PRICING PATH-DEPENDENT OPTIONS UNDER STOCHASTIC VOLATILITY AND FRACTIONAL ENVIRONMENT

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Abstract

This thesis focuses on the evaluation of various path-dependent options, specifically down-and-out put options, floating strike lookback options, and geometric Asian options. We consider a hybrid model with stochastic elasticity of variance and stochastic volatility as the driving factors for the underlying asset. It is well-known that obtaining closed-form solutions for these path-dependent options under stochastic volatility models is challenging.

To address this issue, we employ an asymptotic expansion approach and the Mellin transform method. By utilizing these techniques, we are able to derive explicit closed-form formulas for both the zero-order and the first-order correction terms. These formulas provide valuable insights into the pricing of the options and allow for a more comprehensive analysis.

Furthermore, we conduct a sensitivity analysis on the asymptotic terms obtained from our pricing formulas. This analysis helps us understand the impact of various factors on the option prices. Additionally, we compare the option prices calculated using our derived formulas with those obtained from Monte-Carlo simulations and the binomial tree method.

By comparing the prices derived from different models such as Black-Scholes, CEV, and SVCEV, we demonstrate the accuracy and effectiveness of our pricing formulas. The numerical comparisons highlight the strengths of our approach and emphasize the practical relevance of our findings.

In summary, this thesis contributes to the research field by providing explicit closed-form formulas for path-dependent options under a hybrid model with stochastic elasticity of variance and stochastic volatility or a fractional Brownian motion model. Through numerical analysis and comparisons with other pricing methods, we validate the accuracy and robustness of our derived formulas, thereby enhancing the understanding and applicability of option pricing in financial markets.

Journal Publications

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Chapter 1

Introduction

1.1 Background

The movement of stock prices in financial markets or the dynamics of risky investments exhibit stochastic characteristics. The pioneering mathematical model that successfully addressed these scenarios was introduced by Black and Scholes in 1973 (Black & Scholes, 1973). The groundbreaking work of Black and Scholes significantly contributed to the global expansion of option trading, as acknowledged by MacKenzie (2006). By 2007, the international financial system was trading derivatives with a total value of one quadrillion dollars annually (Steward, 2012).

The Black-Scholes model assumes that the returns of a risky investment follow a Brownian motion, which is inspired by the random movement of particles in a liquid. However, this model does not account for changes in the volatility of the risky investment. Recent studies conducted in the aftermath of the 2007-2008 Global Financial Crisis, such as those by Kim, Lee, Zhu and Yu (2014); Kim, Yoon, Lee and Choi (2015), have demonstrated that the classical Black-Scholes framework is inadequate when the failure of financial firms leads to systemic risk that impacts a larger economy. These findings highlight the limitations of the Black-Scholes model in capturing the complexities and risks associated with financial crises.

In the past four decades, mathematicians have made significant advancements in developing sophisticated models to overcome the limitations of the Black-Scholes model. These models have been proposed by various researchers such as Bergomi (2016); Cox (1975), Cox (1996); Derman and Kani (1994); Derman and Miller (2016); Dupire (1994); Heston (1993); Jouini, Cvitanic and Musiela (2001); Lee, Wu and Chen (2004); MacBeth and Merville (1979), MacBeth and Merville (1980), among others. Despite these efforts, due to the complexity of financial markets and the rapid growth of the financial industry, a satisfactory solution has not yet been achieved.

Typically, financial assets are modelled using stochastic differential equations driven by Brownian motion. In all the existing models that incorporate Brownian motions, the smoothness of the volatility's sample path is assumed to be the same as that of a Brownian motion.

Nevertheless, empirical studies have highlighted the long memory property of financial time series, which is incompatible with the standard framework. Recent empirical evidence in option pricing theory suggests that the shape of the volatility structure is associated with fractional Brownian motion, which was introduced by Mandelbrot and Van Ness (1968). Comte and Renault (1998) proposed a pioneering approach to modelling volatility using fractional Brownian motion, ensuring long memory by selecting a Hurst parameter greater than 1/2.

Recent research on option pricing has demonstrated that the volatility smile's shape depends on the Hurst parameter, suggesting the inadequacy of both the standard Brownian motion and the specific specification by Comte et al. Empirical evidence from various major markets supports the use of fractional Brownian motion with H less than 1/2 as a more appropriate framework. Notably, Gatheral, Jaisson and Rosenbaum (2018) illustrated that a stochastic volatility model driven by a fractional Ornstein-Uhlenbeck process with H less than 1/2 generates the correct shape for the implied volatility surface. Furthermore, this model exhibits autocovariance in spot volatility compatible with empirical observations, implying a long memory property despite H being less than 1/2. Such models are referred to as rough stochastic volatility models. Specifically, Gatheral et al. (2018) demonstrated, based on data from major markets including S&P500, DAX, Bund, and NASDAQ, that a stochastic volatility model driven by fractional Brownian motion with a Hurst parameter less than 1/2 accurately captures the volatility structure. This important work has been followed by many other researchers, such as Bayer, Friz and Gatheral (2016); Bennedsen, Lunde and Pakkanen (2017); Cheridito, Kawaguchi and Maejima (2003); Comte, Coutin and Renault (2012); Da Fonseca and Zhang (2019); El Euch and Rosenbaum (2018), El Euch and Rosenbaum (2019); El Euch, Gatheral and Rosenbaum (2019); Funahashi and Kijima (2011), Funahashi and Kijima (2017); Funahashi (2017); Garnier and Sølna (2017), Garnier and Sølna (2018), Garnier and Sølna (2018); Guennoun, Jacquier, Roome and Shi (2018); Rosenbaum (2008), among others. Subsequent works have further confirmed the prevalence of this "rough" property in various assets. For instance, Bennedsen et al. (2017) extensively analysed the U.S. equity market's S&P 500 index and corroborated the findings of Gatheral et al. (2018).

In a recent study extending the work of Cox (1975), Cox (1996); Cao, Kim, Kim and Zhang (2020) investigated the stochastic behaviour of the elasticity of variance of returns for the S&P 500 and developed a fractional stochastic model called the rough stochastic elasticity of variance model. They successfully applied this model to price European-style options. However, to the best of our knowledge, no research has been conducted on other important types of options, such as path-dependent options, under this framework.

This thesis aims to address these gaps by exploring the challenges associated with pricing path-dependent options in the context of stochastic volatility or fractional stochastic environments. We study the problems of pricing path-dependent options under the framework of stochastic volatility or fractional stochastic environments. Our research contributes to both the academic community and the financial industry by developing novel mathematical models and evaluation formulas specifically tailored for these path-dependent options.

1.2 Literature Review

1.2.1 Path-dependent Options

Path-dependent options possess various intriguing characteristics. Their evaluation can be more challenging, and hedging the risks associated with these unique contracts tends to be more difficult in practical scenarios. Path-dependent options are sometimes referred to as exotic options in the literature.

Barrier options gained significant traction in the over-the-counter markets during the late 1960s, leading to a surge of academic interest in their valuation. Initial studies primarily focused on valuing single barrier options based on assets with lognormal dynamics and continuous monitoring of the barrier. Merton (1973) derived a closed-form pricing formula for the popular down-and-out barrier call options. Reiner and Rubinstein (1991) provided analytical solutions for all eight types of single barrier European options. Broadie and Glasserman (1997) developed methods to incorporate continuity correction for barrier options with discrete barrier monitoring, thereby aligning the valuation problem with the classical setting.

While researchers have solved the valuation of more complex barrier options through various approaches (e.g., Carr, Ellis and Gupta (1998)), there have also been studies exploring valuation under alternative stochastic processes (e.g., Mitov, Rachev, Kim and Fabozzi (2009)).

A barrier option's payoff depends on the underlying asset reaching a specific level, known as the barrier, before its expiration. Barrier options can be classified into two main types: "in" barrier options (knock-in) and "out" barrier options (knock-out). The former yields a payoff only if the barrier level is reached before expiry, while the latter provides a payoff only if the barrier is not reached before expiry. These options exhibit weak path dependence, meaning that their prices are determined solely by the asset's current level and the time remaining until expiration. They satisfy the same equation as vanilla options but with specific boundary conditions.

Lookback options are a type of path-dependent options that derive their payoffs from the maximum or minimum price achieved by an underlying security during a specific period of time. Some of these options are already traded on specialized markets, such as foreign exchange markets, while others are still in the process of development.

The concept of "standard lookback call" involves the right to purchase the underlying asset at its lowest historical price within a given time frame. Initial work on standard lookback options was conducted by Goldman, Sosin and Gatto (1979) and Goldman, Sosin and Shepp (1979). A "call on maximum" resembles a regular call option, but the underlying asset's price is replaced by its maximum value. The advantage of this option is that it behaves similarly to a standard call option exercised at the optimal date, ensuring that its intrinsic value is always non-decreasing.

A "limited risk call" exhibits the same payoff as a call option on the underlying security, except when the historical maximum surpasses a specified level, resulting in a zero payoff. This option allows an investor to sell a call without being exposed to significant upward movements in the security's price. Merton (1973) referred to a related option as a "down and out" option.

A "partial lookback call" grants the holder the right to purchase the underlying asset at a percentage above its historical minimum. It serves as a more affordable version of the "standard lookback call".

The payoff structure of a lookback option depends on the realized maximum or minimum value of the underlying asset during a specific period before the option's expiration. Lookback options generally carry higher premiums compared to vanilla options. The determination of the maximum or minimum value of the underlying asset can be done continuously or discretely. In practice, the realized maximum or minimum value is typically measured discretely.

During the 1980s, Mark Standish and David Spaughton were both associated with Bankers Trust in London. Mark Standish focused on fixed income derivatives and proprietary arbitrage trading, while David Spaughton worked as a systems analyst in the financial markets. Bankers Trust had obtained licenses from the Bank of England in 1984 to engage in foreign exchange options trading in the London market.

In 1987, Standish and Spaughton happened to be in Tokyo for business purposes when they jointly developed the first pricing formula for options linked to the average price of crude oil. This pricing formula was a breakthrough in the options market and was eventually adopted for commercial use. They named this new type of options the "Asian options" since they were in Asia when they devised it.

The payoff of an Asian option depends on the average value of the underlying asset over a specified period leading up to its expiration. Unlike standard options, an Asian option is path dependent, meaning that its value is not solely determined by the final position of the underlying asset but also by the specific path it follows. This introduces the need to incorporate the average value of the asset as an additional state variable. The calculation of the option's payoff can involve various methods of determining the average, such as using arithmetic or geometric averaging.

Path-dependent options are often used by investors and institutions to hedge against specific risks. Driven by the innovative derivative products, financial institutions are compelled to expand their offerings of structured products. These products are carefully designed to cater for the unique requirements of customers. Among the various types of structured products, there is a growing demand for path dependent options due to their distinctive payoff structures, which are directly linked to the historical price movement of the underlying asset throughout the entire duration of the option.

Due to their complexity, the use of path-dependent options requires a deep understanding of financial markets and derivatives pricing. The value of such an option depends not only just on the terminal price of the underlying asset at expiration, but also on the specific path of the underlying asset's price over time.

1.2.2 Valuation of Path-dependent Options

In this thesis, our primary focus revolves around barrier options, lookback options, and Asian options. We aim to present analytical formulas for valuing these options. Due to the path dependent nature of these options, the asset price movement is continuously monitored throughout the contract's lifespan, either to identify breaches of barrier levels, observe new extreme values, or sampling of asset prices for calculating average values. However, in practical scenarios, these monitoring procedures are typically conducted at discrete intervals rather than continuously. Nevertheless, most pricing models for path dependent options assume continuous monitoring of asset prices to achieve better analytical tractability. Therefore, it is desirable to establish analytic price formulas for commonly encountered types of barrier options, lookback options, and geometric averaged Asian options that incorporate continuous monitoring.

Barrier options represent the simplest form of path dependent options. Their key characteristic is that the payoff is influenced not only by the final price of the underlying asset but also by whether the asset price reaches a predefined barrier level during the option's lifetime. An out-barrier option is knocked out before expiration if the underlying asset price crosses the barrier, while an in-barrier option is knocked in only if the asset price crosses the specified in-barrier level. If the barrier is positioned above the asset price, the barrier option is referred to as an up-option, and if it is below, it is called a down-option.

Since 1967, the U.S. market has offered a down-and-out call option, which shares similarities with a vanilla call option, except for the condition that it becomes knocked out if the asset price falls below a predetermined downstream knock-out level. As an extension to the classical European vanilla option pricing formula presented in Black and Scholes (1973), Merton (1973) derived a closed-form pricing formula specifically for down-and-out barrier call options under the framework of geometric Brownian motion. Reiner and Rubinstein (1991) provided explicit formulas for evaluating other types of barrier options.

Closed-form price formulas for barrier options can also be derived for other types of diffusion processes followed by the underlying asset price. Lo, Yuen and Hui (2001) derived price formulas for barrier options using the square root constant elasticity of variance process, while Sepp (2004) utilized the Laplace transform method to obtain evaluation formulas for barrier options under the double exponential jump diffusion process.

A lookback option is a type of path-dependent option that offers a payoff based on the maximum or minimum value attained by the underlying asset price during a specific period known as the lookback period. In this context, we focus on lookback options where the entire lifespan of the options serves as the lookback period. The modelling of lookback options demonstrates a close connection to dynamic investment fund protection.

Closed-form pricing formulas for lookback options under the Black-Scholes framework were provided by Goldman, Sosin and Gatto (1979), Goldman, Sosin and Shepp (1979) and Conze and Vishwanathan (1991).

An Asian option is a type of option where the payoff is determined by an average of the underlying asset's price over a specific period or the entire duration of the option. Traders often find themselves in market situations where they seek to hedge against the average price of a commodity over a given time frame, rather than relying solely on the end-of-period price. For instance, imagine a manufacturer who anticipates making a series of copper purchases for their factory within a fixed time horizon. In such a scenario, the company would be interested in obtaining price protection tied to the average price over that period. Averaging options are proved particularly valuable in businesses dealing with infrequently traded commodities.

The most commonly employed method for averaging is the discrete arithmetic average. Mark Standish and David Spaughton developed the first commercially available option pricing formula related to the average crude oil price during a joint business trip in Tokyo in 1987. Consequently, this type of path-dependent options came to be known as Asian options due to their origin in Asia. Curran (1994) and Rogers and Shi (1995) introduced a conditional approach to establish a lower bound on the price of an Asian option.

The Black-Scholes framework assumes constant volatility in the underlying equity, which makes it possible to derive closed-form formulas for path-dependent options. A comprehensive overview of research on path-dependent options under the Black-Scholes framework can be found in Clewlow, Llanos and Strickland (1994). However, empirical evidence demonstrates that volatility is highly variable and unpredictable over time. Thus, it is more realistic to treat volatility as a stochastic factor. Two popular continuous-time approaches, namely "local volatility" and "stochastic volatility", have been developed to extend the classical geometric Brownian motion model.

The constant elasticity of variance (CEV) model, introduced by Cox (1975), is a widely used local volatility model in the literature. In Boyle and Tian (1999), the authors evaluated lookback options and barrier options for CEV models by approximating the CEV process with a trinomial process. Explicit pricing formulas for barrier options and lookback options under the CEV process were derived in Davydov and Linetsky (2001).

Generally, the pricing problem for barrier and lookback options does not have analytical solutions under stochastic volatility. Chiarella, Kang and Meyer (2012) employed a numerical algorithm to approximate barrier option prices when the equity dynamics follow the Heston model (Heston, 1993) with stochastic volatility. Park and Kim (2013) investigated a semi-analytic pricing method for lookback options under a general stochastic volatility framework. The resulting formula is closely connected to the Black-Scholes price, which represents the first term of a series expansion. In addition, they provided a convergence condition for the expansion along with an error bound. Leung (2013) and Wirtu, Ngare and Kube (2017) derived an analytic pricing formula for floating strike lookback options under the Heston model using the homotopy analysis method. Kato, Takahashi and Yamada (2013) derived a semi-closed-form formula for up-and-out barrier options under a class of stochastic volatility models, including the SABR model introduced by Hagan, Kumar, Lesniewski and Woodward (2002). Saporito (2020) derived an efficient approximation for the price of path-dependent derivatives under the multiscale stochastic volatility models. Cai, Li and Shi (2021) proposed a novel approach that combines Malliavin expansions with Hilbert transform techniques to evaluate discretely barrier options under jump diffusion models with stochastic volatility. Tissot-Daguette (2022) investigated the approximation of path-dependent options using of the Karhunen–Loeve expansion.

1.3 Research Questions

Our main objective in this study is to evaluate the fair prices of several path-dependent options by developing novel mathematical models with stochastic volatility or fractional Brownian motion. Related to this objective, we consider the following questions:

Question 1. What is the fair price of a barrier option or a lookback option under stochastic volatility models?

Question 2. What is the fair price of a barrier option or a lookback option under hybrid models of constant elasticity of variance and stochastic volatility?

Question 3. What is the fair price of an Asian option under models with fractional Brownian motion?

1.4 Research Contributions and Thesis Organization

The primary contribution of this research lies in developing efficient approximate closed-form pricing formulas for various path-dependent options including barrier options, lookback options and Asian options under hybrid stochastic volatility and local volatility models. Through numerical experimentation, we demonstrate that our evaluation formulas surpass traditional methods in terms of accuracy and efficiency.

From a modelling perspective, hybrid models can effectively capture most market phenomena. However, they may fall short in describing sudden market changes, such as financial crises. To address this limitation, we introduce hybrid models incorporating fractional Brownian motions to accommodate newly observed market behaviours.

The remainder of this thesis is structured as follows: Chapter 2 provides an overview of the mathematical tools, financial modelling approaches, and numerical methods employed to address our research questions. Chapter 3 studies the fair price of a barrier option or a lookback option under a stochastic volatility model. Chapter 4 derives the fair pricing formula for a barrier option or a lookback option under a hybrid model that combines constant elasticity of variance and stochastic volatility. Chapter 5 investigates the fair price of a path-dependent Asian option under a model incorporating fractional Brownian motion. Finally, Chapter 6 summarizes the thesis and outlines potential avenues for future research.

Chapter 2

Preliminaries

The utilization of modern probability theory has become a fundamental tool in quantifying a wide range of financial products. In particular, the risk-neutral valuation approach has emerged as a crucial technique for assessing the value of path-dependent options. Additionally, numerical calculation methods have been proven to be highly effective mathematical tools, particularly for solving complex differential equations which don't usually have closed-form solutions. This chapter serves to establish the theoretical foundation for our research, providing the necessary groundwork for our investigation.

This chapter draws on several key references, including Hildebrand (1987), Rossi (2018), Schoutens (2003), Cont and Tankov (2004), Jacod and Protter (2004), Øksendal and Sulem (2005), and Shreve (2008).

2.1 Mathematical Tools

This section serves to introduce mathematical concepts and key results in probability theory that will be utilized in the forthcoming chapters. It lays the foundation for a comprehensive understanding of the subject matter.

2.1.1 Probability Measures

Measures play a fundamental role in the definition of stochastic processes. In the context of probability theory, a measure on a set can be regarded as the size of that set (Cont & Tankov, 2004). This theory comprises three essential elements: the state space, events, and probability distribution. The following sub-section focuses on these three elements.

The state space represents the collection of all possible outcomes of an experiment and is denoted by Ω. The power set of Ω, denoted by 2^{Ω}, is the set of all subsets of Ω including Ω itself and the null or empty set $\varnothing$. Let $\mathcal{F}$ be a collections of subsets of Ω, i.e, $\mathcal{F} \subseteq 2^{\Omega}$. Elements within $\mathcal{F}$ are referred to as events. We introduce the concept of a σ-algebra as follows, see Shreve (2008) for more details.

Definition 2.1.1 (σ-algebra) *A collection $\mathcal{F}$ is a σ-algebra if it satisfies*

1. *The collection $\mathcal{F}$ contains the empty set $\varnothing \in \mathcal{F}$.*
2. *If the collection $\mathcal{F}$ contains a set $A \in \mathcal{F}$, then the collection $\mathcal{F}$ contains its complement $\Omega \setminus A \in \mathcal{F}$.*
3. *If the collection $\mathcal{F}$ contains a sequence of sets $A_1, A_2, A_3, ...$, then the collection $\mathcal{F}$ contains their union $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$.*

Based on properties 2 and 3 in Definition 2.1.1, we can infer that if $A_1, A_2, \ldots$ is a sequence of events in $\mathcal{F}$, then $\bigcap_{i=1}^{\infty} A_i \in \mathcal{F}$. Furthermore, the state space Ω is also an element of $\mathcal{F}$ since Ω is the complement of the empty set. The pair $(\Omega, \mathcal{F})$ is referred to as a measurable space. With this understanding, we now introduce the definition of a probability measure on the measurable space $(\Omega, \mathcal{F})$.

Definition 2.1.2 (Probability measure) *A probability measure defined on the measurable space $(\Omega, \mathcal{F})$ is a function $\mathbb{P} : \mathcal{F} \to [0, 1]$ that satisfies the following properties:*

1. $\mathbb{P}(\Omega) = 1$.

2. *For every sequence* $(A_i)_{i\geq 1}$ *of pairwise disjoint elements of* $\mathcal{F}$ *(i.e.,* $A_n \cap A_m = \varnothing$ *whenever* $n \neq m$*), the probability measure satisfies*

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i). \tag{2.1}$$

In Definition 2.1.2, the value of $\mathbb{P}(A)$ represents the probability associated with the event A. The triple $(\Omega, \mathcal{F}, \mathbb{P})$ is referred to as a probability space. Notably, based on the second property in Definition 2.1.2, we can deduce that $\mathbb{P}(\varnothing) = 0$.

Consider two probability measures, $\mathbb{P}$ and $\mathbb{Q}$, defined on the same measurable space $(\Omega, \mathcal{F})$. The measure $\mathbb{P}$ is said to be absolutely continuous with respect to $\mathbb{Q}$ (Cont & Tankov, 2004) if, for any measurable set $A \in \mathcal{F}$, the following implication holds:

$$\mathbb{Q}(A) = 0 \Rightarrow \mathbb{P}(A) = 0. \tag{2.2}$$

This condition indicates that if $\mathbb{Q}$ assigns zero probability to a particular set A, then $\mathbb{P}$ also assigns zero probability to the same set A.

Theorem 2.1.1 (Radon-Nikodym theorem) *If* $\mathbb{P}$ *is absolutely continuous with respect to* $\mathbb{Q}$*, then there exists a* $\mathbb{Q}$*-measurable function* $Z : \mathcal{F} \to [0, \infty)$ *such that for any measurable set* A*, the following equation holds:*

$$\mathbb{P}(A) = \int_A Z \, d\mathbb{Q}. \tag{2.3}$$

The function Z *is commonly referred to as the Radon-Nikodym derivative of* $\mathbb{P}$ *with respect to* $\mathbb{Q}$ *and is denoted as* $d\mathbb{P}/d\mathbb{Q}$.

The integral in Eq. (2.3) is a Lebesgue integral, which is defined using measurable functions.

Two probability measures, $\mathbb{P}$ and $\mathbb{Q}$, are considered equivalent if they are absolutely continuous with respect to each other. This can be denoted as:

$$\mathbb{P} \sim \mathbb{Q} \iff [\forall A \in \mathcal{F}, \mathbb{P}(A) = 0 \iff \mathbb{Q}(A) = 0]. \tag{2.4}$$

This equivalence implies that $\mathbb{P}$ and $\mathbb{Q}$ assign zero probability to the same measurable sets A. In other words, the two measures have the same null sets and capture the same probabilistic information.

2.1.2 Random Variables

A random variable represents an unknown value that varies based on the occurrence of random events. In the context of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a real-valued random variable is a measurable function X defined on Ω. It possesses the property that for each Borel subset $B \subseteq \mathbb{R}$, the set defined by

$$\{X \in B\} = \{\omega \in \Omega; X(\omega) \in B\} \tag{2.5}$$

belongs to the σ-algebra $\mathcal{F}$ (Shreve, 2008). A scenario of randomness is denoted by $\omega \in \Omega$. In this case, $X(\omega)$ represents the outcome of the random variable when the scenario ω occurs (Cont & Tankov, 2004).

The distribution measure of a random variable X, denoted as μ_X, is defined as follows:

$$\mu_X(B) = \mathbb{P}\{X \in B\}, \tag{2.6}$$

where $B \subseteq \mathbb{R}$ and B can represent either a single number or a set of real numbers, and $\mu_X(B)$ represents the probability that the random variable X falls within the set B (Shreve, 2008).

A real-valued random variable X can be characterized by its cumulative distribution function (CDF), denoted as $F_X(x)$, defined by:

$$F_X(x) = \mathbb{P}\{X \le x\} = \mu_X(-\infty, x], \quad x \in \mathbb{R}. \tag{2.7}$$

Hence, if the CDF F_X is known, we can express the probability measure $\mu_X(x, y]$ for any $x \in \mathbb{R}$, $y \in \mathbb{R}$, and $x < y$ as $\mu_X(x, y] = F_X(y) - F_X(x)$.

Furthermore, the relationship between the CDF and the probability density function (PDF) of a random variable X is given by:

$$\mu_X[a, b] = \mathbb{P}\{a \le X \le b\} = F_X(b) - F_X(a) = \int_a^b f_X(x)\, dx, \tag{2.8}$$

where $-\infty < a \le b < \infty$, and $f_X(x)$ is the PDF, which is a non-negative function.

The following properties hold for random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ (Schoutens, 2003).

Theorem 2.1.2 *Let X be a random variable on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$.*

(a) If X takes only finitely many values $y_0, y_1, \dots, y_n$, then

$$\int_\Omega X(\omega)\, d\mathbb{P}(\omega) = \sum_{k=0}^{n} \mathbb{P}\{X = y_k\}. \tag{2.9}$$

*(b) (**Integrability**) X is integrable if and only if*

$$\int_\Omega X(\omega)\, d\mathbb{P}(\omega) < \infty. \tag{2.10}$$

Now let Y be another random variable on $(\Omega, \mathcal{F}, \mathbb{P})$.

*(c) (**Comparison**) If $X < Y$ almost surely (i.e., $\mathbb{P}\{X < Y\} = 1$) and if $\int_\Omega X(\omega)\, d\mathbb{P}(\omega)$*

and $\int_\Omega Y(\omega)\, d\mathbb{P}(\omega)$ *are defined, then*

$$\int_\Omega X(\omega)\, d\mathbb{P}(\omega) < \int_\Omega Y(\omega)\, d\mathbb{P}(\omega). \tag{2.11}$$

In particular, if $X = Y$ *almost surely and one of the integrals is defined, then they are both defined and*

$$\int_\Omega X(\omega)\, d\mathbb{P}(\omega) = \int_\Omega Y(\omega)\, d\mathbb{P}(\omega). \tag{2.12}$$

*(d) (**Linearity**) If* α *and* β *are real constants and* X *and* Y *are integrable, or if* α *and* β *are non-negative constants and* X *and* Y *are non-negative, then*

$$\int_\Omega \left(\alpha X(\omega) + \beta Y(\omega)\right)\, d\mathbb{P}(\omega) = \alpha \int_\Omega X(\omega)\, d\mathbb{P}(\omega) + \beta \int_\Omega Y(\omega)\, d\mathbb{P}(\omega). \tag{2.13}$$

As mentioned earlier in this section, the value of a random variable is unknown due to its random nature. However, we can compute the average value of a random variable, which is commonly referred to as the expectation of the random variable.

