kappa^* t_{i-1}}}{c_i-\widetilde{D}(\Delta t)}\widetilde{D}(\Delta t)v_0}\left(\frac{c_i}{c_i-\widetilde{D}(\Delta t)}\right)^{\frac{2\kappa^*\theta^*}{\sigma_V^2}} + e^{-r\Delta t} - 2 \tag{B.2}$$

where $c_i = \frac{2\kappa^*}{\sigma_V^2(1-e^{-\kappa^* t_{i-1}})}$. Hence,

$$E_0^Q[(\frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}})^2] = e^{r\Delta t} f_i(v_0) \tag{B.3}$$

Appendix C

Now, we prove Equation (3.4). Using l'Hôpital's rule, one can easily verify that

$$\lim_{\Delta t\to 0} \widetilde{C}(\Delta t) = 0, \qquad \lim_{\Delta t\to 0} \widetilde{D}(\Delta t) = 0 \tag{C.1}$$

and

$$\lim_{\Delta t\to 0} e^{\widetilde{C}(\Delta t)+\widetilde{D}(\Delta t)v_0} + e^{-r\Delta t} - 2 = 0 \tag{C.2}$$

$$\lim_{\Delta t\to 0} \frac{e^{\widetilde{C}(\Delta t)+\widetilde{D}(\Delta t)v_0} + e^{-r\Delta t} - 2}{\Delta t} = v_0 \tag{C.3}$$

$$\lim_{\Delta t\to 0} \frac{f_i(v_0)}{\Delta t} = v_0 e^{-\kappa^*(i-1)\Delta t} + \theta^*(1 - e^{-\kappa^*(i-1)\Delta t}) \tag{C.4}$$

Therefore,

$$\begin{aligned}
&\lim_{\Delta t\to 0} \frac{e^{r\Delta t}}{T}[f(v_0) + \sum_{i=2}^{N} f_i(v_0)] = \frac{1}{T}\lim_{\Delta t\to 0}\sum_{i=2}^{N}\Delta t(v_0 + \frac{f_i(v_0)}{\Delta t}) \\
&= \frac{1}{T}\lim_{\Delta t\to 0}\sum_{i=1}^{N}\Delta t[v_0 e^{-\kappa^*(i-1)\Delta t} + \theta^*(1 - e^{-\kappa^*(i-1)\Delta t})] \\
&= \frac{1}{T}\int_0^T [v_0 e^{-\kappa^* t} + \theta^*(1 - e^{-\kappa^* t})]dt \\
&= v_0\frac{1 - e^{-\kappa^* T}}{\kappa^* T} + \theta^*(1 - \frac{1 - e^{-\kappa^* T}}{\kappa^* T})
\end{aligned} \tag{C.5}$$

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