Pricing volatility derivatives with stochastic volatility

Guanghua Lian
University of Wollongong

Pricing Volatility Derivatives with Stochastic Volatility

A thesis submitted in fulfillment of the

requirements for the award of the degree of

Doctor of Philosophy

from

University of Wollongong

by

Guanghua Lian, B.Sc. (Sichuan University)

M.A. (Huazhong University of Science and Technology)

School of Mathematics and Applied Statistics

2010

CERTIFICATION

I, Guanghua Lian, declare that this thesis, submitted in fulfilment of the requirements for the award of Doctor of Philosophy, in the School of Mathematics and Applied Statistics, University of Wollongong, is wholly my own work unless otherwise referenced or acknowledged. The document has not been submitted for qualifications at any other academic institution.

Guanghua Lian

March, 2010

Acknowledgements

I would like to express my sincerest gratitude and appreciation to my supervisor, Professor Song-Ping Zhu, for his insightful guidance and substantial advice throughout the research. I, in particular, appreciate him for introducing me into this wonderful research area and inspiring my numerous research ideas. His high professional standard and rigorous attitude towards research have greatly influenced me and become my principle that I will abide by in all of my life. It is he who transformed me from a raw beginner into an active researcher, which fulfills my dream of pursuing research in mathematical finance.

Also, I am especially grateful to Dr. Xiao-Ping Lu for her constant encouragements and warm care to me during my study. Without her help this thesis could not have reached its present form. I wish to thank all the fellow friends in the Center of Financial Mathematics and the School of Mathematics and Applied Statistics in University of Wollongong, particularly Professor Matt Wand and Dr. Pam Davy, who taught me statistics and Markov Chain Monte Carlo method, Professor Timothy Marchant and Dr. Mark Nelson for their encouragement to me and valuable advice for my research and career development. I trust that all other people whom I have not specifically mentioned here are aware of my deep appreciation.

Finally, the financial support from the University of Wollongong with HDR tuition scholarship and University Postgraduate Research Award is also gratefully acknowledged. I thank my parents and family for their years of dedication and support to me and for their sincere concern about my life.

Abstract

Volatility derivatives are products where the volatility is the main underlying notion. These products are particularly important for market investors as they use them to have insight into the level of volatility to efficiently manage the market volatility risk. This thesis makes a contribution to literature by presenting a set of closed-form exact solutions for the pricing of volatility derivatives.

The first issue is the pricing of variance swaps, which is discussed in Chapter 2, 3, and 4. We first present an approach to solve the partial differential equation (PDE), based on the Heston (1993) two-factor stochastic volatility, to obtain closed-form exact solutions to price variance swaps with discrete sampling times. We then extend our approach to price forward-start variance swaps to obtain closed-form exact solutions. Finally, our approach is extended to price discretely-sampled variance by further including random jumps in the return and volatility processes. We show that our solutions can substantially improve the pricing accuracy in comparison with those approximations in literature. Our approach is also very versatile in terms of treating the pricing problem of variance swaps with different definitions of discretely-sampled realized variance in a highly unified way.

The second issue, which is covered in Chapter 5, and 6, is the pricing method for volatility swaps. Papers focusing on analytically pricing discretely-sampled volatility swaps are rare in literature, mainly due to the inherent difficulty associated with the nonlinearity in the pay-off function. We present a closed-form exact solution for the pricing of discretely-sampled volatility swaps, under the framework of Heston (1993) stochastic volatility model, based on the definition of the so-called average of realized volatility. Our closed-form exact solution for discretely-sampled volatility swaps can significantly reduce the computational time in obtaining numerical values for the discretely-sampled volatility swaps, and substantially improve the computational accuracy of discretely-sampled volatility swaps, comparing with the continuous sampling approximation. We also investigate the accuracy of the well-known convexity correction approximation in pricing volatility swaps. Through both theoretical analysis and numerical examples,

we show that the convexity correction approximation would result in significantly large errors on some specifical parameters. The validity condition of the convexity correction approximation and a new improved approximation are also presented.

The last issue, which is covered in Chapter 7 and 8, is the pricing of VIX futures and options. We derive closed-form exact solutions for the fair value of VIX futures and VIX options, under stochastic volatility model with simultaneous jumps in the asset price and volatility processes. As for the pricing of VIX futures, we show that our exact solution can substantially improve the pricing accuracy in comparison with the approximation in literature. We then demonstrate how to estimate model parameters, using the Markov Chain Monte Carlo (MCMC) method to analyze a set of coupled VIX and S&P500 data. We also conduct empirical studies to examine the performance of the four different stochastic volatility models with or without jumps. Our empirical studies show that the Heston stochastic volatility model can well capture the dynamics of S&P500 already and is a good candidate for the pricing of VIX futures. Incorporating jumps into the underlying price can indeed further improve the pricing the VIX futures. However, jumps added in the volatility process appear to add little improvement for pricing VIX futures. As for the pricing of VIX options, we point out the solution procedure of Lin & Chang (2009)'s pricing formula for VIX options is wrong, and alert the research community that this formula should not be further used. More importantly, we present a new closed-form pricing formula for VIX options and demonstrate its high efficiency in computing the numerical values of the price of a VIX option. The numerical examples show that results obtained from our formula consistently match up with those obtained from Monte Carlo simulation perfectly, verifying the correctness of our formula; while the results obtained from Lin & Chang (2009)'s pricing formula significantly differ from those from Monte Carlo simulation. Some other important and distinct properties of the VIX options (e.g., put-call parity, the hedging ratios) have also been discussed.

Contents

List of Figures

List of Tables

Chapter 1

Introduction and Background

1.1 Volatility Derivatives

Volatility derivatives are special financial derivatives whose values depend on the future level of volatility. While volatility is traditionally viewed as a measure of variability, or risk, of an underlying asset, the rapid development of trading volatility derivatives introduces a new view on volatility not only as a measure of volatility risk, but also an independent asset class. Hereby, by trading volatility derivatives, volatility, like any other asset, can be used by itself in a variety of trading strategies. Even though it is also possible to obtain the exposure to volatility before the introduction of volatility derivatives by taking and delta-hedging the positions in vanilla options, this alternative approach however has an obvious weakness- the necessity of continuous delta-hedging. The frequent re-balancing to keep the options portfolio delta-neutral, as required by the delta-hedging (constant buying/ selling of underlying), generates transaction costs and can be connected with liquidity problems: some stocks and indices can be expensive to trade or they may lack liquidity. On the contrary, volatility derivatives do not have this drawback; they offer straightforward and pure exposure to the volatility of the underlying asset (see e.g., Carr & Madan 1998).

By providing a more efficient solution to obtain pure exposure to volatility alone, trading volatility derivatives has been growing rapidly in the last decades. Investors in the market use volatility derivatives to have an insight into the dy-

namics of volatility, which empirical evidence shows not to be constant. For this reason, an investor who thinks current level of volatility is low, may want to take a position that profits if volatility rises. As illustrated by Demeterfi et al. (1999), there are at least three reasons for trading volatility. Firstly, one may want to take a long or short position simply due to a personal directional view of the future volatility level. Secondly, speculators may want to trade the spread between the realized volatility and the implied volatility. These two reasons involve direct speculation on the future trend of stock or index volatility. Thirdly, one may need to hedge against volatility risk of his portfolios. This is a more important reason for trading volatility since bad estimation or inefficient hedging of volatility risk might result in financial disasters. In practice, derivative products related to volatility and variance have been experiencing sharp increases in trading volume recently. Jung (2006) showed that there was still growing interest in volatility products, such as conditional and corridor variance swaps, among hedge funds and proprietary desks.

Generally, there are two types of volatility derivatives (see, Dupire 2005). Each type of volatility derivative is associated with a particular measure of volatility. The two parties of the contract define, at the beginning of the contract, the specific measure of the realized volatility to be considered. The first type of volatility derivatives is historical volatility- or variance-based products, the payoff function of which is based on the realized volatility or variance discretely sampled at some pre-specified sampling points over the time of returns on the stock price. Most products of this type are over-the-counter (OTC) contracts, such as volatility swaps, variance swap, corridor variance swap, and options on volatility/variance. There are some listed products of this kind as well, such as futures on realized variance, which are in essence "exchange-listed" version of OTC variance swaps. For example, Chicago Board Options Exchange (CBOE) launched 3-month variance futures on S&P 500 in May 2004, and 12-month variance futures in March 2006. In September 2006, New York Stock Exchange (NYSE) Euronext also started to

offer the cleared-only, on-exchange solution for variance futures on FTSE 100, CAC 40 and AEX indices. The second type of volatility derivatives is future implied-volatility based products. A lot of implied volatility indices have been launched in the major security exchanges to reflect the near-term market implied volatility, e.g., VIX index in the Chicago Board Options Exchange (CBOE) on the volatility of S&P500, VSTOXX on Dow Jones EURO STOXX50 volatility, VDAX on the volatility of DAX published by the German exchange Deutsche Böers, VX1 and VX6 published by the French exchange MONEP, etc. These indices are often used as a benchmark of equity market risk and contain expectation of option market about future volatility. The introduction of these implied volatility indices has laid a good foundation for constructing tradable volatility products and thus facilitating the hedging against volatility risk and speculating in volatility derivatives. In CBOE, a set of volatility derivatives based on the implied volatility index (VIX) has already been launched very recently, such as VIX futures in 2004, VIX options in 2006, Binary options on VIX in 2008, Mini-VIX futures 2009, etc.

1.1.1 Variance Swaps

The most common claim of volatility derivatives is variance swaps. First variance swap contracts were traded in late 1998. For the relatively short period of time, trading interest of variance swaps have been experiencing rapid growth and these OTC derivatives have developed from simple contracts on future variance to much more sophisticated products. And today we already observe the emergence of the 3-rd generation of variance swaps: gamma swaps, corridor variance swaps and conditional variance swaps.

Variance swaps are essentially forward contracts on the future realized variance of the returns of the specified underlying asset. The payoff at expiry for the long position of a variance swap is equal to the annualized realized variance over

a pre-specified period minus a pre-set delivery price of the contract multiplied by a notional amount of the swap in dollars per annualized volatility point, whereas the short position is just the opposite. Thus it can be easily used for investors to trade future realized variance against the current impled variance (the strike price of the variance swaps), gaining exposure to the so-called volatility risk. There is no cost to enter these contracts as they are essentially forward contracts. The payoff at expiry for the long position of a volatility or variance swap is equal to the realized volatility or variance over a pre-specified period minus a pre-set delivery price of the contract multiplied by a notional amount of the swap in dollars per annualized volatility point. A report* from CBOE indicates that "a recent estimate from *risk* magazine placed the daily volume in variance swaps on the major equity-indices to be US$5M vega (or dollar volatility risk per percentage point change in volatility) in the OTC markets. Furthermore, variance trading has roughly doubled every year for the past few years". Broadie & Jain (2008*a*) even estimated that daily trading volume on indices was in the region of $30 million to $35 million notional. The interest in trading volatility-based financial derivatives, such as variance swaps, seems to be still strongly growing among hedge funds and proprietary desks as Jung (2006) pointed out. It can be imagined that recent market turmoil due to the US subprime crisis would further enhance the trading of volatility-based financial derivatives, and thus greatly promote research in this area.

More specifically, the value of a variance swap at expiry can be written as $(RV - K_{var}) \times L$, where the RV is the annualized realized variance over the contract life $[0, T]$, K_{var} is the annualized delivery price for the variance swap, which is set to make the value of a variance swap equal to zero for both long and short positions at the time the contract is initially entered. To a certain extent, it reflects market's expectation of the realized variance in the future. T is the life time of the contract and L is the notional amount of the swap in dollars per

*http://cfe.cboe.com/education/finaleuromoneyvarpaper.pdf

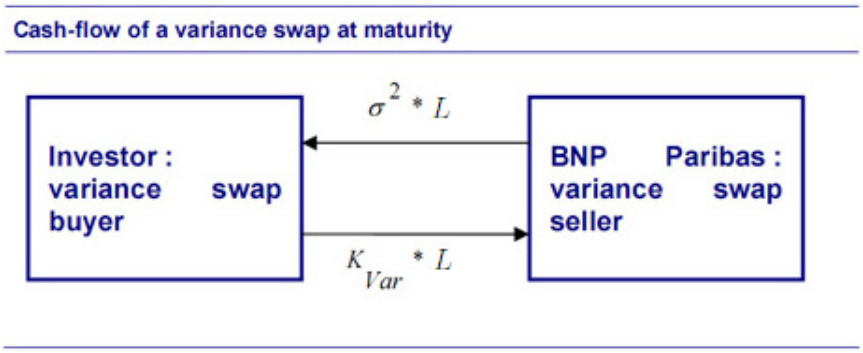

Figure 1.1: The cash flow of a variance swap at maturity

annualized variance point (i.e., the square of volatility point), representing the amount that the holder receives at maturity if the realized variance RV exceeds the strike K_{var} by one unit. The unit of L is dollar per unit variance point; for example $L = 25,000/(\text{variance point})$. A sample term sheet of a variance swap written on the realized variance of S&P500 is shown in Appendix A, and Figure 1.1 demonstrates the cash flow of a variance swap at maturity. For more details about the variance swaps and variance futures, readers are referred to the web sites of CBOE[†] or NYSE Euronext[‡].

One of the most important concepts associated with the variance swaps is the measurement of realized variance. At the beginning of a contract, it is clearly specified the details of how the realized variance should be calculated. Important factors contributing to the calculation of the realized variance include underlying asset(s), the observation frequency of the price of the underlying asset(s), the annualization factor, the contract lifetime, the method of calculating the variance. Some typical formulae (Howison et al. 2004; Little & Pant 2001) for the measure of realized variance are

$$RV_{d1}(0, N, T) = \frac{AF}{N} \sum_{i=1}^{N} \log^2\left(\frac{S_{t_i}}{S_{t_{i-1}}}\right) \times 100^2 \tag{1.1}$$

[†]http://cfe.cboe.com/Products/Spec_VT.aspx
[‡]http://www.euronext.com/fic/000/010/990/109901.ppt

or

$$RV_{d2}(0,N,T) = \frac{AF}{N}\sum_{i=1}^{N}(\frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}})^2 \times 100^2 \tag{1.2}$$

where S_{t_i} is the closing price of the underlying asset at the i-th observation time t_i, and there are altogether N observations. AF is the annualized factor converting this expression to an annualized variance. If the sampling frequency is every trading day, then $AF = 252$, assuming there are 252 trading days in one year, if every week then $AF = 52$, if every month then $AF = 12$ and so on. We assume equally-spaced discrete observations in this thesis so that the annualized factor is of a simple expression $AF = \frac{1}{\Delta t} = \frac{N}{T}$.

As shown by Jacod & Protter (1998), when the sampling frequency increases to infinity, the discretely-sampled realized variance approaches the continuously-sampled realized variance, $V_c(0,T)$, that is:

$$RV_c(0,T) = \lim_{N\to\infty} RV_{d1}(0,N,T) = \frac{1}{T}\int_0^T \sigma_t^2 dt \times 100^2 \tag{1.3}$$

where σ_t is the so-called instantaneous volatility of the underlying. Of course, if there is no assumption on the stochastic nature of the volatility itself, instantaneous volatility is nothing but local volatility as stated in Little & Pant (2001). Since in practice the measure of realized variance is always done discretely, pricing approach for variance swaps based on this continuously-sampled realized variance will result in a systematic bias, as discussed in this thesis.

1.1.2 Volatility Swaps

A volatility swap is also a forward contract on the future realized volatility of the stock price. This contract is similar to and works exactly as a variance swap except that the traded ("swapped") asset here instead of the variance, is directly the volatility. The notional amount L of the payoff is now in dollar per unit volatility point. From now on, we distinguish two kinds of volatility swaps: the

standard deviation swap and the volatility-average swap.

The measure of the volatility in the case of standard deviation swap is the square root of the variance; that is the standard deviation of the returns on the underlying stock price over the contract lift. In this case, K_{vol} denotes the strike of a standard deviation volatility swap and the payoff of the standard deviation volatility swap is

$$(RV_{d1}(0, N, T) - K_{vol}) \times L \tag{1.4}$$

where $RV_{d1}(0, N, T)$ is the discretely-sampled realized volatility defined by the standard deviation, i.e.,

$$RV_{d1}(0, N, T) = \sqrt{\frac{AF}{N} \sum_{i=1}^{N} \left(\frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}} \right)^2} \times 100 \tag{1.5}$$

When the sampling frequency increases to infinity, this discretely-sampled realized volatility approaches to a continuous sampled realized volatility

$$RV_{c1}(0, T) = \lim_{N \to \infty} \sqrt{\frac{AF}{N} \sum_{i=1}^{N} \left(\frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}} \right)^2} \times 100 = \sqrt{\frac{1}{T} \int_0^T \sigma_t^2 dt} \times 100 \tag{1.6}$$

The second type of volatility swap is the volatility-average swap in which the measure $RV_{d2}(0, N, T)$ of the realized volatility is simply the average over time of the absolute returns on the stock price. In discrete time that is

$$RV_{d2}(0, N, T) = \sqrt{\frac{\pi}{2NT}} \sum_{i=1}^{N} \left| \frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}} \right| \times 100 \tag{1.7}$$

where N is the total number of sampling times over the contract life $[0, T]$, S_{t_i} is the stock price at time t_i, and $|\frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}}|$ is the return on the stock price at time t_i. In continuous time when the sampling frequency increases to infinity,

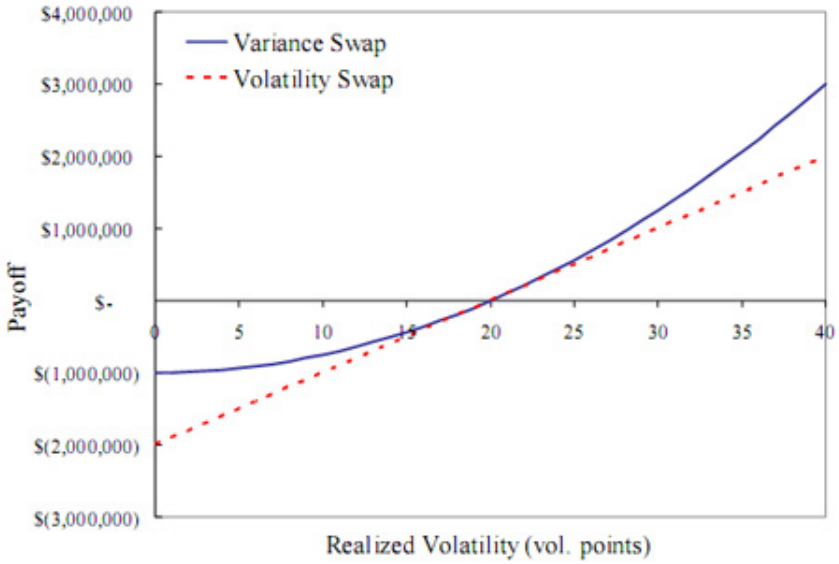

Figure 1.2: The payoffs of variance and volatility swaps for long position with strike=20 volatility points and notional amount L=2,000,000.¶

this discrete measure of realized volatility can be approximated by

$$RV_{c2}(0,T) = \lim_{N\to\infty} \sqrt{\frac{\pi}{2NT}} \sum_{i=1}^{N} \left| \frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}} \right| \times 100 = \frac{1}{T} \int_0^T \sigma_t dt \times 100 \qquad (1.8)$$

The payoff of a volatility swap is directly proportional to realized volatility; the profitability of a variance swap, however, has a quadratic relationship to realized volatility, as shown in Figure 1.2. Since a long position of a variance swap gains more than a simple volatility swap when volatility increases and loss less than a volatility swap when volatility decreases, variance swap levels are typically quoted above the expected level of the future realized volatility (i.e., above the option-implied volatility). This spread between variance and volatility swaps is call convexity.

1.1.3 VIX Futures and Options

The Volatility Index (VIX) is a volatility index launched by the CBOE (Chicago Board Options Exchange) in 1993 to replicate the one-month implied volatility of

¶Source: Bear Stearns Equity Derivatives Strategy, Bloomberg.

the S&P 100 index. In 2003, the calculation method was changed and expanded to replicate the S&P500. Since its introduction, VIX has been considered to be the world's benchmark for stock market volatility. The new definition of VIX is based on a model-free formula and computed from a portfolio of 30-calendar-day out-of-money options written on S&P500 (SPX). This new definition reflects the market's expectation of the 30-day forward S&P500 index volatility and serves as a proxy for investor sentiment, rising when investors are anxious or uncertain about the market and falling during times of confidence. This VIX index, often referred to as the "investor fear gauge", is therefore closely monitored by active traders, financial analysts as well as the media for insight into the financial market. Some other major security markets have also developed volatility indices to measure the market volatility risk, e.g., VDAX published by the German exchange Deutsche Böers, VX1 and VX6 published by the French exchange MONEP, etc.

The introduction of VIX has laid a good foundation for constructing tradable volatility products and thus facilitating the hedging against volatility risk and speculating in volatility derivatives. For instance, on March 26, 2004, the CBOE launched a new exchange, the CBOE Futures Exchange (CFE) to start trading VIX futures, which is a type of new futures written on the new definition of VIX. On February 24, 2006, CBOE started the trading of VIX options to enlarge the family of volatility derivatives. Since its inception, the VIX futures and options market has been rapidly growing. For example, according to the CBOE Futures Exchange press release on Jul. 11, 2007, in June 2007 the average daily volume of VIX option was 95,283 contracts, making the VIX the second most actively traded index and the fifth most actively traded product on the CBOE. On July 11, open interest in VIX options stood a 1,845,820 contracts (1,324,775 calls and 521,045 puts). In the same month, the VIX futures totalled 78,578 contracts traded with open interest at 49,894 contracts at the end of June. Being warmly welcome by the financial market, these volatility derivatives were awarded the

most innovative index derivative products[‖].

1.2 Mathematical Background

One of the key problems in mathematical finance is how to derive the fair value of a financial contract (e.g., options, futures etc.). To consider this kind of evaluation problems from a modeling point of view, we now introduce the fundamental background of mathematical knowledge.

1.2.1 Fundamental Pricing Theorems

We denote the deterministic risk-free interest rate by $r(t)$. The discount factor for the present value at time t of one currency unit of a risk-free cash flow at time T is denoted by $DF(t,T)$ and defined by:

$$DF(t,T) = e^{-\int_t^T r(s)ds} \tag{1.9}$$

and the money market account is defined by:

$$DF^{-1}(0,t) = e^{\int_0^t r(s)ds} \tag{1.10}$$

We now introduce the fundamental pricing framework, as stated in the following two theorems. Before stating them, we need the probability space $(\Omega, \mathscr{F}_t, \mathbb{Q})$. Here, Ω is the samples pace, $\mathscr{F}_t$ is the filtration representing the information flow of asset prices up to time t, and $\mathbb{Q}$ is a probability measure. Subsequently, all expectations are taken with respect to the measure $\mathbb{Q}$.

A special and important probability measure is the martingale pricing measure $\mathbb{Q}$ under which the asset price process $S(t)$, adopted to $\mathscr{F}_t$, satisfies the following

[‖]http://www.cboe.com/AboutCBOE/ShowDocument.aspx?DIR=ACNews&FILE=20061205.doc

martingale properties:

$$\begin{aligned} &i) \quad E^{\mathbb{Q}}[|S(t)|] < \infty, \\ &ii) \quad E^{\mathbb{Q}}[DF(t,T)S(T)|S(t)] = S(t) \end{aligned} \tag{1.11}$$

Such a martingale pricing measure $\mathbb{Q}$ is also called pricing or risk-neutral measure. In mathematical finance terms, we mean by no arbitrage value of the contingent claim its fair value under a martingale pricing measure $\mathbb{Q}$. If the derivative claim is sold by its fair value then the expected returns on both investment strategies - buying the derivative security or replicating it by trading in the underlying security and money market account - are equal to the risk-free rate of return. Although a smart investor may seek and grab such a riskless way of making profits, it would only be a transient opportunity. Once more investors and traders jump in to share the "free lunch", prices of the securities would change immediately. Hence the old equilibrium would break down and be replaced by a new equilibrium, i.e. arbitrage opportunities would vanish. That is why our discussions of pricing derivatives are based on no arbitrage. It is also an implication of the efficient market hypothesis.

The next two theorems are fundamental to calculate fair values of contingent claims. In a general sense, they establish a relationship between arbitrage opportunities with the risk-neutral measure (see, e.g., Harrison & Kreps 1979; Harrison & Pliska 1981; Delbaen & Schachermayer 1994).

Theorem 1 (First Fundamental Theorem of Asset Pricing) *The existence of a martingale pricing measure $\mathbb{Q}$ that satisfies the requirements i) and ii) in Eq. (1.11) implies the absence of risk-free arbitrage opportunity in the market. With the existence of a martingale pricing measure $\mathbb{Q}$, the discounted no-arbitrage price processes of all contingent claims are martingales under the measure $\mathbb{Q}$.*

The next theorem postulates the existence of a unique replicating strategy for derivative securities.

Theorem 2 (Second Fundamental Theorem of Asset Pricing) *If and only if there exists a unique martingale measure $\mathbb{Q}$ that satisfies the requirements i) and ii) in Eq. (1.11), the financial market is complete, i.e., every financial contingent claim on asset $S(t)$ is uniquely replicable by a hedging portfolio consisting of positions in asset $S(t)$ and in money market account.*

We note that finding a unique measure $\mathbb{Q}$ that satisfies the requirements i) and ii) is extremely involved when there is a few risky factors. However for practical purposes, we can assume that the measure $\mathbb{Q}$ is already fixed by market participants and it is reflected in market prices of traded derivative securities. Accordingly, the problem of finding measure $\mathbb{Q}$ comes down to enforcing the martingale condition for the model implied evolution of the asset price process under the measure $\mathbb{Q}$ and calibrating parameters of our chosen pricing model to market prices of traded securities. This estimated measure $\mathbb{Q}$ is sometimes called empirical or pricing martingale measure, and this approach to specify $\mathbb{Q}$ is used by a majority of market participants of mark-to-market and risk-manager positions. The replication strategy (under the measure $\mathbb{Q}$) is typically achieved by assembling many (hundreds of) individual option contracts in a portfolio (the so-called option book) and then hedging aggregated risks of these portfolios (books).

1.2.2 Stochastic Calculus

Now, we introduce some important modeling tools to study the problem of pricing and hedging financial derivative securities.

We assume a stochastic process $S(t)$ is driven by the following stochastic differential equation (SDE):

$$dS_t = \mu(t, S_{t-})dt + \sigma(t, S_{t-})dW(t) + j(t, S_{t-}, J)dN(t) \tag{1.12}$$

where S_{t-} stands for the value of the process S_t just before jump J occurs. $W(t)$ is a standard Wiener process and $N(t)$ is Poisson process with stochastic intensity

$\gamma(t, S_t)$. Processes $W(t)$ and $N(t)$ are assumed to be independent and adopted to $\mathscr{F}_t$. The random variable J is measurable on $\mathscr{F}_t$ with a probability density function $\omega(J)$ describing the magnitude of the jump when it occurs, and $j(t, S_{t-}, J)$ maps the jump size to post-jump value of S_t.