Definition 2.1.3 (Expectation) *Let* X *be a random variable on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$. *The expectation of* X*, denoted as* $\mathbb{E}^{\mathbb{P}}[X]$*, is defined by*

$$\mathbb{E}^{\mathbb{P}}[X] = \int_\Omega X(\omega)\, d\mathbb{P}(\omega). \tag{2.14}$$

2.1.3 Conditional Expectation

In a random experiment, the outcome ω may not be fully known but can be inferred to some extent based on available information. We can narrow down the possibility of ω occurring based on this information. The information can progressively increase as the experiment unfolds over time. To capture this increasing knowledge, we consider a sequence of σ-algebras $\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_m, \dots, \mathcal{F}_n$, indexed by time, where $m < n$. Each $\mathcal{F}_m$ represents the available information at time m, and as time progresses, more information becomes available, leading to $\mathcal{F}_n$ containing the most comprehensive information among all σ-algebras in the sequence $(\mathcal{F}_0, \mathcal{F}_1, \dots, \mathcal{F}_{n-1})$. This sequence of σ-algebras, $\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_n$, is referred to as a filtration. We provide the formal definition of a filtration below.

Definition 2.1.4 (Filtration) *Consider a nonempty set Ω. Let T be a fixed positive number, and for each t in the interval $[0,T]$, let $\mathcal{F}_t$ be a σ-algebra. Furthermore, assume that if $s \leq t$, then every set belonging to $\mathcal{F}_s$ is also included in $\mathcal{F}_t$. In this case, we refer to the collection of σ-algebras $\{\mathcal{F}_t\}_{0 \leq t \leq T}$ as a filtration.*

Definition 2.1.5 *Consider a random variable X that operates within a nonempty sample space Ω. The σ-algebra produced by X, labeled as $\sigma(X)$, comprises all sets within Ω that take the form $\{\omega \in \Omega; X(\omega) \in B\}$, where B covers the range of Borel subsets of $\mathbb{R}$.*

In certain cases, the information contained in $\mathcal{F}$ may not provide any insights or clues that can be used to assess the random variable X. In such situations, we say that X is independent of the information set $\mathcal{F}$. The formal definitions of independence are presented below (Shreve, 2008).

Definition 2.1.6 (Independence) *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $\mathcal{G} \subset \mathcal{F}$ and $\mathcal{E} \subset \mathcal{F}$ be two σ-algebras. We say that $\mathcal{G}$ and $\mathcal{E}$ are independent if*

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B) \quad \textit{for all } A \in \mathcal{G},\ B \in \mathcal{E}. \tag{2.15}$$

Assume that X and Y are random variables on $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\sigma(X)$ and $\sigma(Y)$ denote the σ-algebras generated by X and Y, respectively. We say that X and Y are *independent if $\sigma(X)$ and $\sigma(Y)$ are independent. If $\sigma(X)$ and $\mathcal{G}$ are independent, then we can say that X is independent of the information contained in $\mathcal{G}$.*

The independence of σ-algebras $\mathcal{G}$ and $\mathcal{E}$ indicates that the probability of the intersection of any event from $\mathcal{G}$ and any event from $\mathcal{E}$ is equal to the product of their individual probabilities.

In certain situations, it is possible to estimate the value of a random variable X based on the available information in $\mathcal{G}$, although not with complete accuracy. This estimation is referred to as the conditional expectation of X. The conditional expectation is defined as follows (Shreve, 2008).

Definition 2.1.7 (Conditional expectation) *Let X be a random variable that is either non-negative or integrable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The conditional expectation of X given $\mathcal{G}$, denoted $\mathbb{E}[X|\mathcal{G}]$, is any random variable that satisfies the following properties:*

*1 **(Measurability)** $\mathbb{E}[X|\mathcal{G}]$ is $\mathcal{G}$-measurable.*

*2 **(Partial averaging)***

$$\int_A \mathbb{E}[X|\mathcal{G}](\omega) d\mathbb{P}(\omega) = \int_A X(\omega) d\mathbb{P}(\omega) \quad \text{for all } A \in \mathcal{G}. \tag{2.16}$$

Now, let's assume that X and Y are integrable random variables on $(\Omega, \mathcal{F}, \mathbb{P})$, and $\mathcal{G} \subseteq \mathcal{F}$. We can establish four fundamental properties of conditional expectations (Shreve, 2008) as listed below:

1 **(Linearity of conditional expectations)** If c_1 and c_2 are constants, then

$$\mathbb{E}[c_1 X + c_2 Y|\mathcal{G}] = c_1 \mathbb{E}[X|\mathcal{G}] + c_2 \mathbb{E}[Y|\mathcal{G}]. \tag{2.17}$$

2 **(Taking out what is known)** If XY are integrable and X is $\mathcal{G}$-measurable, then

$$\mathbb{E}[XY|\mathcal{G}] = X\mathbb{E}[Y|\mathcal{G}]. \tag{2.18}$$

3 **(Iterated conditioning)** If $\mathcal{H}$ is a sub-σ-algebra of $\mathcal{G}$, then

$$\mathbb{E}[\mathbb{E}[X|\mathcal{G}]|\mathcal{H}] = \mathbb{E}[X|\mathcal{H}]. \tag{2.19}$$

4 **(Independence)** If X is independent of $\mathcal{G}$, then

$$\mathbb{E}[X|\mathcal{G}] = \mathbb{E}[X]. \tag{2.20}$$

2.1.4 Stochastic Processes

A stochastic process is a collection of random variables indexed by time, denoted as $\{X_t\}_{t\geq 0}$, where the time variable t can be continuous or discrete (Cont & Tankov, 2004). Essentially, a stochastic process can be seen as a function of both time t and randomness ω. Let $\{\mathcal{F}_t\}_{t\geq 0}$ be a filtration on $(\Omega, \mathscr{F}, \mathbb{P})$ such that $\mathcal{F}_t \subseteq \mathcal{F}$ for all $t \geq 0$. We say this collection of random variables is a stochastic process adapted to $\{\mathcal{F}_t\}_{t\geq 0}$ if, for each t, the random variable X_t is $\mathcal{F}_t$-measurable.

The sample path of a stochastic process is a function of time that represents the values observed for each particular outcome of the randomness ω. In other words, we have:

$$X_{\cdot}(\omega) : t \mapsto X_t(\omega).$$

If we consider random variables defined on a continuous space $C([0,T], \mathbb{R})$, we can construct stochastic processes. The standard norm used for $C([0,T], \mathbb{R})$ (Cont &

Tankov, 2004) is given by:

$$||f||_\infty = \sup_{t \in [0,T]} |f(t)|, \tag{2.21}$$

where $f : [0,T] \to \mathbb{R}$. A classic example of a stochastic process with continuous sample paths is the Wiener process.

Definition 2.1.8 (Wiener process) *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. For each $\omega \in \Omega$, let $W(t)$ be a continuous function of $t \geq 0$ satisfying $W(0) = 0$ and depending on ω. Then $W(t)$, $t \geq 0$, is called a Wiener process if the increments*

$$W(t_1) - W(t_0), W(t_2) - W(t_1), \ldots, W(t_m) - W(t_{m-1}), \tag{2.22}$$

are independent for all $0 = t_0 < t_1 < \cdots < t_m$, and each increment follows a normal distribution with the following properties:

$$\mathbb{E}[W(t_i) - W(t_{i-1})] = 0, \tag{2.23}$$

$$Var[W(t_i) - W(t_{i-1})] = t_i - t_{i-1}. \tag{2.24}$$

Wiener process, also known as Brownian motion in the literature, possesses several important properties (Shreve, 2008):

- The increments of the process are independent and normally distributed with mean given by (2.23) and variance given by (2.24).

- The random variables $W(t_1)$, $W(t_2)$, ..., $W(t_m)$ follow a joint normal distribution with zero mean and a covariance matrix given by

$$\begin{bmatrix} \mathbb{E}[W^2(t_1)] & \dots & \mathbb{E}[W(t_1)]\mathbb{E}[W(t_m)] \\ \mathbb{E}[W(t_2)]\mathbb{E}[W(t_1)] & \dots & \mathbb{E}[W(t_1)]\mathbb{E}[W(t_m)] \\ \vdots & \vdots & \vdots \\ \mathbb{E}[W(t_m)]\mathbb{E}[W(t_1)] & \dots & \mathbb{E}[W^2(t_m)] \end{bmatrix} = \begin{bmatrix} t_1 & t_1 & \cdots & t_1 \\ t_1 & t_2 & \cdots & t_2 \\ \vdots & \vdots & & \vdots \\ t_1 & t_2 & \cdots & t_m \end{bmatrix}. \tag{2.25}$$

2.1.5 Asymptotic Analysis

In this subsection, we present a derivation of an approximation function denoted as $P^{\beta,\varepsilon}$ in the regime where ε and β are small positive independent parameters. We choose to expand first with respect to β and then with respect to ε, as it is a more convenient ordering choice. In our notation, the term $P_{j,k}$ corresponds to the term of order $\beta^j\varepsilon^{k/2}$. We begin by considering an expansion of $P^{\beta,\varepsilon}$ in powers of β:

$$P^{\beta,\varepsilon} = P_0^{\varepsilon} + \beta P_1^{\varepsilon} + O(\beta^2). \tag{2.26}$$

Furthermore, we decompose the terms P_0^{ε} and P_1^{ε} as follows:

$$\begin{aligned} P_0^{\varepsilon} &= P_{0,0} + \sqrt{\varepsilon}P_{0,1} + O(\varepsilon), \\ P_1^{\varepsilon} &= P_{1,0} + \sqrt{\varepsilon}P_{1,1} + O(\varepsilon). \end{aligned}$$

Inserting these decompositions into (2.26) yields:

$$P^{\varepsilon,\beta} = P_{0,0} + \sqrt{\varepsilon}P_{0,1} + \beta P_{1,0} + \beta\sqrt{\varepsilon}P_{1,1} + \dots.$$

The term $P_{0,0}$ is commonly referred to as the zeroth-order term, while the terms

$\sqrt{\varepsilon}P_{0,1}$ and $\beta P_{1,0}$, are known as the first-order correction terms.

2.1.6 Mellin Transform

The Mellin transform is an integral transform that may be regarded as the multiplicative version of the two-sided Laplace transform. It is commonly used in the theory of asymptotic expansions. For a locally Lebesgue integrable function $h : \mathbb{R}^+ \to \mathbb{R}$, the Mellin transform, denoted as $\mathcal{M}h$ or $\hat{h}$, is defined as follows:

$$\hat{h}(w) = (\mathcal{M}h)(w) := \int_0^{+\infty} s^{w-1}h(s)\,ds, \quad w \in \mathbb{C}.$$

Moreover, if there exist $a < \text{Re}(w) < b$ and c such that $a < c < b$, the inverse Mellin transform is given by

$$h(s) = \left(\mathcal{M}^{-1}\hat{h}\right)(s) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} s^{-w}\hat{h}(w)\,dw.$$

Throughout this thesis, we employ the following properties of the Mellin transform in Table 2.1.

Table 2.1: Properties of the Mellin transform

Function	Mellin Transform
h	$\hat{h}$
sh'	$-w\hat{h}$
s^2h''	$w(w+1)\hat{h}$
$s^3h^{(3)}$	$-w(w+1)(w+2)\hat{h}$
$\frac{e^{\delta}s^{\eta}}{2\sqrt{\lambda\pi}}e^{-\frac{1}{4\lambda}(\ln s)^2}$	$e^{\lambda(w+\eta)^2+\delta}$
$sh' + s^2h''$	$w^2\hat{h}$
$-sh' - 3s^2h'' - s^3h^{(3)}$	$w^3\hat{h}$

In Table 2.1, λ, η, and δ are unrelated to w or s, and h', h'', and $h^{(3)}$ represent the first-order, second-order, and third-order derivatives of h, respectively.

2.2 Fundamentals of Quantitative Finance

In this section, we introduce the principles of risk-neutral pricing theory and the techniques used for pricing financial derivatives. Risk-neutral modeling is a fundamental approach in derivative pricing theory, enabling us to derive pricing formulas based on the no-arbitrage assumption. By employing these techniques, we can accurately price a wide range of financial derivatives.

2.2.1 Vanilla Options

A financial derivative is a contractual agreement that derives its value from the performance of an underlying asset, which can include commodities, currencies, market indexes, and more.

A vanilla option is a derivative contract that grants the holder the right, but not the obligation, to buy or sell the underlying asset at a predetermined price before or on the expiration date (Cont & Tankov, 2004). Specifically, a call option gives the holder the right to buy, while a put option gives the holder the right to sell. A European option can only be exercised at the expiration date, whereas an American option can be exercised at any time before the expiration date. Let V_T denote the payoff of an option on the risky asset at its expiration date T. The payoffs of a European call and a European put with a strike price of K at the expiration date T are defined as follows:

European call option payoff

$$V_T = \max\left(S_T - K, 0\right); \tag{2.27}$$

European put option payoff

$$V_T = \max\left(K - S_T, 0\right); \tag{2.28}$$

where S_T represents the value of the underlying asset at the expiration date.

2.2.2 Path-dependent Options

Vanilla options are the most basic and standard types of options traded in financial markets. They have simple payoff structures and standard features. Path-dependent options, on the other hand, are more complex compared to vanilla options. They are designed to meet specific investment objectives or address specific market conditions. Path-dependent options have more complex pricing models and may require advanced mathematical techniques for valuation. Their prices can depend on multiple factors, including asset price, time, volatility, interest rates, and additional parameters specific to the exotic option type.

The payoff of an option on the risky asset at its expiration date T is denoted by V_T. Note that V_T varies depending on the type of options. In Chapters 3 & 4 of this thesis, we consider two types of path-dependent options: down-and-out put options and floating strike lookback put options. In Chapter 5, we focus on another type of path-dependent options: geometric average Asian put options.

Let $\{S_t\}_{0\le t\le T}$ denote the price process of a risky asset between time 0 and time T. For notational convenience, we put $U_t := \min_{0\le u\le t} S_u$ and $Z_t := \max_{0\le u\le t} S_u$. The payoff of a down-and-out put option is given by

$$DOP(T) := \max\{K - S_T, 0\} \times \mathbb{1}_{U_T > B}, \tag{2.29}$$

where K is the strike price, B is the barrier level satisfying $0 < B < K$ and $\mathbb{1}_{U_T>B}$ is the indicator function. For a floating strike lookback put option, its payoff has the following form:

$$LP_{float}(T) := Z_T - S_T. \tag{2.30}$$

The payoff of a European geometric average Asian put option is specified as

$$GAP(T) := K - e^{\frac{\int_0^T \ln(S_t)dt}{T}}. \tag{2.31}$$

2.2.3 Risk-Neutral Pricing

In this sub-section, we introduce Girsanov's theorem, which provides a method for changing between two equivalent probability measures. We consider a non-negative random variable, denoted as Z, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, satisfying the following conditions:

$$\mathbb{E}^{\mathbb{P}}[Z] = 1 \quad \text{and} \quad \mathbb{P}\{Z > 0\} = 1. \tag{2.32}$$

Here, $\mathbb{P}$ is referred to as the physical measure. Now, we introduce another probability measure, denoted as $\mathbb{Q}$, which is equivalent to the measure $\mathbb{P}$. The measure $\mathbb{Q}$ is defined as:

$$\mathbb{Q}(A) = \int_A Z(\omega)\, d\mathbb{P}(\omega) \quad \text{for all } A \in \mathcal{F}. \tag{2.33}$$

We refer to $\mathbb{Q}$ as the risk-neutral measure if the price of an asset is precisely equal to the discounted expectation of the asset price under this measure. We denote the expectation of a random variable X under the physical measure $\mathbb{P}$ as $\mathbb{E}^{\mathbb{P}}[X]$, and the expectation of X under the risk-neutral measure $\mathbb{Q}$ as $\mathbb{E}^{\mathbb{Q}}[X]$. The relationship between these two expectations can be expressed as:

$$\mathbb{E}^{\mathbb{Q}}[X] = \mathbb{E}^{\mathbb{P}}[XZ]. \tag{2.34}$$

We also say that Z is the Radon-Nikodým derivative of $\mathbb{Q}$ with respect to $\mathbb{P}$. In the context of the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\{\mathcal{F}_t\}_{0\le t\le T}$, the Radon-Nikodým derivative process (Shreve, 2008) is defined as:

$$Z(t) = \mathbb{E}^{\mathbb{P}}[Z|\mathcal{F}_t], \quad 0 \le t \le T. \tag{2.35}$$

If X is an $\mathcal{F}_t$-measurable random variable, we have the relationship:

$$\mathbb{E}^{\mathbb{Q}}[X] = \mathbb{E}^{\mathbb{P}}[XZ_t]. \tag{2.36}$$

Theorem 2.2.1 (Girsanov's Theorem) *Consider a Wiener process $\{W_t^{\mathbb{P}}\}_{0\le t\le T}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with respect to a filtration $\{\mathcal{F}_t\}_{0\le t\le T}$, and let Θ_t be an adapted process. Define the processes:*

$$Z_t = e^{\left(-\int_0^t \Theta_s dW_s^{\mathbb{P}} - \frac{1}{2}\int_0^t \Theta_s^2 ds\right)}, \tag{2.37}$$

$$W_t^{\mathbb{Q}} = W_t^{\mathbb{P}} + \int_0^t \Theta_s ds, \tag{2.38}$$

and assume that the condition

$$\mathbb{E}^{\mathbb{P}}\left[\int_0^T \Theta_s^2 Z_s^2 ds\right] < \infty \tag{2.39}$$

holds. Then, we have $\mathbb{E}^{\mathbb{P}}[Z_T] = 1$, and $\{W_t^{\mathbb{Q}}\}_{0\le t\le T}$ is a Wiener process under the probability measure $\mathbb{Q}$ defined in Eq. (2.33).

In the following, we examine an asset price model under the physical measure $\mathbb{P}$. Let $\{S_t\}_{t\ge0}$ represent an asset price process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with respect to a filtration $\{\mathcal{F}_t\}_{t\ge0}$, which satisfies the following stochastic differential equation:

$$dS_t = \mu_t S_t dt + \sigma_t S_t dW_t^{\mathbb{P}},$$

where the mean rate of return $\{\mu_t\}_{t\geq 0}$ and volatility $\{\sigma_t\}_{t\geq 0}$ are adapted processes, and $\{W_t^{\mathbb{P}}\}_{t\geq 0}$ denotes the Wiener process under the measure $\mathbb{P}$. The asset price can be represented as follows:

$$S_t = S_0 \exp\left\{\int_0^t \sigma_s dW_s^{\mathbb{P}} + \int_0^t \left(\mu_s - \frac{1}{2}\sigma_s^2\right) ds\right\}. \tag{2.40}$$

Furthermore, we have an interest rate process $\{R_t\}_{t\geq 0}$ that is also an adapted process. Consequently, the discount process $\{D_t\}_{t\geq 0}$ can be defined as:

$$D_t = \exp\left\{-\int_0^t R_s ds\right\}. \tag{2.41}$$

Hence, the discounted asset price process is given by:

$$D_t S_t = S_0 \exp\left\{\int_0^t \sigma_s dW_s^{\mathbb{P}} + \int_0^t \left(\mu_s - R_s - \frac{1}{2}\sigma_s^2\right) ds\right\}, \tag{2.42}$$

and the differential form of Eq. (2.42) is:

$$\begin{aligned} d(D_t S_t) &= (\mu_t - R_t)\, D_t S_t dt + \sigma_t D_t S_t dW_t^{\mathbb{P}} \\ &= \sigma_t D_t S_t \left[\frac{\mu_t - R_t}{\sigma_t} dt + dW_t^{\mathbb{P}}\right]. \end{aligned} \tag{2.43}$$

Let Θ_t be defined as follows:

$$\Theta_t = \frac{\mu_t - R_t}{\sigma_t}. \tag{2.44}$$

Based on Theorem 2.2.1, we can define the Wiener process $\{W_t^{\mathbb{Q}}\}_{t\geq 0}$ under the probability measure $\mathbb{Q}$ as follows:

$$W_t^{\mathbb{P}} = W_t^{\mathbb{Q}} - \int_0^t \Theta_s ds, \tag{2.45}$$

where Θ_s is given in Eq. (2.44). Consequently, the discounted asset price under the measure $\mathbb{Q}$ can be expressed as:

$$D_tS_t = S_0 + \int_0^t \sigma_s D_s S_s dW_s^{\mathbb{Q}}, \tag{2.46}$$

and $\int_0^t \sigma_s D_s S_s dW_s^{\mathbb{Q}}$ is a martingale under $\mathbb{Q}$. Hence, the price of an asset is equal to the discounted expectation of the asset price under $\mathbb{Q}$, i.e.,

$$\mathbb{E}^{\mathbb{Q}}[D_tS_t] = S_0.$$

The asset price process $\{S_t\}_{t\geq 0}$ under the $\mathbb{Q}$ measure can be represented as:

$$S_t = S_0 \exp\left\{\int_0^t \sigma_s dW_s^{\mathbb{Q}} + \int_0^t \left(R_s - \frac{1}{2}\sigma_s^2\right) ds\right\}. \tag{2.47}$$

Furthermore, the generalized pricing formula of a derivative under the measure $\mathbb{Q}$ can be defined as:

$$V_t = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T R_s ds} V_T | \mathcal{F}_t\right], \qquad 0 \leq t \leq T, \tag{2.48}$$

where $\{V_t\}_{0\leq t\leq T}$ is a payoff process that is $\mathcal{F}_t$-adapted, and V_T represents the payoff of a derivative at expiration time T.

2.3 Numerical Methods

This section presents various numerical methods used for the evaluation of improper integrals, solution of ordinary differential equations and estimation of parameter values in the model.

2.3.1 Gauss-Laguerre Quadrature Method

The Gauss-Laguerre quadrature method, introduced by Burnett (1937), is a commonly used technique for approximating the value of a specific integral over the interval $[0, \infty)$. The integral has the form:

$$\int_0^\infty x^m e^{-x} f(x) dx, \tag{2.49}$$

where $m > 0$. According to Burchnall (1937) and Condon, Caswell, Jaeger, and Melissinos (1963), the integral can be estimated as

$$\int_0^\infty x^m e^{-x} f(x) dx = \sum_{i=1}^{n} W_i f(a_i) + E_n, \tag{2.50}$$

where a_i represents the ith zero of the Laguerre polynomial $L_n^m(a_i)$, W_i is the corresponding weight, and E_n denotes the error term. The Laguerre polynomial $L_n^m(a_i)$ is given by

$$L_n^m(a_i) = \sum_{k=0}^{n} \binom{n+m}{n-k} \frac{(-a_i)^k}{k!},$$

while the weight is represented as

$$W_i = \frac{\Gamma(n+m+1) a_i}{n! \left[(n+1) L_{n+1}^m(a_i)\right]^2}, \quad i = 1, 2, ..., n.$$

The error term E_n is defined as

$$E_n = \frac{n! \Gamma(n+m+1)}{(2n)!} f^{(2n)}(\xi),$$

where $f^{(2n)}(\xi)$ represents the $2n$ th derivative of $f(x)$ evaluated at some point ξ in the integration interval.

2.3.2 Runge-Kutta Method

The Runge-Kutta method is a numerical technique that offers an efficient way for solving ordinary differential equations (ODEs) of the form:

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0. \tag{2.51}$$

In the Runge-Kutta method, Equation (2.51) can be effectively solved by using a Taylor-series expansion (Hildebrand, 1987) and approximating the value of y_{n+1}:

$$y_{n+1} = y_n + h\frac{dy}{dx} + \frac{h^2}{2}\frac{d^2y}{dx^2} + \frac{h^3}{6}\frac{d^3y}{dx^3} + \cdots, \tag{2.52}$$

The Runge-Kutta method employs an approximation of y_{n+1} given by:

$$\begin{aligned} y_{n+1} =& y_n + h\left[\alpha_0 f(x_n, y_n) + \alpha_1 f(x_n + \mu_1 h, y_n + b_1 h)\right. \\ &\left. + \alpha_2 f(x_n + \mu_2 h, y_n + b_2 h) + \cdots + \alpha_P f(x_n + \mu_p h, y_n + b_p h)\right], \end{aligned} \tag{2.53}$$

where α's, μ's, and b's are deterministic coefficients. The advantage of this method is that it can be used even when the function $f(x, y)$ does not have a closed-form expression since it does not require evaluating derivatives of $f(x, y)$.

The classic Runge-Kutta method, also known as the 4th-order Runge-Kutta method, provides an approximation with fourth-order accuracy. Considering a problem with an initial condition specified in Equation (2.51), and choosing a step size $h > 0$, we have (Hildebrand, 1987):

$$\begin{aligned} y_{n+1} &= y_n + \frac{1}{6}(k_0 + 2k_1 + 2k_2 + k_3) + O(h^5), \\ x_{n+1} &= x_n + h, \end{aligned} \tag{2.54}$$

where

$$\begin{aligned} k_0 &= hf(x_n, y_n), \\ k_1 &= hf(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_0), \\ k_2 &= hf(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_1), \\ k_3 &= hf(x_n + h, y_n + k_2). \end{aligned} \tag{2.55}$$

This method can also be applied to a simultaneous system of differential equations in the form:

$$\begin{aligned} \frac{dy}{dx} &= f(x, y, z), \quad y(x_0) = y_0, \\ \frac{dz}{dx} &= g(x, y, z), \quad z(x_0) = z_0. \end{aligned} \tag{2.56}$$

In a similar manner, we can obtain the following estimations:

$$\begin{aligned} y_{n+1} &= y_n + \frac{1}{6}(k_0 + 2k_1 + 2k_2 + k_3) + O(h^5), \\ z_{n+1} &= z_n + \frac{1}{6}(m_0 + 2m_1 + 2m_2 + m_3) + O(h^5), \end{aligned} \tag{2.57}$$

with

$$\begin{aligned} k_0 &= hf(x_n, y_n, z_n), \\ k_1 &= hf(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_0, z_n + \frac{1}{2}m_0), \\ k_2 &= hf(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_1, z_n + \frac{1}{2}m_1), \\ k_3 &= hf(x_n + h, y_n + k_2, z_n + m_2), \end{aligned} \tag{2.58}$$

and

$$\begin{aligned} k_0 &= hg(x_n, y_n, z_n), \\ k_1 &= hg(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_0, z_n + \frac{1}{2}m_0), \\ k_2 &= hg(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_1, z_n + \frac{1}{2}m_1), \\ k_3 &= hg(x_n + h, y_n + k_2, z_n + m_2). \end{aligned} \tag{2.59}$$

2.3.3 Fractional Calculus

Traditional financial models, such as the Black-Scholes model developed by Black and Scholes (1973), commonly utilize Brownian motion, or geometric Brownian motion to model the asset prices. However, motivated by empirical research, there has been interest in exploring financial models driven by fractional Brownian motion. Fractional Brownian motion extends the classical Brownian motion by introducing correlation among its disjoint increments. Fractional stochastic volatility models, as proposed and developed by Comte et al. (2012), offer improved explanations for the long-term behavior of implied volatility.