We assume that J has finite first and second moments and that the coefficients of this SDE satisfy the Lipschitz regularity conditions:

$$\begin{aligned}
&\mathbb{P}^{\mathbb{Q}}\left(\int_0^t \sigma^2(t', S_{t'})dt' < \infty\right) = 1, \forall t, 0 \leq t < \infty; \\
&\mathbb{P}^{\mathbb{Q}}\left(\int_0^t |\mu(t', S_{t'})|dt' < \infty\right) = 1, \forall t, 0 \leq t < \infty; \\
&\mathbb{P}^{\mathbb{Q}}\left(\int_0^t j^2(t', S_{t'})|dt' < \infty\right) = 1, \forall t, 0 \leq t < \infty;
\end{aligned} \tag{1.13}$$

To study the pricing of financial derivatives, the following theorem is fundamental.

Theorem 3 (*Itô* Lemma) *If S_t has a SDE given by Eq. (1.12), and $f(t, y) \in C^{1,2}([0, \infty) \times \mathbb{R})$, then $f = f(t, S_t)$ has a stochastic dynamics given by*

$$\begin{aligned}
df(t, S_t) = &\left(\frac{\partial f}{\partial t} + \mu(t, S_{t-})\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2(t, S_{t-})\frac{\partial^2 f}{\partial S^2}\right)dt \\
&+\sigma(t, S_{t-})\frac{\partial^2 f}{\partial S^2}dW(t) + (f(t, S_{t-} + j(t, S_{t-}, J)) - f(t, S_{t-}))dN(t)
\end{aligned} \tag{1.14}$$

where S_{t-} is the value of the process S_t just before jump J occurs.

1.2.3 Connections Between PDE and SDE

The Feynman-Kac theorem and Kolmogoroff (Fokker-Plank) backward equations are our key tool to study the pricing problem from the P(I)DE standpoint, by relating the expectation of the derivative payoff under the martingale measure $\mathbb{Q}$ with the P(I)DE, which can be solved analytically or numerically.

In general, the backward Kolmogoroff equation is applied by valuing derivative securities, which might also include some optionality features, such as American options which can be exercised by the holder at any time up to maturity time T. For option pricing purposes we state this important result relating expectations with respect to realizations of stochastic processes to specific PIDE-s.

Theorem 4 (Kolmogoro Backward Equation and Feynman-Kac Theorem) *If $\mu(t,S)$ and $\sigma^2(t,S)$ satisfy the Lipschitz condition Eq. (1.13) and $f(t,y) \in C^2([0,\infty)\times\mathbb{R})$ satisfies the following partial integro-differential equations (PIDE)*

$$\begin{aligned}&\frac{\partial f}{\partial t}+\mu(t,S)\frac{\partial f}{\partial S}+\frac{1}{2}\sigma^2(t,S)\frac{\partial^2 f}{\partial S^2}-g(t,S)f(t,S)\\&\quad+\gamma(t,S)\int_{-\infty}^{\infty}[f(t,S+j(t,S,J))-f(t,S)]\omega(J)dJ=0\end{aligned} \tag{1.15}$$

with final condition $f(T,S)=p(S)$, then the solution $f(t,S)$ to the above PIDE has the stochastic expectation representation

$$f(t,S)=\mathbb{E}^{\mathbb{Q}}[e^{-\int_t^T g(t',S_{t'})dt'}p(S)|\mathscr{F}_t],\quad (t\leqslant T) \tag{1.16}$$

where S_t is driven by Eq. (1.12).

1.2.4 Transformations

Transform methods, particularly Fourier transforms, are one of the classical and powerful methods for solving ordinary and partial differential equations as well as integral equations. The idea behind these methods is to transform the problem to a space where the solution is relatively easy to obtain. The corresponding solution is referred to as the solution in the Fourier or Laplace space. The original function can be retrieved either by means of computing the inverse transform analytically or, in complicated cases, by methods of numerical inversion.

The generalized Dirac function and its derivative are important for our developments. Let $\delta_\alpha(t)$ denote the generalized Dirac function, and $\delta_\alpha^{(n)}(t)$ be its n-th order derivative, then for a general smooth function $\phi(t)$:

$$\begin{aligned}&\int_{-\infty}^{\infty}\delta_\alpha(t)\phi(t)dt=\phi(\alpha)\\&\int_{-\infty}^{\infty}\delta_\alpha^{(n)}(t)\phi(t)dt=(-1)^n\phi^{(n)}(\alpha)\end{aligned} \tag{1.17}$$

We now introduce the Fourier transform and its generalization. The basic

definitions of Fourier transform and its inversion are given by

$$\begin{aligned} \mathcal{F}[\phi(t)]|_{\omega} &= \int_{-\infty}^{\infty} \phi(t) e^{-j\omega t} dt, \\ \mathcal{F}^{-1}[\phi(\omega)]|_{t} &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi(\omega) e^{j\omega t} d\omega \end{aligned} \tag{1.18}$$

Unfortunately, with this basic definition of Fourier transform, it is even not possible to perform transform to some fundamental functions, such as the real exponential function e^t, or the payoff function of a vanilla European option $\max(S-K, 0)$. So we need to consider a generalization of Fourier transform (see, e.g., Lewis 2000; Poularikas 2000 for more details). We first define a set of rapidly decreasing test functions Φ that satisfies the following two properties:

1. Each test function in Φ is an analytical test function on the entire complex plane;

2. Each test function, $\phi(x+jy)$, in Φ satisfies

$$\phi(x+jy) = O(e^{-\gamma|x|}) \quad as x \to \pm\infty \tag{1.19}$$

for every real of y and γ.

It can be verified that every rapidly decreasing test function $\phi(t)$ in Φ is classical transformable. The generalized Fourier transform of a function f, $\mathcal{F}[f(t)]|_{\omega}$, is the function that satisfies the following equation

$$\int_{-\infty}^{\infty} \mathcal{F}[f(t)]|_{\omega} \phi(\omega) d\omega = \int_{-\infty}^{\infty} f(y) \mathcal{F}[\phi(t)]|_{y} dy \tag{1.20}$$

for every rapidly decreasing test function $\phi(t)$ in Φ. Likewise, if $G(\omega)$ is a function for which the following equation

$$\int_{-\infty}^{\infty} \mathcal{F}^{-1}[G(\omega)]|_{t} \phi(t) dt = \int_{-\infty}^{\infty} G(y) \mathcal{F}^{-1}[\phi(\omega)]|_{y} dy \tag{1.21}$$

is well defined for every rapidly decreasing test function $\phi(t)$ in Φ, then $\mathcal{F}^{-1}[G(\omega)]|_{t}$

is the generalized inverse Fourier transform of $G(\omega)$.

Using this generalized definition of Fourier transform, it can be shown that for any complex value, $\alpha + j\beta$,

$$\mathcal{F}[e^{j(\alpha+j\beta)t}]|_\omega = 2\pi\delta_{\alpha+j\beta}(\omega) \tag{1.22}$$

and

$$\mathcal{F}[\delta_j(t)]|_\omega = e^\omega \tag{1.23}$$

1.2.5 Characteristic Function

Now we start to introduce the characteristic function, which plays a vital role for a real-valued random variable in probability theory.

The characteristic function of a real-valued random variable S is defined by

$$f(\phi) = E^{\mathbb{Q}}[e^{i\phi S}] = \int_{-\infty}^{\infty} e^{i\phi x} p(x) dx \tag{1.24}$$

Actually, the characteristic function is the Fourier transform of the probability density function $p(x)$ of the random variable S. The characteristic function of a random variable completely characterizes the distribution of a random variable; two variables with the same characteristic function are identical distributed. Furthermore, a characteristic function is always continuous and satisfies $f(0) = 1$. More importantly, the corresponding probability density function $p(x)$ and cumulative density function $P(x)$ can be obtained by inverting the characteristic function $f(\phi)$,

$$p(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\phi x} f(\phi) d\phi \tag{1.25}$$

and

$$P(x) = \text{Prob}(S \leq x) = \frac{1}{2} - \frac{1}{\pi} \int_0^{\infty} \text{Re}\left[\frac{e^{-i\phi x} f(\phi)}{\phi i}\right] d\phi \tag{1.26}$$

The reason that the characteristic function is important in mathematical fi-

nance is the transitional probability density function is usually difficult to be found analytically, whereas its Fourier transform (i.e., the characteristic function), is comparatively easy to be obtained. Since the terminal condition for the characteristic function is the well smooth exponential function, its corresponding PDE is comparatively easier to be solved. With the help of the characteristic function, it is therefore convenient to switch the computation to the frequency domain to solve the option pricing problems. For example, Heston (1993) determined the price of an vanilla European call option by obtaining the explicit solution of the characteristic function, based on a stochastic volatility model.

1.3 Mathematical Models

A good pricing model should produce the price of a financial derivative which are very close to the real market price of the this contract. The prices of exotic options given by models based on Black-Scholes assumptions can be wildly inaccurate because they are frequently even more sensitive to levels of volatility than standard European calls and puts. Therefore currently traders or dealers of these financial instruments are motivated to find models to price options which take the volatility smile and skew in to account. To this extent, stochastic volatility models are partially successful because they can capture, and potentially explain, the smiles, skews and other structures which have been observed in market prices for options. In this section, we shall have an overview of these pricing models for financial derivatives.

1.3.1 Black-Scholes Model

The Black-Scholes exponential Brownian motion model provides an approximate description of the behaviour of asset prices and serves as a benchmark against which other models can be compared. However, volatility does not behave in the way the Black-Scholes equation assumes; it is not constant, it is not predictable, it

is not even directly observable. Plenty of evidence exists that returns on equities, currencies and commodities are not normally distributed, they have higher peaks and fatter tails. Volatility has a key role to play in the determination of risk and in the valuation of derivative securities. This section reviews the Black & Scholes (1973) arbitrage argument from option valuation under constant volatility. This allows us to introduce some frequently used notation and provides a basis for the generalization to stochastic volatility. Black & Scholes (1973) model assumes that the stock price satisfies the following stochastic differential equation(SDE):

$$dS = \mu S dt + \sigma S dW \tag{1.27}$$

where μ is the deterministic instantaneous drift or return of the stock price, and σ is the volatility for the stock price.

In the Black & Scholes (1973), it is also assumed that there is a money market security (bank account) paying continuously compounded annual rate r and security markets are perfect so that one can trade continuously with no transaction costs and no arbitrage opportunities**.

Under these assumptions, one can construct a portfolio consisting of one European option C with arbitrary payoff $C(S,T) = \Psi(S)$ and a number $-\phi$ of an underlying asset. The value of the portfolio at time t is:

$$\Pi = C - \phi S \tag{1.28}$$

where ϕ is a constant and makes Π instantaneously risk-free. The jump of the value of this portfolio in one infinitesimal time step is:

$$d\Pi = dC - \phi dS \tag{1.29}$$

**There are never any opportunities to make an instantaneous risk-free profit. "There is no such things as free lunch".

Hence by the principle of no arbitrage, Π must instantaneously earn the risk-free bank rate r:

$$d\Pi = r\Pi dt \tag{1.30}$$

The central idea of the Black-Scholes argument is to eliminate the stochastic component of risk by making the number of shares equal to:

$$\phi = \frac{\partial C}{\partial S} \tag{1.31}$$

Applying *Itô*'s lemma to $C(S,t)$ and with some substitutions, one gets

$$\frac{\partial C}{\partial t} + \frac{1}{2}S^2\sigma^2\frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} = rC \tag{1.32}$$

This is the Black-Scholes equation and is a linear parabolic partial differential equation. In fact, almost partial dierential equations in finance are of a similar form. One of the attractions of the Black-Scholes equation is that the option price function is independent of the expected return of the stock μ. The Black-Scholes equation was first written down in 1969, but the derivation of the equation was finally published in 1973.

The payoff for a European (vanilla) call option with strike K is $C(S,T) = \max(S-K,0)$, and the option's price at time t has an analytic or closed-form solution (i.e., the Black-Scholes formula) by solving the Black-Scholes equation in the form:

$$C(S,t) = SN(d_1) - Ke^{-r(T-t)N(d_2)} \tag{1.33}$$

where

$$\begin{aligned} d_1 &= \frac{\log\left(\frac{S}{K}\right) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \\ d_2 &= d_1 - \sigma\sqrt{T-t} \end{aligned} \tag{1.34}$$

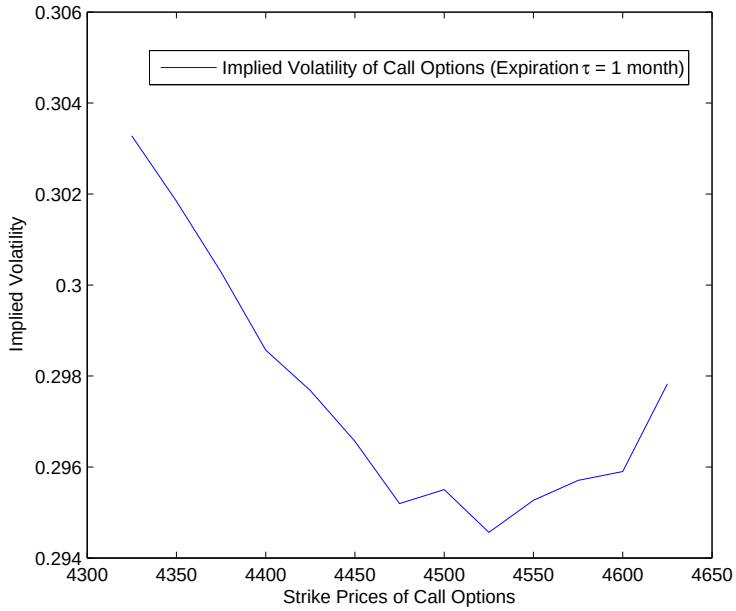

Figure 1.3: The implied volatility of ASX SPI 200 index call options[††]

and $N(d)$ is the standard normal cumulative distribution function

$$N(d) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} e^{-x^2/2} dx \tag{1.35}$$

Conversely, if one knows the market value of the option, one can calculate the volatility for these instruments using the Black-Scholes formula (Eq. (1.33)) and a numerical method that solves to converge to the unique implied volatility for this option price (e.g., use the Newton-Raphson method). This value of volatility obtained from the market option price by conversely solving the Black-Scholes formula is called implied volatility. When the implied volatilities for market prices of options written on the same underlying price are plotted against a range of strikes and maturities, the resulting graph is typically downward sloping for equity markets, or valley-shaped for currency markets, as shown in Fig. 1.3. This observation violates the Black-Scholes model because volatility is not, as assumed

[††]The implied volatility is calculated from the ASX SPI 200 index call options which will expire in one month. Data are obtained from Australia Stock Exchange, on Feb. 8, 2010. The ASX SPI index is 4521 on that date.

in Black-Scholes model, a deterministic quantity. Either the term "volatility smile" or "volatility skew" may be used to refer to the general phenomena of volatilities varying by strikes.

1.3.2 Local Volatility Model

There have been many approaches to remedy the drawback of the constant volatility assumption within the Black-Scholes model. The local volatility model, a concept originated by Derman & Kani (1994), and Dupire (1994), treats volatility as a function of the current asset level S_t and of time t, instead of a constant as assumed in Black-Scholes model. Given the prices of call or put options across all strikes and maturities, one can deduce the local volatility function to match the theoretical option prices with the market prices. Dupire (1994) showed that if the spot price follows a risk-neutral random walk of the form:

$$\frac{dS}{S} = rdt + \sigma(S,t)dW \tag{1.36}$$

and if no-arbitrage market prices for European vanilla options are available for all strikes K and expiries T, then $\sigma_L(K,T)$ can be extracted analytically from these option prices. If $C(S,t,K,T)$ denotes the price of a European call with strike K and expiry T, Dupire's famous equation is obtained:

$$\frac{\partial C}{\partial T} = \sigma_L^2(K,T)\frac{K^2}{2}\frac{\partial^2 C}{\partial K^2} - rK\frac{\partial C}{\partial K} \tag{1.37}$$

Rearranging this equation, the direct expression to calculate the local volatility (Dupire formula) is obtained:

$$\sigma_L^2(K,T) = \sqrt{\frac{\frac{\partial C}{\partial T} + rK\frac{\partial C}{\partial K}}{\frac{K^2}{2}\frac{\partial^2 C}{\partial K^2}}} \tag{1.38}$$

Unlike the naive volatility produced by applying the Black-Scholes formula to market prices, the local volatility is the volatility implied by the market prices and the one factor Black-Scholes. One potential problem of using the Dupire formula (1.38) is that, for some financial instruments, the option prices of different strikes and maturities are not available or not enough to calculate the right local volatility. Another problem is for strikes far in- or out-the-money, the numerator and denominator of this equation may become very small, which could lead to numerical inaccuracies.

1.3.3 Stochastic Volatility Models

Stochastic volatility models are conceptually quite different from the fitting approach of local volatility model. In these models, the volatility is neither a constant as assumed in Black-Scholes model, nor a deterministic function of the current asset level S_t and of time t as assumed in local volatility model. Rather, it is by itself stochastic.

What is happening may be viewed in some different and related ways. Options prices are determined by supply and demand, not by theoretical formula. The traders who are determining the option prices are implicitly modifying the Black-Scholes assumptions to account for volatility that changes both with time and with stock price level. This is contrary to the Black and Scholes (1973) assumptions of constant volatility irrespective of stock price or time to maturity. That is, traders assume $\sigma = \sigma(S_t, t)$, whereas Black-Scholes model assumes that σ is just a constant.

By imposing specific stochastic processes for both the stock price and its instantaneous variance (or volatility), a stochastic volatility model is based on a structural assumption on the underlying stock price. In this way, the stochastic volatility model is to incorporate the empirical observation that volatility appears not to be constant and indeed varies, at least in part, randomly, by making the

volatility itself a stochastic process. The candidate models have generally been motivated by intuition, convenience and a desire for tractability. In particular, the popular Heston (1993) model assumes that the instantaneous variance of the stock price is a square-root diffusion whose increments are correlated to the increments of the return of the stock price. Other popular stochastic volatility models are model by Stein & Stein (1991), model by Schöbel & Zhu (1999), GARCH diusion model (Lewis 2000), to name but a few. In addition, there are also models which incorporate jump processes (see, for example, Merton 1976; Madan et al. 1998) or mixtures of both concepts such as Bates (1996), Duffie et al. (2000). Stochastic volatility on option values is similar to the effect of a jump component: both increase the probability that out-of-the-money options will finish in-the-money and vice versa (Wiggins 1987). Whether the smile is skewed left, skewed right, or symmetrical in a stochastic volatility model depends upon the sign of the correlation between changes in volatility and changes in stock price (Hull & White 1987).

The stochastic nature of the instantaneous variance of the stock price process is particular important if we want to price and hedge heavily volatility-dependent exotic options such as options on realized variance or cliquet-type products. Such products cannot be priced correctly in the BS-model since their very risklies in the movement of volatility (or variance, for that matter) itself.

While Black and Scholes (BS) used only the underlying stock price and the bond to hedge a derivative in their model, this cannot be justified anymore: their model is not able to capture what is today known as the volatility skew or volatility smile, of the implied volatility of traded vanilla options. The root of the discrepancy is that volatility is not, as assumed in BS model, a deterministic quantity. Rather, it is by itself stochastic.

$$\begin{cases} dS_t = \mu S_t dt + \sqrt{V_t} S_t dB_t^S \\ dV_t = \kappa(\theta - V_t)dt + \sigma_V \sqrt{V_t} dB_t^V \end{cases} \tag{1.39}$$

where μ is the deterministic instantaneous drift or return of the stock price, and the variance V is correlated with the stock price by ρ, We cannot hold or "short" volatility, but can hold a position in a second option to do hedging. So let us consider the valuation of the volatility dependent instrument (e.g., volatility swaps), assuming that one can take long or short positions in a second instrument as well as in the underlying.

In the Black-Scholes case, there is only one source of randomness-the stock price, which can be hedged with stock. In the present case, random changes in volatility also need to be hedged in order to form a riskless portfolio. So we setup a portfolio Π containing the option to be priced whose value is denote by $C(S, V, t)$, a quantity $-\phi_1$ of the stock and a quantity $-\phi_2$ of another asset whose value U depends on volatility. We have

$$\Pi = C - \phi_1 S - \phi_2 U \tag{1.40}$$

The change in this portfolio in a time increment dt is given by

$$d\Pi = dC - \phi_1 dS - \phi_2 dU \tag{1.41}$$

As by standard, one applies Ito's Lemma to this portfolio to obtain

$$d\Pi = adS + bdV + cdt \tag{1.42}$$

where

$$\begin{aligned} a &= \frac{\partial C}{\partial S} - \phi_1 - \phi_2 \frac{\partial U}{\partial S}, \quad b = \frac{\partial C}{\partial S} - \phi_1 - \phi_2 \frac{\partial U}{\partial S} \\ c &= (\frac{\partial C}{\partial t} + \frac{1}{2} S^2 V \frac{\partial^2 C}{\partial S^2} + \rho S \sigma_V V \frac{\partial^2 C}{\partial S \partial V} + \frac{1}{2} \sigma_V^2 V \frac{\partial^2 C}{\partial V^2}) \\ &\quad -\phi_2 (\frac{\partial U}{\partial t} + \frac{1}{2} S^2 V \frac{\partial^2 U}{\partial S^2} + \rho S \sigma_V V \frac{\partial^2 U}{\partial S \partial V} + \frac{1}{2} \sigma_V^2 V \frac{\partial^2 U}{\partial V^2}) \end{aligned} \tag{1.43}$$

Clearly we wish to eliminate the stochastic component of risk by setting $a =$

$b = 0$, so one can rearrange the hedge parameters in the form:

$$d\Pi = adS + bdV + cdt \tag{1.44}$$

where

$$\begin{aligned} \phi_1 &= \frac{\partial C}{\partial S} - \phi_2 \frac{\partial U}{\partial S} \\ \phi_2 &= (\frac{\partial C}{\partial V})/(\frac{\partial U}{\partial V}) \end{aligned} \tag{1.45}$$

to eliminate the dS term and the dV term. The avoidance of the arbitrage, once these choices of ϕ_1 and ϕ_2, are made, is the condition:

$$\begin{aligned} d\Pi &= r\Pi dt \\ d\Pi &= r(V - \phi_1 S - \phi_2 U)dt \end{aligned} \tag{1.46}$$

where we have used the fact that the return on a risk-free portfolio must be equal to the risk-free bank rate which we will assume to be deterministic for our purposes. Combining equations (2.14) and (2.15), collecting all C terms on the left hand side and all U terms on the right hand side, one gets:

$$\begin{aligned} &(\frac{\partial C}{\partial t} + \frac{1}{2}VS^2\frac{\partial^2 C}{\partial S^2} + \rho S\sigma_V V\frac{\partial^2 C}{\partial S \partial V} + \frac{1}{2}\sigma_V^2 V\frac{\partial^2 C}{\partial V^2} + rS\frac{\partial C}{\partial S} - rC)/\frac{\partial C}{\partial V} = \\ &(\frac{\partial U}{\partial t} + \frac{1}{2}VS^2\frac{\partial^2 U}{\partial S^2} + \rho S\sigma_V V\frac{\partial^2 U}{\partial S \partial V} + \frac{1}{2}\sigma_V^2 V\frac{\partial^2 U}{\partial V^2} + rS\frac{\partial U}{\partial S} - rU)/\frac{\partial U}{\partial V} \end{aligned} \tag{1.47}$$

The left-hand side is a function of C only and the right-hand side is a function of U only. The only way that this can be is for both sides to be equal to some function depending only on variables S, V and t. So, if one writes both sides as $f(S, V, t)$, in doing so, one arrives at the general PDE for stochastic volatility:

$$(\frac{\partial C}{\partial t} + \frac{1}{2}VS^2\frac{\partial^2 C}{\partial S^2} + \rho\sigma_V SV\frac{\partial^2 C}{\partial S \partial V} + \frac{1}{2}\sigma_V^2 V\frac{\partial^2 C}{\partial V^2} + rS\frac{\partial C}{\partial S} - rC) = -(\alpha - \lambda\beta)\frac{\partial C}{\partial V} \tag{1.48}$$

where, without loss of generality, we have written the arbitrary function f of S, V and t as $(\alpha - \lambda\beta)$. Conventionally, λ is called the market price of volatility risk because it tells us how much of the expected return of C is explained by the risk

(i.e., standard deviation) of V in the Capital Asset Pricing Model framework.

1.4 Literature Review

1.4.1 Variance Swaps and Volatility Swaps

Since the sharp increase in the trading volume of variance swaps recently, it has drawn considerable research interests to develop appropriate valuation approaches for variance swaps. In the literature, there have been two types of valuation approaches, numerical methods and analytical methods.

Of all the analytical methods, there are two subcategories. The most influential ones were proposed by Carr & Madan (1998) and Demeterfi et al. (1999). They have shown how to theoretically replicate a variance swap by a portfolio of standard options. Without requiring to specify the function of volatility process, their models and analytical formulae are indeed very attractive. However, as pointed out by Carr & Corso (2001), the replication strategy has a drawback that the sampling time of a variance swap is assumed to be continuous rather than discrete; such an assumption implies that the results obtained from a continuous model can only be viewed as an approximation for the actual cases in financial practice, in which all contacts are written with the realized variance being evaluated on a set of discrete sampling points. Another drawback is that this strategy also requires options with a continuum of exercise prices, which is not actually available in marketplace. The second kind of analytical methods is the stochastic volatility models. Grunbichler & Longstaff (1996) first developed a pricing model for volatility futures based on mean-reverting squared-root volatility process. Heston (2000) derived an analytical solution for both variance and volatility swaps based on the GARCH volatility process. Javaheri et al. (2004) also discussed the valuation and calibration for variance swaps based on the GARCH(1,1) stochastic volatility model. They used the flexible PDE approach to determine

the first two moments of the realized variance in the context of continuous as well as discrete sampling, and then obtained a closed-form approximate solution after the so-called convexity correction was made. Howison et al. (2004) also considered the valuation of variance swaps and volatility swaps under a variety of diffusion and jump-diffusion models. In their work, approximate solutions of the PDE for pricing volatility-related products are derived. Swishchuk (2004) used an alternative probabilistic approach to value variance and volatility swaps under the Heston (1993) stochastic volatility model. More recently Elliott et al. (2007) proposed a model to evaluate variance swaps and volatility swaps under a continuous-time Markov-modulated version of the stochastic volatility with regime switching, with both probabilistic and PDE approaches being discussed. All these stochastic volatility models, however, are based on the assumption that the realized variance is approximated with a continuously-sampled one, which will result in a systematic bias for the price of a variance swap. As will be shown later, while the approximation methods provide fairly reasonable estimates for the value of variance swaps with high sampling frequencies, they may lead to large relative errors for variance swaps with small sampling frequencies or long tenors.

Various numerical methods, as an alternative to analytical methods, were also intensively developed recently. A typical article in this category belongs to Little & Pant (2001). In their article, it is shown how to price a variance swap using the finite-difference method in an extended Black-Scholes framework, in which the local volatility is assumed to be a known function of time and spot price of the underlying asset. By exploring a dimension reduction technique, their numerical approach achieves high efficiency and accuracy for discretely-sampled variance swaps. Windcliff et al. (2006) also explored a numerical algorithm to evaluate discretely-sampled volatility derivatives using numerical partial-integro differential equation approach. Under this framework, they investigated a variety of modeling assumptions including local volatility models, jump-diffusion models

and models with transaction cost being taken into consideration. Although these two numerical methods evaluate variance swaps based on discretely-sampled realized variance and achieve high accuracy, the major limitation is that their models do not incorporate stochastic volatilities that are the most commonly used to model the dynamics of equity indices. To remedy this drawback, Little & Pant (2001) and Windcliff et al. (2006) pointed out, respectively, in the conclusions of their papers that for better pricing and hedging general variance swaps one needs to adopt an appropriate model that incorporates the stochastic volatility characteristics observed in financial markets.