In this subsection, we present properties of fractional calculus that are relevant to fractional Brownian motion. Let $f : \mathbb{R} \to \mathbb{R}$ denote a continuous function. Its fractional derivative of order α is defined by

$$f^{(\alpha)}(x) := \frac{1}{\Gamma(-\alpha)} \int_0^x (x-t)^{-\alpha-1} f(t)dt \quad \text{for} \quad \alpha < 0.$$

In addition, for any positive α with $n - 1 < \alpha < n$ where $n \in \mathbb{N}$, the derivative of order α is defined by

$$f^{(\alpha)}(x) \quad := \quad (f^{(\alpha-n)})^{(n)}(x).$$

Based on the definition above, we can calculate the fractional derivative of a function $f(x)$ with respect to x for $0 < \alpha < 1$ as the following expression:

$$\begin{aligned}\frac{d^\alpha f}{dx^\alpha} &= f^{(\alpha)}(x) \\ &= (f^{(\alpha-1)})'(x) \\ &= \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_0^x (x-t)^{-\alpha} f(t)dt. \end{aligned} \tag{2.60}$$

Here are some useful formulas derived in Jumarie (2005):

For $0 < \alpha \leq 1$: $d^\alpha f = \Gamma(\alpha+1)df$

For $1 < \alpha \leq 2$: $d^\alpha f = \Gamma(\alpha)d^2 f$

By applying the definition of the fractional derivative of order $\alpha \in (0,1)$ in (2.60) and the useful formulas above to the function $f(x) = x$, we can obtain the following derivation steps:

$$\begin{aligned}\frac{\Gamma(\alpha+1)dx}{dx^\alpha} &= \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_0^x (x-t)^{-\alpha} t dt \\ &= \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_x^0 z^{-\alpha}(x-z)(-dz) \\ &= \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_0^x xz^{-\alpha} - z^{1-\alpha} dz \\ &= \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\left(\frac{xz^{1-\alpha}}{1-\alpha} - \frac{z^{2-\alpha}}{2-\alpha}\right)\Bigg|_{z=0}^{z=x} \\ &= \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\left(\frac{x^{2-\alpha}}{1-\alpha} - \frac{x^{2-\alpha}}{2-\alpha}\right) \\ &= \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\left(\frac{x^{2-\alpha}}{(1-\alpha)(2-\alpha)}\right) \\ &= \frac{1}{\Gamma(1-\alpha)} \cdot \frac{(2-\alpha)x^{1-\alpha}}{(1-\alpha)(2-\alpha)} \\ &= \frac{1}{\Gamma(1-\alpha)} \cdot \frac{x^{1-\alpha}}{1-\alpha} \\ &= \frac{x^{1-\alpha}}{\Gamma(2-\alpha)},\end{aligned}$$

where we substitute $t = x - z$ and $dt = -dz$ in the process. Rearranging the terms in the equation above yields:

$$dx = \frac{x^{1-\alpha}}{\Gamma(2-\alpha)\Gamma(1+\alpha)} dx^{\alpha}. \tag{2.61}$$

Chapter 3

Pricing Path-dependent Options under Stochastic Volatility Model

In this chapter, we derive closed-form formulas of first-order approximation for the prices of down-and-out barrier and floating strike lookback put options under a stochastic volatility model, by using an asymptotic approach. To find the explicit closed-form formulas for the zero-order term and the first-order correction term, we use Mellin transform. We also conduct a sensitivity analysis on these formulas, and compare the option prices calculated by them with those generated by Monte-Carlo simulation.

3.1 SV Model

Let $\{S_t\}_{t\geq 0}$ denote the price process of a risky asset on some filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$, where $\mathbb{P}$ represents the physical probability measure. We assume that $\{S_t\}_{t\geq 0}$ evolves according to the following system of stochastic differential equations:

$$\begin{aligned} dS_t &= \mu S_t dt + f(Y_t) S_t dW_t^s, \\ dY_t &= \alpha (m - Y_t) dt + \beta \left(\rho dW_t^s + \sqrt{1-\rho^2} dW_t^y \right), \end{aligned} \tag{3.1}$$

where μ, $\alpha > 0$, $\beta > 0$ and m are constants, f is a function having non-zero values and specifying the dependence on the hidden process $\{Y_t\}_{t\geq 0}$. The processes $\{W_t^s\}_{t\geq 0}$ and $\{W_t^y\}_{t\geq 0}$ are uncorrelated standard Brownian motions. The constant correlation coefficient ρ with $-1 < \rho < 1$ captures the leverage effect. Here, μ is the drift rate. The mean-reversion process $\{Y_t\}_{t\geq 0}$ given in Eq. (3.1) is characterized by its typical time to obtain back to the mean level m of its long-run distribution. The parameter α determines the speed of mean-reversion and β controls the volatility of $\{Y_t\}_{t\geq 0}$. In the sequel, we shall refer to the above system as the stochastic volatility (abbreviated as SV) model. In Sections 3.1 and 3.2, we will not specify the concrete form of f, but assume that f is bounded and smooth enough, e.g., $f \in C_0^2(\mathbb{R})$. Furthermore, the function f has to satisfy a sufficient growth condition in order to avoid bad behavior such as the non-existence of moments of $\{S_t\}_{t\geq 0}$. For numerical results shown in Section 3.5, we choose f to take a special form as used in Fouque, Papanicolaou and Sircar (2000), Fouque, Papanicolaou, Sircar and Sølna (2011) and Cao, Kim and Zhang (2021).

We apply the well-known Girsanov theorem to change the physical measure $\mathbb{P}$ to a risk-neutral martingale measure $\mathbb{Q}$ by letting

$$dW_t^{s*} = \frac{\mu - r}{f(Y_t)} dt + dW_t^s \quad \text{and} \quad dW_t^{y*} = \xi(Y_t) dt + dW_t^y,$$

where $\xi(Y_t)$ represents the premium of volatility risk. Then the model equations under the measure $\mathbb{Q}$ can be written as

$$
\begin{aligned}
dS_t &= rS_t dt + f(Y_t) S_t dW_t^{s*}, \\
dY_t &= \left[\alpha\,(m - Y_t) - \beta\left(\rho\frac{\mu - r}{f(Y_t)} + \xi(Y_t)\sqrt{1-\rho^2}\right)\right] dt \\
&\quad + \beta\left(\rho dW_t^{s*} + \sqrt{1-\rho^2} dW_t^{y*}\right).
\end{aligned} \tag{3.2}
$$

Note that $\{W_t^{s*}\}_{t\geq 0}$ and $\{W_t^{y*}\}_{t\geq 0}$ are independent standard Brownian motions under $\mathbb{Q}$. As an Ornstein-Uhlenbeck (OU) process, $\{Y_t\}_{t\geq 0}$ in Eq. (3.1) has an invariant distribution, which is normal with mean m and variance $\beta^2/2\alpha$. Thus, we can expect that if mean reversion is very fast, i.e., α goes to infinity, the process $\{S_t\}_{t\geq 0}$ should be close to a geometric Brownian motion. This means that if mean reversion is extremely fast, then the model of Black and Scholes would become a good approximation. In reality, however, it may not be the case. For fast but not extremely fast mean-reversion, the Black-Scholes model needs to be corrected to account for the random characteristics of the volatility of a risky asset. For this purpose, we introduce another small parameter ε defined by $\varepsilon = 1/\alpha$ as done by Fouque et al. (2000). For notational convenience, we put $\nu = \beta/\sqrt{2\alpha}$. With the help of these notations, the model equations under $\mathbb{Q}$ is re-written as

$$
\begin{aligned}
dS_t &= rS_t dt + f\left(Y_t\right) S_t dW_t^{s*}, \\
dY_t &= \left[\frac{1}{\varepsilon}\left(m - Y_t\right) - \frac{\sqrt{2}\nu}{\sqrt{\varepsilon}}\Lambda\left(Y_t\right)\right] dt + \frac{\sqrt{2}\nu}{\sqrt{\varepsilon}} dW_t^{y*},
\end{aligned}
$$

where $\Lambda(\cdot)$, defined by

$$
\Lambda(y) := \rho\frac{\mu - r}{f(y)} + \xi(y)\sqrt{1-\rho^2},
$$

is the combined market price of risk.

Let V_T denote the payoff of a put option on the risky asset at its expiration T. Then its risk-neutral price at time $t \in [0,T]$ under our SV model is given by

$$P(t,s,y) = \mathbb{E}^{\mathbb{Q}}\left(e^{-r(T-t)}V(T) \mid S_t = s, Y_t = y\right).$$

As discussed in 2.2.2, V_T varies depending on the type of options. The payoffs of the two types of path-dependent options considered in this chapter (down-and-out put options and floating strike lookback put options) are specified in (2.29) and (2.30), respectively.

Applying Itô's lemma, we can obtain a partial differential equation (PDE) for $P(t,s,y)$ as follows:

$$\begin{aligned} 0 &= \frac{\partial P}{\partial t} + \frac{1}{2}s^2 f^2(y)\frac{\partial^2 P}{\partial s^2} + r\left(s\frac{\partial P}{\partial s} - P\right) + \frac{\sqrt{2}\rho\nu s}{\sqrt{\varepsilon}} f(y)\frac{\partial^2 P}{\partial s \partial y} \\ &\quad + \frac{\nu^2}{\varepsilon}\frac{\partial^2 P}{\partial y^2} + \left(\frac{1}{\varepsilon}(m-y) - \frac{\sqrt{2}\nu}{\sqrt{\varepsilon}}\Lambda(y)\right)\frac{\partial P}{\partial y}. \end{aligned} \tag{3.3}$$

The boundary conditions for Eq. (3.3) vary depending on the type of options. For example, the boundary conditions for Eq. (3.3) when $V_T = DOP(T)$ are

$$\begin{cases} P(T,s,y) = \max\{K-s,0\}, & s > B, \\ P(t,B,y) = 0, & 0 \le t \le T. \end{cases}$$

When $V_T = LP_{float}(T)$, the boundary conditions become the following

$$\begin{cases} \dfrac{\partial P}{\partial z}(t,z,y,z) = 0, & 0 \le t \le T, z > 0, \\ P(T,s,y,z) = z - s, & 0 \le s \le z. \end{cases}$$

Note that in this case, P is a function of four variables t, s, y and z (here, $Z_t = z$).

3.2 Specified Asymptotic Expansions for the Model

In this section, we apply an asymptotic expansion approach to establish partial differential equations, which will be used to derive an approximate solution to Eq. (3.3) and thus find an approximated value of a put option.

We begin with re-organizing Eq. (3.3) in terms of the orders of ε as follows:

$$\frac{1}{\varepsilon}\mathcal{L}_0 P + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1 P + \mathcal{L}_2 P = 0, \tag{3.4}$$

where the operators $\mathcal{L}_0$, $\mathcal{L}_1$ and $\mathcal{L}_2$ are defined by

$$\begin{aligned}
\mathcal{L}_0 &:= (m-y)\frac{\partial}{\partial y} + \nu^2\frac{\partial^2}{\partial y^2}, \\
\mathcal{L}_1 &:= \sqrt{2}\rho\nu s f(y)\frac{\partial^2}{\partial s\partial y} - \sqrt{2}\nu\Lambda(y)\frac{\partial}{\partial y}, \text{ and} \\
\mathcal{L}_2 &:= \frac{\partial}{\partial t} + \frac{1}{2}s^2 f^2(y)\frac{\partial^2}{\partial s^2} + r\left(s\frac{\partial}{\partial s} - \cdot\right).
\end{aligned}$$

In order to obtain an efficient approximate solution to P, as that in Fouque and Han (2006) and Fouque et al. (2011), we apply the following asymptotic expansion of P as terms with varying orders of ε:

$$P = P_0 + \sqrt{\varepsilon}P_1 + \varepsilon P_2 + \varepsilon\sqrt{\varepsilon}P_3 + \cdots, \tag{3.5}$$

where P_0, P_1, ... are functions corresponding to varying orders of ε. Substituting P in Eq. (3.5) into the Eq.(3.4) and re-organizing terms, we obtain

$$\begin{aligned}
0 = &\frac{1}{\varepsilon}\mathcal{L}_0 P_0 + \frac{1}{\sqrt{\varepsilon}}(\mathcal{L}_1 P_0 + \mathcal{L}_0 P_1) + (\mathcal{L}_0 P_2 + \mathcal{L}_1 P_1 + \mathcal{L}_2 P_0) \\
&+\sqrt{\varepsilon}(\mathcal{L}_0 P_3 + \mathcal{L}_1 P_2 + \mathcal{L}_2 P_1) + \cdots.
\end{aligned} \tag{3.6}$$

Our aim is to find P_0 and P_1.

Firstly, from the $O(1/\varepsilon)$-order term in Eq.(3.6), we get $\mathcal{L}_0 P_0 = 0$. If we assume that P_0 does not grow as fast as $e^{y^2/2}$, we can show that P_0 is independent of y. Secondly, from the $O(1/\sqrt{\varepsilon})$-order term in Eq. (3.6), we can get

$$\mathcal{L}_1 P_0 + \mathcal{L}_0 P_1 = 0.$$

Since P_0 is independent of y, then $\mathcal{L}_1 P_0 = 0$. It follows that $\mathcal{L}_0 P_1 = 0$. Again, if we assume that P_1 does not grow as fast as $e^{y^2/2}$, then we can deduce that P_1 is also independent of y.

Next, from the $O(1)$-order term in Eq.(3.6), we get

$$\mathcal{L}_0 P_2 + \mathcal{L}_1 P_1 + \mathcal{L}_2 P_0 = 0.$$

Since P_1 is independent of y, we have $\mathcal{L}_1 P_1 = 0$ which implies that

$$\mathcal{L}_0 P_2 + \mathcal{L}_2 P_0 = 0. \tag{3.7}$$

Seeing Eq. (3.7) as a Poisson equation for P_2 in y, in order for it to have a solution, it is required to satisfy the centring condition

$$\langle \mathcal{L}_2 P_0 \rangle = \langle \mathcal{L}_2 \rangle P_0 = 0, \tag{3.8}$$

which is equivalent to

$$\frac{\partial P_0}{\partial t} + rs\frac{\partial P_0}{\partial s} + \frac{1}{2}s^2 \langle f^2 \rangle \frac{\partial^2 P_0}{\partial s^2} - rP_0 = 0. \tag{3.9}$$

This is an equation for us to determine P_0 term. Here, $\langle \cdot \rangle$ denotes the expectation with respect to the invariant distribution of the process $\{Y_t\}_{t\geq 0}$, i.e.,

$$\langle h \rangle = \int_{-\infty}^{+\infty} h(y)\Phi(y)dy,$$

where

$$\Phi(y) = \frac{1}{\sqrt{2\pi\nu^2}} e^{-\frac{(y-m)^2}{2\nu^2}}.$$

Note that small ε value corresponds to fast-mean reverting. In this case, Y_t approaches to a constant and $\langle f^2 \rangle$ can be regarded as constant variance and then Eq. (3.9) is the Black-Scholes PDE. Thus, for small ε, P_0 represents the put option price under the Black-Scholes model.

Following Eq. (3.8), we have

$$\mathcal{L}_2 P_0 = \mathcal{L}_2 P_0 - \langle \mathcal{L}_2 \rangle P_0 = \frac{1}{2}\left(f^2 - \langle f^2 \rangle\right) s^2 \frac{\partial^2 P_0}{\partial s^2},$$

which together with Eq. (3.7) implies

$$\mathcal{L}_0 P_2 = -\frac{1}{2}\left(f^2 - \langle f^2 \rangle\right) s^2 \frac{\partial^2 P_0}{\partial s^2}. \tag{3.10}$$

The solution to Eq. (3.10) can be expressed as

$$P_2 = -\frac{1}{2}\left(\phi + c\right) s^2 \frac{\partial^2 P_0}{\partial s^2}, \tag{3.11}$$

where ϕ is a function of y which only satisfies the equation

$$\mathcal{L}_0 \phi = f^2 - \langle f^2 \rangle$$

and c is a function of other variables except y.

To derive an equation for P_1, we consider the $O(\sqrt{\varepsilon})$-term in Eq. (3.6) and obtain

$$\mathcal{L}_0 P_3 + \mathcal{L}_1 P_2 + \mathcal{L}_2 P_1 = 0.$$

This equation can be regarded as a Poisson equation for P_3 in y, and in order for it to have a solution, the following centring condition must be satisfied:

$$\langle \mathcal{L}_1 P_2 + \mathcal{L}_2 P_1 \rangle = 0. \tag{3.12}$$

After we substitute P_2 in Eq. (3.11) into Eq. (3.12) and make simplification, we obtain

$$\frac{\partial P_1}{\partial t} + \frac{1}{2}\langle f^2\rangle s^2 \frac{\partial^2 P_1}{\partial s^2} + rs\frac{\partial P_1}{\partial s} - rP_1 = c_1 s^3 \frac{\partial^3 P_0}{\partial s^3} + c_2 s^2 \frac{\partial^2 P_0}{\partial s^2}, \tag{3.13}$$

where

$$c_1 := \frac{\sqrt{2}}{2}\langle f\phi'\rangle \rho\nu \quad \text{and} \quad c_2 := \frac{\sqrt{2}}{2}\left(2\rho\langle f\phi'\rangle - \langle \Lambda\phi'\rangle\right)\nu. \tag{3.14}$$

This is an equation for us to determine the first correction term P_1.

We summarize the key points in the previous formal analysis as the following theorem.

Theorem 3.2.1 *Under the SV model governed by Eq.* (3.1)*, the risk-neutral value* P *of a path-dependent put option can be approximated by the following formula*

$$P \approx P_0 + \sqrt{\varepsilon} P_1, \tag{3.15}$$

for small ε*, where* P_0 *and* P_1 *are determined by Eq.* (3.9) *and Eq.* (3.13) *with corresponding boundary conditions, respectively.* P_0 *is the put option price under the Black-Scholes model with constant effective volatility* $\sqrt{\langle f^2\rangle}$ *and* P_1 *is the first-order* *correction term.*

Finally, as mentioned in Section 3.1, boundary conditions for Eq. (3.8) and Eq. (3.13) depend on the types of options we consider. We describe the corresponding boundary conditions and solve these equations in the next two sections.

3.3 Analytic Approximate Solutions for Barrier Option Prices

3.3.1 P_0 Term for Down-and-out Put Options

In order to use Mellin transform to calculate the P_0 term for down-and-out put options, noting that P_0 is independent of y under our assumption, we first follow the method in Buchen (2001) and use the boundary condition,

$$P(T,s,y) = \max\{K-s,0\}, \quad \text{for} \quad s > B,$$

to set up the boundary condition of P_0 for $s \geq 0$ as follows:

$$P_0\left(T,s\right) := \left(K-s\right)\mathbb{1}_{B<s<K} - \left(\frac{B}{s}\right)^{k_1-1}\left(K-\frac{B^2}{s}\right)\mathbb{1}_{\frac{B^2}{K}<s<B}, \tag{3.16}$$

where $k_1 = 2r/\langle f^2\rangle$. Now, we apply Mellin transform in Table 2.1 to Eq. (3.9) to convert this PDE into the following ODE:

$$\frac{d\hat{P}_0}{dt} + \left(\frac{1}{2}\langle f^2\rangle(w^2+w) - rw - r\right)\hat{P}_0 = 0. \tag{3.17}$$

The solution to Eq. (3.17) is given by

$$\hat{P}_0\left(t,w\right) = \hat{\theta}(w)e^{\frac{1}{2}\langle f^2\rangle\left(w^2+(1-k_1)w-k_1\right)(T-t)}, \tag{3.18}$$

where $\hat{\theta}$ is a function of w, determined by the boundary condition (3.16).

Next, we take inverse Mellin transform of Eq. (3.18) in Table 2.1 and obtain

$$P_0(t,s) = P_0(T,s) * \mathcal{M}^{-1} e^{\lambda(w+\eta)^2+\delta},$$

where

$$\lambda = \frac{1}{2}\langle f^2\rangle\,(T-t)\,,\ \eta = \frac{1-k_1}{2},\ \delta = -\lambda\eta^2 - r\,(T-t)$$

and the operation $*$ means the convolution.

Using the boundary condition given in Eq. (3.16), we have

$$\begin{aligned} P_0\,(t,s) &= P_0\,(T,s) * \left(\frac{e^{\delta}s^{\eta}}{2\sqrt{\lambda\pi}} e^{-\frac{1}{4\lambda}(\ln s)^2}\right) \\ &= \int_B^K (K-u)\,e^{\delta}\left(\frac{s}{u}\right)^{\eta}\left(\frac{1}{2\sqrt{\lambda\pi}}e^{-\frac{1}{4\lambda}(\ln(\frac{s}{u}))^2}\right)\frac{du}{u} - \\ &\quad \int_{\frac{B^2}{K}}^{B}\left(\frac{B}{u}\right)^{k_1-1}\left(K-\frac{B^2}{u}\right)e^{\delta}\left(\frac{s}{u}\right)^{\eta}\left(\frac{1}{2\sqrt{\lambda\pi}}e^{-\frac{1}{4\lambda}(\ln(\frac{s}{u}))^2}\right)\frac{du}{u}. \end{aligned} \tag{3.19}$$

After some careful calculation, for down-and-out put options, we derive a closed-form expression of the P_0 term as follows:

$$\begin{aligned} P_0(t,s) &= Ke^{-r(T-t)}\left(\Phi\left(-\Delta_-\left(\frac{s}{K}\right)\right) - \Phi\left(-\Delta_-\left(\frac{s}{B}\right)\right)\right) - \\ &\quad s\left(\Phi\left(-\Delta_+\left(\frac{s}{K}\right)\right) - \Phi\left(-\Delta_+\left(\frac{s}{B}\right)\right)\right) - \\ &\quad Ke^{-r(T-t)}\left(\frac{B}{s}\right)^{k_1-1}\left[\Phi\left(\Delta_-\left(\frac{B}{s}\right)\right) - \Phi\left(\Delta_-\left(\frac{B^2}{sK}\right)\right)\right] + \\ &\quad B\left(\frac{B}{s}\right)^{k_1}\left[\Phi\left(\Delta_+\left(\frac{B}{s}\right)\right) - \Phi\left(\Delta_+\left(\frac{B^2}{sK}\right)\right)\right], \end{aligned} \tag{3.20}$$

where $\Phi(\cdot)$ is the CDF of the standard normal distribution and

$$\Delta_{\pm}(x) = \frac{1}{\sqrt{\langle f^2\rangle(T-t)}}\left[\ln(x) + \left(r \pm \frac{1}{2}\langle f^2\rangle\right)(T-t)\right].$$

Note that P_0 given in Eq. (3.20) is precisely the same as the price of a down-and-out put option given in the literature, e.g., Hull (2015) or Haug (2006) , if we let $\sigma^2 = \langle f^2 \rangle$.

From Eq. (3.19), we know that

$$\begin{aligned} P_0(t,s) &= \int_B^K (K-u)\, e^{\delta} \left(\frac{s}{u}\right)^{\eta} \left(\frac{1}{2\sqrt{\lambda\pi}} e^{-\frac{1}{4\lambda}\left(\ln\left(\frac{s}{u}\right)\right)^2}\right) \frac{du}{u} - \\ & \int_{\frac{B^2}{K}}^{B} \left(\frac{B}{u}\right)^{k_1-1} \left(K - \frac{B^2}{u}\right) e^{\delta} \left(\frac{s}{u}\right)^{\eta} \left(\frac{1}{2\sqrt{\lambda\pi}} e^{-\frac{1}{4\lambda}\left(\ln\left(\frac{s}{u}\right)\right)^2}\right) \frac{du}{u}. \end{aligned}$$

By letting $v = \ln u$, we convert the first integral to become

$$\begin{aligned} & \int_{\ln B}^{\ln K} (K - e^v) s^{\eta} e^{\delta} e^{-\eta v} \left(\frac{1}{2\sqrt{\lambda\pi}} e^{-\frac{1}{4\lambda}(\ln s - v)^2}\right) dv \\ = & \frac{s^{\eta} e^{\delta}}{2\sqrt{\lambda\pi}} \left(\int_{\ln B}^{\ln K} K e^{-\frac{1}{4\lambda}(v^2 - 2v\ln s + (\ln s)^2 + 4\lambda\eta v)} dv \right. \\ & \left. - \int_{\ln B}^{\ln K} e^{-\frac{1}{4\lambda}(v^2 - 2v\ln s + (\ln s)^2 + 4\lambda(\eta-1)v)} dv \right) \\ = & \frac{s^{\eta} e^{\delta}}{2\sqrt{\lambda\pi}} \left(\int_{\ln B}^{\ln K} K e^{-\frac{1}{4\lambda}(v - \ln s + 2\lambda\eta)^2 + \lambda\eta^2 - \eta \ln s} dv \right. \\ & \left. - \int_{\ln B}^{\ln K} e^{-\frac{1}{4\lambda}[v - \ln s + 2\lambda(\eta-1)]^2 + \lambda(\eta-1)^2 - (\eta-1)\ln s} dv \right). \end{aligned}$$

We further apply the following changes of variables

$$x' := \frac{v - \ln s + 2\lambda\eta}{\sqrt{2\lambda}} \quad \text{and} \quad x'' := \frac{v - \ln s + 2\lambda(\eta - 1)}{\sqrt{2\lambda}}$$

to get

$$\begin{aligned}
&\int_{\ln B}^{\ln K}(K-e^{v})s^{\eta}e^{\delta}e^{-\eta v}\left(\frac{1}{2\sqrt{\lambda\pi}}e^{-\frac{1}{4\lambda}(\ln s-v)^2}\right)dv\\
=\;&\frac{e^{\delta}}{\sqrt{2\pi}}\left(Ke^{\lambda\eta^2}\int_{\frac{\ln(\frac{B}{s})+2\lambda\eta}{\sqrt{2\lambda}}}^{\frac{\ln(\frac{K}{s})+2\lambda\eta}{\sqrt{2\lambda}}}e^{-\frac{x'^2}{2}}dx'-se^{\lambda(\eta-1)^2}\int_{\frac{\ln(\frac{B}{s})+2\lambda(\eta-1)}{\sqrt{2\lambda}}}^{\frac{\ln(\frac{K}{s})+2\lambda(\eta-1)}{\sqrt{2\lambda}}}e^{-\frac{x''^2}{2}}dx''\right)\\
=\;&Ke^{\delta+\lambda\eta^2}\left[\Phi\left(\frac{\ln(\frac{K}{s})+2\lambda\eta}{\sqrt{2\lambda}}\right)-\Phi\left(\frac{\ln(\frac{B}{s})+2\lambda\eta}{\sqrt{2\lambda}}\right)\right]\\
&-se^{\delta+\lambda(\eta-1)^2}\left[\Phi\left(\frac{\ln(\frac{K}{s})+2\lambda(\eta-1)}{\sqrt{2\lambda}}\right)-\Phi\left(\frac{\ln(\frac{B}{s})+2\lambda(\eta-1)}{\sqrt{2\lambda}}\right)\right].
\end{aligned}$$

Now, if we plug into δ, η and λ into the above formula, we derive

$$\begin{aligned}
&\int_{\ln B}^{\ln K}(K-e^{v})s^{\eta}e^{\delta}e^{-\eta v}\left(\frac{1}{2\sqrt{\lambda\pi}}e^{-\frac{1}{4\lambda}(\ln s-v)^2}\right)dv\\
&=Ke^{-r(T-t)}\left[\Phi\left(-\Delta_-\left(\frac{s}{K}\right)\right)-\Phi\left(-\Delta_-\left(\frac{s}{B}\right)\right)\right]\\
&-s\left[\Phi\left(-\Delta_+\left(\frac{s}{K}\right)\right)-\Phi\left(-\Delta_+\left(\frac{s}{B}\right)\right)\right].
\end{aligned}$$

Similarly, we can evaluate the second integral

$$\int_{\frac{B^2}{K}}^{B}\left(\frac{B}{u}\right)^{k_1-1}\left(K-\frac{B^2}{u}\right)e^{\delta}\left(\frac{s}{u}\right)^{\eta}\left(\frac{1}{2\sqrt{\lambda\pi}}e^{-\frac{1}{4\lambda}(\ln(\frac{s}{u}))^2}\right)\frac{du}{u}$$

to obtain

$$\begin{aligned}
&Ke^{-r(T-t)}\left(\frac{B}{s}\right)^{k_1-1}\left[\Phi\left(\Delta_-\left(\frac{B}{s}\right)\right)-\Phi\left(\Delta_-\left(\frac{B^2}{sK}\right)\right)\right]\\
&-B\left(\frac{B}{s}\right)^{k_1}\left[\Phi\left(\Delta_+\left(\frac{B}{s}\right)\right)-\Phi\left(\Delta_+\left(\frac{B^2}{sK}\right)\right)\right].
\end{aligned}$$

3.3.2 P_1 Term for Down-and-out Put Options

For down-and-out put options, the boundary conditions for P_1 are given as follows:

$$\begin{cases} P_1(T,s) &= 0, \quad for \;\; s \geq B, \\ P_1(t,B) &= 0, \quad for \;\; 0 < t < T. \end{cases}$$

We again follow the method in Buchen (2001) and extend the boundary conditions $P_1(T,s) = 0$, for $s \geq B$ as $P_1(T,s) = 0$ for all $s \geq 0$.

Next, we apply Mellin transform to Eq. (3.13) to get

$$\frac{d\hat{P}_1}{dt} + \left(\frac{1}{2}\langle f^2\rangle\left(w^2+w\right) - rw - r\right)\hat{P}_1 = \left(-c_1 w\left(w+1\right)\left(w+2\right) + c_2 w\left(w+1\right)\right)\hat{P}_0.$$

Solving this equation, we obtain

$$\hat{P}_1\left(t,w\right) = \left[c_1\left(T-t\right)w^3 - \left(c_2-3c_1\right)\left(T-t\right)w^2 - \left(c_2-2c_1\right)\left(T-t\right)w\right]\hat{P}_0\left(t,w\right).$$

Finally, applying inverse Mellin transform, we obtain an explicit closed-form expression of P_1 as follows

$$\begin{aligned} P_1\left(t,s\right) &= \mathcal{M}^{-1}\left(\hat{P}_1\left(t,w\right)\right) \\ &= c_1\left(T-t\right)\left(-s\frac{d}{ds}P_0\left(t,s\right) - 3s^2\frac{d^2}{ds^2}P_0\left(t,s\right) - s^3\frac{d^3}{ds^3}P_0\left(t,s\right)\right) \\ &\quad -\left(c_2-3c_1\right)\left(T-t\right)\left(s\frac{d}{ds}P_0\left(t,s\right) + s^2\frac{d^2}{ds^2}P_0\left(t,s\right)\right) \\ &\quad -\left(c_2-2c_1\right)\left(T-t\right)\left(-s\frac{d}{ds}P_0\left(t,s\right)\right), \end{aligned} \tag{3.21}$$

where P_0 is given in the previous section, c_1 and c_2 are given in Eq. (3.14).

We summarize the above analysis and calculation on down-and-out put options in the following theorem.

Theorem 3.3.1 *Under the SV model governed by Eq.* (3.1)*, the risk-neutral value* P *of a down-and-out put option can be approximated by the following formula*

$$P \approx P_0 + \sqrt{\varepsilon} P_1, \tag{3.22}$$

where P_0 *and* P_1 *are given by Eq.* (3.20) *and Eq.* (3.21)*, respectively.*

3.4 Analytic Approximate Solutions for Lookback Option Prices

3.4.1 P_0 Term for Lookback Put Options

For lookback floating strike put options, the boundary conditions of P_0 are

$$\begin{cases} \dfrac{\partial P_0}{\partial z}(t,z,z) & = & 0, \\ \dfrac{\partial P_0}{\partial z}(T,s,z) & = & 1, \quad for \;\; 0 < s < z. \end{cases}$$

Similar to the case of down-and-out put options, we extend the second boundary condition to $0 < s < \infty$ as follows:

$$\frac{\partial P_0}{\partial z}(T,s,z) := \mathbb{1}_{s<z} - \left(\frac{z}{s}\right)^{k_1-1} \cdot \mathbb{1}_{z<s}, \quad \text{for} \quad 0 < s < \infty.$$

Then, by integrating each side of the last equation, we can obtain

$$P_0(T,s,z) = \int_s^z -\left(\frac{\xi}{s}\right)^{k_1-1} d\xi = -\frac{1}{k_1}\left(\frac{z}{s}\right)^{k_1} s + \frac{1}{k_1} s \tag{3.23}$$

for $s > z$. For convenience, we let $u = s/z$ and $Q_0 = P_0/z$. With these notations, Eq. (3.9) becomes

$$\frac{\partial Q_0}{\partial t} + \frac{1}{2}u^2\langle f^2\rangle\frac{\partial^2 Q_0}{\partial u^2} + ru\frac{\partial Q_0}{\partial u} - rQ_0 = 0, \tag{3.24}$$

with boundary conditions

$$Q_0\left(T,u\right) = -\frac{1}{k_1}u^{1-k_1} + \frac{1}{k_1}u, \quad \text{for} \quad u > 1, \tag{3.25}$$

and $Q_0(T,u) = 1 - u$, for $0 < u < 1$.

Note that except the boundary conditions, Eq. (3.24) is identical to Eq. (3.9). Applying Mellin transform in the same way as that for the case of down-and-out put options, we can derive the solution to Eq. (3.24) as follows:

$$Q_0(t,u) = \hat{\theta}(w) * \mathcal{M}^{-1}e^{\lambda(w+\eta)^2+\delta}.$$

Again, applying Table 2.1 and P_0 given in Eq. (3.16), we have

$$\begin{aligned} Q_0(t,u) &= Q_0(T,u) * e^{\delta}z^{\eta}\left(\frac{1}{2\sqrt{\pi}}\lambda^{-\frac{1}{2}}e^{-\frac{1}{4\lambda}(\ln z)^2}\right) \\ &= \int_0^1 (1-\xi)\,e^{\delta}\left(\frac{u}{\xi}\right)^{\eta}\left(\frac{1}{2\sqrt{\pi}}\lambda^{-\frac{1}{2}}e^{-\frac{1}{4\lambda}\left(\ln\left(\frac{u}{\xi}\right)\right)^2}\right)\frac{d\xi}{\xi} + \\ &\quad \int_1^{\infty}\left(\frac{-1}{k_1}\xi^{1-k_1} + \frac{\xi}{k_1}\right)e^{\delta}\left(\frac{u}{\xi}\right)^{\eta}\left(\frac{1}{2\sqrt{\pi}}\lambda^{-\frac{1}{2}}e^{-\frac{1}{4\lambda}\left(\ln\left(\frac{u}{\xi}\right)\right)^2}\right)\frac{d\xi}{\xi}. \end{aligned} \tag{3.26}$$

After calculating integrals, for floating strike lookback put options, we derive a closed-form expression of the P_0 term as follows:

$$
\begin{aligned}
P_0(t,s,z) &= ze^{-r(T-t)}\Phi\left(-\Delta_-\left(\frac{s}{z}\right)\right) - s\Phi\left(-\Delta_+\left(\frac{s}{z}\right)\right) \\
&\quad -\frac{z}{k_1}\left(\frac{s}{z}\right)^{1-k_1} e^{-r(T-t)}\Phi\left(-\Delta_-\left(\frac{z}{s}\right)\right) + \frac{s}{k_1}\Phi\left(\Delta_+\left(\frac{s}{z}\right)\right),
\end{aligned} \tag{3.27}
$$

where $\Phi(\cdot)$ is the CDF of the standard normal distribution. Note that P_0 given in Eq. (3.27) is precisely the same as the price of a floating strike put option given in the literature, e.g., Hull (2015) or Haug (2006) , if we let $\sigma^2 := \langle f^2 \rangle$.

From Eq. (3.26), we have

$$
\begin{aligned}
Q_0(t,u) &= \int_0^1 (1-\xi)\, e^{\delta}\left(\frac{u}{\xi}\right)^{\eta}\left(\frac{1}{2\sqrt{\lambda\pi}}e^{-\frac{1}{4\lambda}\left(\ln\left(\frac{u}{\xi}\right)\right)^2}\right)\frac{d\xi}{\xi} + \\
&\quad \int_1^{\infty}\left(-\frac{1}{k_1}\xi^{1-k_1} + \frac{\xi}{k_1}\right)e^{\delta}\left(\frac{u}{\xi}\right)^{\eta}\left(\frac{1}{2\sqrt{\lambda\pi}}e^{-\frac{1}{4\lambda}\left(\ln\left(\frac{u}{\xi}\right)\right)^2}\right)\frac{d\xi}{\xi}.
\end{aligned}
$$

We let $v = \ln\xi$. For the first integral, we have

$$
\begin{aligned}
&\int_0^1 (1-\xi)e^{\delta}\left(\frac{u}{\xi}\right)^{\eta}\left(\frac{1}{2\sqrt{\lambda\pi}}e^{-\frac{1}{4\lambda}\left(\ln\left(\frac{u}{\xi}\right)\right)^2}\right)\frac{du}{u} \\
&= \int_{-\infty}^{0} u^{\eta}\left(1-e^{v}\right)e^{\delta-v\eta}\left(\frac{1}{2\sqrt{\lambda\pi}}e^{-\frac{1}{4\lambda}(\ln u - v)^2}\right)dv \\
&= \frac{u^{\eta}e^{\delta}}{2\sqrt{\lambda\pi}}\left(\int_{-\infty}^{0} e^{-\frac{1}{4\lambda}\left(v^2-2v\ln u+(\ln u)^2+4\lambda\eta v\right)}dv\right. \\
&\quad \left. -\int_{-\infty}^{0} e^{-\frac{1}{4\lambda}\left(v^2-2v\ln u+(\ln u)^2+4\lambda(\eta-1)v\right)}dv\right) \\
&= \frac{u^{\eta}e^{\delta}}{2\sqrt{\lambda\pi}}\left(\int_{-\infty}^{0} e^{-\frac{1}{4\lambda}(v-\ln u+2\lambda\eta)^2+\lambda\eta^2-\eta\ln u}dv\right. \\
&\quad \left. -\int_{-\infty}^{0} e^{-\frac{1}{4\lambda}(v-\ln u+2\lambda(\eta-1))^2+\lambda(\eta-1)^2-(\eta-1)\ln u}dv\right).
\end{aligned}
$$

Next, we let

$$v' := \frac{v - \ln u + 2\lambda\eta}{\sqrt{2\lambda}} \quad \text{and} \quad v'' := \frac{v - \ln u + 2\lambda(\eta - 1)}{\sqrt{2\lambda}}.$$

Then, we have

$$\begin{aligned}
&\int_0^1 (1-\xi)e^{\delta}\left(\frac{u}{\xi}\right)^{\eta}\left(\frac{1}{2\sqrt{\lambda\pi}}e^{-\frac{1}{4\lambda}\left(\ln\left(\frac{u}{\xi}\right)\right)^2}\right)\frac{du}{u} \\
=\ & \frac{e^{\delta}}{\sqrt{2\pi}}\left(\int_{-\infty}^{\frac{-\ln u+2\lambda\eta}{\sqrt{2\lambda}}} e^{-\frac{v'^2}{2}+\lambda\eta^2}dv' - u\int_{-\infty}^{\frac{-\ln u+2\lambda(\eta-1)}{\sqrt{2\lambda}}} e^{-\frac{v''^2}{2}+\lambda(\eta-1)^2}dv''\right) \\
=\ & e^{\delta+\lambda\eta^2}\Phi\left(\frac{-\ln u+2\lambda\eta}{\sqrt{2\lambda}}\right) - ue^{\delta+\lambda(\eta-1)^2}\Phi\left(\frac{-\ln u+2\lambda(\eta-1)}{\sqrt{2\lambda}}\right) \\
=\ & e^{-r(T-t)}\Phi\left(-\Delta_-\left(\frac{s}{z}\right)\right) - \left(\frac{s}{z}\right)\Phi\left(-\Delta_+\left(\frac{s}{z}\right)\right).
\end{aligned}$$

For the second integral, we have

$$\begin{aligned}
&\int_1^{\infty}\left(-\frac{1}{k_1}\xi^{1-k_1}+\frac{\xi}{k_1}\right)e^{\delta}\left(\frac{u}{\xi}\right)^{\eta}\left(\frac{1}{2\sqrt{\lambda\pi}}e^{-\frac{1}{4\lambda}\left(\ln\left(\frac{u}{\xi}\right)\right)^2}\right)\frac{d\xi}{\xi} \\
=\ & \int_0^{\infty}\left(-\frac{1}{k_1}e^{(1-k_1)v}+\frac{1}{k_1}e^{v}\right)e^{\delta}u^{\eta}e^{-v\eta}\left(\frac{1}{2\sqrt{\lambda\pi}}e^{-\frac{1}{4\lambda}(\ln u - v)^2}\right)dv \\
=\ & \frac{e^{\delta}u^{\eta}}{2k_1\sqrt{\lambda\pi}}\int_0^{\infty}\left(-e^{\eta v-\frac{1}{4\lambda}(\ln u-v)^2}+e^{v(1-\eta)-\frac{1}{4\lambda}(\ln u-v)^2}\right)dv \\
=\ & \frac{e^{\delta}u^{\eta}}{2k_1\sqrt{\lambda\pi}}\left(\int_0^{\infty}-e^{-\frac{1}{4\lambda}(v-\ln u-2\lambda\eta)^2+\lambda\eta^2+\eta\ln u}dv\right. \\
&\left.+\int_0^{\infty}e^{-\frac{1}{4\lambda}(v-\ln u-2\lambda(1-\eta))^2+\lambda(1-\eta)^2+(1-\eta)\ln u}dv\right),
\end{aligned}$$

where we use the fact that $k_1 - 1 + \eta = -\eta$. Further, we introduce a new variable

$$v''' := \frac{v - \ln u - 2\lambda\eta}{\sqrt{2\lambda}}.$$

Then, we have

$$\begin{aligned}
&\int_1^\infty \left(-\frac{1}{k_1}\xi^{1-k_1} + \frac{\xi}{k_1}\right) e^{\delta}\left(\frac{u}{\xi}\right)^{\eta}\left(\frac{1}{2\sqrt{\lambda\pi}}e^{-\frac{1}{4\lambda}\left(\ln\left(\frac{u}{\xi}\right)\right)^2}\right)\frac{d\xi}{\xi} \\
=\;& \frac{e^{\delta}u^{\eta}}{k_1\sqrt{2\pi}}\left(\int_{\frac{-\ln u-2\lambda\eta}{\sqrt{2\lambda}}}^{\infty} -e^{-\frac{v'''^2}{2}}e^{\lambda\eta^2+\eta\ln u}dv'''\right. \\
&\left. + \int_{\frac{-\ln u+2\lambda(\eta-1)}{\sqrt{2\lambda}}}^{\infty} e^{-\frac{v''^2}{2}}e^{\lambda(\eta-1)^2+(1-\eta)\ln u}dv''\right) \\
=\;& -\frac{1}{k_1}e^{\delta+\lambda\eta^2}u^{1-k_1}\Phi\left(\frac{\ln u + 2\lambda\eta}{\sqrt{2\lambda}}\right) + \frac{1}{k_1}ue^{\delta+\lambda(\eta-1)^2}\Phi\left(\frac{\ln u + 2\lambda\left(1-\eta\right)}{\sqrt{2\lambda}}\right) \\
=\;& -\frac{1}{k_1}\left(\frac{s}{z}\right)^{1-k_1}e^{-r(T-t)}\Phi\left(-\Delta_-\left(\frac{z}{s}\right)\right) + \frac{1}{k_1}\left(\frac{s}{z}\right)\Phi\left(\Delta_+\left(\frac{s}{z}\right)\right).
\end{aligned}$$

3.4.2 P_1 Term for Lookback Put Options

For lookback floating strike put options, the boundary conditions for P_1 are given as follows:

$$\begin{cases}
P_1(T,s,z) = 0, & for\ \ 0 < s < z, \\
\dfrac{\partial P_1}{\partial z}(t,z,z) = 0, & for\ \ 0 < t < T \quad and\ z > 0.
\end{cases}$$

Just like that for the P_0-term for floating strike lookback put options, we let $u = s/z$ and $Q_1 = P_1/z$. With these notation changes, Eq. (3.13) is converted to the following

$$\frac{\partial Q_1}{\partial t} + \frac{1}{2}\langle f^2\rangle u^2\frac{\partial^2 Q_1}{\partial u^2} + ru\frac{\partial Q_1}{\partial u} - rQ_1 = c_1u^3\frac{\partial^3 Q_0}{\partial u^3} + c_2u^2\frac{\partial^2 Q_0}{\partial u^2} \tag{3.28}$$

with $Q_1(T,u) = 0$ for $0 < u < 1$.

Note that Eq. (3.28) is essentially the same as Eq. (3.13), except the notational difference. So, we have

$$\begin{aligned}Q_1(t,u) &= c_1(T-t)\left(-u\frac{d}{du}Q_0(t,u) - 3u^2\frac{d^2}{du^2}Q_0(t,u) - u^3\frac{d^3}{du^3}Q_0(t,u)\right)\\ &\quad -(c_2-3c_1)(T-t)\left(u\frac{d}{du}Q_0(t,u) + u^2\frac{d^2}{dz^2}Q_0(t,u)\right)\\ &\quad -(c_2-2c_1)(T-t)\left(-u\frac{d}{du}Q_0(t,u)\right),\end{aligned} \tag{3.29}$$

where Q_0 is given previously. Consequently, we have

$$\begin{aligned}P_1(t,s,z) &= c_1(T-t)\left(-s\frac{d}{ds}P_0(t,s,z) - 3s^2\frac{d^2}{ds^2}P_0(t,s,z) - s^3\frac{d^3}{ds^3}P_0(t,s,z)\right)\\ &\quad -(c_2-3c_1)(T-t)\left(s\frac{d}{ds}P_0(t,s,z) + s^2\frac{d^2}{ds^2}P_0(t,s,z)\right)\\ &\quad -(c_2-2c_1)(T-t)\left(-s\frac{d}{ds}P_0(t,s,z)\right),\end{aligned} \tag{3.30}$$

where c_1 and c_2 are the same as those defined previously.

We summarize the above analysis and calculation on floating strike lookback put options in the following theorem.

Theorem 3.4.1 *Under the SV model governed by Eq.* (3.1)*, the risk-neutral value* P *of a floating strike lookback put option can be approximated by the following formula*

$$P \approx P_0 + \sqrt{\varepsilon}P_1, \tag{3.31}$$

where P_0 *and* P_1 *are given by Eq.* (3.27) *and Eq.* (3.30)*, respectively.*

3.5 Numerical Analysis

First of all, as done by Fouque et al. (2000), Fouque et al. (2011) and Cao et al. (2021), we choose f to take the following form

$$f(y) = 0.35\left(\tan^{-1}(y) + \frac{\pi}{2}\right)/\pi + 0.05.$$

Secondly, the values of other parameters used in this section are given in Table 3.1, whenever they are required to be fixed.

Table 3.1: The role and numerical value of parameters.

Parameter	Role	Value
r	risk-free interest rate	0.035
B	barrier level	1500
K	put option strike price	2700
c_1	as defined in Section 4.2	-0.004
c_2	as defined in Section 4.2	-0.018

Here, we do not choose precise values of β and ρ, and particular forms of $\xi(y)$ (in Section 3.1) and $\phi(y)$ (in Section 3.2) to calculate the above values of c_1 and c_2. Instead, c_1 and c_2 are calibrated from the term structure of the implied volatility surface as described in the book of Fouque et al. (2000). Specifically, the implied volatility I^ε of a European vallina call option with fast mean-reverting stochastic process can be approximated by the following formula

$$I^\varepsilon = a\frac{\ln(\frac{K}{s})}{T-t} + b + o(\sqrt{\varepsilon})$$

with

$$a = -\frac{c_1}{\langle f^2\rangle^{3/2}} \quad \text{and} \quad b = \sqrt{\langle f^2\rangle} + \frac{c_1}{\langle f^2\rangle^{3/2}}\left(r + \frac{3}{2}\langle f^2\rangle\right) - \frac{c_2}{\sqrt{\langle f^2\rangle}}.$$

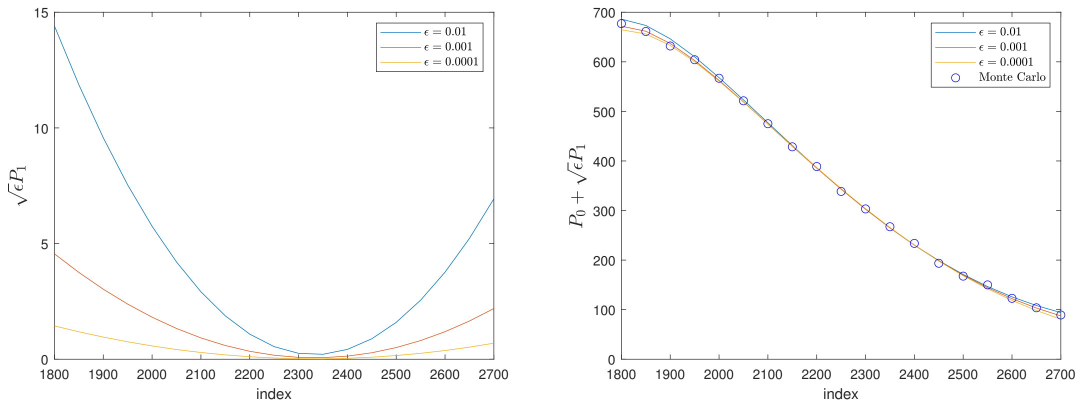
The parameters a and b are estimated as the slope and intercept of the regression fit of the observed implied volatilities as a linear function of logmoneyness-to-maturity-ratio

$\ln(K/s)/(T-t)$. From the calibrated values a and b on the observed implied volatility surface, the parameters c_1 and c_2 are obtained as

$$c_1 = -a\sigma\langle f^2\rangle^{3/2} \quad \text{and} \quad c_2 = \sqrt{\langle f^2\rangle}((\sqrt{\langle f^2\rangle} - b) - a(r + \frac{3}{2}\langle f^2\rangle)).$$

Thirdly, note that when $t = 0$, $s = z$. Hence, in this case, the formula for P_0 given by Eq. (3.27) is simplified.

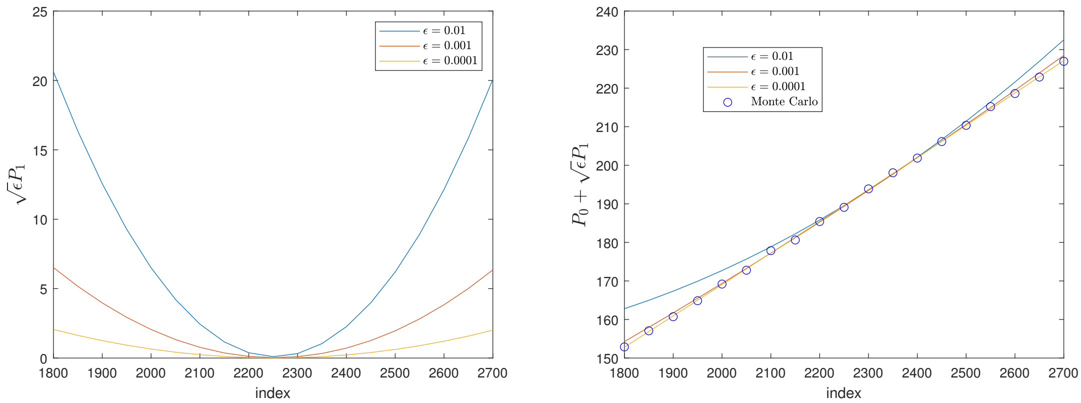
Figure (3.1A) shows how the $\sqrt{\varepsilon}P_1$-term for a down-and-out put option changes with respect to a variation of ε values. As we can see, for fixed ε, when s increases, P_1 decreases first, and then increases after it hits its trough. When ε gets smaller (equivalently, the mean-reverting speed gets larger), $\sqrt{\varepsilon}P_1$ approaches to a zero. Figure (3.1B) shows how the value of $P_0 + \sqrt{\varepsilon}P_1$ for a down-and-out put option varies with respect to the change of ε values. As we can see, when the value of ε changes from 0.01 to 0.0001, the value of $P_0 + \sqrt{\varepsilon}P_1$ does not vary much. In fact, the values of $P_0 + \sqrt{\varepsilon}P_1$ match well with the result of Monte-Carlo simulation in all cases. Furthermore, in all cases, the value of $P_0 + \sqrt{\varepsilon}P_1$ declines as s increases.



(A) (B)

Figure 3.1: Plots of $\sqrt{\varepsilon}P_1$ and $P_0 + \sqrt{\varepsilon}P_1$ against different values of ε for down-and-out put options.

Figure (3.2A) shows how the $\sqrt{\varepsilon}P_1$-term for a floating strike put changes with respect to a variation of ε values. In a similar pattern, for a fixed ε-value, when s increases, P_1 decreases first and then increases after it hits its trough. Similar to the case of down-and-out put options, when ε gets smaller (equivalently, the mean-reverting speed gets larger), $\sqrt{\varepsilon}P_1$ approaches to zero. Figure (3.2B) shows how the value of $P_0 + \sqrt{\varepsilon}P_1$ for a floating strike put varies with respect to the change of ε values. When the value of ε changes from 0.01 to 0.001, the value of $P_0 + \sqrt{\varepsilon}P_1$ varies. But, when the value of ε changes from 0.001 to 0.0001, the value of $P_0 + \sqrt{\varepsilon}P_1$ does not vary much. The values of $P_0 + \sqrt{\varepsilon}P_1$ match well with the result of Monte-Carlo simulation when ε = 0.001 or 0.0001. Furthermore, in all cases, the value of $P_0 + \sqrt{\varepsilon}P_1$ increases as s increases.



(A) (B)

Figure 3.2: Plots of $\sqrt{\varepsilon}P_1$ and $P_0 + \sqrt{\varepsilon}P_1$ against different values of ε for floating strike put options.

3.6 Conclusion

In this chapter, we establishes explicit closed-form solutions for first order approximations of down-and-out barrier and floating strike lookback put option prices under a stochastic volatility model by means of Mellin transform. The zero-order terms in the solutions for the prices of both types of put options coincide with those in Hull (2015) or Haug (2006) under the classical Back-Scholes model. Our numerical analysis shows that the results given by those explicit closed-form solutions match well with those generated by Monte-Carlo simulation. This confirms the accuracy of the approximation. Furthermore, we also discussed the sensitivity of the first-order error terms and the approximation with respect to the underlying asset price and the mean-reverting speed of the OU-process which governs the volatility.

Chapter 4

Pricing Path-dependent Options under Hybrid Constant Elasticity of Variance and Stochastic Volatility Model

In this chapter, we evaluate the price of a down-and-out put option and a floating strike lookback option when the underlying asset is driven by a hybrid model with constant elasticity of variance and stochastic volatility (SVCEV). Usually, it is difficult to get closed-form solutions for those exotic options under stochastic volatility models. Here, we use an asymptotic expansion approach and the Mellin transform method to obtain explicit closed-form formulae for the zero-order and first-order correction terms. In addition, we perform a sensitivity analysis numerically on the asymptotic terms and compare the option prices corresponding to the Black-Scholes, CEV and SVCEV models with those calculated by Monte-Carlo simulations and the binomial tree method to illustrate the accuracy of our pricing formulae.