To properly address this discretely sampling effect, several works have been done very recently. Broadie & Jain (2008*b*) presented a closed-form solution for volatility as well as variance swaps with discrete sampling. They also examined the effects of jumps and stochastic volatility on the price of volatility and variance swaps by comparing calculated prices under various models such as the Black-Scholes model, the Heston stochastic volatility model, the Merton (1973) jump diffusion model and the Bates (1996) and Scott (1997) stochastic volatility and jump model. However, their solution approach is primarily based on integrating the underlying stochastic processes directly and it appears that it can only be used when the realized variance is defined in such a particular form that the stochastic processes assumed for the underlying can happen to be exactly the same as that defined in the calculation of the realized variance. In other words, Broadie & Jain (2008*b*)'s approach can only be used when the realized variance is defined as the average of the squared log returns (Eq. (1.1)), as Zhu & Lian (2009*d*) pointed out.

On the other hand, Zhu & Lian (2009*d*,*f*) presented an completely different approach to obtain two closed-form formulae for variance swaps based on the two different definitions of discretely-sampled realized variance (Eq. (1.1 and (1.2)), under the Heston (1993)'s stochastic volatility model. Unlike Broadie & Jain (2008*b*)'s approach, Zhu & Lian (2009*d*)'s approach of solving the governing

PDE system directly is more versatile in terms of dealing with different forms of realized variance. Moreover, Zhu & Lian (2009*g*,*b*) have shown that they approach can be future extended to price forward-start discretely-sampled variance swaps and to price variance swaps based on a more general framework that allows for stochastic volatility, random jumps in return distribution and random jumps in variance process. Our these papers on the pricing of variance swaps with discrete sampling form the main contents of Chapter 2, 3 and 4 of this thesis.

As for the pricing of volatility swaps, the most popular method is the model-dependent approach. By using Taylor's expansion, Brockhaus & Long (2000) obtained an analytical approximation of the convexity correction for the pricing of volatility swaps. Javaheri et al. (2004) discussed the valuation of volatility swaps in the GARCH(1,1) stochastic volatility model using a partial differential equation approach to determine the first two moments of the realized variance and then adopting Brockhaus & Long (2000)'s convexity approximation approximation to price the volatility swaps. Friz & Gatheral (2005) provided a numerical integration approach for computing fair strikes of volatility swaps in the Heston stochastic volatility model, under the assumption of zero correlation between the underlying and the volatility processes. Broadie & Jain (2008*a*) computed fair volatility strikes by deriving a partial differential equation which exploits a no-arbitrage relationship between variance and volatility swaps. Broadie & Jain (2008*b*) however pointed that Brockhaus & Long (2000)'s approximation is not necessarily accurate in the stochastic volatility model, and presented a closed-form exact solution based on the Heston stochastic volatility model.

Even though most of researchers in this area seem to believe that the pricing and hedging of a volatility swap are, unlike variance swaps, highly model-dependent, Carr & Lee (2005) demonstrated, under the assumption of zero correlation between the asset and its volatility process, as well as the assumption of continuous trading in a continuum of strikes, that a self-finance portfolio has equal value to the continuously sampled volatility swap at expiration time T, and

hence developed model-free trading strategies to price and replicate volatility swaps.

Papers focusing on analytically pricing discretely-sampled volatility swaps are rare in literature, mainly due to the inherent difficulty associated with the non-linearity in the pay-off function. Zhu & Lian (2009*c*) present a closed-form exact solution for the pricing of discretely-sampled volatility swaps, under the framework of Heston (1993) stochastic volatility model, based on the definition of the so-called average of realized volatility (Eq. (1.7)). As for the standard derivation volatility swaps, in which the realized volatility is defined as the square root of the realized variance as shown in Eq. (1.5), there is no exact solution available at all for the discretely-sampled volatility swaps. Broadie & Jain (2008*b*) presented a closed-form approximation to price continuously-sampled standard derivation volatility swaps (Eq. (1.6)), based on the Heston model. A more common approach in literature is Brockhaus & Long (2000)'s convexity correction approximation. Zhu & Lian (2010*e*) systematically investigated the accuracy and the validity condition of the convexity correction approximation, through both theoretical analysis and numerical examples, and found out this approximation on some specifical parameters would result in significantly large errors. They also presented a new approximation, which is an extension of the convexity correction formula, to improve the accuracy. The Chapter 5 and 6 of this thesis are based on these two recent papers of ours (i.e., Zhu & Lian 2009*c*, 2010*e*).

1.4.2 VIX Futures and Options

Given the growing popularity of trading VIX futures, considerable research interests have also been drawn to the development of appropriate pricing models for VIX futures. Grunbichler & Longstaff (1996) first developed a pricing model for volatility futures and volatility options based on a mean-reverting squared-root volatility process. Carr & Wu (2006) presented a lower bound and an upper

bound for the price of VIX futures. By using the Jensen's inequality, they have shown that the lower bound is the forward-starting volatility swap rate (strike price) and the upper bound is the squared root of forward-starting variance swap rate over the period $(t, t + 30/365)$. Dupire (2005) derived the convexity adjustment that needs to be subtracted from the price of forward variance to arrive at the fair value of VIX futures. Zhang & Zhu (2006) proposed an expression for VIX futures, assuming S&P500 is modeled by Heston (1993)'s stochastic volatility. Zhu & Zhang (2007) further derived a no-arbitrage pricing model for VIX futures based on the variance term structure. Lin (2007) presented a convexity adjustment approximation for the value of the VIX futures under various stochastic volatility models with simultaneous jumps both in the asset price and variance processes. Psychoyios et al. (2007) provided a pricing model for both VIX futures and VIX options based on the squared root mean reverting process with jumps. Brenner et al. (2007) used market data to establish the relationship between the VIX futures prices and the VIX itself. They further theoretically explained the relationship between VIX and VIX futures, the valuation of VIX futures and model calibration, based on Heston (1993)'s stochastic volatility model. Sepp (2008*a*, 2008*b*) applied the square root stochastic variance model with variance jumps to describe the evolution of S&P500 volatility, and demonstrated how to apply the model to the pricing and hedging of VIX futures and options. Some other typical recent papers about the VIX and its derivatives (futures and options) include Zhang et al. (2010), Zhang & Huang (2010), Lu & Zhu (2009), Carr & Lee (2009) etc.

Zhu & Lian (2009*a*) recently derived a closed-form exact solution for the fair value of VIX futures under stochastic volatility model with simultaneous jumps in the asset price and volatility processes. With the newly-found pricing formula available, especially with its great computational efficiency, we are also able to conduct empirical studies, aiming at examining the performance of four different stochastic volatility models with or without jumps. More importantly, using the

Markov Chain Monte Carlo (MCMC) method to analyze a set of coupled VIX and S&P500 data, we demonstrate how to estimate model parameters. Through these empirical studies, we are able to compare the pricing performance of four models, of which analytical pricing formulae have been found and presented in this chapter. The Chapter 7 of this thesis is based on our this paper (i.e., Zhu & Lian 2009*a*).

Lin & Chang (2009) presented a closed-form pricing formula for VIX options that reconcile the most general price processes of the S&P500 in the literature: stochastic volatility, price jumps, and volatility jumps. Utilizing this closed-form pricing formula for VIX options, they empirically investigated how much each generalization of the S&P500 price dynamics improves VIX option pricing, and concluded that a model with stochastic volatility and state-dependent correlated jumps in S&P500 returns and volatility (i.e., Duffie et al. 2000) is a better alternative to the others in terms of pricing VIX options. By applying the exactly same pricing formula for VIX options shown in Lin & Chang (2009), Lin & Chang (2010) further studied the relationships among stylized features on S&P 500, VIX and options on VIX, and examined how jump factors impact VIX option pricing and hedging. Zhu & Lian (2010*b*), however, pointed out that the correctness of the formula proposed in Lin & Chang (2009) is in serious doubt. Using a completely different approach from Lin & Chang (2009), they presented an analytical exact solution for the price of VIX options under stochastic volatility model with simultaneous jumps in the asset price and volatility processes. They also offered numerical results to illustrate the correctness of their formula, and the incorrectness of the formula in Lin & Chang (2009).

1.5 Structure of Thesis

In this thesis, we develop some highly efficient approaches to analytically price volatility derivatives. In particular, using our approach, we present a set of closed-

form exact pricing formulae for discretely-sampled variance swaps, forward-start variance swaps, volatility swaps and VIX futures and options.

In Chapter 2, we present two closed-form exact solutions to price variance swaps with discrete sampling times by solving the partial differential equation (PDE) system based on the Heston (1993) two-factor stochastic volatility model, embedded in the framework proposed by Little & Pant (2001). In comparison with all the previous approximation models based on the assumption of continuous sampling time, the current research of working out closed-form exact solutions for variance swaps with discrete sampling times at least serves for two major purposes: (i) to verify the degree of validity of using a continuous-sampling-time approximation for variance swaps of relatively short sampling period; (ii) to demonstrate that significant errors can result from still adopting such an assumption for a variance swap with small sampling frequencies or long tenor. Other key features of our new solution approach include: (a) with the newly found analytic solutions, all the hedging ratios of a variance swap can also be analytically derived; (b) numerical values can be very efficiently computed from the newly found analytic formula.

In Chapter 3, a more general and condense approach is presented to price forward-start variance swaps with discrete sampling times, based on the Heston (1993) two-factor stochastic volatility model. By developing the forward characteristic function, it is shown this approach possesses some great advantages over those in literature: (1) treating the pricing problem of variance swaps with different definitions of discretely-sampled realized variance in a highly unified way; (2) easily obtaining analytical closed-form solutions for forward-start variance swaps with two popularly-used definitions of discretely-sampled realized variance; (3) enabling the investigation of some important properties of the forward-start variance swaps, utilizing the elegant and simple form of the obtained solutions. Thereby, this work represents a substantial progress in the field of pricing variance swaps.

In Chapter 4, we extend the approach in Chapter 3 to price discretely-sampled variance by further including random jumps in the return and volatility processes, and present two new closed-form exact solutions for the prices of variance swaps with discrete sampling times based on the Heston stochastic volatility and random jumps in the return and volatility processes. By working out the closed-form exact solutions for such a general model with jumps being possibly included in both the underlying and the variance, our new formulae for the two most commonly-adopted definitions of discretely-sampled realized variance can serve to improve the computational speed and accuracy in pricing variance swaps as well as in model calibration using stochastic volatility models with jumps. The fact that the newly-derived formulae can cover a wide range of models proposed in the literature, i.e., either with jumps being included in the underlying process or in the variance process or both, further demonstrate that our approach is a highly versatile and unified approach that can be used for pricing discretely-sampled variance swaps. Utilizing our new closed-form exact solutions, we have also conducted some cross-model comparison, examining various parameters involved in the jump processes.

In Chapter 5, we present a closed-form exact solution for the pricing of discretely-sampled volatility swaps, under the framework of Heston (1993) stochastic volatility model, based on the definition of the so-called average of realized volatility. Papers focusing on analytically pricing discretely-sampled volatility swaps are rare in literature, mainly due to the inherent difficulty associated with the nonlinearity in the pay-off function. By working out such a closed-form exact solution for discretely-sampled volatility swaps, this work has: (1) significantly reduced the computational time in obtaining numerical values for the discretely-sampled volatility swaps; (2) substantially improved the computational accuracy of discretely-sampled volatility swaps, comparing with the continuous sampling approximation; (3) enabled all the hedging ratios of a volatility swap to be also analytically derived.

In Chapter 6, we investigate another important issue in pricing volatility swaps. Convexity correction is a well-known approximation technique used in pricing volatility swaps. However, studies focusing on examining the accuracy of the technique itself are rare and the validity condition of this convexity correction approximation was hardly addressed and discussed in literature. In this chapter, we systematically investigate the accuracy and the validity condition of the convexity correction approximation, through both theoretical analysis and numerical examples. Hereby, our study answers the two basic questions in adopting the convexity correction approximation to derive approximate formula for pricing variance or volatility swaps: (a) why and when the convexity correction approximation will result in significantly large errors. In other words, what is the validity condition of applying the convexity correction approximation; (b) a better accuracy cannot be achieved by extending the convexity correction approximation, which is the second-order Taylor expansion, to third order or fourth order Taylor expansions. Some other contributions of this study include: (1) alerting that one should be aware of the inaccuracy of this approximation and be very careful in using it; (2) a new approximation, which is an extension of the convexity correction approximation, has been proposed to improve the accuracy.

In Chapter 7, we price VIX futures by deriving a closed-form exact solution for the fair value of VIX futures under stochastic volatility model with simultaneous jumps in the asset price and volatility processes. Since the inception of the volatility index (VIX) by the CBOE in 1993, in particular, the introduction of the VIX futures by CBOE in 2004, various pricing models with stochastic volatilities have been proposed to value VIX futures. However, rarely could an analytic closed-form solution be found, especially for models that include jumps in both VIX and its volatility. Thus the derivation of this formula for VIX futures with a very general dynamics of VIX represents a substantial progress in identifying and developing more realistic VIX futures models and pricing formulae. With the newly-found pricing formula available, especially with its great computational

efficiency, we are also able to conduct empirical studies, aiming at examining the performance of four different stochastic volatility models with or without jumps. More importantly, using the Markov Chain Monte Carlo (MCMC) method to analyze a set of coupled VIX and S&P500 data, we demonstrate how to estimate model parameters, which is a crucial step for any fancy mathematical model to be of practical use. Through these empirical studies, we are able to compare the pricing performance of four models, of which analytical pricing formulae have been found and presented in this chapter.

In the Chapter 8, we demonstrate the derivation of an analytical exact solution for the price of VIX options under stochastic volatility model with simultaneous jumps in the asset price and volatility processes. We point out that the solution procedure of Lin & Chang (2009)'s pricing formula for VIX options is incorrect. Our approach presented in this chapter is totally different from the approach in Lin & Chang (2009) in obtaining a closed-form pricing formula for VIX options. We then show that the numerical results obtained from our formula consistently match up with those obtained from Monte Carlo simulation perfectly, verifying the correctness of our formula. However the results obtained from Lin & Chang (2009)'s pricing formula significantly differ from those from Monte Carlo simulation, confirming our doubt that their pricing formula is incorrect. It is shown that our pricing formula is very efficient in computing the numerical prices of VIX options. Some important and distinct properties of the VIX options (e.g., put-call parity, the hedging ratios) have also been discussed in this chapter. Therefore, our formula can be a very useful tool in trading practice when there is obviously increasing demand of trading VIX options in financial markets.

Chapter 2

Pricing Variance Swaps with Discrete Sampling

2.1 Introduction

The trading volume of variance swaps has been experiencing a sharp increase recently. This has drawn considerable research interests to develop appropriate valuation approaches for variance swaps. However, most of the studies in literature are based on the assumption that the realized variance is approximated with a continuously-sampled one, as discussed in Chapter 1. In this chapter, we price discretely-sampled variance swaps based on Heston's two-factor stochastic volatility model embedded in Little & Pant's (2001) framework. Unlike Broadie & Jain (2008*b*)'s approach, our approach presented here is much simpler by solving the governing PDE system directly and is more versatile in terms of dealing with different forms of realized variance. In this way, the nature of stochastic volatility is included in the model and most importantly, two closed-form exact solutions can be worked out, even when the sampling times are discrete, for the corresponding two definitions of the discretely-sampled realized variance.

Furthermore, it is shown that our solutions degenerate to continuous sampling model when sampling frequency approaches infinity, as expected. Our explicit

pricing formulae for variance swaps presented here should be valuable in both theoretical and practical senses. Theoretically, although there are many existing models, as mentioned above, to price variance swaps, the closed-form exact solutions for discretely-sampled variance swaps are presented for the first time in the stochastic volatility framework. Secondly, our discrete model can be used to verify the validity of the corresponding continuous models for the specific pay-off discussed here and thus would fill a gap that has been in the field of pricing variance swaps. Thirdly, the Fourier inverse transform in our model has been analytically worked out, which is a significant step forward in the literature of Heston's model. Practically, the final form of our solution is simple enough in a closed form and thus can be easily used by market practitioners. Furthermore, our explicit solution shows substantial advantage, in terms of both accuracy and efficiency, over previous numerical or approximate approaches, and thus it can satisfy the increasing demand of trading variance swaps in financial markets.

This chapter is organized into four sections. In Section 2.2, a detailed description of variance swaps is first provided, followed by our analytical formulae for the variance swaps. In Section 2.3, some numerical examples are given, demonstrating the correctness of our solutions from various aspects. Comparison with continuous sampling models and discussion for other properties of the variance swaps are also carried out. In Section 2.4, a brief summary is provided.

2.2 Pricing Variance Swaps

In this section, we use the Heston (1993) stochastic volatility model to describe the dynamics of the underlying asset. To evaluate the discretely-sampled realized variance swaps, we employ the dimension reduction technique proposed by Little & Pant (2001) to analytically solve the associated PDE and hence obtain closed-form analytical solutions for fair strike prices of variance swaps with discrete sampling.

2.2.1 The Heston Stochastic Volatility Model

It is a well-known fact by now that the Black & Scholes (1973) model may fail to reflect certain features of the reality of financial markets due to some unrealistic assumptions, such as the constant volatility assumption; numerous phenomena such as smile effect (Wilmott 1998), skewness and kurtosis effects (Voit 2005) have been observed and reported, suggesting necessary improvements of the Black-Scholes model.

In the hope of remedying some apparent drawback of the Black-Scholes model, many models have been proposed to incorporate stochastic volatility, stochastic volatility with jump, stochastic volatility and stochastic interest rate (c.f., Stein & Stein 1991; Heston 1993; Scott 1997; Schöbel & Zhu 1999). In order to assess the performance of these models, Bakshi et al. (1997) systematically analyzed the performance of incorporating stochastic volatility, jump diffusion, and stochastic interest rate, and concluded that the most important improvement over the Black-Scholes model was achieved by introducing stochastic volatility into option pricing models. Once this is done, introducing jumps and stochastic interest rate leads to only marginal improvement in option pricing. For this reason, we shall focus on the stochastic volatility model in this chapter, leaving stochastic volatility with jump diffusions model to be discussed in Chapter 4. Among all the stochastic volatility models in the literature, model proposed by Heston (1993) has received the most attention since it can give a satisfactory description of the underlying asset dynamics (Daniel et al. 2005; Silva et al. 2004). In the Heston (1993) model, the underlying asset S_t is modeled by the following diffusion process with a stochastic instantaneous variance V_t.

$$\begin{cases} dS_t = \mu S_t dt + \sqrt{V_t} S_t dB_t^S \\ dV_t = \kappa(\theta - V_t)dt + \sigma_V \sqrt{V_t} dB_t^V \end{cases} \tag{2.1}$$

where μ is the expected return of the underlying asset, θ is the long-term mean

of variance, κ is a mean-reverting speed parameter of the variance, σ_V is the so-called volatility of volatility. The two Wiener processes dB_t^S and dB_t^V describe the random noise in asset and variance respectively. They are assumed to be correlated with a constant correlation coefficient ρ, that is $(dB_t^S, dB_t^V) = \rho dt$. The stochastic volatility process is the familiar squared-root process. To ensure the variance is always positive, it is required that $2\kappa\theta \geq \sigma^2$ (see Cox et al. 1985; Heston 1993; Zhang & Zhu 2006).

According to the existence theorem of equivalent martingale measure, we are able to change the real probability measure to a risk-neutral probability measure and describe the processes as:

$$\begin{cases} dS_t = rS_t dt + \sqrt{V_t} S_t d\widetilde{B}_t^S \\ dV_t = \kappa^{\mathbb{Q}}(\theta^{\mathbb{Q}} - V_t)dt + \sigma_V \sqrt{V_t} d\widetilde{B}_t^V \end{cases} \tag{2.2}$$

where $\kappa^{\mathbb{Q}} = \kappa + \lambda$ and $\theta^{\mathbb{Q}} = \frac{\kappa\theta}{\kappa+\lambda}$ are the risk-neutral parameters, the new parameter λ is the premium of volatility risk (Heston 1993). As illustrated in Heston's paper, applying Breeden (1979)'s consumption-based model yields a volatility risk premium of the form $\lambda(t, S_t, v_t) = \lambda V$ for the CIR square-root process. For the rest of this chapter, our analysis will be based on the risk-neutral probability measure. The conditional expectation at time t is denoted by $E_t^Q = E^Q[\cdot \mid \mathscr{F}_t]$, where $\mathscr{F}_t$ is the filtration up to time t.

2.2.2 Variance Swaps

As discussed in Chapter 1, variance swaps are forward contracts on the future realized variance of the returns of the specified underlying asset. The value of a variance swap at expiry can be written as $(RV - K_{var}) \times L$, where the RV is the annualized realized variance over the contract life $[0, T]$, K_{var} is the annualized delivery price for the variance swap, which is set to make the value of a variance swap equal to zero for both long and short positions at the time the contract

is initially entered. To a certain extent, it reflects market's expectation of the realized variance in the future. L is the notional amount of the swap in dollars per annualized volatility point squared and T is the life time of the contract. For more details about the variance swaps and variance futures, readers are referred to the web sites of CBOE* or NYSE Euronext†.

At the beginning of a contract, it is clearly specified the details of how the realized variance should be calculated. Important factors contributing to the calculation of the realized variance include underlying asset(s), the observation frequency of the price of the underlying asset(s), the annualization factor, the contract lifetime, the method of calculating the variance. Some typical formulae (Howison et al. 2004; Little & Pant 2001) for the measure of realized variance are

$$RV_{d1}(0,N,T) = \frac{AF}{N}\sum_{i=1}^{N}\left(\frac{S_{t_i}-S_{t_{i-1}}}{S_{t_{i-1}}}\right)^2 \times 100^2 \tag{2.3}$$

or

$$RV_{d2}(0,N,T) = \frac{AF}{N}\sum_{i=1}^{N}\log^2\left(\frac{S_{t_i}}{S_{t_{i-1}}}\right) \times 100^2 \tag{2.4}$$

where S_{t_i} is the closing price of the underlying asset at the i-th observation time t_i, and there are altogether N observations. AF is the annualized factor converting this expression to an annualized variance. If the sampling frequency is every trading day, then $AF = 252$, assuming there are 252 trading days in one year, if every week then $AF = 52$, if every month then $AF = 12$ and so on. We assume equally-spaced discrete observations in this thesis so that the annualized factor is of a simple expression $AF = \frac{1}{\Delta t} = \frac{N}{T}$.

In the literature, these two definitions have been alternatingly used to measure the realized variance, even though in practice most of the contracts appear to be drawn up using the definition $RV_{d2}(0,N,T)$ for the realized variance. For example, while Little & Pant (2001) used $RV_{d1}(0,N,T)$ in their numerical method

*http://cfe.cboe.com/Products/Spec_VT.aspx

†http://www.euronext.com/fic/000/010/990/109901.ppt

pricing model for variance swaps, Broadie & Jain (2008*b*) employed $RV_{d2}(0, N, T)$ as the discretely-sampled realized variance to price variance swaps. Zhu & Lian (2009*d*) pointed out that Broadie & Jain (2008*b*)'s approach could be only applied if the realized variance is defined by $RV_{d2}(0, N, T)$ and showed a completely different approach of pricing variance swaps based on the definition $RV_{d1}(0, N, T)$ under the Heston (1993)'s stochastic volatility model. Windcliff et al. (2006) discussed how to numerically price variance swaps using the both definitions as the measurement of realized variance under Black-Scholes framework. They refereed $RV_{d1}(0, N, T)$ and $RV_{d2}(0, N, T)$ as actual-return variance and log-return variance, respectively. Hereafter, definitions $RV_{d1}(0, N, T)$ and $RV_{d2}(0, N, T)$ are referred to as the actual-return realized variance and the log-return realized variance, respectively.

As shown by Jacod & Protter (1998), when the sampling frequency increases to infinity, the discretely-sampled realized variance approaches the continuously-sampled realized variance, $RV_c(0, T)$, that is:

$$RV_c(0, T) = \lim_{N \to \infty} RV_{d1}(0, N, T) = \frac{1}{T} \int_0^T \sigma_t^2 dt \times 100^2 \tag{2.5}$$

where σ_t is the so-called instantaneous volatility of the underlying. Of course, if there is no assumption on the stochastic nature of the volatility itself, instantaneous volatility is nothing but local volatility as stated in Little & Pant (2001).

In the risk-neutral world, the value of a variance swap at time t is the expected present value of the future payoff. This should be zero at the beginning of the contract since there is no cost to enter into a swap. Therefore, the fair variance delivery price can be easily defined as $K_{var} = E_0^Q[RV]$, after setting the initial value of a variance swap to be zero. The variance swap valuation problem is therefore reduced to calculating the expectation value of the future realized variance in the risk-neutral world.

2.2.3 Our Approach to Price Variance Swaps

We first illustrate our approach to obtain the closed-form analytical solution for fair strike price of a variance swap by taking $RV_{d1}(T_s, N, T_e)$ as the definition of the realized variance. For the case of $RV_{d2}(T_s, N, T_e)$, the solution procedure is very similar and the corresponding pricing formula can be easily obtained with little effort, demonstrating the versatility of this approach, as shall be shown in the next subsection.

As illustrated in Eq. (2.4), the expected value of realized variance in the risk neutral world is defined as:

$$E_0^Q[RV_{d1}(0,N,T)] = E_0^Q[\frac{1}{N\Delta t}\sum_{i=1}^{N}(\frac{S_{t_i}-S_{t_{i-1}}}{S_{t_{i-1}}})^2]\times 100^2 = \frac{100^2}{N\Delta t}\sum_{i=1}^{N}E_0^Q[(\frac{S_{t_i}-S_{t_{i-1}}}{S_{t_{i-1}}})^2] \tag{2.6}$$

So the problem of pricing variance swap is reduced to calculating the N expectations in the form of:

$$E_0^Q[(\frac{S_{t_i}-S_{t_{i-1}}}{S_{t_{i-1}}})^2] \tag{2.7}$$

for some fixed equal time period Δt and N different tenors $t_i = i\Delta t$ ($i = 1, \cdots, N$). In the rest of this section, we will focus our main attention on calculating the expectation of this expression. As shall be shown later, we need to consider two cases, $i = 1$ and $i > 1$, due to the difference in the calculation procedures. In the process of calculating of this expectation, i, unless otherwise stated, is regarded as a constant. And hence both t_i and t_{i-1} are regarded as known constants.

Firstly we consider the case $i > 1$. In this case the time $t_{i-1} > 0$ and thus $S_{t_{i-1}}$ is also an unknown at the current time $t = 0$. Therefore, the payoff function depends on two unknown variables $S_{t_{i-1}}$ and S_{t_i} which are the underlying price in the future. This two-dimensional payoff function makes the problem extremely difficult to deal with. We will however show that the problem could be solved by firstly introducing a new variable I_t and then decomposing the original problem

into two one-dimensional problems which could be relatively easier to be solved analytically. This technique was firstly proposed by Little & Pant (2001).

Let us first introduce a new variable I_t

$$I_t = \int_0^t \delta(t_{i-1} - \tau) S_\tau d\tau \tag{2.8}$$

where the $\delta(\cdot)$ is the Dirac delta function. Note that $I_t = 0$ for $t < t_{i-1}$ and $I_t = S_{t_{i-1}}$ for $t \geq t_{i-1}$.

We now consider a contingent claim $U_i = U_i(S_t, v_t, I_t, t)$ whose payoff at expiry t_i is $(\frac{S_{t_i}}{I_{t_i}} - 1)^2$. Following the general asset valuation theory by Garman (1977), or the standard analysis of Asian options with stochastic volatility (Fouque et al. 2000; Wilmott 1998), we obtain the PDE for U_i (subscripts have been omitted in the PDE without ambiguity).

$$\begin{aligned} &\frac{\partial U_i}{\partial t} + \frac{1}{2} V S^2 \frac{\partial U_i^2}{\partial S^2} + \rho \sigma_V V S \frac{\partial U_i^2}{\partial S \partial V} + \frac{1}{2} \sigma_V^2 V \frac{\partial U_i^2}{\partial V^2} + rS \frac{\partial U_i}{\partial S} \\ &+ [\kappa^{\mathbb{Q}}(\theta^{\mathbb{Q}} - V)] \frac{\partial U_i}{\partial V} - r U_i + \delta(t_{i-1} - t) \frac{\partial U_i}{\partial I} = 0 \end{aligned} \tag{2.9}$$

The terminal condition is

$$U_i(S, v, I, t_i) = \left(\frac{S}{I} - 1 \right)^2 \tag{2.10}$$

Howison et al. (2004) also derived a similar PDE based on their model, however, they didn't solve the PDE directly.

The Feynman-Kac theorem (Karatzas et al. 1991) states that the solution of the PDE system satisfies:

$$E_0^Q[(\frac{S_{t_i}}{I_{t_i}} - 1)^2] = e^{rt_i} U_i(S_0, v_0, I_0, 0) \tag{2.11}$$

Thus it is sufficient to solve the PDE (2.9) with terminal condition (2.10) to obtain the expectation (2.7) we require. To solve this PDE system, we need to

utilize the properties of variable I_t and the Dirac delta function in the equation.

The property of Dirac delta function indicates that any time away from t_{i-1} the PDE (2.9) could be reduced as

$$\frac{\partial U_i}{\partial t}+\frac{1}{2}VS^2\frac{\partial U_i^2}{\partial S^2}+\rho\sigma_V VS\frac{\partial U_i^2}{\partial S\partial V}+\frac{1}{2}\sigma_V^2V\frac{\partial U_i^2}{\partial V^2}+rS\frac{\partial U_i}{\partial S}+[\kappa^{\mathbb{Q}}(\theta^{\mathbb{Q}}-V)]\frac{\partial U_i}{\partial v}-rU_i=0 \tag{2.12}$$

This means that we have managed to get rid of variable I_t in the equation except at the time t_{i-1}. However, we cannot declare that we have succeeded in getting rid of one spatial dimension due to the presence of I_t in the terminal condition (2.10). To handle the I_t in the terminal condition, we turn to the so-called jump condition.

As mentioned previously, $I_t = 0, t < t_{i-1}$ and $I_t = S_{t_{i-1}}, t \geq t_{i-1}$. The variable I_t therefore experiences a jump in value across time t_{i-1}. The no-arbitrary pricing theory however requires the claim's value should remain continuous. This leads to an additional jump condition at time t_{i-1} (refer to Wilmott et al. (1993) for a further discussion of jump conditions),

$$\lim_{t\uparrow t_{i-1}} U_i(S,v,I,t) = \lim_{t\downarrow t_{i-1}} U_i(S,v,I,t) \tag{2.13}$$

From this viewpoint, we can equivalently solve the PDE (2.12) with terminal condition (2.10) and jump condition (2.13) in order to obtain the expectation we are interested in. Furthermore, inspired by the property of variable I_t, we consider dividing the time domain $[0, t_i]$ into two parts $[0, t_{i-1}]$ and $[t_{i-1}, t_i]$ since during each of the two time sub-domains, I_t could be regarded as constant. Hence, it is a clever idea to solve the PDE system by two stages, the first stage in $[t_{i-1}, t_i]$ and the second stage in $[0, t_{i-1}]$. During each of the two stages the PDE systems have one dimension less than the original PDE system. The obtained solution of the first stage will provide the terminal condition for PDE system in second stage through the jump condition (2.13). We need to remark that this is one of the key

features of the research in this chapter. Little & Pant (2001) were the first to use the dimension reduction approach which provides many computational benefits in their instantaneous local volatility model. In this study, the approach is applied to the stochastic volatility model and provides us with a closed-form solution.

Now, the PDE system (2.9) could be equivalently expressed by two PDE systems as

$$\begin{cases} \dfrac{\partial U_i}{\partial t} + \dfrac{1}{2}VS^2\dfrac{\partial U_i^2}{\partial S^2} + \rho\sigma_V VS\dfrac{\partial U_i^2}{\partial S\partial V} + \dfrac{1}{2}\sigma_V^2 V\dfrac{\partial U_i^2}{\partial V^2} + rS\dfrac{\partial U_i}{\partial S} + [\kappa^{\mathbb{Q}}(\theta^{\mathbb{Q}} - V)]\dfrac{\partial U_i}{\partial V} - rU_i = 0 \\ U_i(S, V, I, t_i) = (\dfrac{S}{I} - 1)^2 \qquad\qquad t_{i-1} \le t \le t_i \end{cases} \tag{2.14}$$

and

$$\begin{cases} \dfrac{\partial U_i}{\partial t} + \dfrac{1}{2}VS^2\dfrac{\partial U_i^2}{\partial S^2} + \rho\sigma_V VS\dfrac{\partial U_i^2}{\partial S\partial V} + \dfrac{1}{2}\sigma_V^2 V\dfrac{\partial U_i^2}{\partial V^2} + rS\dfrac{\partial U_i}{\partial S} + [\kappa^{\mathbb{Q}}(\theta^{\mathbb{Q}} - v)]\dfrac{\partial U_i}{\partial V} - rU_i = 0 \\ \lim\limits_{t\uparrow t_{i-1}} U_i(S, V, I, t) = \lim\limits_{t\downarrow t_{i-1}} U_i(S, V, I, t) \qquad 0 \le t \le t_{i-1} \end{cases} \tag{2.15}$$

Note that I_t is a fixed number $I_t = S_{t_{i-1}}$ in the domain $t_{i-1} \le t \le t_i$ and $I_t = 0$ in $0 \le t < t_{i-1}$. We firstly analytically solve the PDE system (2.14) using the generalized Fourier transform method (see Lewis 2000; Poularikas 2000).

Proposition 1 *If the underlying asset follows the dynamic process (2.2) and a European-style derivative written on this underlying asset has a payoff function $U(S, V, T) = H(S)$ at expiry T, then the solution of the associated PDE system of the derivative value*

$$\begin{cases} \dfrac{\partial U}{\partial t} + \dfrac{1}{2}VS^2\dfrac{\partial U^2}{\partial S^2} + \rho\sigma_V VS\dfrac{\partial U^2}{\partial S\partial V} + \dfrac{1}{2}\sigma_V^2 V\dfrac{\partial U^2}{\partial V^2} + rS\dfrac{\partial U}{\partial S} + [\kappa^{\mathbb{Q}}(\theta^{\mathbb{Q}} - V)]\dfrac{\partial U}{\partial V} - rU = 0 \\ U(S, V, T) = H(S) \end{cases} \tag{2.16}$$

can be expressed in closed form as:

$$U(x, V, t) = \mathcal{F}^{-1}[e^{C(\omega, T-t) + D(\omega, T-t)V}\mathcal{F}[H(e^x)]] \tag{2.17}$$

using generalized Fourier transform method (see Lewis 2000; Poularikas 2000), where $x = \ln S$, $j = \sqrt{-1}$ *and* ω *is the Fourier transform variable, and*

$$\begin{cases} C(\omega,\tau) = r(\omega j - 1)\tau + \dfrac{\kappa^{\mathbb{Q}}\theta^{\mathbb{Q}}}{\sigma_V^2}[(a+b)\tau - 2\ln(\dfrac{1-ge^{b\tau}}{1-g})] \\ D(\omega,\tau) = \dfrac{a+b}{\sigma_V^2}\dfrac{1-e^{b\tau}}{1-ge^{b\tau}} \\ a = \kappa^{\mathbb{Q}} - \rho\sigma_V\omega j, \quad b = \sqrt{a^2 + \sigma_V^2(\omega^2 + \omega j)}, \quad g = \dfrac{a+b}{a-b} \end{cases} \tag{2.18}$$

The proof of this proposition is left in Appendix B.1.

It should be noted that Formula (2.17) has been deliberately left in a rather general form. This is because the payoff function $H(S)$ hasn't been specified yet. In this most general form, Proposition 1 is applicable to most derivatives whose payoffs depend on spot price S of underlying asset in the framework of Heston's stochastic volatility. The original result of Heston (1993) is actually a special case covered by this proposition.

However, for some payoffs, the Fourier transform in Proposition 1 has to be interpreted as the generalized Fourier transform, which is a useful tool for pricing derivatives. For most popularly used financial derivatives, such as vanilla call options with $H(S) = \max(S - K, 0)$, performing the generalized Fourier transform is straightforward. The main difficulty with this approach, however, is associated with the Fourier inverse transform needed to be performed, if one wishes to reduce the computational time substantially. For our specific case, $H(S) = (\frac{S}{I} - 1)^2$, the Fourier inverse transform could be explicitly worked out and hence the solution can be written in a much simple and elegant form.

Based on the generalized Fourier transform, we can perform the transformation as

$$\mathcal{F}[e^{j\alpha t}] = 2\pi\delta_\alpha(\omega) \tag{2.19}$$

where $j = \sqrt{-1}$, α is any complex number and $\delta_\alpha(\omega)$ is the generalized delta

function satisfying

$$\int_{-\infty}^{\infty} \delta_\alpha(t)\Phi(t)dt = \Phi(\alpha) \tag{2.20}$$

In our specific case PDE (2.14), $H(S) = (\frac{S}{I} - 1)^2$. By setting $x = \ln S$ and noting I a constant, we perform the generalized Fourier transform to the payoff function $H(e^x)$ with regards to x.

$$\mathcal{F}[(\frac{e^x}{I} - 1)^2] = 2\pi[\frac{\delta_{-2j}(\omega)}{I^2} - 2\frac{\delta_{-j}(\omega)}{I} + \delta_0(\omega)] \tag{2.21}$$

Using the Proposition 1, the solution of PDE (2.14) is given by

$$\begin{aligned}
&U_i(S,V,I,t) = \mathcal{F}^{-1}[e^{C(\omega,t_i-t)+D(\omega,t_i-t)V}2\pi[\frac{\delta_{-2j}(\omega)}{I^2} - 2\frac{\delta_{-j}(\omega)}{I} + \delta_0(\omega)]] \\
&= \int_{-\infty}^{\infty} e^{C(\omega,t_i-t)+D(\omega,t_i-t)V}[\frac{\delta_{-2j}(\omega)}{I^2} - 2\frac{\delta_{-j}(\omega)}{I} + \delta_0(\omega)]e^{x\omega j}d\omega \\
&= \frac{1}{I^2}e^{C(\omega,t_i-t)+D(\omega,t_i-t)V+x\omega j}|_{\omega=-2j} - \frac{2}{I}e^{C(\omega,t_i-t)+D(\omega,t_i-t)V+x\omega j}|_{\omega=-j} \\
&\quad + e^{C(\omega,t_i-t)+D(\omega,t_i-t)V+x\omega j}|_{\omega=0} \\
&= \frac{e^{2x}}{I^2}e^{\widetilde{C}(t_i-t)+\widetilde{D}(t_i-t)V} - \frac{2e^x}{I} + e^{-r(t_i-t)}
\end{aligned} \tag{2.22}$$

where $x = \ln S$ and $t_{i-1} \le t \le t_i$, and $\widetilde{C}(\tau)$ and $\widetilde{D}(\tau)$ are equal to $C(-2j,\tau)$, $D(-2j,\tau)$ respectively, and have simple forms as

$$\begin{cases}
\widetilde{C}(\tau) = r\tau + \dfrac{\kappa^{\mathbb{Q}}\theta^{\mathbb{Q}}}{\sigma_V^2}[(\widetilde{a}+\widetilde{b})\tau - 2\ln(\dfrac{1-\widetilde{g}e^{\widetilde{b}\tau}}{1-\widetilde{g}})] \\
\widetilde{D}(\tau) = \dfrac{\widetilde{a}+\widetilde{b}}{\sigma_V^2}(\dfrac{1-e^{\widetilde{b}\tau}}{1-\widetilde{g}e^{\widetilde{b}\tau}}) \\
\widetilde{a} = \kappa^{\mathbb{Q}} - 2\rho\sigma_V, \quad \widetilde{b} = \sqrt{\widetilde{a}^2 - 2\sigma_V^2}, \quad \widetilde{g} = (\dfrac{\widetilde{a}}{\sigma_V})^2 - 1 + (\dfrac{\widetilde{a}}{\sigma_V})\sqrt{(\dfrac{\widetilde{a}}{\sigma_V})^2 - 2}
\end{cases} \tag{2.23}$$

Now, we have succeeded in obtaining the solution for the PDE system (2.14), which is the first stage in calculating $E_0^Q[(\frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}})^2]$. It should be remarked that we have actually solved an option pricing problem based on Heston's stochastic volatility model. The very reason that we have explicitly worked out the Fourier inverse transform so that our final solution (2.22) of the first stage can be

written in such a simple and closed form, whereas the Fourier inverse transform could not be worked out by Heston (1993), is because of the very special form of the payoff function (2.10). One may argue that Heston's solution for a simple European call is still in closed form, because there is only an explicit integral left to be calculated, the same as the calculation of the cumulative distribution function required in using the Black-Scholes formula. But, a sharp difference between the two is that the integrand of the latter is a well-defined and smooth real function whereas the integrand of the former (i.e., the Heston's original solution as well as the solutions presented in many other follow-up papers based on Heston's model, such as Bakshi et al. 1997; Bates 1996; Pan 2002), is a complex-value function, as a result of the Fourier inverse transform not being analytically performed. The main disadvantage of a solution being left in terms of complex-valued integrals is that the numerical calculation of these integrals has to be handled very carefully as they are multi-valued complex functions, which may cause some problems when one needs to decide which root is the correct one to take. There have been examples reported in the literature (e.g., Kahl & Jackel 2005) for the wrong numerical integration that those complex-valued integrand may result in. In comparison with those complicated integral calculations, the advantage of our compact solution (2.22) is obvious. Although our success in analytically performing Fourier inverse transform under the Heston's model may be limited for a special form of payoff function, it made us to believe that there might be other payoff functions, with which the Fourier inverse transform can be worked out analytically as well. This belief has not been clearly articulated in the relevant literature before; all the papers following Heston's work stopped at the same point where Heston did, i.e., did not bother to analytically perform the Fourier inverse transform at all.

To finish off the calculation of $E_0^Q[(\frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}})^2]$, we need to move to the second stage, i.e. solving the PDE system (2.15), after the imposition of the jump condition (2.13). As we shall show later, the simple form of solution (2.22) has paved an easy way of obtaining an analytical solution in the second stage.

By noting the fact that $\lim_{t\downarrow t_{i-1}} \ln S_t = \ln I$ due to the definition of I, we have,

$$\lim_{t\downarrow t_{i-1}} U_i(S,V,I,t) = e^{\widetilde{C}(\Delta t)+\widetilde{D}(\Delta t)V} + e^{-r\Delta t} - 2 \tag{2.24}$$

For the simplicity of notation, the right hand side of above equation is denoted as $f(V)$, i.e.,

$$f(V) = e^{\widetilde{C}(\Delta t)+\widetilde{D}(\Delta t)V} + e^{-r\Delta t} - 2 \tag{2.25}$$

which is now the terminal condition for the PDE system (2.15) in the period $0 \le t \le t_{i-1}$, according to the jump condition (2.13).

It should be noticed that the terminal condition (2.25) for the PDE system (2.15) in the period $0 \le t \le t_{i-1}$ happens to contain one independent variable, V, only. One can thus take the advantage of this fact and solve the problem neatly with the following proposition.

Proposition 2 *If the underlying asset follows the dynamic process (2.2), the derivative written on some stochastic aggregated property of this underlying asset with payoff function depending on the V_T only, i.e., $U(S,V,T) = G(V_T)$ at expiry T will satisfy the PDE*

$$\begin{cases} \dfrac{\partial U}{\partial t} + \dfrac{1}{2}VS^2\dfrac{\partial U^2}{\partial S^2} + \rho\sigma_V VS\dfrac{\partial U^2}{\partial S\partial V} + \dfrac{1}{2}\sigma_V^2 V\dfrac{\partial U^2}{\partial V^2} + rS\dfrac{\partial U}{\partial S} + [\kappa^{\mathbb{Q}}(\theta^{\mathbb{Q}} - V)]\dfrac{\partial U}{\partial V} - rU = 0 \\ U(S,V,T) = G(V) \end{cases} \tag{2.26}$$

The solution of this PDE can be obtained analytically in the form of

$$U(S,V,t) = \int_0^{+\infty} e^{-r(T-t)} G(V_T) p(V_T|V_t) dV_T \tag{2.27}$$

where

$$p(V_T|V_t) = ce^{-W-v}(\frac{v}{W})^{q/2} I_q(2\sqrt{Wv})$$

$$c = \frac{2\kappa^{\mathbb{Q}}}{\sigma_V^2(1-e^{-\kappa^{\mathbb{Q}}(T-t)})}, \qquad W = cV_t e^{-\kappa^{\mathbb{Q}}(T-t)}, \qquad v = cV_T, \qquad q = \frac{2\kappa^{\mathbb{Q}}\theta^{\mathbb{Q}}}{\sigma_V^2} - 1 \tag{2.28}$$

and $I_q(\cdot)$ is the modified Bessel function of the first kind of order q.

The proof of Proposition 2 is trivial, as it is actually implied by the Feynman-Kac formula, which states that the solution of PDE (2.26) can be derived from the conditional expectation of the payoff function under the risk-neutral probability measure. Hence, the solution can be expressed in the form of

$$U(S, V, t) = E_t^Q[e^{-r(T-t)}G(V_T)] \tag{2.29}$$

where the associated two processes S_t and V_t follow the stochastic processes in (2.2), respectively. The expectation is actually not related to the process S since the payoff function is independent of S. The process V_t is the well-known CIR squared-root process (Cox et al. 1985) and the distribution is the noncentral chi-square, $\chi^2(2v; 2q+2, 2W)$, with $2q+2$ degrees of freedom and parameter of non-centrality $2W$ proportional to the current variance, V_t. Once we realized that the needed transition probability density function $p(V_T|V_t)$ has been given in Cox et al. (1985), as shown in Equation (2.28), the proof naturally follows.

Using the Proposition 2, we can express the solution of PDE system (2.15) as

$$U_i(S, V, I, t) = \int_0^\infty e^{-r(t_{i-1}-t)} f(V_{t_{i-1}}) p(V_{t_{i-1}}|V_t) dV_{t_{i-1}} \tag{2.30}$$

where $f(V)$ and $p(V_{t_{i-1}}|V_t)$ are given in Eq. (2.25) and Eq. (2.28) respectively, and $0 \leq t < t_{i-1}$. This means for each $i > 1$ the expectation (2.7) has been found by solving the PDE systems (2.14) and (2.15) in two stages,

$$\begin{aligned} &E_0^Q[(\frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}})^2] = e^{rt_i} U_i(S_0, V_0, I_0, 0) \\ &= \int_0^\infty e^{r\Delta t} f(V_{t_{i-1}}) p(V_{t_{i-1}}|V_0) dV_{t_{i-1}} \end{aligned} \tag{2.31}$$

As Zhang & Zhu (2006) commented in their paper, the integration in the above equation usually cannot be explicitly carried out; we had initially decided

to leave our final solution in this integral form too. However, after a careful examination of the properties of the integrand, we realized that the elegant form of $f(V)$, which is the solution of the first stage, could be explored again. Utilizing the characteristic function of noncentral chi-squared distribution (Johnson et al. 1970), we have successfully carried out the above integral analytically and obtain a fully closed-form solution as our final solution for the price of a variance swap with the realized variance defined by (2.3). This has made our solution in a remarkably simple form as

$$E_0^Q[(\frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}})^2] = e^{r\Delta t} f_i(V_0) \tag{2.32}$$

where

$$\begin{aligned} f_i(V_0) &= \int_0^\infty f(V_{t_{i-1}}) p(V_{t_{i-1}}|V_0) dV_{t_{i-1}} \\ &= e^{\widetilde{C}(\Delta t) + \frac{c_i e^{-\kappa^{\mathbb{Q}} t_{i-1}}}{c_i - \widetilde{D}(\Delta t)} \widetilde{D}(\Delta t) V_0} (\frac{c_i}{c_i - \widetilde{D}(\Delta t)})^{\frac{2\kappa^{\mathbb{Q}}\theta^{\mathbb{Q}}}{\sigma_V^2}} + e^{-r\Delta t} - 2 \end{aligned} \tag{2.33}$$

and $c_i = \frac{2\kappa^{\mathbb{Q}}}{\sigma_V^2(1-e^{-\kappa^{\mathbb{Q}} t_{i-1}})}$. To a certain extent, it is even simpler than that of the classic Black-Scholes formula, because the latter still involves the calculation of the cumulative distribution function, which is an integral of a smooth real-value function, whereas there is no need to calculate any integral at all in our final solution! The details of analytically carrying out the integration in Eq. (2.33) are left in Appendix B.2.

Utilizing (2.32), the summation in (2.6) can now be carried out all the way except for the very first period with $i = 1$.

We need to treat the case $i = 1$, separately, simply because in this case we have $t_{i-1} = 0$ and $S_{t_{i-1}} = S_0$, which is the current underlying asset price and is a known value, instead of an unknown value of $S_{t_{i-1}}$ for any other cases with $i > 1$.

So the expectation that needs to be calculated in this special case is reduced to

$$E_0^Q[(\frac{S_{t_i}}{S_0} - 1)^2] \tag{2.34}$$

which can be easily derived by invoking Proposition 1 directly,

$$E_0^Q[(\frac{S_{t_1}}{S_0} - 1)^2] = e^{r\Delta t} f(V_0) \tag{2.35}$$

Summarizing the calculation procedure discussed above, we finally obtain the fair strike price for the actual-return variance swap as:

$$K_{var} = E_0^Q[RV_{d1}(0, N, T)] = \frac{e^{r\Delta t}}{T}[f(V_0) + \sum_{i=2}^{N} f_i(V_0)] \times 100^2 \tag{2.36}$$

where N is a finite number denoting the total sampling times of the swap contract. This formula is obtained by solving the associated PDEs in two stages. Since we have managed to express the solution of the associated PDEs, in both stages, in terms of simple and elementary functions, we are able to write the fair strike price of an actual-return variance swap with discretely-sampled realized variance defined in Eq. (2.3) in a simple and closed form.

In fact, even for the a log-return variance swap with discretely-sampled realized variance defined in Eq. (2.4), our approach presented here can also be analogically applied to obtain a closed-form exact solution, demonstrating the flexibility of our approach.

As shown previously, the problem of pricing a log-return variance swap is reduced to calculating the N expectations in the form of:

$$E_0^Q \left[\log^2(\frac{S_{t_i}}{S_{t_{i-1}}})\right] \tag{2.37}$$

for some fixed equal time period Δt and N different tenors $t_i = i\Delta t$ ($i =$

$1, \cdots, N$).

This expectation can be carried out by solving the two PDE systems as

$$\begin{cases} \dfrac{\partial U_i}{\partial t} + \dfrac{1}{2}VS^2\dfrac{\partial U_i^2}{\partial S^2} + \rho\sigma_V VS\dfrac{\partial U_i^2}{\partial S\partial V} + \dfrac{1}{2}\sigma_V^2 V\dfrac{\partial U_i^2}{\partial V^2} + rS\dfrac{\partial U_i}{\partial S} + [\kappa^{\mathbb{Q}}(\theta^{\mathbb{Q}} - V)]\dfrac{\partial U_i}{\partial V} - rU_i = 0 \\ U_i(S, V, I, t_i) = \log^2(\dfrac{S}{I}) \qquad t_{i-1} \le t \le t_i \end{cases} \tag{2.38}$$

and

$$\begin{cases} \dfrac{\partial U_i}{\partial t} + \dfrac{1}{2}VS^2\dfrac{\partial U_i^2}{\partial S^2} + \rho\sigma_V VS\dfrac{\partial U_i^2}{\partial S\partial V} + \dfrac{1}{2}\sigma_V^2 V\dfrac{\partial U_i^2}{\partial V^2} + rS\dfrac{\partial U_i}{\partial S} + [\kappa^{\mathbb{Q}}(\theta^{\mathbb{Q}} - V)]\dfrac{\partial U_i}{\partial V} - rU_i = 0 \\ \lim\limits_{t\uparrow t_{i-1}} U_i(S, V, I, t) = \lim\limits_{t\downarrow t_{i-1}} U_i(S, V, I, t) \qquad 0 \le t \le t_{i-1} \end{cases} \tag{2.39}$$

where I_t is a fixed number $I_t = S_{t_{i-1}}$ in the domain $t_{i-1} \le t \le t_i$ and $I_t = 0$ in $0 \le t < t_{i-1}$. The solutions of these two PDE systems are actually implied by the Proposition 1 and Proposition 2.

Specifically, based on the generalized Fourier transform, we can perform the transformation as

$$\mathcal{F}[x^n] = 2\pi j^n \delta^{(n)}(\omega) \tag{2.40}$$

where $j = \sqrt{-1}$, n is any integer and $\delta^{(n)}(\omega)$ is the n-th order derivative of the generalized delta function satisfying

$$\int_{-\infty}^{\infty} \delta^{(n)}(\omega)\Phi(\omega)d\omega = (-1)^n \Phi^{(n)}(0) \tag{2.41}$$

By setting $x = \ln S$ and noting I a constant, we perform the generalized Fourier transform to the payoff function $H(x)$ in PDE (2.38) with regards to x.

$$\mathcal{F}[(x - \log I)^2] = 2\pi[-\delta^{(2)}(\omega) - 2j\delta^{(1)}(\omega)\log(I) + \delta(\omega)\log^2 I] \tag{2.42}$$

Using the Proposition 1, the solution of PDE (2.38) is given by

$$\begin{aligned} U_i(S,V,I,t) &= \mathcal{F}^{-1}[e^{C(\omega,t_i-t)+D(\omega,t_i-t)V} 2\pi[-\delta^{(2)}(\omega) - 2j\delta^{(1)}(\omega)\log(I) + \delta(\omega)\log^2 I] \\ &= \int_{-\infty}^{\infty} e^{C(\omega,t_i-t)+D(\omega,t_i-t)V}[-\delta^{(2)}(\omega) - 2j\delta^{(1)}(\omega)\log(I) + \delta(\omega)\log^2 I]e^{x\omega j}d\omega \\ &= -f^{(2)}(0) + 2jf^{(1)}(0)\log I + f(0)\log^2 I \end{aligned} \tag{2.43}$$

where $f(\omega) = e^{C(\omega,t_i-t)+D(\omega,t_i-t)V+x\omega j}$, with $x = \log S$ and $t_{i-1} \leq t \leq t_i$. The terms $f^{(2)}(0)$ and $f^{(1)}(0)$ can be easily computed, using symbolic calculation packages, such as Maple 10.