4.1 SVCEV Model

Let $\{S_t\}_{t\geq 0}$ represent the price process of a risky asset on some filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$, where $\mathbb{P}$ is the physical probability measure. In this paper, we assume that $\{S_t\}_{t\geq 0}$ follows the system of stochastic differential equations:

$$\begin{aligned} dS_t &= \mu S_t dt + f(Y_t) S_t^{1+\beta} dW_t^s, \\ dY_t &= \alpha (m - Y_t) dt + \gamma \left(\rho dW_t^s + \sqrt{1-\rho^2} dW_t^y \right), \end{aligned} \tag{4.1}$$

where μ, $\alpha > 0$, $\gamma > 0$, m and β are constants, f is a function specifying the volatility dependence on the unobserved process $\{Y_t\}_{t\geq 0}$. The processes $\{W_t^s\}_{t\geq 0}$ and $\{W_t^y\}_{t\geq 0}$ are independent standard Brownian motions under the physical probability measure. The leverage effect between the equity process $\{S_t\}_{t\geq 0}$ and the unobserved process $\{Y_t\}_{t\geq 0}$ is captured by the correlation coefficient ρ with $-1 < \rho < 1$. The drift rate of the equity process is denoted by μ. The mean-reversion process $\{Y_t\}_{t\geq 0}$ given in Eq. (4.1) is characterized by its typical time to revert back to the long-term mean m. The parameter α measures the mean-reverting speed and γ represents the volatility of the hidden process $\{Y_t\}_{t\geq 0}$. The parameter β measures the elasticity of variance of the risky asset, which is small from empirical results as in Kim et al. (2015). The above system is referred to as the stochastic volatility with constant elasticity of variance (abbreviated as SVCEV) model in the paper. We assume that f is bounded and smooth enough. For numerical calculation, we choose f to take a specific form as used in Fouque et al. (2000), Fouque et al. (2011) and Cao et al. (2021).

We apply the well-known Girsanov theorem 2.2.1 to change the physical measure $\mathbb{P}$ to a risk-neutral martingale measure $\mathbb{Q}$ by letting

$$dW_t^{s*} = \frac{\mu - r}{f(Y_t) S_t^{\beta}} dt + dW_t^s \quad \text{and} \quad dW_t^{y*} = \xi(Y_t) dt + dW_t^y,$$

where $\xi(Y_t)$ represents the premium of volatility risk. Then the model equations under the measure $\mathbb{Q}$ can be written as

$$\begin{aligned} dS_t &= rS_t dt + f(Y_t) S_t^{1+\beta} dW_t^{s*}, \\ dY_t &= \left[\alpha(m - Y_t) - \gamma\left(\rho\frac{\mu - r}{f(Y_t)S_t^{\beta}} + \xi(Y_t)\sqrt{1-\rho^2}\right)\right]dt \\ &\quad + \gamma\left(\rho dW_t^{s*} + \sqrt{1-\rho^2}\, dW_t^{y*}\right). \end{aligned} \tag{4.2}$$

Note that the standard Brownian motions $\{W_t^{s*}\}_{t\geq 0}$ and $\{W_t^{y*}\}_{t\geq 0}$ are uncorrelated under the risk-neutral measure $\mathbb{Q}$. The Ornstein-Uhlenbeck (OU) process $\{Y_t\}_{t\geq 0}$ has an invariant distribution, which is a normal distribution with mean m and variance $\gamma^2/2\alpha$. Thus, when the elasticity of variance is very small and the mean reverting speed is very large, i.e., β goes to zero and α goes to infinity, the process $\{S_t\}_{t\geq 0}$ should follow the geometric Brownian motion process. This means that if mean reversion is extremely fast and the elasticity of variance is extremely small, then the Black-Scholes model would become a good approximation. In reality, however, it may not be the case. For fast but not extremely fast mean-reversion and small but not extremely small elasticity of variance, the Black-Scholes model needs to be corrected to account for the stochastic feature of the volatility process of a risky asset. For this purpose, in addition to the small parameter β we introduce another small parameter ε defined by $\varepsilon = 1/\alpha$ as done by Fouque et al. (2000). For notational convenience, we put $\nu = \gamma/\sqrt{2\alpha}$. With the help of these notations, the model equations under $\mathbb{Q}$ can be re-written as

$$\begin{aligned} dS_t &= rS_t dt + f(Y_t) S_t^{1+\beta} dW_t^{s*}, \\ dY_t &= \left[\frac{1}{\varepsilon}(m - Y_t) - \frac{\sqrt{2}\nu}{\sqrt{\varepsilon}}\Lambda(S_t, Y_t)\right]dt + \frac{\sqrt{2}\nu}{\sqrt{\varepsilon}} dW_t^{y*}, \end{aligned}$$

where $\Lambda(\cdot)$ is defined as

$$\Lambda(s,y) := \rho\frac{\mu - r}{f(y)s^{\beta}} + \xi(y)\sqrt{1-\rho^2},$$

is the combined market price of risk.

Let V_T denote the payoff of a put option on the risky asset at its expiration T. Then its risk-neutral price at time $t \in [0,T]$ under our SVCEV model is given by

$$P^{\beta,\varepsilon}(t,s,y) = \mathbb{E}^{\mathbb{Q}}\left(e^{-r(T-t)}V(T) \middle| S_t = s, Y_t = y\right).$$

As discussed in 2.2.2, V_T varies depending on the type of options. The payoffs of the two types of path-dependent options considered in this chapter (down-and-out put options and floating strike lookback put options) are specified in Eqs. (2.29) and (2.30), respectively.

Applying Itô's lemma, we can obtain a partial differential equation (PDE) for $P^{\beta,\varepsilon}(t,s,y)$ as follows:

$$\begin{aligned} 0 \;=\; & \frac{\partial P^{\beta,\varepsilon}}{\partial t} + \frac{1}{2}s^{2(1+\beta)}f^2(y)\frac{\partial^2 P^{\beta,\varepsilon}}{\partial s^2} + r\left(s\frac{\partial P^{\beta,\varepsilon}}{\partial s} - P^{\beta,\varepsilon}\right) + \frac{\sqrt{2}\rho\nu s^{1+\beta}}{\sqrt{\varepsilon}}f(y)\frac{\partial^2 P^{\beta,\varepsilon}}{\partial s\partial y} \\ & + \frac{\nu^2}{\varepsilon}\frac{\partial^2 P^{\beta,\varepsilon}}{\partial y^2} + \left(\frac{1}{\varepsilon}(m-y) - \frac{\sqrt{2}\nu}{\sqrt{\varepsilon}}\Lambda(y)\right)\frac{\partial P^{\beta,\varepsilon}}{\partial y}. \end{aligned} \tag{4.3}$$

The boundary conditions for Eq. (4.3) vary depending on the type of options. For example, the boundary conditions for Eq. (4.3) when $V_T = DOP(T)$ are

$$\begin{cases} P^{\beta,\varepsilon}(T,s,y) = \max\{K - s, 0\}, & s > B, \\ P^{\beta,\varepsilon}(t,B,y) = 0, & 0 \le t \le T. \end{cases}$$

When $V_T = LP_{float}(T)$, the boundary conditions become the following

$$\begin{cases} \dfrac{\partial P}{\partial z}(t,z,y,z) = 0, & 0 \le t \le T, z > 0, \\ P^{\beta,\varepsilon}(T,s,y,z) = z - s, & 0 \le s \le z. \end{cases}$$

Note that in this case, $P^{\beta,\varepsilon}$ is a function of four variables t, s, y and z (here, $Z_t = z$).

4.2 Specified Asymptotic Expansions

We begin with re-organizing Eq. (4.3) in terms of varying orders of ε:

$$\left(\frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1^\beta + \mathcal{L}_2^\beta\right) P^{\beta,\varepsilon} = 0, \tag{4.4}$$

where the operators $\mathcal{L}_0$, $\mathcal{L}_1$ and $\mathcal{L}_2$ are defined by

$$\begin{aligned} \mathcal{L}_0 &:= (m-y)\frac{\partial}{\partial y} + \nu^2 \frac{\partial^2}{\partial y^2}, \\ \mathcal{L}_1^\beta &:= \sqrt{2}\rho\nu s^{1+\beta} f(y) \frac{\partial^2}{\partial s \partial y} - \sqrt{2}\nu\Lambda(s,y)\frac{\partial}{\partial y}, \text{ and} \\ \mathcal{L}_2^\beta &:= \frac{\partial}{\partial t} + \frac{1}{2}s^{2(1+\beta)} f^2(y) \frac{\partial^2}{\partial s^2} + r\left(s\frac{\partial}{\partial s} - \cdot\right). \end{aligned}$$

Note that the operators $\mathcal{L}_0$ and $\mathcal{L}_1^\beta$ have partial differentiation with respect to y.

For the β-dependence of $\mathcal{L}_1^\beta$ and $\mathcal{L}_2^\beta$, we apply a Taylor expansion of the term $s^{1+\beta}$ in the operators with respect to β to obtain

$$\begin{aligned} \mathcal{L}_1^\beta &= \mathcal{L}_{10} + \beta\mathcal{L}_{11} + \beta^2\mathcal{L}_{12} + O\left(\beta^3\right), \\ \mathcal{L}_2^\beta &= \mathcal{L}_{20} + \beta\mathcal{L}_{21} + \beta^2\mathcal{L}_{22} + O\left(\beta^3\right), \end{aligned} \tag{4.5}$$

where

$$\begin{aligned}
\mathcal{L}_{10} &:= \sqrt{2}\rho\nu s f(y) \frac{\partial^2}{\partial s \partial y} - \sqrt{2}\nu\Lambda(s,y)\frac{\partial}{\partial y}, \\
\mathcal{L}_{11} &:= \sqrt{2}\rho\nu s \ln s\, f(y) \frac{\partial^2}{\partial s \partial y}, \\
\mathcal{L}_{12} &:= \frac{\sqrt{2}}{2}\rho\nu s (\ln s)^2 f(y) \frac{\partial^2}{\partial s \partial y},
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{L}_{20} &:= \frac{\partial}{\partial t} + \frac{1}{2}s^2 f^2(y)\frac{\partial^2}{\partial s^2} + r\left(s\frac{\partial}{\partial s} - \cdot\right), \\
\mathcal{L}_{21} &:= s^2 \ln s\, f^2(y)\frac{\partial^2}{\partial s^2}, \\
\mathcal{L}_{22} &:= s^2 (\ln s)^2 f^2(y)\frac{\partial^2}{\partial s^2}.
\end{aligned}$$

In order to obtain an efficient approximate solution to $P^{\beta,\varepsilon}$, as that in Cao et al. (2021), we apply the following asymptotic expansion of $P^{\beta,\varepsilon}$:

$$P^{\beta,\varepsilon} = P_0^{\varepsilon} + \beta P_1^{\varepsilon} + \beta^2 P_2^{\varepsilon} + \cdots, \tag{4.6}$$

where P_0^{ε}, P_1^{ε}, ... are functions corresponding to varying orders of β. By substituting the expansion forms Eqs. (4.5) and (4.6) of $P^{\beta,\varepsilon}$ into Eq. (4.4) and re-organizing terms, we obtain

$$\begin{aligned} 0 &= \left(\frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1^{\beta} + \mathcal{L}_2^{\beta}\right)\left(P_0^{\varepsilon} + \beta P_1^{\varepsilon} + \beta^2 P_2^{\varepsilon} + ...\right) \\ &= \left(\frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_{10} + \mathcal{L}_{20}\right)P_0^{\varepsilon} \\ &\quad +\beta\left[\left(\frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_{10} + \mathcal{L}_{20}\right)P_1^{\varepsilon} + \left(\frac{1}{\sqrt{\varepsilon}}\mathcal{L}_{11} + \mathcal{L}_{21}\right)P_0^{\varepsilon}\right] \\ &\quad +\beta^2\left[(\frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_{10} + \mathcal{L}_{20})P_2^{\varepsilon} + \left(\frac{1}{\sqrt{\varepsilon}}\mathcal{L}_{11} + \mathcal{L}_{21}\right)P_1^{\varepsilon} + \left(\frac{1}{\sqrt{\varepsilon}}\mathcal{L}_{12} + \mathcal{L}_{22}\right)P_0^{\varepsilon}\right] \\ &\quad +O\left(\beta^3\right) \end{aligned} \tag{4.7}$$

4.3 Analytic Approximate Solutions for Barrier Option Prices

4.3.1 Zero-order Term

Firstly, from the leading order $O(1)$ term with respect to β in Eq. (4.7), we have the following PDE for the zero-order terms

$$\left(\frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_{10} + \mathcal{L}_{20}\right)P_0^{\varepsilon} = 0.$$

If we apply the asymptotic expansion $P_0^{\varepsilon} = P_{0,0} + \sqrt{\varepsilon}P_{0,1} + \varepsilon P_{0,2} + \sqrt{\varepsilon}P_{0,3}\cdots$ to this PDE, then $P_{0,k}$ satisfies the PDE

$$\mathcal{L}_0 P_{0,k} + \mathcal{L}_{10}P_{0,k-1} + \mathcal{L}_{20}P_{0,k-2} = 0,$$

for all $k = 0, 1, 2...$, where $P_{0,-2} = P_{0,-1} = 0$.

For the case $k = 0$, we have the equation $\mathcal{L}_0 P_{0,0} = 0$. As discussed in (Cao et al., 2021), if we assume that $P_{0,0}$ does not grow as fast as $e^{y^2/2}$, we can show that $P_{0,0}$ is independent of y.

For the case $k = 1$, we get the equation $\mathcal{L}_{10} P_{0,0} + \mathcal{L}_0 P_{0,1} = 0$. Since $P_{0,0}$ is independent of y, then $\mathcal{L}_{10} P_{0,0} = 0$. It follows that $\mathcal{L}_0 P_{0,1} = 0$. Again, if we assume that $P_{0,1}$ does not grow as fast as $e^{y^2/2}$, then we can deduce that $P_{0,1}$ is also independent of y.

For the case $k = 2$, we get

$$\mathcal{L}_0 P_{0,2} + \mathcal{L}_{10} P_{0,1} + \mathcal{L}_{20} P_{0,0} = 0.$$

Since $P_{0,1}$ is independent of y, we have $\mathcal{L}_{10} P_{0,1} = 0$ which implies that

$$\mathcal{L}_0 P_{0,2} + \mathcal{L}_{20} P_{0,0} = 0. \tag{4.8}$$

Regarding Eq. (4.8) as a Poisson equation for $P_{0,2}$ in y, in order for it to have a solution, it is required to satisfy the centring condition

$$\langle \mathcal{L}_{20} P_{0,0} \rangle = \langle \mathcal{L}_{20} \rangle P_{0,0} = 0, \tag{4.9}$$

which is equivalent to

$$\frac{\partial P_{0,0}}{\partial t} + rs \frac{\partial P_{0,0}}{\partial s} + \frac{1}{2} s^2 \langle f^2 \rangle \frac{\partial^2 P_{0,0}}{\partial s^2} - r P_{0,0} = 0. \tag{4.10}$$

This is an equation for us to determine $P_{0,0}$ term. Here, $\langle \cdot \rangle$ denotes the expectation with respect to the invariant distribution of the process $\{Y_t\}_{t \geq 0}$, i.e.,

$$\langle h \rangle = \int_{-\infty}^{+\infty} h(y) \Phi(y) dy,$$

where

$$\Phi(y) = \frac{1}{\sqrt{2\pi\nu^2}} e^{-\frac{(y-m)^2}{2\nu^2}}.$$

Since $\langle f^2 \rangle$ is a constant, Eq. (4.10) is the classical Black-Scholes PDE with constant volatility $\sqrt{\langle f^2 \rangle}$. Thus, for small ε, $P_{0,0}$ represents the European put option price under the Black-Scholes model.

Mellin transform could be used to derive an analytical expression of the $P_{0,0}$ term for a down-and-out put option. Noting that $P_{0,0}$ is independent of y under our assumption, we first follow the method in Buchen (2001) and transform the boundary condition,

$$P_{0,0}(T, s, y) = \max\{K - s, 0\}, \quad \text{for} \quad s > B,$$

to set up the boundary condition of $P_{0,0}$ for $s \geq 0$ as follows:

$$P_{0,0}(T, s) := (K - s)\, \mathbb{1}_{B<s<K} - \left(\frac{B}{s}\right)^{k_1 - 1} \left(K - \frac{B^2}{s}\right) \mathbb{1}_{\frac{B^2}{K}<s<B}, \tag{4.11}$$

where $k_1 = 2r/\langle f^2 \rangle$. Now, we apply Mellin transform in Table 2.1 to Eq. (4.10) to convert this PDE into the following ODE:

$$\frac{d\hat{P}_{0,0}}{dt} + \left(\frac{1}{2}\langle f^2 \rangle (w^2 + w) - rw - r\right) \hat{P}_{0,0} = 0. \tag{4.12}$$

The solution to Eq. (4.12) is given by

$$\hat{P}_{0,0}(t, w) = \hat{\theta}(w) e^{\frac{1}{2}\langle f^2 \rangle \left(w^2 + (1-k_1)w - k_1\right)(T-t)}, \tag{4.13}$$

where $\hat{\theta}$ is a function of w, determined by the boundary condition (4.11).

Next, we take inverse Mellin transform of Eq. (4.13) in Table 2.1 and obtain the expression

$$P_{0,0}(t,s) = P_{0,0}(T,s) * \mathcal{M}^{-1} e^{\lambda(w+\eta)^2+\delta},$$

where

$$\lambda = \frac{1}{2}\langle f^2\rangle (T-t), \eta = \frac{1-k_1}{2}, \delta = -\lambda\eta^2 - r(T-t) \tag{4.14}$$

and the operation $*$ means the convolution.

Using the boundary condition given in Eq. (4.11), we can deduce that

$$\begin{aligned} P_{0,0}(t,s) &= P_{0,0}(T,s) * \left(\frac{e^{\delta}s^{\eta}}{2\sqrt{\lambda\pi}} e^{-\frac{1}{4\lambda}(\ln s)^2}\right) \\ &= \int_B^K (K-u) e^{\delta} \left(\frac{s}{u}\right)^{\eta} \left(\frac{1}{2\sqrt{\lambda\pi}} e^{-\frac{1}{4\lambda}\left(\ln\left(\frac{s}{u}\right)\right)^2}\right) \frac{du}{u} - \\ & \int_{\frac{B^2}{K}}^{B} \left(\frac{B}{u}\right)^{k_1-1} \left(K - \frac{B^2}{u}\right) e^{\delta} \left(\frac{s}{u}\right)^{\eta} \left(\frac{1}{2\sqrt{\lambda\pi}} e^{-\frac{1}{4\lambda}\left(\ln\left(\frac{s}{u}\right)\right)^2}\right) \frac{du}{u}. \end{aligned} \tag{4.15}$$

The closed-form expression of the $P_{0,0}$ term of a down-and-out put option is obtained as:

$$\begin{aligned} P_{0,0}(t,s) &= Ke^{-r(T-t)} \left(\Phi\left(-\Delta_-\left(\frac{s}{K}\right)\right) - \Phi\left(-\Delta_-\left(\frac{s}{B}\right)\right)\right) - \\ & s\left(\Phi\left(-\Delta_+\left(\frac{s}{K}\right)\right) - \Phi\left(-\Delta_+\left(\frac{s}{B}\right)\right)\right) - \\ & Ke^{-r(T-t)} \left(\frac{B}{s}\right)^{k_1-1} \left[\Phi\left(\Delta_-\left(\frac{B}{s}\right)\right) - \Phi\left(\Delta_-\left(\frac{B^2}{sK}\right)\right)\right] + \\ & B\left(\frac{B}{s}\right)^{k_1} \left[\Phi\left(\Delta_+\left(\frac{B}{s}\right)\right) - \Phi\left(\Delta_+\left(\frac{B^2}{sK}\right)\right)\right], \end{aligned} \tag{4.16}$$

where $k_1 = 2r/\langle f^2\rangle$, $\Phi(\cdot)$ is the CDF of the standard normal distribution and

$$\Delta_{\pm}(x) = \frac{1}{\sqrt{\langle f^2\rangle (T-t)}} \left[\ln(x) + \left(r \pm \frac{1}{2}\langle f^2\rangle\right)(T-t)\right].$$

Note that $P_{0,0}$ given in Eq. (4.16) is precisely the same as the price of a down-and-out put option given in the literature, e.g., Hull (2015) or Haug (2006) for letting the variance to be $\langle f^2 \rangle$. We give detailed derivation of Eq. (4.16) as follows. From Eq. (4.15), we know that

$$\begin{aligned} P_{0,0}(t,s) &= \int_B^K (K-u)\, e^{\delta} \left(\frac{s}{u}\right)^{\eta} \left(\frac{1}{2\sqrt{\lambda\pi}} e^{-\frac{1}{4\lambda}\left(\ln\left(\frac{s}{u}\right)\right)^2}\right) \frac{du}{u} - \\ & \int_{\frac{B^2}{K}}^{B} \left(\frac{B}{u}\right)^{k_1-1} \left(K - \frac{B^2}{u}\right) e^{\delta} \left(\frac{s}{u}\right)^{\eta} \left(\frac{1}{2\sqrt{\lambda\pi}} e^{-\frac{1}{4\lambda}\left(\ln\left(\frac{s}{u}\right)\right)^2}\right) \frac{du}{u}. \end{aligned}$$

By letting $v = \ln u$, we convert the first integral to

$$\begin{aligned} & \int_{\ln B}^{\ln K} (K - e^v) s^{\eta} e^{\delta} e^{-\eta v} \left(\frac{1}{2\sqrt{\lambda\pi}} e^{-\frac{1}{4\lambda}(\ln s - v)^2}\right) dv \\ =\; & \frac{s^{\eta} e^{\delta}}{2\sqrt{\lambda\pi}} \left(\int_{\ln B}^{\ln K} K e^{-\frac{1}{4\lambda}(v^2 - 2v\ln s + (\ln s)^2 + 4\lambda\eta v)} dv \right. \\ & \left. - \int_{\ln B}^{\ln K} e^{-\frac{1}{4\lambda}(v^2 - 2v\ln s + (\ln s)^2 + 4\lambda(\eta-1)v)} dv \right) \\ =\; & \frac{s^{\eta} e^{\delta}}{2\sqrt{\lambda\pi}} \left(\int_{\ln B}^{\ln K} K e^{-\frac{1}{4\lambda}(v - \ln s + 2\lambda\eta)^2 + \lambda\eta^2 - \eta\ln s} dv \right. \\ & \left. - \int_{\ln B}^{\ln K} e^{-\frac{1}{4\lambda}[v - \ln s + 2\lambda(\eta-1)]^2 + \lambda(\eta-1)^2 - (\eta-1)\ln s} dv \right). \end{aligned}$$

We further apply the following changes of variables

$$x' := \frac{v - \ln s + 2\lambda\eta}{\sqrt{2\lambda}} \quad \text{and} \quad x'' := \frac{v - \ln s + 2\lambda(\eta - 1)}{\sqrt{2\lambda}}$$

to get

$$\begin{aligned}
&\int_{\ln B}^{\ln K}(K-e^v)s^\eta e^\delta e^{-\eta v}\left(\frac{1}{2\sqrt{\lambda\pi}}e^{-\frac{1}{4\lambda}(\ln s-v)^2}\right)dv\\
=\ & \frac{e^\delta}{\sqrt{2\pi}}\left(Ke^{\lambda\eta^2}\int_{\frac{\ln(\frac{B}{s})+2\lambda\eta}{\sqrt{2\lambda}}}^{\frac{\ln(\frac{K}{s})+2\lambda\eta}{\sqrt{2\lambda}}} e^{-\frac{x'^2}{2}}dx' - se^{\lambda(\eta-1)^2}\int_{\frac{\ln(\frac{B}{s})+2\lambda(\eta-1)}{\sqrt{2\lambda}}}^{\frac{\ln(\frac{K}{s})+2\lambda(\eta-1)}{\sqrt{2\lambda}}} e^{-\frac{x''^2}{2}}dx''\right)\\
=\ & Ke^{\delta+\lambda\eta^2}\left[\Phi\left(\frac{\ln(\frac{K}{s})+2\lambda\eta}{\sqrt{2\lambda}}\right)-\Phi\left(\frac{\ln(\frac{B}{s})+2\lambda\eta}{\sqrt{2\lambda}}\right)\right]\\
&-se^{\delta+\lambda(\eta-1)^2}\left[\Phi\left(\frac{\ln(\frac{K}{s})+2\lambda(\eta-1)}{\sqrt{2\lambda}}\right)-\Phi\left(\frac{\ln(\frac{B}{s})+2\lambda(\eta-1)}{\sqrt{2\lambda}}\right)\right].
\end{aligned}$$

Now, if we plug Eq. (4.14) into δ, η and λ into the above formula, we derive

$$\begin{aligned}
&\int_{\ln B}^{\ln K}(K-e^v)s^\eta e^\delta e^{-\eta v}\left(\frac{1}{2\sqrt{\lambda\pi}}e^{-\frac{1}{4\lambda}(\ln s-v)^2}\right)dv\\
&= Ke^{-r(T-t)}\left[\Phi\left(-\Delta_-\left(\frac{s}{K}\right)\right)-\Phi\left(-\Delta_-\left(\frac{s}{B}\right)\right)\right]\\
&-s\left[\Phi\left(-\Delta_+\left(\frac{s}{K}\right)\right)-\Phi\left(-\Delta_+\left(\frac{s}{B}\right)\right)\right].
\end{aligned}$$

Similarly, we can evaluate the second integral

$$\int_{\frac{B^2}{K}}^{B}\left(\frac{B}{u}\right)^{k_1-1}\left(K-\frac{B^2}{u}\right)e^\delta\left(\frac{s}{u}\right)^\eta\left(\frac{1}{2\sqrt{\lambda\pi}}e^{-\frac{1}{4\lambda}(\ln(\frac{s}{u}))^2}\right)\frac{du}{u}$$

to obtain

$$\begin{aligned}
&Ke^{-r(T-t)}\left(\frac{B}{s}\right)^{k_1-1}\left[\Phi\left(\Delta_-\left(\frac{B}{s}\right)\right)-\Phi\left(\Delta_-\left(\frac{B^2}{sK}\right)\right)\right]\\
&-B\left(\frac{B}{s}\right)^{k_1}\left[\Phi\left(\Delta_+\left(\frac{B}{s}\right)\right)-\Phi\left(\Delta_+\left(\frac{B^2}{sK}\right)\right)\right].
\end{aligned}$$

Putting these two integrals together yields the formula (4.16).

4.3.2 Correction Terms

For the case $k = 3$, we have $\mathcal{L}_{20}P_{0,1} + \mathcal{L}_0 P_{0,3} = 0$, which implies that

$$\langle \mathcal{L}_{20}P_{0,1}\rangle = \langle \mathcal{L}_{20}\rangle P_{0,1} = 0.$$

This equation with the terminal condition $P_{0,1}(T,s) = 0$ and the boundary condition $P_{0,1}(t,B) = 0$ yields that $P_{0,1} = 0$. Following similar steps, we can deduce that $P_{0,i} = 0$ for $i > 1$.