To finish off the calculation of $E_0^Q[\log^2(\frac{S_{t_i}}{S_{t_{i-1}}})]$, we need to move to the second stage, i.e. solving the PDE system (2.39), after the imposition of the jump condition (2.13). By noting the fact that $\lim_{t\downarrow t_{i-1}} \log S_t = \log I$ due to the definition of I, we obtain

$$\lim_{t\downarrow t_{i-1}} U_i(S,V,I,t) = e^{-r\Delta t} g(V) \tag{2.44}$$

where $g(V)$ is the expression

$$g(V) = (D^{(1)})^2V^2 + (2C^{(1)}D^{(1)} - D^{(2)})V + (C^{(1)})^2 - C^{(2)} \tag{2.45}$$

resulting from computing all the derivatives in (2.43) with $C^{(1)} = \frac{\partial C(\omega,\Delta t)}{\partial \omega}|_{\omega=0}$, $C^{(2)} = \frac{\partial^2 C(\omega,\Delta t)}{\partial \omega^2}|_{\omega=0}$. $D^{(1)}$ and $D^{(2)}$ are defined similarly. $C(\omega,\tau)$ and $D(\omega,\tau)$ are given in Eq. (2.18).

Eq. (2.44) is now the terminal condition for the PDE system (2.39) in the period $0 \leq t \leq t_{i-1}$, according to the jump condition (2.13).

Using the Proposition 2, we can express the solution of PDE system (2.39) as

$$U_i(S,V,I,t) = \int_0^{\infty} e^{-r(t_{i-1}-t)} e^{-r\Delta t} g(V_{t_{i-1}}) p(V_{t_{i-1}}|V_t) dV_{t_{i-1}} \tag{2.46}$$

where $0 \leq t < t_{i-1}$, $g(V_{t_{i-1}})$ and $p(V_{t_{i-1}}|V_t)$ are given in Equation (2.45) and Equation (2.28) respectively. This means for each $i > 1$ the expectation (2.37)

has been found by solving the PDE systems (2.38) and (2.39) in two stages,

$$E_0^Q[\log^2(\frac{S_{t_i}}{S_{t_{i-1}}})] = e^{rt_i}U_i(S_0, V_0, I_0, 0)$$
$$= \int_0^\infty g(V_{t_{i-1}})p(V_{t_{i-1}}|V_0)dV_{t_{i-1}} \tag{2.47}$$

Utilizing the characteristic function of noncentral chi-squared distribution (Johnson et al. 1970), we have successfully carried out the above integral analytically and obtain a fully closed-form solution as our final solution for the price of a variance swap with the realized variance defined by (2.4). This has made our solution in a remarkably simple form as

$$E_0^Q[\log^2(\frac{S_{t_i}}{S_{t_{i-1}}})] = g_i(V_0) \tag{2.48}$$

where

$$g_i(V_0) = \int_0^\infty g(V_{t_{i-1}})p(V_{t_{i-1}}|V_0)dv_{t_{i-1}}$$
$$= (D^{(1)})^2(\frac{\widetilde{q} + 2W_i + (\widetilde{q} + W_i)^2}{c_i^2})$$
$$+(2C^{(1)}D^{(1)} - D^{(2)})(\frac{\widetilde{q} + W_i}{c_i}) + (C^{(1)})^2 - C^{(2)} \tag{2.49}$$

$c_i = \frac{2\kappa^{\mathbb{Q}}}{\sigma_V^2(1-e^{-\kappa^{\mathbb{Q}}t_{i-1}})}$, $W_i = c_i V_0 e^{-\kappa^{\mathbb{Q}}t_{i-1}}$ and $\widetilde{q} = \frac{2\kappa^{\mathbb{Q}}\theta^{\mathbb{Q}}}{\sigma_V^2}$.

Utilizing (2.48), the summation in (2.4) can now be carried out all the way except for the very first period with $i = 1$, which can be easily derived by invoking Proposition 1 directly,

$$E_0^Q[\log^2(\frac{S_{t_i}}{S_{t_{i-1}}})] = g(V_0) \tag{2.50}$$

Summarizing the calculation procedure discussed above, we finally obtain the fair strike price for the log-return variance swap as:

$$K_{var} = E_0^Q[V_{d2}(0, N, T)] = \frac{1}{T}[g(V_0) + \sum_{i=2}^{N} g_i(V_0)] \times 100^2 \tag{2.51}$$

where N is a finite number denoting the total sampling times of the swap contract. The above equation gives a fair strike price for log-return variance swaps in a simple and closed-form solution.

One may wonder why not use the Feynman-Kac formula to calculate the expectation of the payoff function directly instead of painfully detouring around to solve a PDE (2.14) in Stage 1 first and then using the Feynman-Kac formula in Stage 2. This is actually due to the dimensionality of the payoff functions $(\frac{S_{t_i}-S_{t_{i-1}}}{S_{t_{i-1}}})^2$ and $\log^2(\frac{S_{t_i}}{S_{t_{i-1}}})$, that involves two stochastic variables, S_{t_i} and $S_{t_{i-1}}$. To use the Feynman-Kac formula for this two dimensional payoff function, one needs to find the joint transition probability function of the two stochastic variables, which is a very difficult task, and even if it could be successfully found, there are still difficulties involved in the numerical computation of the resulted two-dimensional integral. This is why we chose to use this two-stage approach to reduce the dimensionality of solving the original problem with the Feynman-Kac formula directly. The great benefit of using these analytic formulae for the prices of variance swaps with the realized variance being defined in Eq. (2.3) and (2.4) is illustrated in the next section through some examples.

2.3 Numerical Examples and Discussions

In this section, we show some numerical examples for illustration purposes. Although theoretically there would be no need to discuss the accuracy of a closed-form exact solution and present numerical results, some comparisons with the Monte Carlo (MC) simulations may give readers a sense of verification for the newly found solution. This is particularly so for some market practitioners who are very used to MC simulations and would not trust analytical solutions that may contain algebraic errors unless they have seen numerical evidence of such a comparison. In addition, comparisons with the previous continuous sampling model will also help readers understand the improvement in accuracy with our

exact solution. Furthermore, we shall discuss some essential properties of variance swaps as well, utilizing the newly found analytical solutions.

To achieve these purposes, we use the following parameters (unless otherwise stated): $v_0 = 0.04$, $\theta^{\mathbb{Q}} = 0.022$, $\kappa^{\mathbb{Q}} = 11.35$, $\rho = -0.64$, $\sigma_V = 0.618$, $r = 0.1$, $T = 1$ in this section. This set of parameters for the square root process was also adopted by Dragulescu & Yakovenko (2002). As for the MC simulations, we took asset price $S_0 = 1$ and the number of the paths $N = 200,000$ for all the simulation results presented here. All the numerical values of variance swaps presented in this section are quoted in variance points (the square of volatility points).

2.3.1 Monte Carlo Simulations

Our MC simulations are based on a simple simulation of the CIR variance process, which is anything but straightforward. Glasserman (2003) proposed a method to simulate the square-root process by sampling the transition density function. Broadie & Kaya (2006) developed an approach for exact simulation of Heston dynamical process. Andreasen (2006) also suggested a method using log-normal approximation for the transition density of the variance with matched first two moments. Higham & Mao (2005) proved that the Euler-Maruyama discretization is an attractive approach, providing qualitatively correct approximations. Since our aim is primarily to obtain some benchmark values for our solutions Eq. (2.36) and Eq. (2.51), we will not focus our attention on the use of other variance reduction techniques that could further enhance the computational efficiency. In our MC simulations, we have employed the simple Euler-Maruyama discretization for the Heston model

$$\begin{cases} S_t = S_{t-1} + rS_{t-1}\Delta t + \sqrt{|V_{t-1}|}S_{t-1}\sqrt{\Delta t}W_t^1 \\ V_t = V_{t-1} + \kappa^{\mathbb{Q}}(\theta^{\mathbb{Q}} - V_{t-1})\Delta t + \sigma\sqrt{|V_{t-1}|}\sqrt{\Delta t}(\rho W_t^1 + \sqrt{1-\rho^2}W_t^2) \end{cases} \tag{2.52}$$

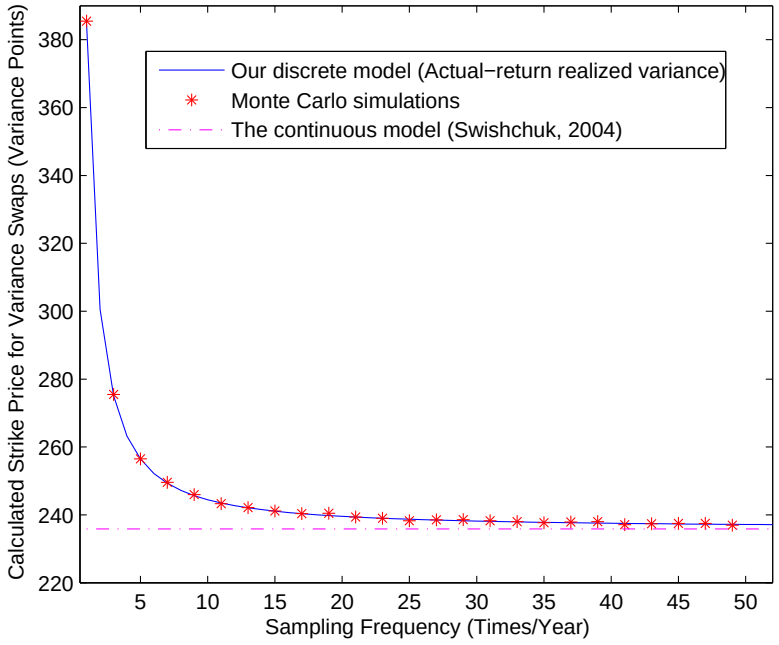

Figure 2.1: A comparison of fair strike values of actual-return variance swaps obtained from our closed-form solution, the continuous approximation and the Monte Carlo simulations, based on the Heston stochastic volatility model

where W_t^1 and W_t^2 are two independent standard normal random variables.

Shown in Fig. 2.1, as well as in Table 2.1, are three sets of data, for the strike price of actual-return variance swaps obtained with the numerical implementation of Formula (2.36), those from MC simulations (2.52) and the numerical results obtained from the continuously-sampled realized variance Formula (2.54). And shown in Fig. 2.2 is a comparison of the strike prices of log-return variance swaps obtained with the numerical implementation of Formula (2.51), those from Monte Carlo simulations (2.52) and the numerical results obtained from the continuously-sampled realized variance Formula (2.54).

One can clearly observe that the results from our exact solution perfectly match the results from the MC simulations. To make sure that readers have some quantitative concept of how large the difference between the results from our exact solution and those from the MC simulations, we have also tabulated the relative difference of the two as a function of the number of paths, using our exact

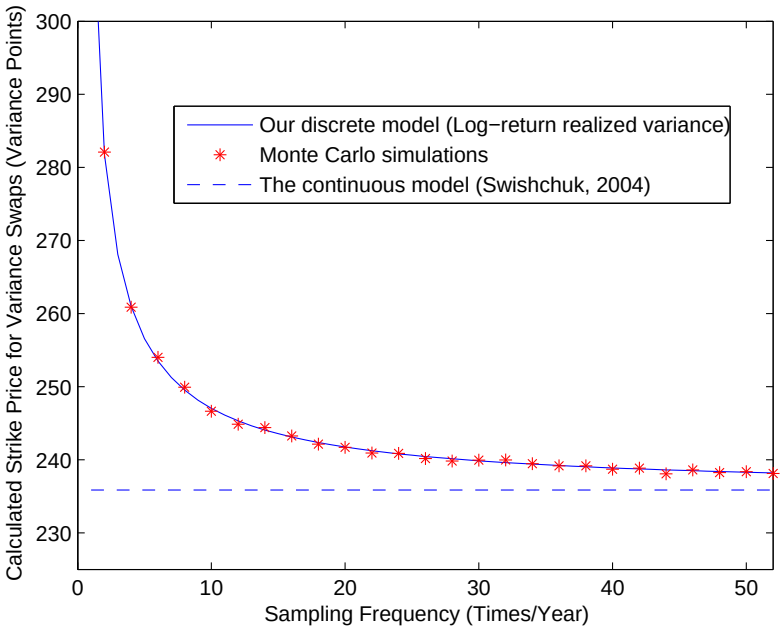

Figure 2.2: A comparison of fair strike values of log-return variance swaps obtained from our closed-form solution, the continuous approximation and the Monte Carlo simulations, based on the Heston stochastic volatility model

solution (2.36) as the reference in the calculation, in Table 2.2. Clearly, when the number of paths reaches 200,000 in MC simulations, the relative difference of the two is less than 0.1% already. Such a relative difference is further reduced when the number of paths is increased; demonstrating the convergence of the MC simulations towards our exact solution.

On the other hand, in terms of computational time, the MC simulations take a much longer time than our analytical solutions do. To illustrate it clearly,

Table 2.1: The strike prices of discretely-sampled actual-return variance swaps obtained from our closed-form solution Eq. (2.36), the continuous approximation and MC simulations

Sampling Frequency	Discrete Model	Continuous Model	MC Simulations
Quarterly(N=4)	267.6	235.9	267.3
Monthly(N=12)	242.7	235.9	243.2
Fortnightly(N=26)	238.6	235.9	238.1
Weekly(N=52)	237.1	235.9	237.4
Daily(N=252)	236.1	235.9	

Table 2.2: Relative errors and computational time of MC simulations in calculating the strike prices of actual-Return variance swaps

Path Numbers of the MC	Relative Error %	Computational Time(s)
10,000	0.233	5.126
100,000	0.191	89.549
200,000	0.074	360.268
500,000	0.012	2,184.239

we compare the computational times of implementing Formula (2.36) and the MC simulations with sampling frequency for the realized variance equalling to 5 times per year. Table 2.2 shows the computational times for different path numbers in the MC simulations. In contrast to a formidable computational time of 2,184.239 seconds using the MC simulations with 500,000 paths, implementing Formula (2.36) just consumed 0.011 seconds; a roughly 200 thousands folds of reduction in computational time for one data point. The difference is even more significant when the sampling frequency is increased; we had to abandon the calculation when the sampling frequency became daily as it just simply took too long to finish off the calculation on our PC (as a result, one cell in Table 2.1 is left empty). This is not surprising at all since time-consuming is a well-known drawback of MC simulations.

2.3.2 The Validity of the Continuous Approximation

In the literature, many researchers, such as Swishchuk (2004), Zhang & Zhu (2006), have proposed continuous sampling models for variance swaps based on the Heston model. In their papers, the realized variance (2.4) is approximated by

$$RV_c(0,T) = \frac{1}{T}\int_0^T V_t dt \times 100^2 \tag{2.53}$$

for the convenience of calculation. This is because Swishchuk (2004) has shown that once the realized variance is defined in terms of an integral, the expectation

of this continuous integral can be easily obtained, utilizing the second stochastic process defined in (2.2). The resulting fair delivery price for the variance swap is thus written as

$$E_0^Q[RV_c(0,T)] = [V_0\frac{1-e^{-\kappa^{\mathbb{Q}}T}}{\kappa^{\mathbb{Q}}T} + \theta^{\mathbb{Q}}(1-\frac{1-e^{-\kappa^{\mathbb{Q}}T}}{\kappa^{\mathbb{Q}}T})] \times 100^2 \tag{2.54}$$

which can be interpreted as a weighted average of the spot variance v_0 and the long-term mean of variance $\theta^{\mathbb{Q}}$. Indeed, this formula is very simple and can be easily implemented in calculating the numerical value of $E_0^Q[RV_c(0,T)]$. For the convenience of referencing, this formula will be referred to as the Swishchuk formula hereafter, although many others also derived this formula.

Due to the lack of exact solution, in the past, for pricing a variance swap with discrete sampling, the Swishchuk formula was primarily used in pricing variance swaps, based on the assumption that the sampling period, such as daily sampling, is short enough so that the result obtained from the continuous model should be close to that without the continuum assumption of the sampling period. However, no one knew exactly how close the results were because there was no exact solution as a pricing formula for the case of discrete sampling times. Nor did any one know when the Swishchuk formula starts to yield large errors when the sampling time is large enough. In other words, there is a validity issue for the Swishchuk formula, since it is nevertheless an approximation in the trading practice where the sampling time, no matter how small, is always discrete. Our newly-derived formulae can now be used not only as pricing formulae for any discrete sample period, but also as a validation tool for checking the accuracy level that the Swishchuk formula yields as a function of the sampling period.

In Fig. 2.1 and Fig. 2.2, we illustrate the numerical results of the Swishchuk formula (2.54) which is obtained from the continuous approximation model. From these figures, one can clearly see that the values of our discrete formulae asymptotically approach the values of the continuous approximation model when the

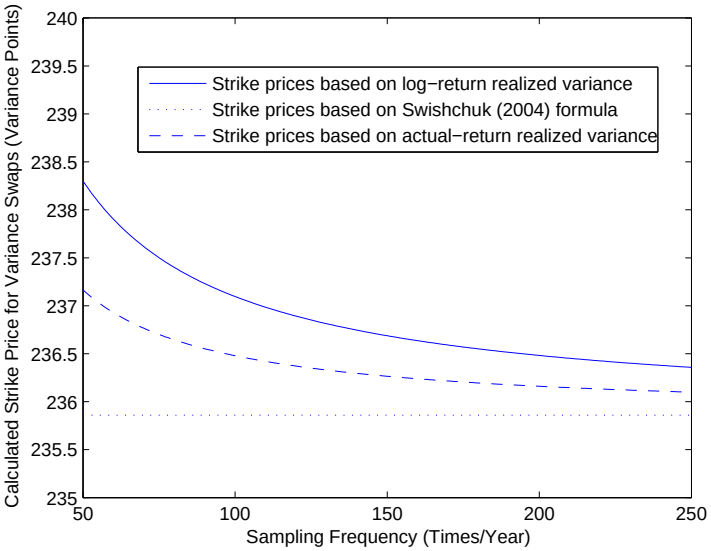

Figure 2.3: Calculated fair strike values of actual-return and log-return variance swaps as a function of sampling frequency

sampling frequency increases; the realized variance defined in (2.53) appears to be the limit of the realized variance defined in Eq. (2.4) and Eq. (2.3) as $\Delta t \to 0$. Of course, one can theoretically prove that our solutions (2.36) and (2.51) indeed approach the Formula (2.54) when the discrete sampling period approaches zero, i.e.,

$$\lim_{\Delta t \to 0} \frac{e^{r\Delta t}}{T}[f(V_0) + \sum_{i=2}^{N} f_i(V_0)] = V_0 \frac{1 - e^{-\kappa^{\mathbb{Q}} T}}{\kappa^{\mathbb{Q}} T} + \theta^{\mathbb{Q}}(1 - \frac{1 - e^{-\kappa^{\mathbb{Q}} T}}{\kappa^{\mathbb{Q}} T}) \qquad (2.55)$$

With the proof of this limit, which is left in Appendix B.3, our solution is once again verified as the correct solution for the discrete sampling cases, taking the continuous sampling case as a special case with the sampling period shrinking down to zero.

On the other hand, we now can use our discrete model to check the validity of the continuous model as an approximation. Shown in Fig. 2.3 is a refined plot of Fig. 2.1 and Fig. 2.2, in order to compare the degree of approximation between daily and weekly sampling. With the daily sampling, the relative difference between the results of the actual-return variance swap and the continuous model is

0.101%, whereas it has increased to 0.530% for weekly sampling. For log-return variance swaps, the relative difference is even greater, with 0.201% for daily sampling and 1.00% for weekly sampling. If the long-term mean variance is further reduced to $\theta^{\mathbb{Q}} = 0.01$ from $\theta^{\mathbb{Q}} = 0.022$ while the other parameters are held the same, the relative difference between the results of variance swaps of weekly sampling and the continuous model becomes more than doubled to reach 1.226% for actual-return variance swaps, and 1.70% for log-return variance swaps of weekly sampling. With a relative difference of the order of one percent, adopting the continuous model as an approximation to price variance swaps with weekly sampling is clearly not justifiable. For example, when the error level reaches more than 0.5%, Little & Pant (2001) has already concluded, within the Black-Scholes framework, that such an error is "fairly large" so that adopting the continuous model might not be so justifiable any more. Our current findings not only confirm Little & Pant (2001)'s conclusion, but also show that, under the Heston model, the difference between the continuous model and the discrete model will exponentially grow, when the sampling frequency is reduced, as shown in Fig. 2.1. Of course, contracts with sampling frequency higher than weekly are very rare in practice. However, specially designed over-the-counter (OTC) contacts of long tenor may still have sampling frequencies small enough to not warrant the realized variance being calculated with the continuous model.

The effect of contract lifetime has been demonstrated in Fig. 2.4, in which the calculated fair strike price is plotted as a function of the tenor of a swap contract. Clearly, all models show that the fair strike price falls as tenor increases. However, the difference between the two becomes larger and larger as tenor increases, further demonstrating the need of using the correct formula presented in this chapter for the discrete sampling case, rather than using the continuous model as an approximation.

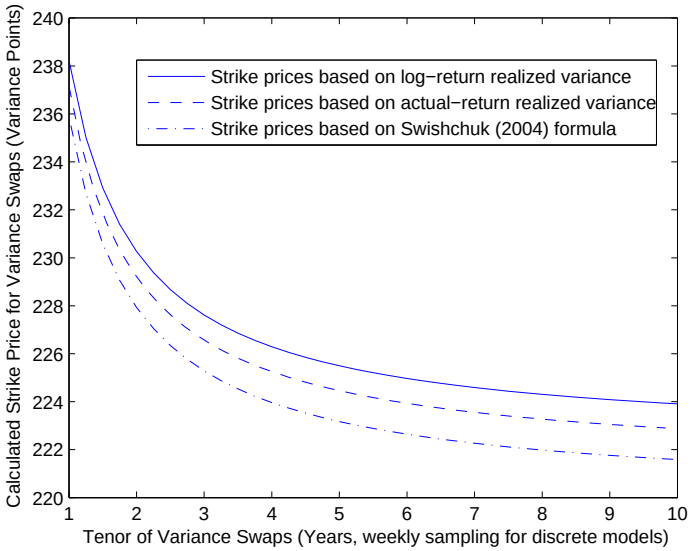

Figure 2.4: Calculated fair strike values of actual-return and log-return variance swaps as a function of tenor

2.3.3 Comparison with Other Solutions

Recently, Broadie & Jain (2008*b*) presented a closed-form formula for log-return variance swaps as that presented in this chapter. However, their solution approach is totally different from ours as what we have presented here. While Broadie & Jain (2008*b*) derived the discrete variance strike for variance swaps by integrating the underlying stochastic processes directly, we have directly solved the governing PDEs that were derived based on the same underlying stochastic processes. The two resulting formulae appear to be quite different in form. It is therefore quite interesting to compare the prices of discretely-sampled variance swaps obtained from these two formulae.

One should notice that the definition of the realized variance defined in Broadie & Jain (2008*b*) is slightly different from Eq. (2.4) used in this thesis. However, the difference between the two is so trivial that all we needed to do was to re-scale the calculated results by a constant, in order to make a good comparison with the results presented in Broadie & Jain (2008*b*). In other words, the results we have re-calculated from Broadie & Jain (2008*b*)'s formula were obtained with their Eq. (45) being re-scaled by a factor of $\frac{N-1}{N}$, where N is the number of sampling points

defined in Eq. (2.4). These results are presented in Fig. 2.5 later.

The main advantage of our approach is its versatility in dealing with different definitions of the realized variance in the payoff function, namely our approach can be easily applied to price variance swaps for both actual-return-based realized variance and the log-return realized variance, and may even possibly be extended to price other volatility- or variance-based derivatives, since our approach does not depend on if the payoff is of such a particular form that the underlying stochastic differential equations can be directly integrated. For example, the approach presented in Broadie & Jain (2008*b*) cannot be used to price variance swaps with the actual-return-based realized variance (Eq. (2.3)) defined in the payoff as pointed out by Zhu & Lian (2009*d*). However, when the payoff is defined by the log-return-based realized variance, Eq. (2.3), which is exactly the same as that defined in Broadie & Jain (2008*b*), we should have every reason to believe that the pricing formulae should yield the same numerical values, although they may appear in different analytical forms.

Shown in Fig. 2.5 is a comparison of the fair delivery prices of variance swaps obtained from these two formulae as a function of sampling frequency. The numerical results calculated from both formulae appear to agree with each other perfectly, as they should be. It is also shown that numerical results calculated from both analytical pricing formulae match up with those obtained from the implementation of Monte Carlo simulation, providing a verification of the correctness of the newly-derived analytical pricing formula presented here as well as that presented in Broadie & Jain (2008*b*).

With the newly found closed-form formulae for the both cases when the realized variance is defined by $RV_{d1}(0, N, T)$ (the actual-return realized variance) and $RV_{d2}(0, N, T)$ (the log-return-based realized variance), we can also make a comparison of the price difference for two swap contracts being identical except the payoff involving these two most frequently used definitions of realized variance. Such a comparison should be very interesting, because intuitively the

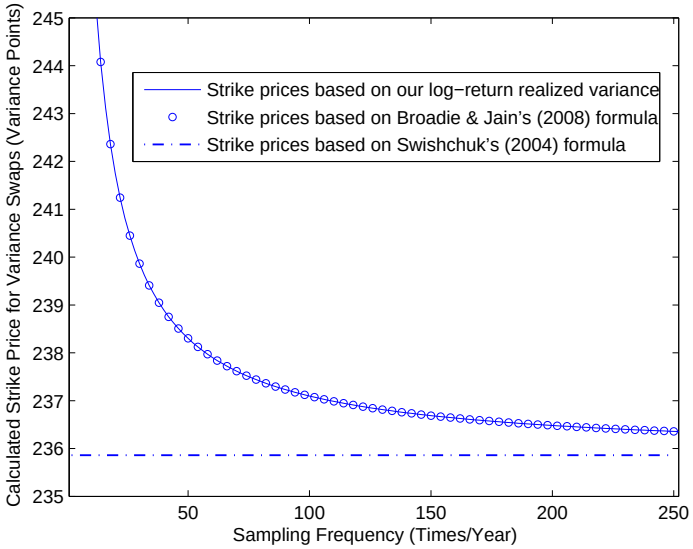

Figure 2.5: The comparison of our results with those of Broadie & Jain (2008) for log-return variance swaps

realized variance defined by the actual-return variance, $RV_{d1}(0, N, T)$, should be a more straightforward definition with a direct financial interpretation than that defined through the log-return realized variance, $RV_{d2}(0, N, T)$. However, the latter seems to be always more popular in practice, perhaps dues to the mathematical tractability it leads to. One naturally wonders if they would lead to quite difference prices if other terms are otherwise identically given.

Fig. 2.6 displays the variance strike prices computed using the two definitions of realized variance, $RV_{d1}(0, N, T)$ and $RV_{d2}(0, N, T)$, as a function of different sampling frequencies. The results show that the strike price associated with a log-return realized variance $RV_{d2}(0, N, T)$ is less than that associated with the actual-return realized variance $RV_{d1}(0, N, T)$ for low sampling frequencies. However, when the sampling frequency is increased beyond about 5 times per annum, the strike price for a variance swap contract with the log-return realized variance defined in its payoff becomes greater than that with the actual-return realized variance. Given that most of variance swaps have a sampling frequency much higher than 5 times per annum, we may conjecture that variance swaps associated with the log-return realized variance should have a higher strike price than those

with the actual-return variance realized variance in practice. The reason that we call this a conjecture rather than a conclusion is that to draw a conclusion, one really needs to test if this is true for all other parameters in the parameter space. Such tests must be thorough, and thus would be quite time consuming. Therefore, we have decided to leave it for future research. On the other hand, our conjecture is indeed reenforced by the results presented in Fig. 2.3 and Fig. 2.4, which further demonstrate that both strike prices associated with the log-return realized variance and the actual-return variance realized variance are higher than those associated with the continuously-sampling approximation, but strike prices associated with the log-return is usually higher than those associated with the actual return, at least for the most common sampling frequencies used in financial practice. In practice most of the contracts sampling is done daily or weekly and sometimes monthly (very rarely). For weekly sampling realized variance, there is a 0.46% difference between the strike prices calculated with the two definitions of realized variance. The effect of discreteness further decreases as sampling frequencies increases further; the strike prices obtained with two formulae for discretely-sampled variance swaps do approach to that of the continuous-sampled variance swaps, as one would have expected.

A couple of more points should be remarked before leaving this section. Firstly, with the newly found analytic solution, all the hedging ratios of a variance swap can also be analytically derived by taking partial derivatives against various parameters in the model. With symbolic calculation packages, such Mathematica or Maple, widely available to researchers and market practitioners, these partial derivatives can be readily calculated and thus omitted here. However, to demonstrate how sensitive the strike price is to the change of the key parameters in the model, we performed some sensitivity tests for the example presented in this section. Shown in Table 2.3 are the results of the percentage change of the strike price when a model parameter is given a 1% change from its base value used in the example presented in this Section. Clearly, the strike price of a variance swap

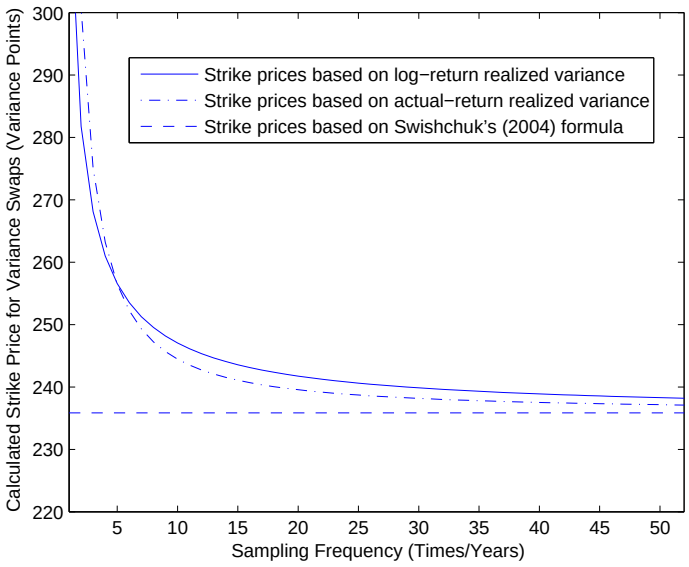

Figure 2.6: The effect of alternative measures of realized variance

appears to be most sensible to the long-term mean variance $\theta^{\mathbb{Q}}$ for the case studied. On the other hand, the spot variance V_0 may also have significant influence in terms of the sensitivity of the strike price. Secondly, due to the notational amount factor L and the size of the contract traded per order, the 1% or 2% relative error may result in a considerable amount of absolute loss if the formula based on the continuous approximation is adopted. Combining these two points together, one may conclude that even with a relatively high sampling frequency, such as daily sampling, the approximation based on the continuous model could still lead to larger errors for a certain combination of parameter values. Thereby, having closed-form formulae for the case of discrete sampling would enable us to completely abandon the approximation formula based on the continuous model; whether the sampling period is small or not, the computational time of adopting our newly-derived formulae, Eq. (2.36), and Eq. (2.51) is virtually the same as that of adopting the traditional formula, Eq. (2.54).

Table 2.3: The sensitivity of strike price of variance swap (daily sampling)

Name	Value	Sensitivity
$\kappa^{\mathbb{Q}}$	11.35	-0.066%
$\theta^{\mathbb{Q}}$	0.022	0.85%
σ_V	0.618	-0.0015%
V_0	0.04	0.15%

2.4 Conclusion

In this chapter, we have applied the Heston stochastic volatility model to describe the underlying asset price and its volatility, and obtained two closed-form exact solutions for discretely-sampled variance swaps with the actual-return and log-return realized variance. This can be viewed as a substantial progress made in developing a more realistic pricing model for variance swaps. Through numerical examples, we have shown that the our formulae can improve the accuracy in pricing variance swaps. We have compared the results produced from our new solutions with those produced by the MC simulations for the validation purposes and found that our results agree with those from the MC simulations perfectly.

The significance of our work can be illustrated in two aspects. Theoretically, our discrete model can be used to verify the validity of the corresponding continuous models. Our study has demonstrated that the well-known continuous approximation in the literature for variance swaps leads to an error exponentially growing with the inverse of the sampling frequencies. The study of this chapter thus would fill a gap that has been in the field of pricing variance swaps. Fourier inverse transform in our model has been analytically worked out, which is a significant step forward in the literature of Heston's model. Practically, the computational efficiency is enormously enhanced in terms of assisting practitioners to price variance swaps, and thus it can be a very useful tool in trading practice when there is obviously increasing demand of trading variance swaps in financial markets.

Chapter 3

Pricing Forward-Start Variance Swaps

3.1 Introduction

In Chapter 2, we have discussed the pricing of discretely-sampled variance swaps, based on the Heston stochastic volatility model, to improve the pricing accuracy the continuously-sampling approximations in literature. This chapter will address the pricing problem of forward-start variance swaps, as most of those traded variance swaps in markets or even some over-the-counter ones are of a forward-start nature, characterized by the starting time of the sampling period being a future date.

Forward-start variance swaps are a kind of variance swaps whose annualized realized variance is measured between two future dates T_s and T_e, where $0 < T_s < T_e$, with $t = 0$ being the current time. Even though forward-start variance swaps seem to be a simple and natural extension to the normally defined variance swaps with the sampling period covering all the time between now and a future time T, the introduction of the forward-start feature can increase the flexibility of variance swaps in hedging risk, and hence greatly promote the trading of variance swaps. This is indeed the case as variance futures are indeed listed as

standardized forward-start variance swaps in some stock exchanges. For example, Chicago Board Options Exchange (CBOE) launched 3-month and 12-month variance futures on S&P 500 in May 2004 and March 2006, respectively. New York Stock Exchange (NYSE) Euronext also started to offer variance futures on FTSE 100, CAC 40 and AEX indices in September 2006. All those listed variance futures are nothing but forward-start variance swaps.

However, up to now, none in literature has taken the forward-start feature into consideration which is usually imbedded in most of traded variance swaps, in the context of stochastic volatility and discretely-sampled realized variance. In this chapter, we present an approach to price discretely-sampled forward-start variance swaps based on Heston's two-factor stochastic volatility model. In this way, the nature of stochastic volatility is included in the model and most importantly, two closed-form exact solutions can be worked out for forward-start variance swaps with the two alternative definitions of realized variance, respectively. The main contributions of this study can be summarized in the following aspects. Firstly, by developing a forward characteristic function, we demonstrate a more versatile approach to deal with the issue of pricing forward-start variance swaps under stochastic volatility, and obtain two close-form exact solutions for the price of forward-start variance swaps based on two different definitions of realized variance. Secondly, this study also contributes to the literature in that the new approach presented in this chapter is applicable to the both definitions of realized variance. It actually handles the pricing of different definitions of realized variance in a highly unified and consistent way, which can been viewed as an advantage over those in the literature (e.g., Broadie & Jain 2008*b*; Zhu & Lian 2009*d*). Thirdly, with closed-form exact solutions obtained from the newly-developed approach available to us, we can easily investigate some important properties of variance swaps, by examining the effect of the forward-start feature on the values of variance swaps, discussing the continuously sampling approximation and the effect of sampling frequency to the prices of variance swaps, and comparing the

difference between the two alternative definitions of discretely-sampled realized variance, i.e., the realized variance defined as the sum of log-return of the underlying asset or defined as the sum of relative percentage return of the underlying asset.

The rest of this chapter is organized into four sections. In Section 3.2, a detailed description of forward-start variance swaps is first provided, followed by our solution approach and analytical formulae for the variance swaps. In Section 3.3, discussions about the forward-start feature, effects of sampling frequency and other properties are carried out. Some numerical examples are also given, demonstrating the correctness of our solutions. In Section 3.4, a brief summary is provided.

3.2 Our Solution Approach

In this Chapter, we still use the Heston (1993) stochastic volatility model to describe the dynamics of the underlying asset. To price forward-start variance swaps with discretely-sampled realized variance, we explore a new approach by developing a forward characteristic function first and then using it to obtain closed-form exact solutions.

In the Heston (1993) model, the underlying asset S_t and its stochastic instantaneous variance V_t are modeled by the following diffusion processes, in the risk-neutral probability measure $\mathbb{Q}$:

$$\begin{cases} dS_t = rS_t dt + \sqrt{V_t} S_t d\widetilde{B}_t^S \\ dV_t = \kappa^{\mathbb{Q}}(\theta^{\mathbb{Q}} - V_t)dt + \sigma_V \sqrt{V_t} d\widetilde{B}_t^V \end{cases} \tag{3.1}$$

3.2.1 Forward-Start Variance Swaps

The most difference between a forward-start variance swap and a normally-defined variance swap discussed in Chapter 2 is that the starting point T_s of the *total*

sampling period $[T_s, T_e]$ $(0 < T_s < T_e)$, over which the realized variance is discretely sampled, is not the current time 0, when the forward-start variance swap is initially entered. We refer to $[T_s, T_e]$ as the total sampling period, in comparison with the sampling period that is used to define the time span between two sampling points within the total sampling period.

When the starting point of the sampling period $T_s = 0$, that is the case that has been discussed in Chapter 2 (also see Broadie & Jain (2008*b*) and Zhu & Lian (2009*d*)) already and the variance swap will be referred to as a normally-defined variance swap hereafter. If $T_s > 0$, a variance swap has a forward-start feature and can thus be called a forward-start variance swap. This would add an additional dimension of complexity, in comparison with the case with $T_s = 0$, because additional unknowns of $S_{T_s} > 0$ and $\sigma_{T_s} > 0$ will be present in the calculation of the realized variance defined over a future time period $[T_s, T_e]$. The additional complexity has to be dealt with because most of the actually traded variance swaps in practice all into the category of $T_s > 0$. For example, all variance futures contracts listed in CBOE are forward-start variance swaps*.

As the normally-defined variance swaps, some typical formulae for the measure of realized variance, $RV(T_s, N, T_e)$, are

$$RV_{d1}(T_s, N, T_e) = \frac{AF}{N} \sum_{i=1}^{N} \left(\frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}} \right)^2 \times 100^2 \tag{3.2}$$

or

$$RV_{d2}(T_s, N, T_e) = \frac{AF}{N} \sum_{i=1}^{N} \log^2 \left(\frac{S_{t_i}}{S_{t_{i-1}}} \right) \times 100^2 \tag{3.3}$$

where $t_i, i = 0...N$, is the i-th observation time of the realized variance in the pre-specified time period $[T_s, T_e]$, and $t_0 = T_s, t_N = T_e$. S_{t_i} is the closing price of the underlying asset at the i-th observation time t_i, and there are altogether N observations. AF is the annualized factor converting this expression to an annualized variance. For most of the traded variance swaps, or even over-the-

*http://cfe.cboe.com/education/VT_IUGRUG.aspx

counter ones, the sampling period is usually constant to make the calculation of the realized variance easier. Therefore, we also assume equally-spaced discrete observations in the period $[T_s, T_e]$ as well. As a result, the annualized factor is of a simple expression $AF = \frac{1}{\Delta t} = \frac{N}{T_e - T_s}$. Clearly, with these notations, normally-defined variance swaps would be those with their realized variance defined as either $RV_{d1}(0, N, T_e)$ or $RV_{d2}(0, N, T_e)$, while forward-start variance swaps are just those with their realized variance defined as $RV_{d1}(T_s, N, T_e)$ or $RV_{d2}(T_s, N, T_e)$, assuming the current time is 0 and $T_s > 0$. Hereafter, definitions $RV_{d1}(T_s, N, T_e)$ and $RV_{d2}(T_s, N, T_e)$ are referred to as the actual-return realized variance and the log-return realized variance, respectively. For more details about the variance swaps and variance futures, readers are referred to the web sites of CBOE[†] or NYSE Euronext[‡].

When the sampling frequency increases to infinity, the discretely-sampled realized variance approaches the continuously-sampled realized variance, $RV_c(T_s, T_e)$, i.e.,

$$RV_c(T_s, T_e) = \lim_{N\to\infty} RV_{d1}(T_s, N, T_e) = \lim_{N\to\infty} RV_{d2}(T_s, N, T_e) = \frac{1}{T_e - T_s}\int_{T_s}^{T_e} \sigma_t^2 dt \times 100^2 \tag{3.4}$$

where σ_t is the so-called instantaneous volatility of the underlying, a concept that is associated with any stochastic volatility model.

In the risk-neutral world, the fair variance delivery price of a variance swap at time 0, when the contract is initially entered, is the expectation value of the future realized variance, i.e., $K_{var} = E_0^Q[RV(T_s, N, T_e)]$. In this chapter, we shall develop a more general and versatile approach to obtain closed-form solutions for the prices of forward-start variance swap with the realized variance defined by $RV_{d1}(T_s, N, T_e)$ and $RV_{d2}(T_s, N, T_e)$, respectively, in a unified way, as demonstrated in the next section.

[†]http://cfe.cboe.com/Products/Spec_VT.aspx
[‡]http://www.euronext.com/fic/000/010/990/109901.ppt

3.2.2 Forward Characteristic Function

As the normally-defined variance swaps discussed in Chapter 2, the value of a forward-start variance swap at time 0 is the expected present value of the future payoff. This should be zero at the beginning of the contract since there is no cost to enter into a swap. Therefore, the fair variance delivery price can be easily defined as $K_{var} = E_0^Q[RV(T_s, N, T_e)]$, after setting the initial value of variance swaps to be zero. The variance swap valuation problem is therefore reduced to calculating the expectation value of the future realized variance in the risk-neutral world.

Pricing formula for variance swaps with the realized variance defined by $RV_{d1}(0, N, T_e)$ has been presented by Zhu & Lian (2009*d*), and pricing formulae for variance swaps with the realized variance defined by $RV_{d2}(0, N, T_e)$ have been respectively presented by Zhu & Lian (2009*f*) and Broadie & Jain (2008*b*). Here in this chapter, we shall develop a more general and versatile approach to obtain closed-form solutions for the prices of forward-start variance swap with the realized variance defined by $RV_{d1}(T_s, N, T_e)$ and $RV_{d2}(T_s, N, T_e)$, respectively, in a unified way, as demonstrated in the next section.

To pave the way of obtaining analytical solutions for the pricing of forward-start variance swaps, we demonstrate in this subsection the derivation of the so-called forward characteristic function.

Assuming the current time is 0, we let $y_{t,T} = \log S_T - \log S_t$ $(t < T)$ and define the forward characteristic function $f(\phi; t, T, V_0)$ of the stochastic variable $y_{t,T}$ as the Fourier transform of the probability density function of $y_{t,T}$, i.e.,

$$f(\phi; t, T, V_0) = E^{\mathbb{Q}}[e^{\phi y_{t,T}} | y_0, V_0], \quad t < T \tag{3.5}$$

It should be noted that the imaginary unit $j = \sqrt{-1}$ has been deliberately absorbed into the parameter ϕ of the Fourier transform. In some references (e.g., Cont & Tankov 2004), the Fourier transform of the probability density function

of $y_{t,T}$ without the explicit use of j is called a moment generating function. Here, for simplicity, we shall still call it the forward characteristic function because the explicit exposition of j does not alter the essence of this function at all. What's more important is to search for an explicit and analytical expression of this expectation, which forms the core of this chapter as shown in the following proposition:

Proposition 3 *If the underlying asset follows the dynamics (3.1), then the forward characteristic function of the stochastic variable* $y_{t,T} = \log S_T - \log S_t$ $(t < T)$ *is given by:*

$$f(\phi; t, T, V_0) = e^{C(\phi, T-t)} g(D(\phi, T-t); t, V_0) \tag{3.6}$$

where

$$\begin{cases} C(\phi, \tau) = r\phi\tau + \dfrac{\kappa^{\mathbb{Q}}\theta^{\mathbb{Q}}}{\sigma_V^2}[(a+b)\tau - 2\ln(\dfrac{1-ge^{b\tau}}{1-g})] \\ D(\phi, \tau) = \dfrac{a+b}{\sigma_V^2}\dfrac{1-e^{b\tau}}{1-ge^{b\tau}} \\ a = \kappa^{\mathbb{Q}} - \rho\sigma_V\phi, \quad b = \sqrt{a^2 + \sigma_V^2(\phi - \phi^2)}, \quad g = \dfrac{a+b}{a-b} \end{cases} \tag{3.7}$$

and

$$g(\phi; \tau, V) = \exp\left(\frac{-2\kappa\theta}{\sigma_V^2}\ln\left(1 + \frac{\sigma_V^2\phi}{2\kappa^{\mathbb{Q}}}(e^{-\kappa^{\mathbb{Q}}\tau} - 1)\right) + \frac{2\kappa^{\mathbb{Q}}\phi}{\sigma_V^2\phi + (2\kappa^{\mathbb{Q}} - \sigma_V^2\phi)e^{\kappa^{\mathbb{Q}}\tau}}V\right) \tag{3.8}$$

The proof of this proposition is left in Appendix C.

Several unique features of this function could be remarked. Firstly, comparison with the normally defined characteristic functions of stochastic variable $\log S_T$ (such as the one presented in Heston (1993)), this expression of forward characteristic function is in a more general form which covers the normal one as a special case; by settling $t = 0$ in the Eq. (3.6), the forward characteristic function degenerates to the normally defined one (by the difference of a factor $e^{\phi \log S_0}$). For example, the characteristic function of stochastic variable $y_T = \log S_T$, which was first presented by Heston (1993) as a useful tool to obtain

closed-form solutions for options with stochastic volatility, can be easily found out to be $e^{\phi \log S_0} f(\phi; 0, T, V_0)$, utilizing the Proposition 3. Secondly, even though the forward characteristic function appears to be a simple extension to the normally defined characteristic function, the derivation procedure of the former is far more involving than that of the latter. We had to analytically solve two associated PDEs successively in two steps for the former case, whereas only first step of these two steps was needed to obtain the latter. Thirdly, it is important to notice that the key step in pricing a forward-start variance swap is the calculation of expectation of payoff function depending on two stochastic variables, S_t and S_T, and thus the forward characteristic function presented here can be used to price derivatives whose payoff function depends on two stochastic variables, a case that cannot be handled by the normally defined characteristic function. Finally, the forward characteristic function no longer depends on the stock price but only on the instantaneous variance and the time to maturity. This is because of a very special feature of the Heston model, in which the stochastic process of V_t is independent of S. As a result, the quotient of $\frac{S_T}{S_t}$, which is used in the calculation of the $E_0^Q[RV_{d1}(T_s, N, T_e)]$ and $E_0^Q[RV_{d2}(T_s, N, T_e)]$, is independent of the price S_t.

3.2.3 Pricing Forward-Start Variance Swaps

With the availability of forward characteristic function, we now proceed to pricing a forward-start variance swap. As discussed above, the fair strike price of a variance swap can be defined as $K_{var} = E_0[RV(T_s, N, T_e)]$, once the detailed definition of the realized variance, $RV(T_s, N, T_e)$, is specified. In this chapter, we will concentrate on the two alternative definitions of realized variance, $RV_{d1}(T_s, N, T_e)$ specified in Eq. (3.2) and $RV_{d2}(T_s, N, T_e)$ specified in Eq. (3.3).

We first illustrate our approach to obtain the closed-form analytical solution for fair strike price of a variance swap by taking $RV_{d1}(T_s, N, T_e)$ as the definition of the realized variance. For the case of $RV_{d2}(T_s, N, T_e)$ the solution procedure is

very similar and the corresponding pricing formula can be easily obtained with little effort, demonstrating the versatility of this approach.

As illustrated in Eq. (3.2), the expected value of realized variance in the risk-neutral world is defined as:

$$\begin{aligned} K_{var} &= E_0^Q[RV_{d1}(T_s, N, T_e)] = E_0^Q[\frac{1}{N\Delta t}\sum_{i=1}^{N}\left(\frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}}\right)^2] \times 100^2 \\ &= \frac{1}{N\Delta t}\sum_{i=1}^{N} E_0^Q\left[\left(\frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}}\right)^2\right] \times 100^2 \end{aligned} \tag{3.9}$$

where N is a finite number denoting the total sampling times of the swap contract. So the problem of pricing variance swap is reduced to calculating the N expectations in the form of:

$$E_0^Q\left[\left(\frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}}\right)^2\right] \tag{3.10}$$

for some fixed equal time interval Δt and N different tenors $t_i = T_s + i\Delta t$ ($i = 1, \cdots, N$). Once the details of the variance swaps are specified (and hence a specific discretization along the time axis $[T_s, T_e]$ is made), all the sampling points t_i ($i = 1, \cdots, N$) are fixed points and hence can be regarded as known constants.

For each i ($i = 1, \cdots, N$), t_i and t_{i-1} are two constants future time points (assuming the current time is 0), and hence S_{t_i} and $S_{t_{i-1}}$ in the expression $\left(\frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}}\right)^2$ are two stochastic variables. This is a pricing problem whose payoff depends on two stochastic variables and we need to use the forward characteristic function presented in Proposition 3, i.e.,

$$\begin{aligned} &E_0^Q[(\frac{S_{t_i}}{S_{t_{i-1}}} - 1)^2] = E_0^Q[(e^{2y_{t_{i-1},t_i}} - 2e^{y_{t_{i-1},t_i}} + 1)] \\ &= f(2; t_{i-1}, t_i, V_0) - 2f(1; t_{i-1}, t_i, V_0) + 1 \end{aligned} \tag{3.11}$$

where $y_{t_{i-1},t_i} = \log S_{t_i} - \log S_{t_{i-1}}$ and function $f(\phi; t_{i-1}, t_i, V_0)$ is given in Eq. (3.6).

Following this procedure, the summation in Eq. (3.9) can now be carried out all the way with i ranging from 1 to N, and we finally obtain the fair strike price for the variance swap in the form of:

$$K_{var} = E_0^Q[RV_{d1}(T_s, N, T_e)] = \frac{1}{T_e - T_s}\sum_{i=1}^{N}[f(2; t_{i-1}, t_i, V_0) - 2f(1; t_{i-1}, t_i, V_0) + 1] \times 100^2 \tag{3.12}$$

Eq. (3.12) is a simple and closed-form solution for the fair strike price of a discretely-sampled forward-start variance swap. To a certain extent, it is even simpler than that of the classic Black-Scholes formula, because the latter still involves the calculation of the cumulative distribution function, which is an integral of a smooth real-value function, whereas there is no need to calculate any integral at all in our final solution! Furthermore, the whole derivation procedure as shown above is much simpler than those in literature. For example, Zhu & Lian (2009*d*) and Zhu & Lian (2009*d*) obtained the final solutions of variance swaps by painfully solving two associated PDEs, which correspond to the two steps involved in the current approach. Broadie & Jain (2008*b*)'s approach appears to have involved an even terribly long and tedious derivation.

More importantly, the derivation procedure presented here is so versatile that it can be analogically applied to the case of $RV_{d2}(T_s, N, T_e)$ with hardly any addition effort. The key step of obtaining a closed-form pricing formula for variance swaps in this case is the calculations of the N expectations in the form of:

$$E_0^Q\left[\left(\log\frac{S_{t_i}}{S_{t_{i-1}}}\right)^2\right] \tag{3.13}$$

for some fixed equal time interval Δt and N different sampling points $t_i = T_s + i\Delta t$ $(i = 1, \cdots, N)$. Again, all the sampling points t_i $(i = 1, \cdots, N)$ are fixed points and hence can be regarded as known constants, once the details of the variance swaps are specified.

For each i $(i = 1, \cdots, N)$, this expectation can be analytically carried out by

utilizing the forward characteristic function, i.e.,

$$E_0^Q\left[\left(\log\frac{S_{t_i}}{S_{t_{i-1}}}\right)^2\right] = E_0^Q[(y_{t_{i-1},t_i})^2] = f^{(2)}(0;t_{i-1},t_i,V_0) \tag{3.14}$$

where $f^{(2)}(0;t_{i-1},t_i,V_0) = \frac{\partial^2 f(\phi;t_{i-1},t_i,V_0)}{\partial\phi^2}|_{\phi=0}$, i.e., the second order derivative of the characteristic function given in Eq. (3.6) with $\phi = 0$, which can be easily computed, using any symbolic calculation package, such as Maple.

In this way, the fair value of a variance swap is equal to the sum of the N expectations and hence can be given in the form of:

$$K_{var} = E_0^Q[RV_{d2}(T_s,N,T_e)] = \frac{1}{T_e - T_s}\sum_{i=1}^{N} f^{(2)}(0;t_{i-1},t_i,V_0)\times 100^2 \tag{3.15}$$

Now, we have succeeded in obtaining the two solutions, Eq. (3.12) and Eq. (3.15), for the pricing of forward-start variance swaps based on a stochastic volatility model (Heston model). It should be remarked that both formulae are obtained in a neat and closed form; they are actually simpler than the Black-Scholes formula, in the sense that there is no need of calculating any integral at all. By developing the forward characteristic function, the whole derivation procedures of the two formulae become very simple and easy. An even more noticeable advantage of this approach is that it unifies the pricing procedures of the two definitions of realized variance associated within the variance swaps, whereas the approach in Broadie & Jain (2008*b*) is so limited that it is incapable of dealing with the definition of $RV_{d1}(T_s,N,T_e)$. It should be noted the main difficulty associated with our approach lies in the derivation of the forward characteristic function, which involves two steps of solving PDEs in order to analytically carry out the calculation for the expectation. After obtaining the useful forward characteristic function, the rest calculations of variance swaps are straightforward. In the next section, through some examples, we demonstrate some great benefits of using these analytic formulae for the price of forward-start variance swaps.

3.3 Numerical Results and Discussions

In this section, we first present some numerical examples to illustrate the correctness of our closed-form exact solutions by comparing with Monte Carlo (MC) simulations. We then show some comparisons with the previous continuous sampling model to help readers to understand the improvement in accuracy with our exact solutions. We shall also discuss the effects of alternative measures of realized variance and the effects of forward-start features imbedded in forward-start variance swaps, utilizing the newly found analytical solutions.

To achieve these purposes, we use the following parameters (unless otherwise stated) to specify the underlying process: $V_0 = (20\%)^2$, $\theta^{\mathbb{Q}} = (14.83\%)^2$, $\kappa^{\mathbb{Q}} = 11.35$, $\rho = -0.64$, $\sigma_V = 0.618$, $r = 10\%$ in this section. As for the MC simulations, we took asset price $S_0 = 1$ and the number of the paths $N = 200,000$ for all the simulation results presented here. Following the definition of 12-month variance futures in CBOE, we choose the total sampling period of realized variance to be 12 months in future, $[T_s, T_s + 12/12]$, in the calculation of forward-start variance swaps, with T_s being specified later. Following the quotation rules of variance futures in CBOE, all the numerical values of variance swaps presented in this section are quoted in terms of variance points (the square of volatility points), which are defined as realized variance multiplied by 10,000.

3.3.1 Continuous Sampling Approximation

Before performing the Monte Carlo simulations, we also worked out, for the comparison purpose, the corresponding pricing formula based on the continuous sampling approximation.