Secondly, we get the following PDE for the first-order terms from $O(\beta)$ term in Eq. (4.7).

$$\left(\frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_{10} + \mathcal{L}_{20}\right)P_1^\varepsilon + \left(\frac{1}{\sqrt{\varepsilon}}\mathcal{L}_{11} + \mathcal{L}_{21}\right)P_0^\varepsilon = 0. \tag{4.17}$$

Similar to the expansion for P_0^ε, we consider the following asymptotic expansion for P_1^ε

$$P_1^\varepsilon = P_{1,0} + \sqrt{\varepsilon}P_{1,1} + \varepsilon P_{1,2} + \varepsilon\sqrt{\varepsilon}P_{1,3} + \cdots,$$

and then substitute the expansion to Eq. (4.17) yielding

$$\begin{aligned} 0 = {} & \frac{1}{\varepsilon}\mathcal{L}_0 P_{1,0} + \frac{1}{\sqrt{\varepsilon}}(\mathcal{L}_0 P_{1,1} + \mathcal{L}_{10}P_{1,0} + \mathcal{L}_{11}P_{0,0}) + (\mathcal{L}_0 P_{1,2} + \mathcal{L}_{10}P_{1,1} + \mathcal{L}_{20}P_{1,0} \\ & + \mathcal{L}_{21}P_{0,0}) + \sqrt{\varepsilon}(\mathcal{L}_0 P_{1,3} + \mathcal{L}_{10}P_{1,2} + \mathcal{L}_{20}P_{1,1}) + O(\varepsilon) \end{aligned} \tag{4.18}$$

From the leading $O(\frac{1}{\varepsilon})$ term in Eq. (4.18), we get $\mathcal{L}_0 P_{1,0} = 0$. If we assume that $P_{1,0}$ does not grow as fast as $e^{y^2/2}$, we know that $P_{1,0}$ is independent of y.

From the $O(\frac{1}{\sqrt{\varepsilon}})$ term in Eq. (4.18), we get $\mathcal{L}_0 P_{1,1} + \mathcal{L}_{10}P_{1,0} + \mathcal{L}_{11}P_{0,0} = 0$ and then $\mathcal{L}_0 P_{1,1} = 0$, which implies that $P_{1,1}$ is independent of y.

From the $O(1)$ term in Eq. (4.17), we get

$$\mathcal{L}_0 P_{1,2} + \mathcal{L}_{10} P_{1,1} + \mathcal{L}_{20} P_{1,0} + \mathcal{L}_{21} P_{0,0} = 0,$$

which implies

$$\mathcal{L}_0 P_{1,2} + \mathcal{L}_{20} P_{1,0} + \mathcal{L}_{21} P_{0,0} = 0. \tag{4.19}$$

From the existence of solution to the Poisson equation for $P_{1,2}$ in Eq. (4.19), we obtain the centering condition

$$\langle \mathcal{L}_{20} P_{1,0} + \mathcal{L}_{21} P_{0,0} \rangle = \langle \mathcal{L}_{20} \rangle P_{1,0} + \langle \mathcal{L}_{21} \rangle P_{0,0} = 0,$$

which is equivalent to

$$\begin{aligned} \langle \mathcal{L}_{20} \rangle P_{1,0}(t,s) &= -\langle \mathcal{L}_{21} \rangle P_{0,0}(t,s) = -s^2 \ln s \left\langle f^2 \right\rangle \frac{\partial^2 P_{0,0}}{\partial s^2}, \ s > B, 0 < t < T \\ P_{1,0}(T,s) &= 0, \\ P_{1,0}(t,B) &= 0. \end{aligned} \tag{4.20}$$

One could solve the PDE problem Eq. (4.20) for $P_{1,0}$ by employing the Duhamel principle for barrier options.

Proposition 4.3.1 *The solution to the PDE problem* (4.20) *is given by*

$$\begin{aligned} P_{1,0}(t,s) &= \left(\frac{B}{s}\right)^{\frac{1}{2}(k_1-1)} \frac{\ln\left(\frac{s}{B}\right)}{\sqrt{2\pi\langle f\rangle^2}} \int_0^t \int_{\tau'}^T (t'-\tau')^{\frac{-3}{2}} \\ &\cdot \exp\left(-\frac{\left(\ln\left(\frac{s}{B}\right)\right)^2}{2\langle f\rangle^2 (t'-\tau')} - \frac{1}{2}\langle f\rangle^2 \left(\frac{k_1+1}{2}\right)^2 (t'-\tau')\right) \\ &\langle \mathcal{L}_{21}\rangle P_{0,0}(\tau',B)\, dt' d\tau'. \end{aligned} \tag{4.21}$$

Proof: Consider a PDE problem of $U_{1,0}(t,s)$ given by

$$\begin{aligned} \langle \mathcal{L}_{20}\rangle U_{1,0}(t,s) &= 0,\ s>B,\ 0<t<T, \\ U_{1,0}(T,s) &= 0, \\ U_{1,0}(t,B) &= -\langle \mathcal{L}_{21}\rangle P_{0,0}(t,B) := G(t). \end{aligned} \tag{4.22}$$

Note that the PDE (4.22) has no inhomogeneous term while there is a boundary condition at the barrier B. By using the change of variables given by

$$\begin{aligned} U_{1,0}(t,s) &= Ke^{ay+b\tau}u(\tau,w), \\ G(t) &= Ke^{ay+b\tau}g(\tau), \\ \alpha &= \frac{-1}{2}(k_1-1), \\ \beta &= \frac{-1}{2}\langle f\rangle^2\left(\frac{k_1+1}{2}\right)^2, \\ w &= \ln\left(\frac{s}{K}\right), \\ w_0 &= \ln\left(\frac{B}{K}\right), \\ \tau &= T-t, \end{aligned}$$

one can convert the PDE (4.22) into a heat equation as follows:

$$\begin{aligned} \frac{\partial u}{\partial \tau} - \frac{1}{2}\langle f\rangle^2 \frac{\partial^2 u}{\partial w^2} &= 0, w > w_0,\ 0 < \tau < T, \\ u(0,w) &= 0, \\ u(\tau, w_0) &= g(\tau). \end{aligned}$$

The solution to this heat equation is given by

$$u(\tau,w) = \frac{w-w_0}{\sqrt{2\pi\langle f\rangle^2}} \int_0^{\tau} (\tau-\tau')^{\frac{-3}{2}} \exp\left(\frac{-(w-w_0)^2}{2\langle f\rangle^2(\tau-\tau')}\right) g(\tau')\, d\tau'$$

and transforming back to the original variables provides

$$\begin{aligned} U_{1,0}(t,s) &= Ke^{\alpha\ln\left(\frac{s}{K}\right)+\beta(T-t)} \frac{\ln\left(\frac{s}{B}\right)}{\sqrt{2\pi\langle f\rangle^2}} \int_t^T (t'-t)^{\frac{-3}{2}} \\ &\cdot \exp\left(-\frac{\left(\ln\left(\frac{s}{B}\right)\right)^2}{2\langle f\rangle^2(t'-t)}\right)\left(\frac{1}{K}e^{\alpha\ln\left(\frac{s}{K}\right)+\beta(T-t)}G(t')\right)dt' \\ &= \left(\frac{B}{s}\right)^{\frac{1}{2}(k_1-1)} \frac{\ln\left(\frac{s}{B}\right)}{\sqrt{2\pi\langle f\rangle^2}} \int_t^T (t'-t)^{\frac{-3}{2}} \\ &\cdot \exp\left(-\frac{\left(\ln\left(\frac{s}{B}\right)\right)^2}{2\langle f\rangle^2(t'-t)} - \frac{1}{2}\langle f\rangle^2\left(\frac{k_1+1}{2}\right)^2(t'-t)\right)\langle\mathcal{L}_{21}\rangle P_{0,0}(t',B)\, dt'. \end{aligned}$$

By the Duhamel principle, the expression of $P_{1,0}$ can be represented in terms of $U_{1,0}$ as

$$P_{1,0}(t,s) = \int_0^t U_{1,0}(\tau',s)d\tau'.$$

Next, we find an expression for $P_{1,1}$. From the $O(\sqrt{\varepsilon})$ term in Eq. (4.19), we get

$$\mathcal{L}_{20}P_{1,1} + \mathcal{L}_{10}P_{1,2} + \mathcal{L}_0 P_{1,3} = 0.$$

The existence of solution for $P_{1,3}$ requires $\langle \mathcal{L}_{20} P_{1,1} + \mathcal{L}_{10} P_{1,2} \rangle = 0$. We find a solution for $P_{1,2}$ by solving $\mathcal{L}_0 P_{1,2} = \left(\langle \mathcal{L}_{21} \rangle - \mathcal{L}_{21} \right) P_{0,0}$ to obtain

$$P_{1,2} = s^2 \ln s \frac{\partial^2 P_{0,0}}{\partial s^2} \mathcal{L}_0^{-1} \left(f^2 - \langle f^2 \rangle \right).$$

Then we can obtain the solution for $P_{1,1}$ by solving the equation

$$\begin{aligned} \langle \mathcal{L}_{20} \rangle P_{1,1} &= -\langle \mathcal{L}_{10} P_{1,2} \rangle = -\sqrt{2} \rho v s \langle f \psi' \rangle s \frac{\partial}{\partial s} \left(s^2 \ln s \frac{\partial^2 P_{0,0}}{\partial s^2} \right) \\ &\quad + \sqrt{2} v \langle \Lambda \psi' \rangle s^2 \ln s \frac{\partial^2 P_{0,0}}{\partial s^2}, \qquad (4.23) \\ P_{1,1}(T, s) &= 0, \\ P_{1,1}(t, B) &= 0, \end{aligned}$$

where ψ is the solution of $\mathcal{L}_0 \psi = f^2 - \langle f^2 \rangle$. Employing the Duhamel's principle similar to the one in Proposition 1, one can obtain the solution $P_{1,1}$ of the PDE problem (4.23) as follows:

$$\begin{aligned} P_{1,1}(t, s) &= \frac{\left(\frac{B}{s} \right)^{\frac{1}{2}(k_1 - 1)} \ln \left(\frac{s}{B} \right)}{\sqrt{2\pi \langle f \rangle^2}} \int_0^t \int_{\tau'}^T (t' - \tau')^{\frac{-3}{2}} \\ &\quad \exp \left(-\frac{\left(\ln \left(\frac{s}{B} \right) \right)^2}{2 \langle f \rangle^2 (t' - \tau')} - \frac{\langle f \rangle^2 (t' - \tau')}{2} \left(\frac{k_1 + 1}{2} \right)^2 \right) \\ &\quad \cdot \left(\sqrt{2} \rho v s \langle f \psi' \rangle s \frac{\partial}{\partial s} \left(s^2 \ln s \frac{\partial^2 P_{0,0}}{\partial s^2} \right) - \sqrt{2} v \langle \Lambda \psi' \rangle s^2 \ln s \frac{\partial^2 P_{0,0}}{\partial s^2} \right) dt' d\tau'. \end{aligned} \qquad (4.24)$$

We summarize the key points in the previous asymptotic analysis as the following theorem.

Theorem 4.3.1 *Under the SVCEV model governed by Eq.* (4.1)*, the risk-neutral value*

P of a down-and-out put option can be approximated by the following formula

$$P^{\beta,\varepsilon} = P_{0,0} + \beta\left(P_{1,0} + \sqrt{\varepsilon}P_{1,1}\right) + O(\beta^2)$$

for small β and ε, where $P_{0,0}$, $P_{1,0}$ and $P_{1,1}$ are given in Eqs. (4.16)*,* (4.21) *and* (4.24) *respectively. Note that $P_{0,0}$ is the down-and-out put option price under the Black-Scholes model with constant effective volatility $\sqrt{\langle f^2\rangle}$, $\beta\left(P_{1,0} + \sqrt{\varepsilon}P_{1,1}\right)$ is the correction term.*

4.4 Analytic Approximate Solutions for Lookback Option Prices

4.4.1 Zero-order Term

Using a method similar to that for the case of barrier options, we consider floating strike lookback put options. For a floating strike lookback put option, its payoff has the following form:

$$LP_{float}(T) := Z_T - S_T.$$

The risk-neutral price at time $t \in [0, T]$ under the SVCEV model is denoted by

$$P^{\beta,\varepsilon}(t, s, y, z) = \mathbb{E}^{\mathbb{Q}}\left(e^{-r(T-t)}LP(S_T)\right| S_t = s, Y_t = y, Z_t = z\right).$$

Applying Itô's lemma, we can obtain a partial differential equation (PDE) for $P^{\beta,\varepsilon}(t, s, y, z)$ with the same form as Eq. (4.3) but with different terminal and boundary conditions for the lookback put option as

$$\begin{cases} \dfrac{\partial P^{\beta,\varepsilon}}{\partial z}(t, z, y, z) = 0, & 0 \le t \le T, z > 0, \\ P^{\beta,\varepsilon}(T, s, y, z) = z - s, & 0 \le s \le z. \end{cases}$$

Note that in this case, $P^{\beta,\varepsilon}$ is a function of four variables t, s, y and z.

For lookback floating strike put options, the boundary conditions of $P_{0,0}$ are

$$\begin{cases} \dfrac{\partial P_{0,0}}{\partial z}(t,z,z) & = & 0, \\ \dfrac{\partial P_{0,0}}{\partial z}(T,s,z) & = & 1, \quad for \;\; 0 < s < z. \end{cases}$$

For convenience, we let $u = s/z$ and $Q_{0,0} = P_{0,0}/z$. With these notations, Eq. (4.10) becomes

$$\frac{\partial Q_{0,0}}{\partial t} + \frac{1}{2}u^2\langle f^2\rangle\frac{\partial^2 Q_{0,0}}{\partial u^2} + ru\frac{\partial Q_{0,0}}{\partial u} - rQ_{0,0} = 0 \tag{4.25}$$

with boundary conditions

$$Q_{0,0}(T,u) = -\frac{1}{k_1}u^{1-k_1} + \frac{1}{k_1}u, \quad \text{for} \quad u > 1, \tag{4.26}$$

and $Q_{0,0}(T,u) = 1 - u$, for $0 < u < 1$.

Note that except the boundary conditions, Eq. (4.25) is identical to Eq. (4.10). Applying Mellin transform in the same way as that for the case of down-and-out put options, we can derive the solution to Eq. (4.25) as follows:

$$Q_{0,0}(t,u) = \hat{\theta}(w) * \mathcal{M}^{-1}e^{\lambda(w+\eta)^2+\delta}.$$

Again, applying Table 1 and $Q_{0,0}(T,u)$ given in Eq. (4.26), we have

$$\begin{aligned} Q_{0,0}(t,u) &= Q_{0,0}(T,u) * e^{\delta} z^{\eta} \left(\frac{1}{2\sqrt{\pi}} \lambda^{-\frac{1}{2}} e^{-\frac{1}{4\lambda}(\ln z)^2} \right) \\ &= \int_0^1 (1-\xi)\, e^{\delta} \left(\frac{u}{\xi}\right)^{\eta} \left(\frac{1}{2\sqrt{\pi}} \lambda^{-\frac{1}{2}} e^{-\frac{1}{4\lambda}\left(\ln\left(\frac{u}{\xi}\right)\right)^2} \right) \frac{d\xi}{\xi} + \\ &\quad \int_1^{\infty} \left(\frac{-1}{k_1} \xi^{1-k_1} + \frac{\xi}{k_1} \right) e^{\delta} \left(\frac{u}{\xi}\right)^{\eta} \left(\frac{1}{2\sqrt{\pi}} \lambda^{-\frac{1}{2}} e^{-\frac{1}{4\lambda}\left(\ln\left(\frac{u}{\xi}\right)\right)^2} \right) \frac{d\xi}{\xi}. \end{aligned} \tag{4.27}$$

Similar to barrier options, we can derive a closed-form expression of the $P_{0,0}$ term for a floating strike lookback put option as follows:

$$\begin{aligned} P_{0,0}(t,s,z) &= z e^{-r(T-t)} \Phi\left(-\Delta_- \left(\frac{s}{z}\right)\right) - s\Phi\left(-\Delta_+\left(\frac{s}{z}\right)\right) \\ &\quad - \frac{z}{k_1}\left(\frac{s}{z}\right)^{1-k_1} e^{-r(T-t)} \Phi\left(-\Delta_-\left(\frac{z}{s}\right)\right) + \frac{s}{k_1}\Phi\left(\Delta_+\left(\frac{s}{z}\right)\right). \end{aligned} \tag{4.28}$$

Note that $P_{0,0}$ given in Eq. (4.28) is precisely the same as the price of a floating strike put option given in the literature, e.g., Hull (2015) or Haug (2006) , if we let $\sigma^2 := \langle f^2 \rangle$. The derivation of this formula is given below. We let $v = \ln \xi$. For the first integral, we have

$$\begin{aligned} & \int_0^1 (1-\xi) e^{\delta} \left(\frac{u}{\xi}\right)^{\eta} \left(\frac{1}{2\sqrt{\lambda\pi}} e^{-\frac{1}{4\lambda}\left(\ln\left(\frac{u}{\xi}\right)\right)^2} \right) \frac{du}{u} \\ = & \int_{-\infty}^{0} u^{\eta} (1-e^{v})\, e^{\delta - v\eta} \left(\frac{1}{2\sqrt{\lambda\pi}} e^{-\frac{1}{4\lambda}(\ln u - v)^2} \right) dv \\ = & \frac{u^{\eta} e^{\delta}}{2\sqrt{\lambda\pi}} \left(\int_{-\infty}^{0} e^{-\frac{1}{4\lambda}\left(v^2 - 2v\ln u + (\ln u)^2 + 4\lambda\eta v\right)} dv \right. \\ & \left. - \int_{-\infty}^{0} e^{-\frac{1}{4\lambda}\left(v^2 - 2v\ln u + (\ln u)^2 + 4\lambda(\eta-1)v\right)} dv \right) \\ = & \frac{u^{\eta} e^{\delta}}{2\sqrt{\lambda\pi}} \left(\int_{-\infty}^{0} e^{-\frac{1}{4\lambda}(v - \ln u + 2\lambda\eta)^2 + \lambda\eta^2 - \eta\ln u} dv \right. \\ & \left. - \int_{-\infty}^{0} e^{-\frac{1}{4\lambda}(v - \ln u + 2\lambda(\eta-1))^2 + \lambda(\eta-1)^2 - (\eta-1)\ln u} dv \right). \end{aligned}$$

Next, we let

$$v' := \frac{v - \ln u + 2\lambda\eta}{\sqrt{2\lambda}} \quad \text{and} \quad v'' := \frac{v - \ln u + 2\lambda(\eta - 1)}{\sqrt{2\lambda}}.$$

Then, we have

$$\begin{aligned}
&\int_0^1 (1-\xi)e^{\delta}\left(\frac{u}{\xi}\right)^{\eta}\left(\frac{1}{2\sqrt{\lambda\pi}}e^{-\frac{1}{4\lambda}\left(\ln\left(\frac{u}{\xi}\right)\right)^2}\right)\frac{du}{u} \\
= \;& \frac{e^{\delta}}{\sqrt{2\pi}}\left(\int_{-\infty}^{\frac{-\ln u + 2\lambda\eta}{\sqrt{2\lambda}}} e^{-\frac{v'^2}{2}+\lambda\eta^2}dv' - u\int_{-\infty}^{\frac{-\ln u+2\lambda(\eta-1)}{\sqrt{2\lambda}}} e^{-\frac{v''^2}{2}+\lambda(\eta-1)^2}dv''\right) \\
= \;& e^{\delta+\lambda\eta^2}\Phi\left(\frac{-\ln u + 2\lambda\eta}{\sqrt{2\lambda}}\right) - ue^{\delta+\lambda(\eta-1)^2}\Phi\left(\frac{-\ln u + 2\lambda(\eta-1)}{\sqrt{2\lambda}}\right) \\
= \;& e^{-r(T-t)}\Phi\left(-\Delta_-\left(\frac{s}{z}\right)\right) - \left(\frac{s}{z}\right)\Phi\left(-\Delta_+\left(\frac{s}{z}\right)\right).
\end{aligned}$$

For the second integral, we have

$$\begin{aligned}
&\int_1^{\infty}\left(-\frac{1}{k_1}\xi^{1-k_1} + \frac{\xi}{k_1}\right)e^{\delta}\left(\frac{u}{\xi}\right)^{\eta}\left(\frac{1}{2\sqrt{\lambda\pi}}e^{-\frac{1}{4\lambda}\left(\ln\left(\frac{u}{\xi}\right)\right)^2}\right)\frac{d\xi}{\xi} \\
= \;& \int_0^{\infty}\left(-\frac{1}{k_1}e^{(1-k_1)v} + \frac{1}{k_1}e^{v}\right)e^{\delta}u^{\eta}e^{-v\eta}\left(\frac{1}{2\sqrt{\lambda\pi}}e^{-\frac{1}{4\lambda}(\ln u - v)^2}\right)dv \\
= \;& \frac{e^{\delta}u^{\eta}}{2k_1\sqrt{\lambda\pi}}\int_0^{\infty}\left(-e^{\eta v - \frac{1}{4\lambda}(\ln u - v)^2} + e^{v(1-\eta)-\frac{1}{4\lambda}(\ln u - v)^2}\right)dv \\
= \;& \frac{e^{\delta}u^{\eta}}{2k_1\sqrt{\lambda\pi}}\left(\int_0^{\infty} -e^{-\frac{1}{4\lambda}(v-\ln u - 2\lambda\eta)^2+\lambda\eta^2+\eta\ln u}dv\right. \\
&\left. + \int_0^{\infty} e^{-\frac{1}{4\lambda}(v-\ln u-2\lambda(1-\eta))^2+\lambda(1-\eta)^2+(1-\eta)\ln u}dv\right),
\end{aligned}$$

where we use the fact that $k_1 - 1 + \eta = -\eta$. Further, we introduce a new variable

$$v''' := \frac{v - \ln u - 2\lambda\eta}{\sqrt{2\lambda}}.$$

Then, we have

$$\begin{aligned}
&\int_1^\infty \left(-\frac{1}{k_1}\xi^{1-k_1} + \frac{\xi}{k_1}\right) e^\delta \left(\frac{u}{\xi}\right)^\eta \left(\frac{1}{2\sqrt{\lambda\pi}} e^{-\frac{1}{4\lambda}\left(\ln\left(\frac{u}{\xi}\right)\right)^2}\right)\frac{d\xi}{\xi} \\
= \ & \frac{e^\delta u^\eta}{k_1\sqrt{2\pi}}\left(\int_{\frac{-\ln u - 2\lambda\eta}{\sqrt{2\lambda}}}^\infty -e^{-\frac{v'''^2}{2}} e^{\lambda\eta^2+\eta\ln u} dv''' \right.\\
&\left. + \int_{\frac{-\ln u+2\lambda(\eta-1)}{\sqrt{2\lambda}}}^\infty e^{-\frac{v''^2}{2}} e^{\lambda(\eta-1)^2+(1-\eta)\ln u} dv''\right) \\
= \ & -\frac{1}{k_1} e^{\delta+\lambda\eta^2} u^{1-k_1}\Phi\left(\frac{\ln u + 2\lambda\eta}{\sqrt{2\lambda}}\right) + \frac{1}{k_1} u e^{\delta+\lambda(\eta-1)^2}\Phi\left(\frac{\ln u + 2\lambda\left(1-\eta\right)}{\sqrt{2\lambda}}\right) \\
= \ & -\frac{1}{k_1}\left(\frac{s}{z}\right)^{1-k_1} e^{-r(T-t)}\Phi\left(-\Delta_-\left(\frac{z}{s}\right)\right) + \frac{1}{k_1}\left(\frac{s}{z}\right)\Phi\left(\Delta_+\left(\frac{s}{z}\right)\right).
\end{aligned}$$

Putting these two integrals together and using the fact that $P_0 = zQ_0$, we can obtain our formula (4.28).

4.4.2 Correction Terms

For the case $k = 3$, we have $\mathcal{L}_{20}P_{0,1} + \mathcal{L}_0 P_{0,3} = 0$, which implies that

$$\langle \mathcal{L}_{20}P_{0,1}\rangle = \langle \mathcal{L}_{20}\rangle P_{0,1} = 0.$$

This equation with the terminal condition $P_{0,1}(T, s) = 0$ and the boundary condition $P_{0,1}(t, B) = 0$ yields that $P_{0,1} = 0$. Following similar steps, we can deduce that $P_{0,i} = 0$ for $i > 1$.

Secondly, we get the following PDE for the first-order terms from the $O(\beta)$ term in Eq. (4.7):

$$\left(\frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_{10} + \mathcal{L}_{20}\right)P_1^\varepsilon + \left(\frac{1}{\sqrt{\varepsilon}}\mathcal{L}_{11} + \mathcal{L}_{21}\right)P_0^\varepsilon \quad = \quad 0. \qquad (4.29)$$

Similar to the expansion for P_0^ε, we consider the following asymptotic expansion for P_1^ε

$$P_1^\varepsilon = P_{1,0} + \sqrt{\varepsilon}P_{1,1} + \varepsilon P_{1,2} + \varepsilon\sqrt{\varepsilon}P_{1,3} + \cdots,$$

and then substitute the expansion to Eq. (4.29) yielding

$$\begin{aligned} 0 &= \frac{1}{\varepsilon}\mathcal{L}_0 P_{1,0} + \frac{1}{\sqrt{\varepsilon}}(\mathcal{L}_0 P_{1,1} + \mathcal{L}_{10}P_{1,0} + \mathcal{L}_{11}P_{0,0}) \\ &\quad +(\mathcal{L}_0 P_{1,2} + \mathcal{L}_{10}P_{1,1} + \mathcal{L}_{20}P_{1,0} + \mathcal{L}_{21}P_{0,0}) \\ &\quad +\sqrt{\varepsilon}(\mathcal{L}_0 P_{1,3} + \mathcal{L}_{10}P_{1,2} + \mathcal{L}_{20}P_{1,1}) + O(\varepsilon) \end{aligned} \tag{4.30}$$

From the leading $O(\frac{1}{\varepsilon})$ term in Eq. (4.30), we get $\mathcal{L}_0 P_{1,0} = 0$. If we assume that $P_{1,0}$ does not grow as fast as $e^{y^2/2}$, we know that $P_{1,0}$ is independent of y.

From the $O(\frac{1}{\sqrt{\varepsilon}})$ term in Eq. (4.30), we get $\mathcal{L}_0 P_{1,1} + \mathcal{L}_{10}P_{1,0} + \mathcal{L}_{11}P_{0,0} = 0$, which implies $\mathcal{L}_0 P_{1,1} = 0$. Thus, we conclude that $P_{1,1}$ is independent of y.