In the literature, many researchers (i.e., Swishchuk 2004) have proposed a continuous sampling approximation for realized variance to price the normally defined variance swaps, based on Heston model. In Chapter 2, we have pointed out that adopting such a continuous sampling approximation for a normally defined

variance swap with small sampling frequencies or long tenor can result in significant pricing errors, comparing with the exact value of the discretely-sampled variance swap. For the case of forward-start variance swaps, a similar approximation pricing formula can also be obtained by carrying out the expectation of the continuously-sampled realized variance, i.e.,

$$\begin{aligned} E_0^Q[RV_c(T_s,T_e)] &= E_0^Q[\frac{1}{T_e-T_s}\int_{T_s}^{T_e} V_t dt \times 100^2] \\ &= [V_0(\frac{e^{-\kappa^{\mathbb{Q}}T_s}-e^{-\kappa^{\mathbb{Q}}T_e}}{\kappa^{\mathbb{Q}}(T_e-T_s)}) + \theta^{\mathbb{Q}}(1-\frac{e^{-\kappa^{\mathbb{Q}}T_s}-e^{-\kappa^{\mathbb{Q}}T_e}}{\kappa^{\mathbb{Q}}(T_e-T_s)})] \times 100^2 \end{aligned} \tag{3.16}$$

where V_t is the instantaneous variance (which is the square of the instantaneous volatility defined in our Eq. (3.4), i.e., $\sigma_t^2 = V_t$). This formula is very simple and can be easily implemented in calculating the numerical value of $E_0^Q[RV_c(T_s,T_e)]$. However, similar to the question raised by Zhu & Lian (2009*d*) for the normally defined variance swaps, there is also a validity issue for this formula, since it is nevertheless an approximation of the true value of the actually traded variance swaps where the sampling time, no matter how small, is always discrete. A naturally raised question is how close the results of the approximation and the true values are. One would also like to know when the approximation formula starts to yield large errors when the sampling time is large enough. To address this question, we compare the numerical results obtained from this approximation formula, the newly developed analytical formulae for discretely sampled realized variance and the Monte Carlo simulations.

3.3.2 Monte Carlo Simulations

Our MC simulations are based on a simple Euler-Maruyama discretization for the Heston model

$$\begin{cases} S_t = S_{t-1} + rS_{t-1}\Delta t + \sqrt{|V_{t-1}|}S_{t-1}\sqrt{\Delta t}W_t^1 \\ V_t = V_{t-1} + \kappa^{\mathbb{Q}}(\theta^{\mathbb{Q}} - V_{t-1})\Delta t + \sigma\sqrt{|V_{t-1}|}\sqrt{\Delta t}(\rho W_t^1 + \sqrt{1-\rho^2}W_t^2) \end{cases} \tag{3.17}$$

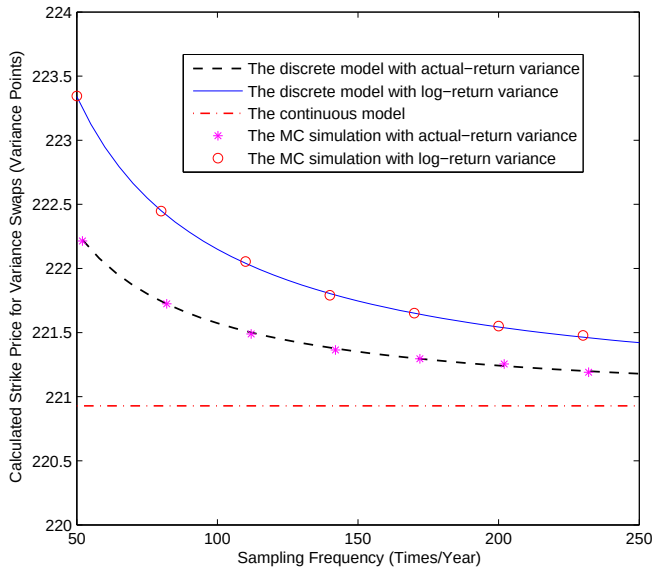

Figure 3.1: Calculated fair strike values as a function of sampling frequency

where W_t^1 and W_t^2 are two independent standard normal random variables.

Shown in Fig. 3.1, as well as in Table 3.1, are the comparison of five sets of data for the strike price of the variance swap. These data were obtained from the numerical calculation of Eq. (3.12) and Eq. (3.15), the MC simulations (3.17) for the corresponding two definitions, and the numerical calculation of the continuously-sampled realized variance Eq. (3.16), respectively. The starting time of the sampling period is set to be 3 months (i.e., $T_s = 1/3$) in the calculation of the forward-start variance swaps. One can clearly observe that the results from our exact solution perfectly match the results from the MC simulations. For example, for the forward-start variance swaps with actual-return realized variance $RV_{d1}(T_s, N, T_e)$, the relative difference between numerical results obtained from the Eq. (3.12) and the MC simulations is less than 0.1% already, when the number of paths reaches 200,000 in MC simulations. Such a relative difference is further reduced when the number of paths is increased; demonstrating the convergence of the MC simulations towards our exact solution and hence to a certain extent providing a verification of the correctness of our exact solutions.

Table 3.1: The numerical results of discrete model, continuous model and MC simulations

Sampling Frequency	Monthly(N=12)	Weekly(N=52)	Daily(N=252)
$RV_{d1}(T_s, N, T_e)$	227.9	222.2	221.1
$RV_{d2}(T_s, N, T_e)$	230.3	223.2	221.4
$RV_c(T_s, T_e)$	220.9	220.9	220.9
MC for $RV_{d1}(T_s, N, T_e)$	227.2	222.9	221.5
MC for $RV_{d2}(T_s, N, T_e)$	230.0	223.8	221.2

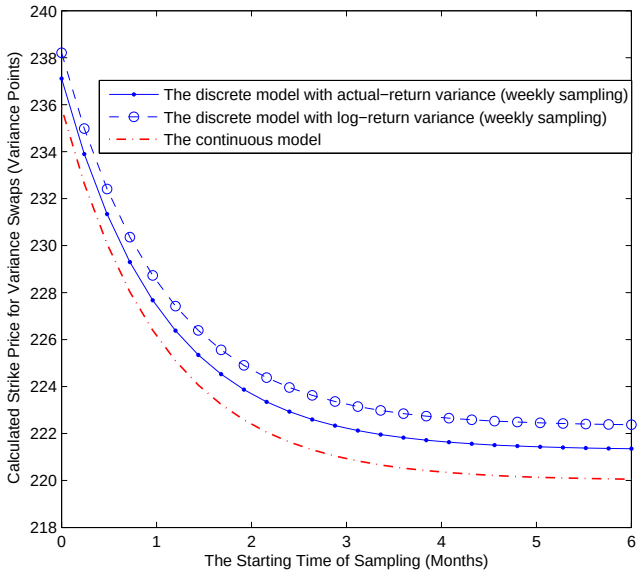

Figure 3.2: Calculated fair strike values as a function of the starting time of sampling while the total sampling period is held as a constant, $T_e - T_s = 1$

3.3.3 The Effect of Forward Start

Although most practically traded variance swaps (e.g., variance futures) have imbedded the forward-start feature, however only few papers in the literature have considered this important feature. With the explicit closed-form solutions available to us, it is interesting to investigate the effects of this forward-start feature on the pricing of variance swaps.

Plotted in Fig. 3.2 are three sets of data, which represent the fair price of variance swaps calculated from Eq. (3.12), Eq. (3.15) and Eq. (3.16), respectively, with the starting time of sampling period, T_s, varying from 0 to 6 months while

the total sampling period $T_e - T_s$ is held as a constant. Of course, this means that the total tenor of a contract, T_e, is varying too. When the starting time of sampling is equal to 0, the variance swap degenerates to a normally defined variance swap. It can be observed that the price of variance swaps decreases when the starting time of sampling increases, showing that the forward-start feature imbedded in variance swaps may significantly alter the the value of variance swaps. For example, comparing with the normally defined variance swaps with $T_s = 0$, the value of a swap contract with starting time of sampling being 6 months in future (i.e., $T_s = 6/12$) have decreased by 7.6%! One can also observe that as the sampling window of a constant width $T_e - T_s$ is shifted along with the time axis when the start of the sampling T_s is increased, the price of variance swaps tends to approach the long-term mean of variance, which is set to be 220 variance points in this example (i.e., $\theta^{\mathbb{Q}} \times 100^2$). This asymptotic trend of the fair strike value towards the long-term mean of variance can be explained by a close examination of the continuous case Eq. (3.16), in which the final swap price can be viewed as a weighted average of current instantaneous variance V_0 and long-term mean of variance $\theta^{\mathbb{Q}}$, with the weights being $W_{V_0} = \frac{e^{-\kappa^{\mathbb{Q}} T_s} - e^{-\kappa^{\mathbb{Q}} T_e}}{\kappa^{\mathbb{Q}}(T_e - T_s)}$ and $W_{\theta^{\mathbb{Q}}} = 1 - W_{V_0}$, respectively. As the starting time of sampling increases, the spot variance, V_0, is weighted less and less on the values of variance swaps, while the long-term mean of variance, $\theta^{\mathbb{Q}}$, is gaining more weights. In the discretely sampling cases, one may not be able to rewrite the Eq. (3.12) and Eq. (3.15) in terms of two weight functions W_{V_0} and $W_{\theta^{\mathbb{Q}}}$ that are totally independent of the V_0 and $\theta^{\mathbb{Q}}$. But the trend of V_0 and $\theta^{\mathbb{Q}}$ being weighted by two functions of two almost monotonicity but opposite rate of change (one increasing and one decreasing) can certainly been seen from Fig. 3.2, in which the discretely-sampled realized variance eventually approaches a constant, that is greater than the long-term mean of variance, when the start of the sampling T_s is increased.

We have also examined the case when the terminating time of sampling is held as a constant, while the starting time of sampling is increased. Of course, the

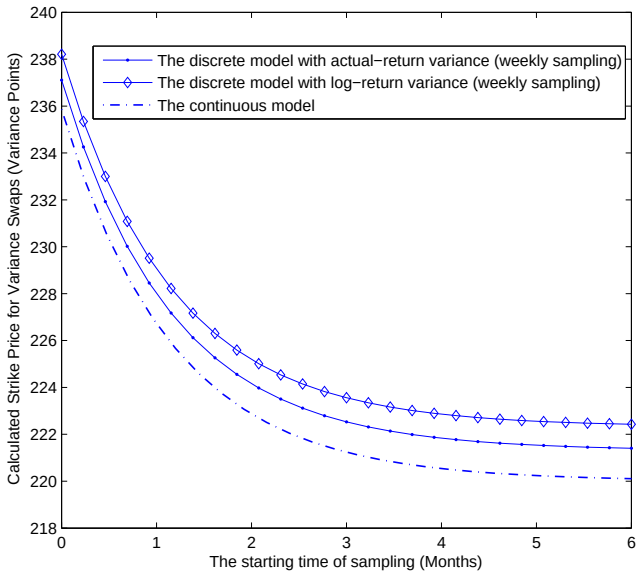

Figure 3.3: Calculated fair strike values as a function of the starting time of sampling while the terminating time of sampling is held as a constant, $T_e = 1$

width of the sampling window $T_e - T_s$ now varies with the change of the start of the sampling T_s. But, the trend displayed in Fig. 3.3 appears to be very similar to that displayed in Fig. 3.2. Therefore, we can conclude that if the start of the sampling time is too far away from the current time, one may use the long-term mean of variance as a good approximation for the expectation of the realized variance, regardless of the width of the sampling window $T_e - T_s$ being a fixed constant or not.

3.3.4 The Effect of Mean-reverting Speed

$\kappa^{\mathbb{Q}}$ is the parameter controlling the speed of mean reversion from the spot variance. Again, using the continuous sampling case, one can easily understand how $\kappa^{\mathbb{Q}}$ controls the speed of mean reversion towards the long-term mean of variance. From Eq. (3.16), we can see that W_{V_0} is a decreasing function of the parameter $\kappa^{\mathbb{Q}}$, which means a greater value of $\kappa^{\mathbb{Q}}$ reduces the weight of V_0 on the total value of variance swap, while increasing the weight of $\theta^{\mathbb{Q}}$ at the same time. Therefore, a higher value of $\kappa^{\mathbb{Q}}$ means the variance V_t approaches to the long-term mean of

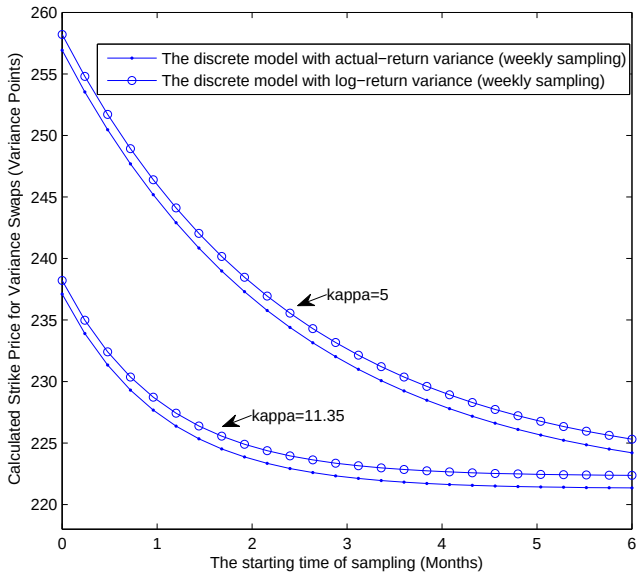

Figure 3.4: Calculated fair strike values as a function of the starting time of sampling while the total sampling period is held as a constant, $T_e - T_s = 1$

variance $\theta^{\mathbb{Q}}$ more quickly, and as a result, $\theta^{\mathbb{Q}}$ naturally gains more weight on the value of the variance swap.

Demonstrated in Fig. 3.4 is the effect of κ on the prices of variance swaps. When κ is specified to be 11.35, the value of a 3-month forward-start variance swap ($T_s = 3/12$) is 222 for $RV_{d1}(3/12, 52, 15/12)$ and 223 for $RV_{d2}(3/12, 52, 15/12)$, respectively; whereas, when $\kappa^{\mathbb{Q}}$ is reduced to be 5 and other parameters are holden the same, the value of a 3-month forward-start variance swap increases to be 231 and 232 for the two corresponding definitions of the realized variance. One should also notice that the value of forward-start variance swap with a larger $\kappa^{\mathbb{Q}}$ value is consistently lower than that with smaller $\kappa^{\mathbb{Q}}$. This is because a larger $\kappa^{\mathbb{Q}}$ value has made $\theta^{\mathbb{Q}}$ being weighted much more than V_0, although the former is specified smaller than latter in this example, resulting a lower price of variance swap for all the starting time that has been examined and displayed in Fig. 3.4. If the specification of $\theta^{\mathbb{Q}}$ and V_0 is reversed, i.e., with $\theta^{\mathbb{Q}} > V_0$, the prices of variance swaps with different $\kappa^{\mathbb{Q}}$ values should be reversed too.

3.3.5 The Effect of Realized-Variance Definitions

As mentioned above, the two definitions, $RV_{d1}(T_s, N, T_e)$ and $RV_{d2}(T_s, N, T_e)$, have been alternatively used as the realized variance in the literature. With the newly found closed-form formulae, Eq. (3.12) and Eq. (3.15), for the corresponding two different definitions of realized variance available to us, we can make a comparison of the price difference for two swap contracts being identical except the payoff involving these two most frequently used definitions of realized variance. Such a comparison should be very interesting, because intuitively the realized variance defined by the actual-return variance, $RV_{d1}(T_s, N, T_e)$, should be a more straightforward definition with a direct financial interpretation than the log-return realized variance, $RV_{d2}(T_s, N, T_e)$. However, the latter seems to be always more popular in practice, perhaps due to the mathematical tractability it leads to. One naturally wonders if they would lead to quite difference prices if other terms are otherwise identically given.

Fig. 3.2 displays the variance strike prices computed using the two definitions of realized variance with weekly sampling, $RV_{d1}(T_s, N, T_e)$ and $RV_{d2}(T_s, N, T_e)$, as a function of the starting time of sampling. The results show that the strike price associated with an actual-return realized $RV_{d1}(T_s, N, T_e)$ is consistently less than that associated with the log-return realized variance variance $RV_{d2}(T_s, N, T_e)$. This finding serves, to a certain extent, as a confirmation of the conjecture raised by Zhu & Lian (2009*f*) that variance swaps associated with the log-return realized variance should have a higher strike price than those with the actual-return variance realized variance in practice, even though our finding in this chapter is based on the forward-start variance swaps and the conjecture then was made for the normally defined variance swaps. This conjecture is also verified by Fig. 3.1, which displays the values of forward-start variance swaps as a function of sampling frequency.

As shown in Fig. 3.1, there is a difference of 0.50% between the strike prices

calculated with the two definitions of realized variance, for weekly sampling frequency. The effect of discreteness decreases as sampling frequencies increases; the strike prices obtained with two formulae for discretely-sampled variance swaps do approach to that of the continuous-sampled variance swaps, as one would have expected.

3.3.6 The Effect of Sampling Frequencies

In Fig. 3.1 and Fig. 3.2, we have also shown the numerical results obtained from the continuous approximation, Eq. (3.16). From Fig. 3.1, one can clearly see that the values of our discretely sampling model asymptotically approach the values of the continuous approximation model when the sampling frequency increases; the continuously-sampled realized variance (Eq. (3.4)) appears to be the limit of the both discretely-sampled realized variance, Eq. (3.2) and Eq. (3.3), as $\Delta t \to 0$. Of course, one can theoretically prove that our solutions Eq. (3.12) and Eq. (3.15) indeed approaches the formula (3.16) when the discrete sampling time approaches zero. With the proof of this limit, our solution is once again verified as the correct solution for the discrete sampling cases, taking the continuous sampling case as a special case with the sampling interval shrinking down to zero.

On the other hand, with the daily sampling, there is a relative difference of 0.11% between the results of the actual-return variance model ($RV_{d1}(4/12, 252, 16/12)$) and the continuous model ($RV_c(4/12, 16/12)$), and a relative difference of 0.22% between the results of the log-return variance model ($RV_{d2}(4/12, 252, 16/12)$) and the continuous model ($RV_c(4/12, 16/12)$). When the sampling frequency becomes weekly sampling (52 sampling times/years), these corresponding relative differences have increased to 0.59% and 1.09%, respectively. If the long-term variance is reduced to $\theta^{\mathbb{Q}} = 0.01$ while the other parameters are held the same, those relative differences would be further enlarged. With a relative difference of the order of one percent, adopting the continuous model as an approximation to price

Table 3.2: The sensitivity of strike price of variance swap (daily sampling)

Name	Value	Sensitivity
$\kappa^{\mathbb{Q}}$	11.35	-0.066%
$\theta^{\mathbb{Q}}$	0.022	0.85%
σ_V	0.618	-0.0015%
V_0	0.04	0.15%

variance swaps with weekly sampling may not be acceptable already, as Little & Pant (2001) has already concluded that an error level reaching more than 0.5% is "fairly large" so that adopting the continuous model may not be so justifiable any more. Of course, when the sampling frequency is further reduced, the difference between the continuous model and the discrete model will exponentially grow.

With the newly-found analytic solutions, all the hedging ratios of a variance swap can also be analytically derived by taking partial derivatives against various parameters in the model, which are omitted here since these partial derivatives can be readily calculated using symbolic calculation packages. To demonstrate how sensitive the strike price is to the change of the key parameters in the model, we performed some sensitivity tests for the example presented in this section. Shown in Table 2 are the results of the percentage change of the strike price when a model parameter is given a 1% change from its base value used in the example presented in this Section. Clearly, the strike price of a variance swap appears to be most sensible to the long-term mean variance $\theta^{\mathbb{Q}}$ for the case studied. On the other hand, the spot variance V_0 may also have significant influence in terms of the sensitivity of the strike price.

As shown in Table 3.2, the effect of the "*vol. of vol.*", σ_V, on the price of a discretely-sampled variance swap appears to be very small, if we are confined to the case of daily or even weekly sampling. However, the "*vol. of vol.*" nevertheless remains in our pricing formulae Eq. (3.12) and Eq. (3.15). This is no longer the case in the continuous sampling approximation, in which σ_V has completely disappeared! This can be clearly seen from our Eq. (3.16) in Section 3.1 and

papers published before by other authors (Howison et al. 2004; Swishchuk 2004; Javaheri et al. 2004; Elliott et al. 2007). This is an interesting as well as amazing observation as it implies that making a continuous approximation in terms of sampling period totally negates the initial motivation of adopting a stochastic volatility model as the final pricing formulae do not depend of the fluctuation of the assumed stochastic volatility anyway; one might as well just use a deterministic local volatility function to begin with. This observation of course further strengthens the case we present here, i.e., abandoning the continuous approximation and developing closed-form exact solutions for discrete sampling cases is the only consistent approach to adopt in dealing with discretely-sampled variance swaps.

3.4 Conclusion

In this chapter, a substantial progress has been made in the field of pricing forward-start variance swaps, by developing a new approach that possesses some great advantages over those in the literature. We have applied the Heston stochastic volatility model to describe the underlying asset price and its volatility, and obtained two closed-form exact solutions for discretely-sampled variance swaps with the different popularly-used definitions of realized variance. It has been shown how to handle the pricing of different definitions of realized variance in a highly unified way, which can been seen as a great advantage over those in literature. By taking the forward-start variance swaps into consideration, this study has also filled a gap in the field of variance swaps pricing. Using the newly-found solutions, we have investigated some important properties of variance swaps, by examining the effect of forward-start feature and the mean-reversion speed on the values of variance swaps, discussing the continuously sampling approximation and the effect of sampling frequency to the prices of variance swaps, and comparing the difference between the two alternative definitions of discretely-sampled real-

ized variance.

Chapter 4

Pricing Variance Swaps with Stochastic Volatility and Random Jumps

4.1 Introduction

In the previous two chapters, we have demonstrated how to analytically price discretely-sampled variance swaps (with or without forward-start feature), under the Heston (1993) stochastic volatility model. In this chapter, we further extend the approach presented in Chapter 3 to a general framework that allows for stochastic volatility, random jumps in return distribution and random jumps in variance process, to obtain closed-form exact solutions for the two popularly-used discretely-sampled realized variance.

This general specification, which will be refereed to as the SVJJ model hereafter, is general enough to cover most of the already-known alternative models as its special cases, including (i) the Heston stochastic volatility (SV) model, (ii) the stochastic volatility with jumps in asset return (SVJ) model, (iii) the stochastic volatility model with jumps in variance process (SVVJ) model, and (iv) the stochastic volatility, random jumps in both return distribution and variance pro-

cess (SVJJ) model. The Heston SV model has the advantage of non-negative variance, easily capturing volatility smile as well as the mean-reverting feature observed in options market. Bates (1996) and Bakshi et al. (1997) extended the SV model to the SVJ model, which was found to be extremely useful in improving the performance of pricing short-term options. However, researchers found strong evidence for model mis-specification in the SVJ model framework, and hence called for further extension models, such as adding jumps in the variance process. The further inclusion of jumps in the variance process leads to the so-called SVVJ model and the SVJJ model (see, e.g., Duffie et al. 2000; Pan 2002; Eraker 2004).

There are several reasons that we believe such an extension of finding the most general closed-form solution to cover all four different stochastic volatility models will benefit the research community as well as market practitioners. Firstly, the newly-found analytic solutions would cover a wide range of stochastic volatility, with or without jumps being included in either the return distribution or the variance process or even both. Since such closed-form solutions were not available for the SVJJ model in the literature, this study fills a gap that has been in the field of pricing variance swaps. Secondly, this study also demonstrates that the versatility of the approach proposed by Zhu & Lian (2009*d*), as it can also be applied to price variance swaps under the SVJJ model, dealing with the both different definitions of realized variance in a highly unified way. Our approach has a clear advantage over Broadie & Jain (2008*b*)'s approach, which is primarily based on integrating the underlying stochastic processes directly and it is not possible to be extended to the SVJJ model. Even under the SV or SVJ model, their approach could only be applied when the realized variance is defined as the average of the squared log return of the underlying asset, leaving the case of the realized variance being defined as the average of the squared relative percentage increment of the underlying price unsolvable. Thirdly, having worked out the closed-form exact solutions for the most general SVJJ model enables us to

not only carry out some cross-model and cross-payoff comparisons for discretely-sampled variance swaps, but also examine some important properties such as the effect of the sampling periods and ultimately the accuracy of the extreme case when the continuously-sampling approximation is adopted as an alternative of pricing discretely-sampled variance swaps.

The rest of this chapter is organized into four sections. In Section 4.2, a detailed description of variance swaps is first provided, followed by the discussion of the SVJJ model. We then present our solution approach and analytical formulae for the discretely-sampled variance swaps under the SVJJ model. In Section 4.3, utilizing the newly-discovered analytical formulae, we discuss the effect of jumps on the prices of variance swaps as well as the effects of sampling frequency and other properties. Some numerical examples are also given in this section, demonstrating the correctness of our solutions. In Section 4.4, a brief summary is provided.

4.2 Our Solution Approach

In this section, we use the framework of stochastic volatility with jump diffusions to describe the dynamics of the underlying asset. This general pricing framework that leads to the SVJJ model takes all SV, SVJ and SVVJ as special cases. Based on this general model, we present our approach to obtain two closed-form exact solutions for the pricing of variance swaps for the two definitions of discretely-sampled realized variance.

The definitions of variance swaps are the same as those discussed in Section 3.2.1. Specifically, we still price forward-start variance swaps with discrete sampling, for the actual-return realized variance $RV_{d1}(T_s, N, T_e)$ and the log-return realized variance $RV_{d2}(T_s, N, T_e)$. In the risk-neutral world, the value of a variance swap at time 0 is the expected present value of the future payoff. This should be zero at the beginning of the contract since there is no cost to enter

into a swap. Therefore, the fair variance delivery price can be easily defined as $K_{var} = E_0^Q[RV(T_s, N, T_e)]$, after setting the initial value of variance swaps to be zero. The variance swap valuation problem is therefore reduced to calculating the expectation value of the realized variance in the risk-neutral world.

4.2.1 Affine Model Specification

Our general analysis model in this chapter incorporates stochastic volatility characteristic and simultaneous jumps in asset price and volatility process. This general model was initially proposed by Duffie et al. (2000). Under the risk-neutral probability measure $\mathbb{Q}$, the underlying asset, denoted by S_t, is assumed to follow the process

$$\begin{cases} d\log S_t = (r - \overline{\mu}^{\mathbb{Q}} - \frac{1}{2}V_t)dt + \sqrt{V_t}dW_t^S(\mathbb{Q}) + d\left(\sum_{n=1}^{N_t(\mathbb{Q})} Z_n^S(\mathbb{Q})\right) \\ dV_t = \kappa^{\mathbb{Q}}(\theta^{\mathbb{Q}} - V_t)dt + \sigma_V\sqrt{V_t}dW_t^V(\mathbb{Q}) + d\left(\sum_{n=1}^{N_t(\mathbb{Q})} Z_n^V(\mathbb{Q})\right) \end{cases} \tag{4.1}$$

where:

r_t *is the constant spot interest rate;*

V *is the diffusion component of the variance of the underlying asset dynamics (conditional on no jumps occurring);*

$dW_t^S(\mathbb{Q})$ *and* $dW_t^V(\mathbb{Q})$ *are two standard Brownian motions correlated with* $E[dW_t^S, dW_t^V] = \rho dt$*;*

$\kappa^{\mathbb{Q}}$, $\theta^{\mathbb{Q}}$ *and* σ_V *are respectively the mean-reverting speed parameter, long-term mean, and variance coefficient of the diffusion* V_t*;*

N_t *is the independent Poisson process with intensity* λ*, that is,* $Pr\{N_{t+dt} - N_t = 1\} = \lambda dt$ *and* $Pr\{N_{t+dt} - N_t = 0\} = 1 - \lambda dt$*. The jumps happen simultaneously in underlying dynamics* S_t *and variance process* V_t*;*

The jump sizes are assumed to be $Z_n^V \sim \exp(\mu_V)$*, and* $Z_n^S | Z_n^V \sim N(\mu_S^{\mathbb{Q}} + \rho_J Z_n^V, \sigma_S^2)$*;*

$\overline{\mu} = \lambda(\frac{\exp(\mu_S^{\mathbb{Q}} + \frac{1}{2}\sigma_S^2)}{1 - \rho_J\mu_V} - 1)$ *is the risk premium of the jump term in the process to*

compensate the jump component.