From the $O(1)$ term in Eq. (4.29), we get

$$\mathcal{L}_0 P_{1,2} + \mathcal{L}_{10}P_{1,1} + \mathcal{L}_{20}P_{1,0} + \mathcal{L}_{21}P_{0,0} = 0, \tag{4.31}$$

which is reduced to $\mathcal{L}_0 P_{1,2} + \mathcal{L}_{20}P_{1,0} + \mathcal{L}_{21}P_{0,0} = 0$ as $P_{1,1}$ is independent of y. From the existence of solution to the Poisson equation for $P_{1,2}$ in Eq. (4.31), we obtain the centering condition

$$\langle \mathcal{L}_{20}P_{1,0} + \mathcal{L}_{21}P_{0,0}\rangle = \langle \mathcal{L}_{20}\rangle P_{1,0} + \langle \mathcal{L}_{21}\rangle P_{0,0} = 0,$$

which is equivalent to

$$
\begin{aligned}
\langle \mathcal{L}_{20} \rangle P_{1,0}(t,s) &= -\langle \mathcal{L}_{21} \rangle P_{0,0}(t,s) = -s^2 \ln s \left\langle f^2 \right\rangle \frac{\partial^2 P_{0,0}}{\partial s^2}, \\
& \quad s > B, -0 < t < T, \qquad (4.32) \\
P_{1,0}(T,s) &= 0, \\
P_{1,0}(t,B) &= 0.
\end{aligned}
$$

A floating strike lookback option can be transformed into a barrier option, so the first order correction term for a floating strike lookback option can be derived in the same way as that of a barrier option. As a result,

$$
\begin{aligned}
P_{1,0}(t,s,z) &= \left(\frac{z}{s}\right)^{\frac{1}{2}(k_1-1)} \frac{\ln\left(\frac{s}{z}\right)}{\sqrt{2\pi \langle f \rangle^2}} \int_0^t \int_{\tau'}^T (t'-\tau')^{\frac{-3}{2}} \qquad (4.33) \\
& \cdot \exp\left(-\frac{\left(\ln\left(\frac{s}{z}\right)\right)^2}{2\langle f \rangle^2 (t'-\tau')} - \frac{1}{2}\langle f \rangle^2 \left(\frac{k_1+1}{2}\right)^2 (t'-\tau')\right) \\
& \langle \mathcal{L}_{21} \rangle P_{0,0}(\tau', z, z)\, dt' d\tau',
\end{aligned}
$$

and

$$
\begin{aligned}
P_{1,1}(t,s,z) &= \frac{\left(\frac{z}{s}\right)^{\frac{1}{2}(k_1-1)} \ln\left(\frac{s}{z}\right)}{\sqrt{2\pi \langle f \rangle^2}} \int_0^t \int_{\tau'}^T (t'-\tau')^{\frac{-3}{2}} \\
& \exp\left(-\frac{\left(\ln\left(\frac{s}{z}\right)\right)^2}{2\langle f \rangle^2 (t'-\tau')} - \frac{\langle f \rangle^2 (t'-\tau')}{2}\left(\frac{k_1+1}{2}\right)^2\right) \\
& \cdot \left(\sqrt{2}\rho v s \langle f\psi' \rangle s \frac{\partial}{\partial s}\left(s^2 \ln s \frac{\partial^2 P_{0,0}}{\partial s^2}\right) - \sqrt{2} v \langle \Lambda \psi' \rangle s^2 \ln s \frac{\partial^2 P_{0,0}}{\partial s^2}\right) dt' d\tau'.
\end{aligned}
\qquad (4.34)
$$

where $P_{0,0}(t,s,z)$ is given in Eq. (4.28). We summarize the asymptotic expansion of floating strike lookback options as the following theorem.

Theorem 4.4.1 *Under the SVCEV model governed by Eq.* (4.1)*, the risk-neutral value P of a floating strike put option can be approximated by the following formula*

$$P^{\beta,\varepsilon} = P_{0,0} + \beta\left(P_{1,0} + \sqrt{\varepsilon}P_{1,1}\right) + O(\beta^2)$$

for small β and ε, where $P_{0,0}$, $P_{1,0}$ and $P_{1,1}$ are given in Eqs. (4.28), (4.33) *and* (4.34) *respectively. Note that $P_{0,0}$ is the price of a floating strike lookback put option under the Black-Scholes model with constant effective volatility $\sqrt{\langle f^2\rangle}$, $\beta\left(P_{1,0} + \sqrt{\varepsilon}P_{1,1}\right)$ the correction term.*

4.5 Numerical Analysis

In this section, we perform numerical experiments to investigate the sensitivity of the first-order correction term $\beta\left(P_{1,0} + \sqrt{\varepsilon}P_{1,1}\right)$ and the approximation results $P_{0,0} + \beta\left(P_{1,0} + \sqrt{\varepsilon}P_{1,1}\right)$ with respect to various initial values of the underlying equity. We also compare the numerical results obtained by the closed-form formulas with those obtained by the Monte-Carlo simulation and the binomial tree method used in Peng and Peng (2010). Firstly, we choose the function f to take the specific form as in Fouque et al. (2000), Fouque et al. (2011) and Cao et al. (2021),

$$f(y) = 0.35\left(\tan^{-1}(y) + \frac{\pi}{2}\right)/\pi + 0.05.$$

Secondly, the values of other parameters used in the numerical computation are specified in Table 4.1. Note that the correction terms $P_{1,0}$ and $P_{1,1}$ depend only on the group parameters $\langle f\psi'\rangle\rho\nu$ and $\langle\Lambda\psi'\rangle\nu$, so we do not need to choose specific values of γ and ρ, and particular forms of $\xi(y)$ and $\psi(y)$ for numerical calculation.

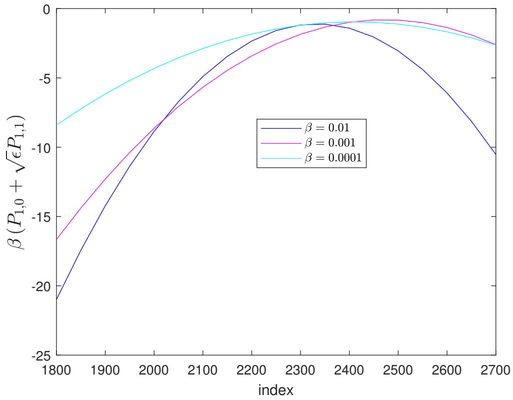
Table 4.1: The role and numerical value of parameters.

Parameter	Role	Value
r	risk-free interest rate	0.035
B	barrier level	1500
K	put option strike price	2700
$\langle f\psi' \rangle \rho\nu$	model group parameter	-0.005
$\langle \Lambda\psi' \rangle \nu$	model group parameter	-0.02

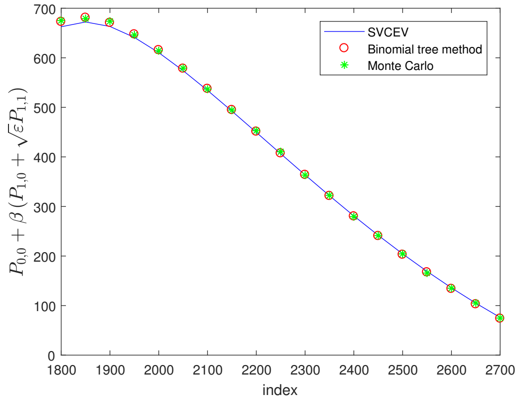
We obtain benchmark results of option prices by two methods, Monte Carlo simulation and the binomial tree method used in Peng and Peng (2010). For all the experiments below, we can observe that the two methods provide very similar results for the prices of both down-and-out and floating strike lookback put options.

Figure (4.1A) shows how the $\beta\left(P_{1,0} + \sqrt{\varepsilon}P_{1,1}\right)$-term for a down-and-out put option changes with respect to a variation of β values. As we can see, for fixed β, when s increases, the correction term $\beta\left(P_{1,0} + \sqrt{\varepsilon}P_{1,1}\right)$ increases first, and then decreases after it hits its peak. When β gets closer to zero (equivalently, the elasticity of variance gets smaller), $\beta\left(P_{1,0} + \sqrt{\varepsilon}P_{1,1}\right)$ approaches to a zero. Figure (4.1B) shows how the results of $P_{0,0} + \beta\left(P_{1,0} + \sqrt{\varepsilon}P_{1,1}\right)$ compare with the benchmark results from Monte Carlo simulation and the binomial tree method for down-and-out put options. As observed from the figure, the values of the approximate formula $P_{0,0} + \beta\left(P_{1,0} + \sqrt{\varepsilon}P_{1,1}\right)$ match well with the corresponding results of Monte-Carlo simulation and the binomial tree method in all cases. Furthermore, in all cases, the value $P_{0,0}+\beta\left(P_{1,0} + \sqrt{\varepsilon}P_{1,1}\right)$ declines as s increases.

Figure (4.2A) displays the $\beta\left(P_{1,0} + \sqrt{\varepsilon}P_{1,1}\right)$-term for a floating strike lookback put option for various β values. Similar to the down-and-out put option case, for fixed β, when s increases, the correction term $\beta\left(P_{1,0} + \sqrt{\varepsilon}P_{1,1}\right)$ increases first, and then decreases after reaching the peak. When β gets closer to zero (equivalently, the elasticity of variance gets smaller), $\beta\left(P_{1,0} + \sqrt{\varepsilon}P_{1,1}\right)$ approaches to a zero. Figure (4.2B) shows how the value of $P_{0,0} + \beta\left(P_{1,0} + \sqrt{\varepsilon}P_{1,1}\right)$ for a floating strike lookback



(A)



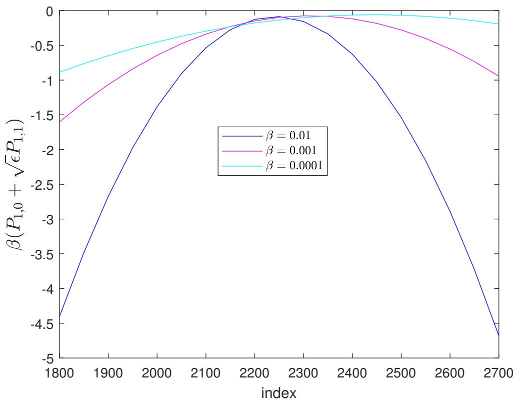
(B)

Figure 4.1: Plots of $\beta\left(P_{1,0}+\sqrt{\varepsilon}P_{1,1}\right)$ for different β values and $P_{0,0}+\beta\left(P_{1,0}+\sqrt{\varepsilon}P_{1,1}\right)$ against Monte Carlo and the binomial tree method results for down-and-out put options.

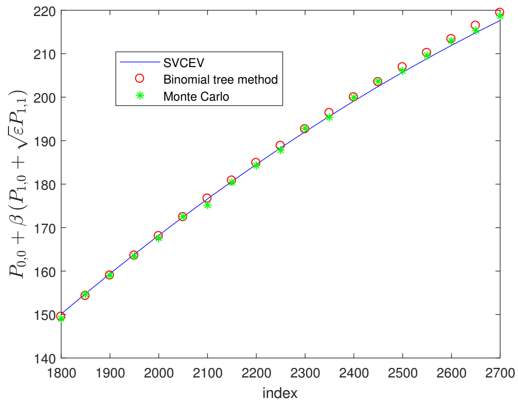
put option is compared to benchmark results. The values of the approximate formula $P_{0,0}+\beta\left(P_{1,0}+\sqrt{\varepsilon}P_{1,1}\right)$ again match well with the corresponding results of Monte-Carlo simulation and the binomial tree method in all cases. The value $P_{0,0}+\beta\left(P_{1,0}+\sqrt{\varepsilon}P_{1,1}\right)$ goes up as s increases.

Figure (4.3) and Figure (4.4) illustrate the improvement of down-and-out and floating strike lookback put options pricing formulae of the SVCEV model compared to those of CEV and Black-Scholes models. We compare the benchmark results against the results from the pricing formulae of the Black-Scholes model ($\beta = 0$, $\varepsilon = 0$), the CEV model ($\varepsilon = 0$) and the SVCEV model (both β and ε nonzero). From the comparison, we see that the results of down-and-out and floating strike lookback put options pricing formulae of the SVCEV model are much closer to the benchmark results compared to those of the CEV and Black-Scholes models. Therefore, it is demonstrated that the down-and-out and floating strike lookback put options pricing formulae of the SVCEV model are more accurate.

In addition, we also summarize the computational cost of the closed-form approximation formulae, Monte-Carlo simulation and the binomial tree method in terms of running time in Table 4.2. We can see that the closed-form approximation formulae are



(A)



(B)

Figure 4.2: Plots of $\beta\left(P_{1,0}+\sqrt{\varepsilon}P_{1,1}\right)$ for different β values and $P_{0,0}+\beta\left(P_{1,0}+\sqrt{\varepsilon}P_{1,1}\right)$ against Monte Carlo and the binomial tree method results for floating strike lookback put options.

more efficient compared to both Monte-Carlo simulation and the binomial tree method.

Table 4.2: Comparison of running time.

Method	Option	Computation time
Approximation formula	barrier option	0.3 seconds
Approximation formula	lookback option	0.3 seconds
Binomial tree method	barrier option	4.8 seconds
Binomial tree method	lookback option	5.2 seconds
Monte Carlo simulation	barrier option	32.9 seconds
Monte Carlo simulation	lookback option	37.6 seconds

4.6 Conclusion

In this chapter, we obtain closed-form formulas for the first-order approximations to the prices of the down-and-out and floating strike lookback put options under the SVCEV model. The zero-order terms are the classical solution under the geometric Brownian motion framework, which match with those given in the literature. In addition, we perform numerical analysis to illustrate that the results from the explicit closed-form formulas are quite close to those generated by Monte-Carlo simulation and the binomial

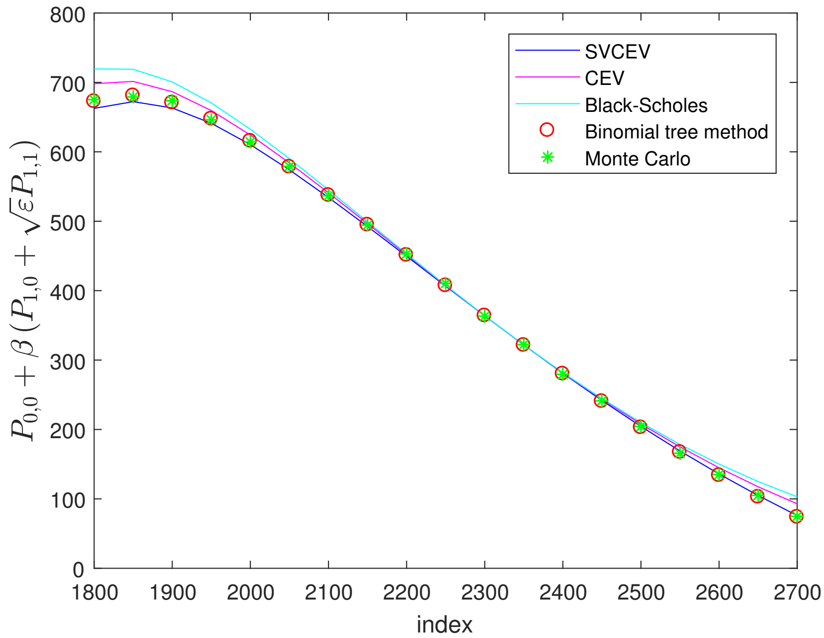


Figure 4.3: Plots of Black-Scholes, CEV and SVCEV price approximations against Monte Carlo simulations and the binomial tree method results for different values of β for down-and-out put options.

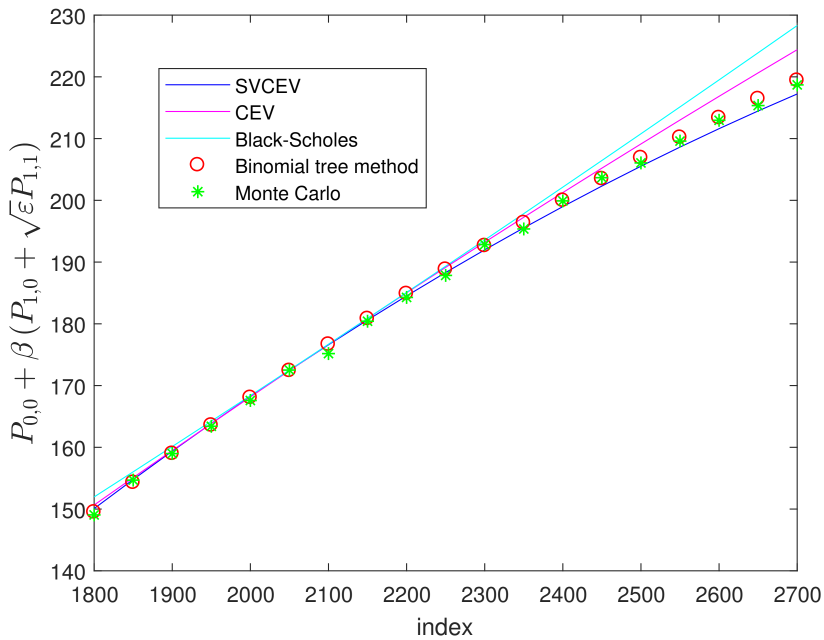


Figure 4.4: Plots of Black-Scholes, CEV and SVCEV price approximations against Monte Carlo simulations and the binomial tree method results for different values of β for floating strike lookback put options.

tree method. Our results on down-and-out put options could be easily extended to other types of barrier options.

Chapter 5

Pricing Path-dependent Options under Stochastic Volatility Model with Fractional Brownian Motions

In this chapter, we employ an asymptotic approach to derive closed-form formulas for the first-order approximation to the price of a geometric Asian put option under a fractional stochastic volatility model. We utilize the Mellin transform to obtain explicit closed-form expressions for both the zero-order term and the first-order correction term. Additionally, we perform a sensitivity analysis on these formulas and compare the resulting option prices with those obtained through Monte-Carlo simulation.

5.1 A Fractional Stochastic Volatility Model

Let $\{S_t\}_{t\geq 0}$ denote the price process of a risky asset on some filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{Q})$, where $\mathbb{Q}$ is the risk-neutral probability measure. In this chapter, we assume that $\{S_t\}_{t\geq 0}$ follows the following system of stochastic differential equations

$$\begin{aligned} dS_t &= rS_t dt + f\left(Y_t\right) S_t W_t(dt)^H \\ dY_t &= \alpha((m - Y_t) - \beta\Lambda(Y_t))dt + \beta dZ_t^Y \\ dG_t &= \ln\left(S_t\right) dt \end{aligned} \tag{5.1}$$

where r, $\alpha > 0$, $\beta > 0$ and m are constants, f is a function specifying the dependence on the hidden process $\{Y_t\}_{t\geq 0}$, $\{W_t(dt)^H\}_{t\geq 0}$ represents a fractional Brownian motion and it is uncorrelated to a standard Brownian motions $\{Z_t^Y\}_{t\geq 0}$. The Hurst parameter of the fractional Brownian motion is denoted by H, and r represents the risk-free rate. The mean-reversion process $\{Y_t\}_{t\geq 0}$ given in Eq. (5.1) is characterized by its typical time to obtain back to the mean level m of its long-run distribution. The parameter α determines the speed of mean-reversion and β controls the volatility of $\{Y_t\}_{t\geq 0}$ and $\Lambda\left(Y_t\right)$ represents the combined market price of risk. The process $\{G_t\}_{t\geq 0}$ represents the realized geometric mean of the risky asset prices in the interval $[0, t]$. We define new variables ε and ν as

$$\varepsilon = \frac{1}{\alpha}, \;\; \nu = \frac{\beta\sqrt{\varepsilon}}{\sqrt{2}},$$

then the model equations can be rewritten as

$$\begin{aligned} dS_t &= rS_t dt + f\left(Y_t\right) S_t W_t(dt)^H \\ dY_t &= \left(\frac{1}{\varepsilon}\left(m - Y_t\right) - \frac{\sqrt{2}\nu}{\sqrt{\varepsilon}}\Lambda\left(Y_t\right)\right) dt + \frac{\sqrt{2}\nu}{\sqrt{\varepsilon}} dZ_t^Y \\ dG_t &= \ln\left(S_t\right) dt. \end{aligned} \tag{5.2}$$

As discussed in Section 2.2.2, the price of a European geometric Asian put option is given by

$$P^{\varepsilon}(t,s,y,g) = \mathbb{E}^{Q}\left(e^{-r(T-t)}\left(K - e^{\frac{g}{t}}\right)^{+} | S_t = s, Y_t = y, G_t = g\right).$$

Applying the Feynman-Kac theorem, we obtain a partial differential equation for P^{ε} from Eq. (5.2) as follows

$$\begin{aligned} 0 &= \frac{\partial P^{\varepsilon}}{\partial t}dt + \frac{1}{2}s^2 f^2(y)\frac{\partial^2 P^{\varepsilon}}{\partial s^2}dt^{(2H)} + r\left(s\frac{\partial P^{\varepsilon}}{\partial s} - P^{\varepsilon}\right)dt \\ &\quad + \ln(s)\frac{\partial P^{\varepsilon}}{\partial g}dt + \frac{\nu^2}{\varepsilon}\frac{\partial^2 P^{\varepsilon}}{\partial y^2}dt + \left(\frac{1}{\varepsilon}(m-y) - \frac{\sqrt{2}\nu}{\sqrt{\varepsilon}}\Lambda(y)\right)\frac{\partial P^{\varepsilon}}{\partial y}dt, \end{aligned} \tag{5.3}$$

with the terminal condition

$$P^{\varepsilon}(T,s,y,g) = \left(K - e^{\frac{g}{T}}\right)^{+}.$$

Applying the fractional calculus relation Eq. (2.61) derived in Section 2.3.3, we can rewrite the PDE (5.3) as

$$\begin{aligned} 0 &= \frac{\partial P^{\varepsilon}}{\partial t} + \frac{1}{2}s^2 f^2(y)\frac{\Gamma(1+2H)\Gamma(2-2H)}{t^{1-2H}}\frac{\partial^2 P^{\varepsilon}}{\partial s^2} + r\left(s\frac{\partial P^{\varepsilon}}{\partial s} - P^{\varepsilon}\right) \\ &\quad + \ln(s)\frac{\partial P^{\varepsilon}}{\partial g} + \frac{\nu^2}{\varepsilon}\frac{\partial^2 P^{\varepsilon}}{\partial y^2} + \left(\frac{1}{\varepsilon}(m-y) - \frac{\sqrt{2}\nu}{\sqrt{\varepsilon}}\Lambda(y)\right)\frac{\partial P^{\varepsilon}}{\partial y}, \end{aligned} \tag{5.4}$$

with the terminal condition

$$P^{\varepsilon}(T,s,y,g) = \left(K - e^{\frac{g}{T}}\right)^{+}.$$

We can then reorganize the different terms of Eq. (5.4) with varying orders of ε as follows:

$$
\begin{aligned}
0 &= \frac{1}{\varepsilon}\left((m-y)\frac{\partial}{\partial y}+\nu^2\frac{\partial^2}{\partial y^2}\right)P^\varepsilon \\
&+\frac{1}{\sqrt{\varepsilon}}\left(-\sqrt{2}\nu\Lambda(y)\frac{\partial}{\partial y}\right)P^\varepsilon \\
&+\left(\frac{\partial}{\partial t}+\frac{1}{2}s^2f^2(y)\frac{\Gamma(1+2H)\Gamma(2-2H)}{t^{1-2H}}\frac{\partial^2}{\partial s^2}+rs\frac{\partial}{\partial s}-r+\ln(s)\frac{\partial}{\partial g}\right)P^\varepsilon,
\end{aligned}
$$

where we can define the following operators

$$
\begin{aligned}
\mathcal{L}_0 &= (m-y)\frac{\partial}{\partial y}+\nu^2\frac{\partial^2}{\partial y^2}, \\
\mathcal{L}_1 &= -\sqrt{2}\nu\Lambda(y)\frac{\partial}{\partial y}, \\
\mathcal{L}_2 &= \frac{\partial}{\partial t}+\frac{1}{2}s^2f^2(y)\frac{\Gamma(1+2H)\Gamma(2-2H)}{t^{1-2H}}\frac{\partial^2}{\partial s^2}+rs\frac{\partial}{\partial s}-r+\ln(s)\frac{\partial}{\partial g}.
\end{aligned}
$$

5.2 Asymptotic Approach with Mellin Transform

In this section, we apply an asymptotic expansion approach to establish partial differential equations, which will be used to derive an approximate solution to Eq. (5.4) and thus find an approximate value of a geometric Asian put option.

Following Fouque and Han (2006) and Fouque et al. (2011), we expand the function P^ε as follows

$$P^\varepsilon = P_0+\sqrt{\varepsilon}P_1+\varepsilon P_2+\varepsilon\sqrt{\varepsilon}P_3+\dots, \tag{5.5}$$

where P_0, P_1, ... are functions corresponding to varying orders of ε. Then we can substitute Eq. (5.5) into the Eq. (5.4) to obtain

$$\begin{aligned} 0 &= \left(\frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1 + \mathcal{L}_2\right)\left(P_0 + \sqrt{\varepsilon}P_1 + \varepsilon P_2 + \varepsilon\sqrt{\varepsilon}P_3 + ...\right) \\ &= \frac{1}{\varepsilon}L_0 P_0 + \frac{1}{\sqrt{\varepsilon}}\left(\mathcal{L}_1 P_0 + \mathcal{L}_0 P_1\right) + \left(\mathcal{L}_0 P_2 + \mathcal{L}_1 P_1 + \mathcal{L}_2 P_0\right) \\ &\quad + \sqrt{\varepsilon}\left(\mathcal{L}_0 P_3 + \mathcal{L}_1 P_2 + \mathcal{L}_2 P_1\right) + ... \end{aligned} \tag{5.6}$$

From the order $O(\frac{1}{\varepsilon})$ term in Eq. (5.6), we get

$$\mathcal{L}_0 P_0 = 0,$$

so, if P_0 does not grow as fast as $e^{\frac{y^2}{2}}$, then P_0 is independent of y.

From the the order $O(\frac{1}{\sqrt{\varepsilon}})$ terms in Eq. (5.6), we get

$$\mathcal{L}_1 P_0 + \mathcal{L}_0 P_1 = 0,$$

which implies

$$\mathcal{L}_0 P_1 = 0,$$

since P_0 is independent of y implying $\mathcal{L}_1 P_0 = 0$. Assuming P_1 does not grow as fast as $e^{\frac{y^2}{2}}$, we can also deduce that P_1 is independent of y.

From the order $O(1)$ term in Eq. (5.6), we get

$$\mathcal{L}_0 P_2 + \mathcal{L}_1 P_1 + \mathcal{L}_2 P_0 = 0,$$

which implies

$$\langle \mathcal{L}_2 P_0 \rangle = \langle \mathcal{L}_2 \rangle P_0 = 0$$

and

$$\left(\frac{\partial}{\partial t} + \frac{1}{2} s^2 \bar{f}^2 \frac{\Gamma(1+2H)\Gamma(2-2H)}{t^{1-2H}} \frac{\partial^2}{\partial s^2} + rs\frac{\partial}{\partial s} - r + \ln s \frac{\partial}{\partial g} \right) P_0 = 0, \tag{5.7}$$

where $\bar{f} = \langle f(y) \rangle$.

In order to solve the Eq. (5.7) for P_0, we need to change the independent variables several times. Firstly we change the independent variable s to x by letting $x = \ln s$, then Eq. (5.7) becomes

$$\left(\frac{\partial}{\partial t} + \frac{1}{2} \bar{f}^2 \frac{\Gamma(1+2H)\Gamma(2-2H)}{t^{1-2H}} \frac{\partial^2}{\partial x^2} + \left(r - \frac{1}{2}\bar{f}^2 \right) \frac{\partial}{\partial x} - r + x \frac{\partial}{\partial g} \right) P_0 = 0. \tag{5.8}$$

We further let

$$x' = g + (T-t)\, x$$

then the Eq. (5.8) can be updated as

$$\left(\frac{\partial}{\partial t} + \frac{\bar{f}^2 (T-t)^2}{2} \frac{\Gamma(1+2H)\Gamma(2-2H)}{t^{1-2H}} \frac{\partial^2}{\partial x'^2} + \left(r - \frac{\bar{f}^2}{2} \right) (T-t) \frac{\partial}{\partial x'} - r \right) P_0 = 0, \tag{5.9}$$

with the terminal condition

$$P_0(T, x') = \left(K - e^{\frac{x'}{T}} \right)^+ .$$

Next, we introduce a new variable $s' = e^{\frac{x'}{T}}$, and the Eq. (5.9) becomes

$$\left(\frac{\partial}{\partial t} + \frac{\bar{f}^2\left(T-t\right)^2}{2T^2}\frac{\Gamma(1+2H)\Gamma(2-2H)}{t^{1-2H}}\left(s'\frac{\partial}{\partial s'} + s'^2\frac{\partial^2}{\partial s'^2}\right) + \left(r - \frac{1}{2}\bar{f}^2\right)\frac{(T-t)}{T}s'\frac{\partial}{\partial s'}\right)P_0 = rP_0 \tag{5.10}$$

with the terminal condition

$$P_0\left(s', T\right) = \max\left(K - s', 0\right).$$

5.3 Analytic Approximate Solutions for Geometric Asian Option Prices

5.3.1 The P_0 Term for Geometric Asian Put Options

We apply the Mellin transform in Table 2.1 to Eq. (5.10) to convert this PDE into the following ODE:

$$\frac{d\hat{P}_0}{dt} + \left(\frac{\bar{f}^2\left(T-t\right)^2}{2T^2}\frac{\Gamma(1+2H)\Gamma(2-2H)}{t^{1-2H}}w^2 - \frac{\left(r-\frac{1}{2}\bar{f}^2\right)(T-t)}{T}w - r\right)\hat{P}_0 = 0.$$

Solving the above ODE by separating variables, we obtain

$$\ln\hat{P}_0(t,w) = \ln\hat{P}_0\left(T,w\right) + \frac{\bar{\sigma}^2(t)}{2}\left(T-t\right)w^2 - \frac{\left(r-\frac{1}{2}\bar{f}^2\right)\left(T-t\right)^2}{2T}w - r\left(T-t\right),$$

where

$$\bar{\sigma}^2(t) = \frac{\bar{f}^2\Gamma(1+2H)\Gamma(2-2H)}{T^2}\left(\frac{T^2(T^{2H}-t^{2H})}{2H} - \frac{2T(T^{2H+1}-t^{2H+1})}{2H+1} + \frac{T^{2H+2}-t^{2H+2}}{2H+2}\right)$$

which is equal to $\frac{\bar{f}^2}{3}\frac{(T-t)^3}{T^2}$ for $H = 1/2$. Then we can apply the technique of completing the squares for the equation above to obtain the following form

$$\ln \hat{P}_0(t,w) = \ln \hat{P}_0(T,w) + \lambda(w+\eta)^2 + \delta,$$

which yields

$$\hat{P}_0(t,w) = \hat{P}_0(T,w)\, e^{\lambda(w+\eta)^2+\delta},$$

where

$$\begin{aligned} \lambda &= \frac{\bar{\sigma}^2}{2}(T-t) \\ 2\lambda\eta &= -\frac{\left(r-\frac{1}{2}\bar{f}^2\right)(T-t)^2}{2T} \\ \lambda\eta^2+\delta &= -r(T-t). \end{aligned}$$

For the next step, we can apply the inverse Mellin transform specified in Table 2.1 to obtain

$$\begin{aligned} P_0(t,s') &= P_0(T,s') * M^{-1}\left(e^{\lambda(w+\eta)^2+\delta}\right) \\ &= P_0(T,s') * e^{\delta}(s')^{\eta}\left(\frac{1}{2}\frac{1}{\sqrt{\pi}}\lambda^{\frac{-1}{2}} e^{\frac{-1}{4\lambda}(\ln s')^2}\right) \\ &= \int_0^K (K-u)\, e^{\delta}\left(\frac{s'}{u}\right)^{\eta}\left(\frac{1}{2}\frac{1}{\sqrt{\pi}}\lambda^{\frac{-1}{2}} e^{\frac{-1}{4\lambda}\left(\ln\left(\frac{s'}{u}\right)\right)^2}\right)\frac{du}{u} \end{aligned}$$

In order to evaluate the integral, we change the variable from u to $v = \ln u$, and the integral becomes

$$\begin{aligned}
P_0(t,s') &= \int_{-\infty}^{\ln K} (K - e^v)\, e^{\delta}\, (s')^{\eta}\, e^{-\eta v} \left(\frac{1}{2} \frac{1}{\sqrt{\pi}} \lambda^{\frac{-1}{2}} e^{\frac{-1}{4\lambda}(\ln s' - v)^2} \right) dv \\
&= e^{\delta}\, (s')^{\eta}\, \frac{1}{2\sqrt{\pi}} \lambda^{\frac{-1}{2}} \int_{-\infty}^{\ln K} K e^{\frac{-1}{4\lambda}\left((v - \ln s' + 2\lambda\eta)^2 - 4\lambda^2\eta^2 + 4\lambda\eta \ln s'\right)} \\
&\quad - e^{\frac{-1}{4\lambda}\left((v - \ln s' + 2\lambda(\eta-1))^2 - 4\lambda^2(\eta-1)^2 + 4\lambda(\eta-1)\ln s'\right)} d\nu, \\
&= e^{\delta}\, (s')^{\eta}\, K N\left(\frac{\ln \frac{K}{s'} + 2\lambda\eta}{\sqrt{2\lambda}} \right) e^{\lambda\eta^2 - \eta \ln s'} \\
&\quad - e^{\delta} (s')^{\eta} e^{\lambda\eta^2 - 2\lambda\eta + \lambda + (\eta-1)\ln s'} N\left(\frac{\ln \frac{K}{s'} + 2\lambda\eta}{\sqrt{2\lambda}} - \sqrt{2\lambda} \right).
\end{aligned}$$

We can simplify the above expression using the following relations

$$\begin{aligned}
\lambda\eta^2 + \delta &= -r\,(T-t) \\
2\lambda\eta &= \frac{-\left(r - \frac{1}{2}\bar{f}^2\right)(T-t)^2}{2T} \\
\lambda &= \frac{\bar{\sigma}^2}{2}\,(T-t) \\
\bar{\sigma}^2(t) &= \frac{\bar{f}^2 \Gamma(1+2H)\Gamma(2-2H)}{T^2} \left(\frac{T^2(T^{2H} - t^{2H})}{2H} - \frac{2T(T^{2H+1} - t^{2H+1})}{2H+1} + \frac{T^{2H+2} - t^{2H+2}}{2H+2} \right)
\end{aligned}$$

and then obtain

$$\begin{aligned}
P_0\,(t,s') &= e^{-r(T-t)} \left[K N\left(\frac{-\ln \frac{s'}{K} - \frac{\left(r - \frac{1}{2}\bar{f}^2\right)(T-t)^2}{2T}}{\sqrt{\bar{\sigma}^2\,(T-t)}} \right) \right. \\
&\quad \left. - s' e^{\frac{\left(r - \frac{1}{2}\bar{f}^2\right)(T-t)^2}{2T} + \frac{\bar{\sigma}^2}{2}(T-t)} N\left(\frac{-\ln \frac{s'}{K} - \frac{\left(r - \frac{1}{2}\bar{f}^2\right)(T-t)^2}{2T}}{\sqrt{\bar{\sigma}^2\,(T-t)}} - \bar{\sigma}\sqrt{T-t} \right) \right]
\end{aligned} \tag{5.11}$$

5.3.2 The P_1 Term for Geometric Asian Put Options

Note that the centering condition in Eq. (5.7) is

$$\langle \mathcal{L}_2 P_0 \rangle = 0.$$

Since P_0 does not depend on y, it implies $\langle \mathcal{L}_2 \rangle P_0 = 0$. As the centering condition is satisfied, we can deduce that

$$-\mathcal{L}_0 P_2 = \mathcal{L}_2 P_0 = \mathcal{L}_2 P_0 - \langle \mathcal{L}_2 P_0 \rangle = \frac{1}{2} s^2 \left(f(y)^2 - \bar{f}^2 \right) \frac{\Gamma(1+2H)\Gamma(2-2H)}{t^{1-2H}} \frac{\partial^2 P_0}{\partial s^2}.$$

Hence, the second-order correction P_2 is a solution of the Poisson equation given by the following form

$$\begin{aligned} P_2(t,s,y,g) &= -\frac{1}{2} \mathcal{L}_0^{-1} \left(f(y)^2 - \bar{f}^2 \right) s^2 \frac{\Gamma(1+2H)\Gamma(2-2H)}{t^{1-2H}} \frac{\partial^2 P_0}{\partial s^2} \\ &= -\frac{1}{2} \left(\phi(y) + c(t,s,g) \right) s^2 \frac{\Gamma(1+2H)\Gamma(2-2H)}{t^{1-2H}} \frac{\partial^2 P_0}{\partial s^2}, \end{aligned}$$

where $\phi(y)$ is a solution of the following Poisson equation

$$\mathcal{L}_0 \phi = f(y)^2 - \langle f^2 \rangle$$

and $c(t,s,g)$ is independent of y that may depend on variables t, s and g.

From the order $O(\sqrt{\varepsilon})$ term in Eq. (5.6), we get

$$\mathcal{L}_0 P_3 + \mathcal{L}_1 P_2 + \mathcal{L}_2 P_1 = 0.$$

This is again a Poisson equation for P_3 with respect to the operator $\mathcal{L}_0$, which requires the centering condition

$$\langle \mathcal{L}_1 P_2 + \mathcal{L}_2 P_1 \rangle = 0.$$

Using the result for P_2, the fact that P_1 does not depend on y, we deduce that

$$\langle \mathcal{L}_2 \rangle P_1 = \frac{1}{2} \langle \mathcal{L}_1 \phi(y) \rangle s^2 \frac{\Gamma(1+2H)\Gamma(2-2H)}{t^{1-2H}} \frac{\partial^2 P_0}{\partial s^2}.$$

Notice that $\mathcal{L}_1 c(t,s,g) = 0$, since $\mathcal{L}_1$ takes derivatives with respect to y and $c(t,s,g)$ is independent of y. We can compute the term $\langle \mathcal{L}_1 \phi(y) \rangle$ as

$$\langle \mathcal{L}_1 \phi(y) \rangle = -\sqrt{2}\nu \langle \Lambda(y) \phi'(y) \rangle .$$

Then the equation for P_1 can be expressed as:

$$\langle \mathcal{L}_2 \rangle P_1 = \frac{\Gamma(1+2H)\Gamma(2-2H)}{t^{1-2H}} \left(-\frac{\sqrt{2}}{2} \nu \langle \Lambda \phi' \rangle s^2 \frac{\partial^2 P_0}{\partial s^2} \right),$$

with the terminal condition

$$P_1(T,s,g) = 0.$$

Therefore, the correction term P_1 satisfies the PDE

$$\begin{aligned} &\left(\frac{\partial}{\partial t} + \frac{1}{2} s^2 \bar{f}^2 \frac{\Gamma(1+2H)\Gamma(2-2H)}{t^{1-2H}} \frac{\partial^2}{\partial s^2} + rs \frac{\partial}{\partial s} - r + \ln s \frac{\partial}{\partial g} \right) P_1 \\ = \quad & \frac{\Gamma(1+2H)\Gamma(2-2H)}{t^{1-2H}} \left(-\frac{\sqrt{2}}{2} \nu \langle \Lambda \phi' \rangle s^2 \frac{\partial^2 P_0}{\partial s^2} \right). \end{aligned}$$

Using similar steps by changing variables and the Mellin transform, we get the ODE for $\hat{P}_1(t,w)$ as

$$\begin{aligned} &\frac{d\hat{P}_1}{dt} + \left(\frac{1}{2} \bar{f}^2 \frac{(T-t)^2}{T^2} \frac{\Gamma(1+2H)\Gamma(2-2H)}{t^{1-2H}} w^2 - \left(r - \frac{1}{2}\bar{f}^2 \right) \frac{(T-t)}{T} w - r \right) \hat{P}_1(t,w) \\ = \quad & -\frac{\sqrt{2}}{2} \nu \langle \Lambda \phi' \rangle \frac{\Gamma(1+2H)\Gamma(2-2H)}{t^{1-2H}} \left(\frac{(T-t)^2}{T^2} w^2 + \frac{(T-t)}{T} w \right) \hat{P}_0. \end{aligned}$$

Using the integrating factor method and $\hat{P}_1(T,w)=0$, we obtain

$$
\begin{aligned}
&\hat{P}_1(t,w)\\
&= -\frac{\sqrt{2}}{2}\nu\langle\Lambda\phi'\rangle\hat{P}_0(t,w)\int_T^t \frac{\Gamma(1+2H)\Gamma(2-2H)}{\tau^{1-2H}}\left(\frac{(T-\tau)^2}{T^2}w^2+\frac{(T-\tau)}{T}w\right)d\tau,\\
&= -\frac{\sqrt{2}}{2}\nu\langle\Lambda\phi'\rangle\hat{P}_0(t,w)\left(A_1(t)w+A_2(t)w^2\right),
\end{aligned}
$$

where

$$
\begin{aligned}
A_1(t) &= \int_T^t \frac{\Gamma(1+2H)\Gamma(2-2H)}{\tau^{1-2H}}\frac{(T-\tau)}{T}d\tau \\
&= \frac{\Gamma(1+2H)\Gamma(2-2H)}{T}\left(\frac{t^{2H}T-T^{2H+1}}{2H}-\frac{t^{2H+1}-T^{2H+1}}{2H+1}\right)\\
A_2(t) &= \int_T^t \frac{\Gamma(1+2H)\Gamma(2-2H)}{\tau^{1-2H}}\frac{(T-\tau)^2}{T^2}d\tau\\
&= \frac{\Gamma(1+2H)\Gamma(2-2H)}{T^2}\left(\frac{t^{2H}T^2-T^{2H+2}}{2H}-\frac{2t^{2H+1}T-2T^{2H+2}}{2H+1}+\frac{t^{2H+2}-T^{2H+2}}{2H+2}\right)
\end{aligned}
\tag{5.12}
$$

Note that $A_1(t)$ and $A_2(t)$ have closed forms which can be derived from standard calculus techniques.

The inverse Mellin transform of $\hat{P}_1(t,w)$ can be expressed as

$$
\begin{aligned}
P_1(t,s') &= M^{-1}\left(\hat{P}_1(t,w)\right)\\
&= -\frac{\sqrt{2}}{2}\nu\langle\Lambda\phi'\rangle\left(A_2(t)M^{-1}\left(w^2\hat{P}_0(t,w)\right)+A_1(t)M^{-1}\left(w\hat{P}_0(t,w)\right)\right).
\end{aligned}
\tag{5.13}
$$

One of the basic property of the Mellin transform is

$$M^{-1}(w\hat{f}(w)) = s'\frac{d}{ds'}f(s').$$

In particular, we can deduce that

$$\begin{aligned} M^{-1}\left(w\hat{P}_0(t,w)\right) &= -s'\frac{d}{ds'}P_0(t,s') \\ M^{-1}\left(w^2\hat{P}_0(t,w)\right) &= s'\frac{d}{ds'}\left(s'\frac{d}{ds'}P_0(t,s')\right) \\ &= s'\frac{d}{ds'}P_0(t,s') + s'^2\frac{d}{ds'^2}P_0(t,s') \end{aligned}$$

Hence, we obtain an explicit closed-form expression for P_1 as follows

$$\begin{aligned} P_1(t,s') &= M^{-1}\left(\hat{P}_1(t,w)\right) \\ &= -\frac{\sqrt{2}}{2}\nu\langle\Lambda\phi'\rangle\left(A_2(t)M^{-1}\left(w^2\hat{P}_0(t,w)\right) + A_1(t)M^{-1}\left(w\hat{P}_0(t,w)\right)\right). \\ &= -\frac{\sqrt{2}}{2}\nu\langle\Lambda\phi'\rangle\left(A_2(t)\left(s'\frac{d}{ds'}P_0(t,s') + s'^2\frac{d}{ds'^2}P_0(t,s')\right) - A_1(t)s'\frac{d}{ds'}P_0(t,s')\right), \end{aligned} \tag{5.14}$$

where $A_1(t)$ and $A_2(t)$ are given in Eq. (5.12).

We summarize the above analysis and calculation on geometric Asian put options in the following theorem.

Theorem 5.3.1 *Under the fSV model governed by Eq.* (5.1)*, the risk-neutral value* P^ε *of a geometric Asian put option can be approximated by the following formula*

$$P^\varepsilon \approx P_0 + \sqrt{\varepsilon}P_1, \tag{5.15}$$

where P_0 *and* P_1 *are given by Eq.* (5.11) *and Eq.* (5.14)*, respectively.*

5.4 Numerical Analysis

Following Fouque et al. (2000), Fouque et al. (2011) and Cao et al. (2021), we choose f to take the following form

$$f(y) = 0.35\left(\tan^{-1}(y) + \frac{\pi}{2}\right)/\pi + 0.05.$$

The values of other parameters used in this section are similar to those used in Wong and Cheung (2004). Table 5.1 specify the model parameters, whenever they are required to be fixed. As done in in Wong and Cheung (2004), we choose 10 different values of

Table 5.1: The role and numerical value of parameters.

Parameter	Role	Value
r	risk-free interest rate	0.03
t	time when the geometric Asian option is evaluated	1
S_t	asset price at time t	15
T	time when the geometric Asian option is expired	2
ε	small parameter indicating the mean reverting speed of Y_t	0.01
H	Hurst parameter	0.3

moneyness G_t/S_t between 0.5 and 1.5 (inclusive). For each selected value of moneyness, we apply Theorem 5.3.1 to obtain the approximation to the price of a for the geometric Asian option. It takes less than 1 second to find the numerical solutions. The results are shown in the following figures.

Figure (5.1) shows how the value of $P_0 + \sqrt{\varepsilon}P_1$ for a geometric Asian put option varies with respect to the change of ε values. As we can see, when the value of ε changes from 0.01 to 0.0001, the value of $P_0 + \sqrt{\varepsilon}P_1$ does not vary much. In fact, the values of $P_0 + \sqrt{\varepsilon}P_1$ match well with the results of Monte-Carlo simulation in all cases. Furthermore, in all cases, the values of $P_0 + \sqrt{\varepsilon}P_1$ decline as s increases.

Figure (5.2) illustrates the dependence of the value of $P_0 + \sqrt{\varepsilon}P_1$ for a geometric Asian put option on different values of the Hurst parameter H. It is evident that a

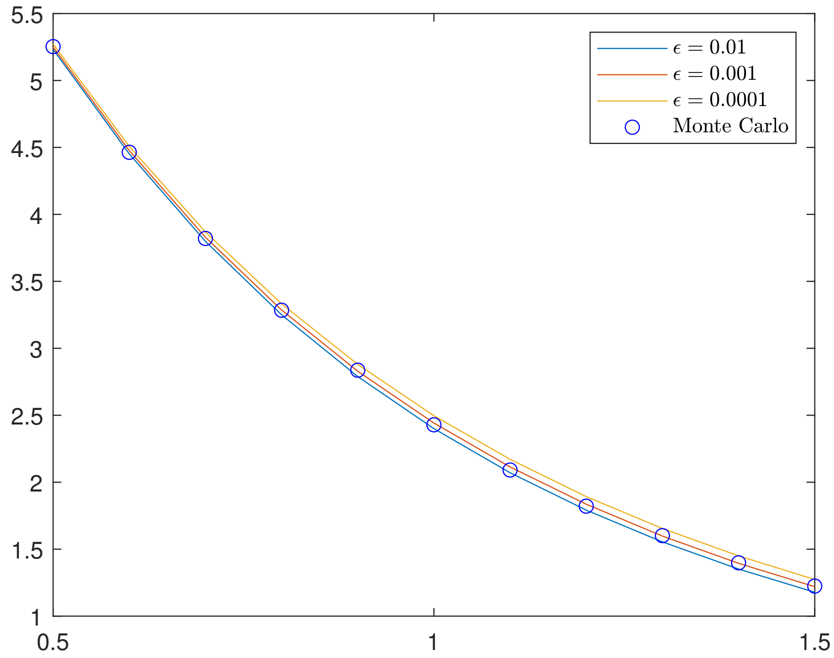


Figure 5.1: Plot of $P_0 + \sqrt{\varepsilon}P_1$ against different values of ε for a geometric Asian put option.

change in H from 0.3 to 0.7 leads to a notable variation in the value of $P_0 + \sqrt{\varepsilon}P_1$. This observation highlights the significant impact of fractional Brownian motion on the pricing of geometric Asian options. Furthermore, for all cases, the values of $P_0 + \sqrt{\varepsilon}P_1$ decline as s increases.

5.5 Conclusion

In this chapter, we derive an explicit closed-form solution for the first-order approximation of a geometric Asian put option price under a fractional stochastic volatility model. We apply an asymptotic expansion method and the Mellin transform to obtain this solution. Notably, the zero-order term in our solution coincides with those obtained in previous studies such as Hull (2015) or Haug (2006) when the Hurst parameter H is set to 1/2, corresponding to the classical Black-Scholes model.

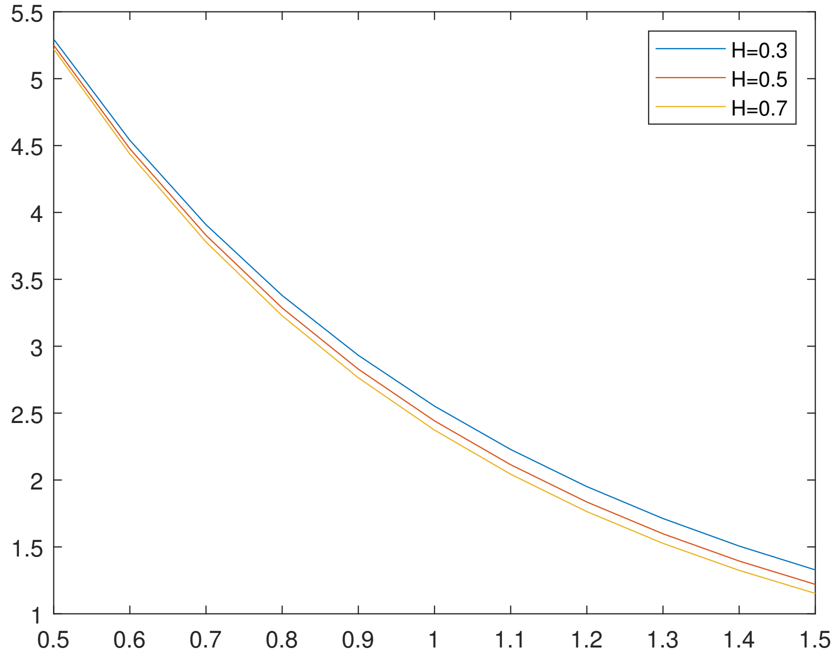


Figure 5.2: Plot of $P_0 + \sqrt{\varepsilon}P_1$ against different values of Hurst parameters H for a geometric Asian put option.

To validate the accuracy of our approximation, we conduct a numerical analysis and compare the results obtained from our explicit closed-form solution with those generated through Monte Carlo simulation. The close agreement between the two sets of results confirms the accuracy of our approximation.

The numerical analysis demonstrates the variation of geometric Asian put option prices with respect to different the Hurst parameters H. We observe a substantial change in the geometric Asian put option values as the Hurst parameter varies. This indicates the significance of fractional Brownian motion's impact on pricing geometric Asian options.

Chapter 6

Conclusion and Future Works

In this thesis, we aim to evaluate the fair prices of several path-dependent options by developing new mathematical models under stochastic volatility or fractional Brownian motion. In this chapter, we summarize the major results of our research and give some suggestions which can be possible used to improve our works in the future.

6.1 Conclusion

The main contributions of this research are to provide efficient approximate closed-form pricing formulas for several path-dependent options under various hybrid stochastic volatility and local volatility models. We have conducted numerical experiments to demonstrate that our pricing formulas outperform the traditional evaluation methods in terms of accuracy and efficiency. In terms of modeling, the hybrid models are capable to capture most phenomena observed in the market. However, the sudden changes observed in the market are not captured well in these hybrid models. Therefore, we introduce a hybrid model with fractional Brownian motion to account for the sudden changes observed in the market.

Chapter 1 gives an overview of existing literature and outlines the structure of the thesis. Chapter 2 provides preliminary knowledge and necessary tools for the study of the path-dependent option evaluation problems.

In Chapter 3 we derive the fair price of a barrier option or a lookback option under a stochastic volatility model. We establish explicit closed-form formulas for first order approximations to the prices of down-and-out barrier and floating strike lookback put options under a stochastic volatility model by means of Mellin transform. The zero-order terms in the solutions for the prices of both types of put options coincide with those in Hull (2015) or Haug (2006) under the classical Back-Scholes model. Our numerical analysis shows that the results given by those explicit closed-form solutions match well with those generated by Monte-Carlo simulation. This confirms the accuracy of the approximation. Furthermore, we also discuss the sensitivity of the first-order error terms and the approximation with respect to the underlying asset price and the mean-reverting speed of the OU-process which governs the volatility.

In Chapter 4, we obtain the fair price formula of a barrier option or a lookback option under a hybrid model of stochastic volatility (SV) and constant elasticity of variance (CEV). We provide closed-form formulas for the first-order approximations to the prices of the down-and-out and floating strike lookback put options under the SVCEV model. The zero-order terms are the classical solutions under the geometric Brownian motion framework, which match with those given in the literature. In addition, we perform numerical analysis to illustrate that the results from the explicit closed-form formulas are quite close to those generated by Monte-Carlo simulation and the binomial tree method. Our results on the down-and-out put options could be easily extended to other types of barrier options.

Chapter 5 is devoted to investigate the fair price of an geometric Asian option under a hybrid model with fractional Brownian motion. We provide closed-form formula for the first-order approximation to the prices of the geometric Asian put options under the fSV model. The zero-order term is the classical solution of a geometric Asian put option under the geometric Brownian motion framework, which matches with that given in the literature. In addition, we perform numerical analysis to illustrate that the results from the explicit closed-form formula are quite close to those generated by Monte-Carlo simulation. Our results on the geometric Asian put option could be easily extended to other types of geometric Asian options and path-dependent options as well.

6.2 Future Works

The research presented in this thesis has raised some new issues. Many different experiments and algorithm implementations have been left for the future due to the time and resource limitations. In the future research, it is worth to conduct deeper level analysis of the evaluation of other path-dependent options. To address this point, we will try to use new theoretical methods and new available resources. The possible future works are listed below,

1. In Chapters 3 and 4, the calibration of the SV and SVCEV models can be performed using the vanilla option data. However, without the path-dependent option data, it is impossible to validate the calibration accuracy of the SV and SVCEV models. We are in search of some relevant path-dependent option data. After data are collected, we can estimate parameters of the SV and SVCEV models which can be applied to the path-dependent option pricing formulas. In this way, we can compare our pricing formulas with other existing evaluation methods using the market data.

2. In Chapter 5, we provide a pricing formula for geometric Asian put options under a fSV model. There is another way to derive pricing formulas under fSV using the Wick product method (Øksendal & Sulem, 2005). We plan to undertake research in this direction and compare results with those obtained in this thesis. Since the

application of fractional Brownian motion in finance is an active research field. More theoretical results will be expected in the near future.

3. We also plan to carry out systematic research on valuations of other types of path-dependent options under the fractional environment in the future.

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