This general model, combining both the stochastic volatility and jump diffusions characteristics, takes the four models (SV, SVJ, SVVJ and SVJJ) as special cases according to the specification of jump components in Eq. (4.1).

As the complexity of these models progresses with jump terms being added to various stochastic processes, so does the degree of difficulty involved in searching for an analytic closed-form solution. This may explain why no one has taken the SVJJ model into consideration in the pricing of discrete sampling variance swaps.

4.2.2 Pricing Variance Swaps

We now discuss our analytical solution approach for the determination of the fair price of a variance swap, under the general SVJJ model, which incorporates not only the Heston stochastic volatility but also random jumps in return and volatility processes.

As discussed in Section 2.1, the fair strike price of a variance swap can been defined as $K_{var} = E_0[RV(T_s, N, T_e)]$, after the details of the realized variance, $RV(T_s, N, T_e)$, is specified. We shall illustrate our approach to obtain an analytical closed-form solution for fair strike price of a variance swap by taking $RV_{d1}(T_s, N, T_e)$ as the definition of the realized variance. For the case of $RV_{d2}(T_s, N, T_e)$ the solution procedure is very similar and the corresponding pricing formula can be easily obtained with little effort, demonstrating the versatility of this approach. If the realized variance in a variance swap contract is defined in Eq. (3.2), the expected value of realized variance in the risk-neutral world is then:

$$\begin{aligned} K_{var} = E_0^Q[V_{d1}(T_s, N, T_e)] &= E_0^Q\left[\frac{1}{N\Delta t}\sum_{i=1}^{N}\left(\frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}}\right)^2\right] \times 100^2 \\ &= \frac{1}{N\Delta t}\sum_{i=1}^{N} E_0^Q\left[\left(\frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}}\right)^2\right] \times 100^2 \end{aligned} \tag{4.2}$$

where N is a finite number denoting the total sampling times of the swap contract. So the problem of pricing variance swap is reduced to calculating the N expectations in the form of:

$$E_0^Q\left[\left(\frac{S_{t_i}-S_{t_{i-1}}}{S_{t_{i-1}}}\right)^2\right] \tag{4.3}$$

for some fixed equal time interval Δt and N different tenors $t_i = T_s + i\Delta t$ ($i = 1, \cdots, N$). Once the details of the variance swaps are specified (and hence a specific discretization along the time axis $[T_s, T_e]$ is made), all the sampling points t_i ($i = 1, \cdots, N$) are fixed points and hence can be regarded as known constants.

The main difficulty associated with this pricing problem is the fact that two stochastic variables, S_{t_i} and $S_{t_{i-1}}$, concurrently exist inside of the expectation operator in Eq. (4.3) as they are the underlying prices at two future sampling points t_i and t_{i-1} for each i ($i = 1, \cdots, N$) (assuming the current time is 0). Zhu & Lian (2009*d*,*f*) have shown an approach to handle this difficulty by solving the governing PDE in two steps, based on the special SV model. In this chapter, we present a further extension of the approach shown in Chapter 3 to price discretely-sampled variance swaps based on the SVJJ model, with the condensed and more systematic approach of directly utilizing the forward characteristic function. This approach is versatile enough to handle these two definitions in a highly unified manner.

We first demonstrate the derivation of the forward characteristic function in the SVJJ model. Assuming the current time is 0, we let $y_{t,T} = \log S_T - \log S_t$ ($t < T$) and define the forward characteristic function $f(\phi; t, T, V_0)$ of the stochastic variable $y_{t,T}$ as the Fourier transform of the probability density function of $y_{t,T}$, i.e.,

$$f(\phi; t, T, V_0) = E^{\mathbb{Q}}[e^{\phi y_{t,T}} | y_0, V_0], \quad t < T \tag{4.4}$$

The imaginary unit $j = \sqrt{-1}$ has been deliberately absorbed into the parameter ϕ of the Fourier transform. This forward characteristic function in the SVJJ model

can be carried out explicitly as:

Proposition 4 *If the underlying asset follows the dynamics (4.1), then the forward characteristic function of the stochastic variable* $y_{t,T} = \log S_T - \log S_t$ $(t < T)$ *is given by:*

$$f(\phi; t, T, V_0) = e^{C(\phi,T-t)+A(\phi,T-t)} g(D(\phi, T-t); t, V_0) \tag{4.5}$$

where

$$\begin{cases} C(\phi,\tau) = (r-\overline{\mu})\phi\tau + \dfrac{\kappa^{\mathbb{Q}}\theta^{\mathbb{Q}}}{\sigma_V^2}[(a+b)\tau - 2\log(\dfrac{1-ge^{b\tau}}{1-g})] \\ D(\phi,\tau) = \dfrac{a+b}{\sigma_V^2}\dfrac{1-e^{b\tau}}{1-ge^{b\tau}} \\ A(\phi,\tau) = \lambda\left(\exp(\mu_S\phi + \dfrac{1}{2}\sigma_S^2\phi^2)\right)\left(\dfrac{(a+b)\tau}{c(a+b)+\mu_V\widetilde{\phi}} + \dfrac{2\mu_V\widetilde{\phi}}{(ac+\mu_V\widetilde{\phi})^2-(bc)^2}\log B\right) \\ \qquad -\lambda\tau \\ B = 1 + \dfrac{c(b-a)-\mu_V\widetilde{\phi}}{2bc}(e^{-b\tau}-1) \\ a = \kappa^{\mathbb{Q}} - \rho\sigma_V\phi, \quad b = \sqrt{a^2+\sigma_V^2\widetilde{\phi}}, \quad g = \dfrac{a+b}{a-b}, \quad c = 1-\rho_J\mu_V\phi, \quad \widetilde{\phi} = \phi(1-\phi) \\ \bar{\mu} = \lambda(\dfrac{\exp(\mu_S+\frac{1}{2}\sigma_S^2)}{1-\rho_J\mu_V} - 1) \end{cases} \tag{4.6}$$

and

$$g(\phi; \tau, V) = e^{E(\phi,\tau)+F(\phi,\tau)+G(\phi,\tau)V} \tag{4.7}$$

where

$$\begin{cases} E(\phi,\tau) = \dfrac{2\mu_V\lambda}{2\mu_V\kappa^{\mathbb{Q}}-\sigma_V^2}\log\left(1+\dfrac{\phi(\sigma_V^2-2\mu_V\kappa^{\mathbb{Q}})(e^{-\kappa^{\mathbb{Q}}\tau}-1)}{2\kappa^{\mathbb{Q}}(1-\mu_V\phi)}\right) \\ F(\phi,\tau) = \dfrac{-2\kappa^{\mathbb{Q}}\theta^{\mathbb{Q}}}{\sigma_V^2}\log\left(1+\dfrac{\sigma_V^2\phi}{2\kappa^{\mathbb{Q}}}(e^{-\kappa^{\mathbb{Q}}\tau}-1)\right) \\ G(\phi,\tau) = \frac{2\kappa^{\mathbb{Q}}\phi}{\sigma_V^2\phi+(2\kappa^{\mathbb{Q}}-\sigma_V^2\phi)e^{\kappa^{\mathbb{Q}}\tau}} \end{cases}$$

The proof of this proposition is left in Appendix C.

A comparison with the characteristic function defined in Heston (1993) for the stochastic variable $\log S_T$ shows that the forward characteristic function, Eq.

(4.5), is of a more general form than that defined in Heston (1993). That is, the latter is, up to a difference of a factor $e^{\phi \log S_0}$, a special case with t and λ in the former being both set to zero. In other words, the characteristic function of the stochastic variable $y_T = \log S_T$, which was first presented by Heston (1993) as a useful tool to obtain closed-form solutions for options with stochastic volatility, can be easily found to be $e^{\phi \log S_0} f(\phi; 0, T, V_0)$, utilizing Proposition 4.

Having worked out the needed forward characteristic function, Eq. (4.3) can be written in terms of the spot variance V_0 as

$$\begin{aligned} &E_0^Q[(\frac{S_{t_i}}{S_{t_{i-1}}} - 1)^2] = E_0^Q[(e^{2y_{t_{i-1},t_i}} - 2e^{y_{t_{i-1},t_i}} + 1)] \\ &= f(2; t_{i-1}, t_i, V_0) - 2f(1; t_{i-1}, t_i, V_0) + 1 \end{aligned} \tag{4.8}$$

where $y_{t_{i-1},t_i} = \log S_{t_i} - \log S_{t_{i-1}}$ and function $f(\phi; t_{i-1}, t_i, V_0)$ is given in Eq. (4.5). Consequently, the summation in Eq. (4.2) can be carried out all the way with i ranging from 1 to N, leading to the fair strike price for the variance swap being worked out in terms of the spot variance V_0 as

$$K_{var} = E_0^Q[RV_{d1}(T_s, N, T_e)] = \frac{1}{T_e - T_s} \sum_{i=1}^{N} [f(2; t_{i-1}, t_i, V_0) - 2f(1; t_{i-1}, t_i, V_0) + 1] \times 100^2 \tag{4.9}$$

Since the forward characteristic function is obtained for the most general SVJJ model, Eq. (4.9) is a simple and closed-form solution for the fair strike price of a discretely-sampled variance swap with the market volatility being calibrated with any of the four different stochastic processes (with or without jumps). It is amazing that with a much more complicated dynamics used to model both the underlying and the variance than that adopted in the Black-Scholes model, this formula is even simpler, to a certain extent, than that of the classic Black-Scholes formula, because the latter still involves the calculation of the cumulative distribution function, which is an integral of a smooth real-value function, whereas there is no need to calculate any integral at all in our final solution!

More importantly, the derivation procedure presented here is so versatile that it can be analogically applied to the case of $RV_{d2}(T_s, N, T_e)$ with hardly any additional effort. The key step of obtaining a closed-form pricing formula for variance swaps in this case is the calculation of the N expectations in the form of:

$$E_0^Q\left[\log^2\left(\frac{S_{t_i}}{S_{t_{i-1}}}\right)\right] \tag{4.10}$$

for some fixed equal time interval Δt and N different sampling points $t_i = T_s + i\Delta t$ $(i = 1, \cdots, N)$. Again, all the sampling points t_i $(i = 1, \cdots, N)$ are fixed points and hence can be regarded as known constants, once the details of the variance swaps are specified.

For each i $(i = 1, \cdots, N)$, this expectation can be analytically carried out by utilizing the forward characteristic function, i.e.,

$$E_0^Q\left[\log^2\left(\frac{S_{t_i}}{S_{t_{i-1}}}\right)\right] = E_0^Q[(y_{t_{i-1},t_i})^2] = f^{(2)}(0; t_{i-1}, t_i, V_0) \tag{4.11}$$

where $f^{(2)}(0; t_{i-1}, t_i, V_0) = \frac{\partial^2 f(\phi; t_{i-1}, t_i, V_0)}{\partial \phi^2}|_{\phi=0}$, i.e., the second-order derivative of the characteristic function given in Eq. (4.5) with $\phi = 0$, which can be easily computed, using any symbolic calculation package, such as Maple. Therefore, the fair value of a variance swap with the payoff defined by $RV_{d2}(T_s, N, T_e)$ can now be easily worked out as:

$$K_{var} = E_0^Q[RV_{d2}(T_s, N, T_e)] = \frac{1}{T_e - T_s}\sum_{i=1}^{N} f^{(2)}(0; t_{i-1}, t_i, V_0) \times 100^2 \tag{4.12}$$

Again, this exact formula for a variance swap with the payoff defined by $RV_{d2}(T_s, N, T_e)$ is of amazing simplicity too as the one presented in Eq. (4.9) for a variance swap with the payoff defined by $RV_{d1}(T_s, N, T_e)$.

Before we demonstrate some great advantages of using these analytic formulae to price variance swaps, through some examples in the next section, the subtle difference between this approach and that shown in Zhu & Lian (2009*d*) should

be pointed out. At the first glance, they appear to be different in form. However, a scrutiny reveals that the two approaches in essence are the same. As shown in the proof of the Proposition 4, working out the forward characteristic function under the SVJJ model has actually involved two steps, with two corresponding PDEs being successively solved. This procedure is just equivalent to the one demonstrated in Zhu & Lian (2009*d*) by solving the governing PDEs directly in two steps. In this sense, this study is an extension of the approach presented by Zhu & Lian (2009*d*), demonstrating the high versatility of their approach which is applicable not only for the SV model but also for the SVJJ model. By developing the forward characteristic function and hence integrating the two steps into one proposition in this chapter, the whole derivation procedure of the two formulae becomes simpler and easier. The pricing procedures of the two definitions of realized variance are also highly unified in this way. In contrast, the approach presented by Broadie & Jain (2008*b*) is limited in the sense that it is incapable of dealing with the definition of $RV_{d1}(T_s, N, T_e)$ based on the SV model. Moreover, their approach appears to be more difficult in handling the pricing problem of variance swaps based on the SVJJ model, no matter which definition of discretely-sampled realized variance, $RV_{d1}(T_s, N, T_e)$ or $RV_{d2}(T_s, N, T_e)$, is adopted. It should be stressed the main difficulty associated with our approach lies in the derivation of the forward characteristic function, which involves two steps of solving PDEs in order to analytically carry out the calculation for the expectation.

4.3 Numerical Results and Discussions

In this section, we firstly present some numerical examples for illustration purposes. Although theoretically there would be no need to discuss the accuracy of a closed-form exact solution and present numerical results, some comparisons with the Monte Carlo (MC) simulations may give readers a sense of verification for the

newly found solution. This is particularly so for some market practitioners who are very used to MC simulations and would not trust analytical solutions that may contain algebraic errors unless they have seen numerical evidence of such a comparison. In addition, comparisons with the continuous sampling model will also help readers to understand the improvement in accuracy with our exact solution of discretely-sampled realized variance. We shall also discuss the effects of alternative measures of realized variance in variance swaps, utilizing the newly-found analytical solutions.

To achieve these purposes, we use the parameters (unless otherwise stated) reported in Duffie et al. (2000) that were founded by minimizing the mean-squared differences between models and the market S&P500 options prices on November 2, 1993, i.e., $\sqrt{V_0} = (8.7\%)$, $\theta^{\mathbb{Q}} = (8.94\%)^2$, $\kappa^{\mathbb{Q}} = 3.46$, $\sigma_V = 0.14$, $\rho = -0.82$, $\lambda = 0.47$, $\mu_V = 0.05$, $\mu_S = -0.086^*$, $\sigma_S = 0.0001$, $\rho_J = -0.38$, $r = 3.19\%$. This set of parameters was also adopted by Broadie & Jain (2008*a*). As for the MC simulations, we took asset price $S_0 = 1$ and the number of the paths $N = 500,000$ for all the simulation results presented here. Following the quotation rules of variance futures in CBOE, all the numerical values of variance swaps presented in this section are quoted in terms of variance points (the square of volatility points), which are defined as realized variance multiplied by 10,000.

4.3.1 Continuous Sampling Approximation

Before performing the Monte Carlo simulations, we have also worked out the corresponding pricing formula based on the continuous sampling approximation, under the framework of SVJJ model.

In the literature, many researchers (i.e., Swishchuk 2004) have proposed continuous sampling approximations for realized variance to price the variance swaps, based on the Heston stochastic volatility model. Some others (e.g., Little & Pant

*The value of μ_S is backward calculated by using $\overline{\mu} = \theta(1,0) - 1$ with $\overline{\mu} = -0.10$ in Duffie et al. (2000).

2001, Broadie & Jain 2008*b*, Zhu & Lian 2009*d*,*f*) however pointed out that adopting such a continuous sampling approximation under the SV model for a variance swap with small sampling frequencies or long tenor can result in significant pricing errors, comparing with the exact value of the discretely-sampled variance swap. As for the framework of SVJJ model, the continuous sampling approximations become somewhat more complicate, due to the fact that there exist several versions of continuously-sampled realized variance.

Corresponding to the definition of actual-return realized variance, Eq. (3.2), the continuously-sampled realized variance is denoted by the $RV_{c1}(T_s, T_e)$ and given by:

$$\begin{aligned} RV_{c1}(T_s, T_e) &= \lim_{N\to\infty} \frac{AF}{N} \sum_{i=1}^{N} \left(\frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}} \right)^2 \times 100^2 \\ &= \left(\frac{1}{T_e - T_s} \int_{T_s}^{T_e} V_t dt + \sum_{k=N(T_s)}^{N(T_e)} (e^{Z_k^s} - 1)^2 \right) \times 100^2 \end{aligned} \tag{4.13}$$

The expectation of this expression can be carried out and hence the approximation pricing formula for a variance swap based on this continuously-sampled realized variance is obtained,

$$E_0^Q[RV_{c1}(T_s, T_e)] = [AV_0 + (1 - A)(\theta^{\mathbb{Q}} + \frac{\lambda\mu_V}{\kappa^{\mathbb{Q}}}) + \lambda C_1] \times 100^2 \tag{4.14}$$

with

$$\begin{aligned} A &= \frac{e^{-\kappa^{\mathbb{Q}} T_s} - e^{-\kappa^{\mathbb{Q}} T_e}}{\kappa^{\mathbb{Q}}(T_e - T_s)} \\ C_1 &= \frac{\exp(2\mu_S + 2\sigma_S^2)}{1 - 2\rho_J\mu_V} - 2\frac{\exp(\mu_S + \frac{\sigma_S^2}{2})}{1 - \rho_J\mu_V} + 1 \end{aligned} \tag{4.15}$$

Zhu & Lian (2009*f*) presented numerical examples based on the SV model to demonstrate that the prices of variance swaps obtained from the two discretely-sampled variance swaps asymptotically approach to the value of continuous-sampled variance swaps when the sampling frequency increases to infinity. Their work shows that the effect of discreteness resulted from the different definition of discretely-sampled realized variance decreases as sampling frequencies increases

and the two definitions of discretely-sampled realized variance have the same limiting value when the discrete sampling time approaches zero. Under the SVJJ model, however, the continuous sampling approximations for the discretely-sampled actual-return or log-return realized variance (i.e., the limiting value of Eq. (3.2) and Eq. (3.3) when the sampling frequency increases to infinity) are not the same any more. We used the $RV_{c2}(T_s, T_e)$ to denote the continuously-sampled realized variance corresponding to the log-return realized variance, Eq. (3.3), i.e.,

$$\begin{aligned} RV_{c2}(T_s, T_e) &= \lim_{N\to\infty} \frac{AF}{N} \sum_{i=1}^{N} \log^2 \left(\frac{S_{t_i}}{S_{t_{i-1}}} \right) \times 100^2 \\ &= \left(\frac{1}{T_e - T_s} \int_{T_s}^{T_e} V_t dt + \sum_{k=N(T_s)}^{N(T_e)} (Z_k^s)^2 \right) \times 100^2 \end{aligned} \tag{4.16}$$

The expectation of this expression can also be carried out and the approximation pricing formula for a variance swap based on this continuously-sampled realized variance is obtained as,

$$E_0^Q[RV_{c2}(T_s, T_e)] = [AV_0 + (1 - A)(\theta^{\mathbb{Q}} + \frac{\lambda\mu_V}{\kappa^{\mathbb{Q}}}) + \lambda C_2] \times 100^2 \tag{4.17}$$

with

$$\begin{aligned} A &= \frac{e^{-\kappa^{\mathbb{Q}} T_s} - e^{-\kappa^{\mathbb{Q}} T_e}}{\kappa^{\mathbb{Q}}(T_e - T_s)} \\ C_2 &= \sigma_S^2 + 2\rho_J^2 \mu_V^2 + 2\rho_J \mu_V \mu_S + \mu_S^2 \end{aligned} \tag{4.18}$$

These two versions (i.e., Eq. (4.14) and Eq. (4.17)) of continuously-sampled realized variance degenerate to exactly the same one if no jumps are assumed within the underlying process (i.e., the SV and SVVJ models), as can be clearly observed from these two formulae (by setting $Z^S = 0$, $Z^V = 0$ for SV model and $Z^S = 0$ and $\rho_J = 0$ for SVVJ models, respectively). When the jumps in the underlying process are taken into consideration (the SVJ and SVJJ models), it is not the case any more, and the issues on choosing the appropriate continuous sampling approximations deserve to have some clarification. For example, Bakshi et al. (1997) chose Eq. (4.14) as the approximation formula to calculate the

continuously-sampled variance under the SVJ model (their Eq. (4)), while Sepp (2008*a*) and Broadie & Jain (2008*b*) believed the continuously-sampled variance should be calculated using Eq. (4.17). The fact is that both of these claims are correct, depending on the definition of the discretely-sampled realized variance they are approximating. With the discretely-sampled realized variance being defined in Eq. (3.2), the continuous sampling variance naturally corresponds to Eq. (4.14). On the other hand, if the discretely-sampled realized variance is measured with Eq. (3.3), then the continuous sampling variance should be Eq. (4.17).

Besides the two versions of continuous sampling approximations discussed above, there is another continuous sampling approximation that can be used to price variance swaps. With no jumps assumed in the underlying asset price (i.e., in the SV and SVVJ models), Carr & Madan (1998) and Demeterfi et al. (1999) respectively demonstrated that the continuously-sampled realized variance can be replicated by a portfolio of out-of-the-money options. This replication strategy has also been applied to introduce the new definition of the VIX (cf. Carr & Wu 2006). Within this approach, the continuously-sampled realized variance is given by

$$RV_{c3}(T_s, T_e) = \frac{2}{T_s - T_e}\left(\int_{T_s}^{T_e} \frac{dS_t}{S_t} - \log\left(\frac{S_{T_e}}{S_{T_s}}\right)\right) \times 100^2 \tag{4.19}$$

The explicit pricing formula for variance swaps can be obtained by carrying out the expectation of this continuously-sampled realized variance in the form of

$$E_0^Q[RV_{c3}(T_s, T_e)] = [AV_0 + (1 - A)(\theta^{\mathbb{Q}} + \frac{\lambda\mu_V}{\kappa^{\mathbb{Q}}}) + \lambda C_3] \times 100^2 \tag{4.20}$$

with

$$\begin{aligned} A &= \frac{e^{-\kappa^{\mathbb{Q}} T_s} - e^{-\kappa^{\mathbb{Q}} T_e}}{\kappa^{\mathbb{Q}}(T_e - T_s)} \\ C_3 &= 2[\frac{\exp(\mu_S^{\mathbb{Q}} + \frac{1}{2}\sigma_S^2)}{1 - \rho_J\mu_V} - (\mu_S^{\mathbb{Q}} + \rho_J\mu_V) - 1] \end{aligned} \tag{4.21}$$

Two naturally raised questions are how close the results of the above three

approximations are when there are jumps specified in the underlying process and how much each of them deviates from the true values of a discretely-sampled variance swap under different definitions of the realized variance, particularly when the sampling period is large. To properly address these questions, we compare the numerical results obtained from these approximation formulae, the newly-developed analytical formulae for discretely-sampled realized variance and the Monte Carlo simulations.

4.3.2 Monte Carlo Simulations

Our MC simulations are based on a simple Euler-Maruyama discretization for the SVJJ model

$$\begin{cases} V_{t_i} = V_{t_{i-1}} + \kappa^{\mathbb{Q}}(\theta^{\mathbb{Q}} - V_{t_{i-1}})\Delta t + \sigma\sqrt{|V_{t_{i-1}}|}\sqrt{\Delta t}(\rho W_t^1 + \sqrt{1-\rho^2}W_t^2) + \sum_{j=N(t_i)}^{N(t_{i+1})} Z_j^V(\mathbb{Q}) \\ \log S_{t_i} = \log S_{t_{i-1}} + (r - \overline{\mu} - 0.5V_{t_i})\Delta t + \sqrt{|V_{t_i}|}\sqrt{\Delta t}W_t^1 + \sum_{j=N(t_i)}^{N(t_{i+1})} Z_j^S(\mathbb{Q}) \end{cases} \tag{4.22}$$

where W_t^1 and W_t^2 are two independent standard normal random variables, and $N(t_i)$ refers to total jumps in time $[0, t_i]$.

Shown in Fig. 4.1, as well as in Table 4.1, are the comparison of six sets of data for the strike price of the variance swap. These data were obtained from the numerical calculation of Eq. (4.9) and Eq. (4.12), the MC simulations (4.22) for the corresponding two definitions, and the numerical calculation of the continuously-sampled realized variance Eq. (4.14) and Eq. (4.17) respectively. In Table 4.1, we also displayed the numerical values obtained from the calculation of Eq. (4.20).

One can clearly observe that the results from our exact solution perfectly match the results from the MC simulations. For example, for the weekly-sampled variance swaps with actual-return realized variance $RV_{d1}(0, 52, 1)$, the relative difference between numerical results obtained from the Eq. (4.9) and the MC

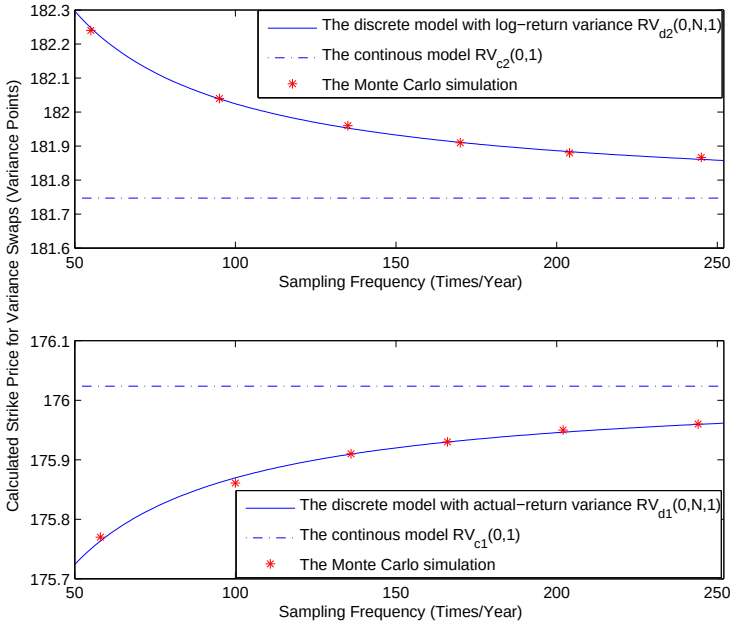

Figure 4.1: Calculated fair strike values in the SVJJ model as a function of the sampling frequency, which ranges from weekly (N=52) to daily (N=252)

simulations is less than 0.1% already, when the number of paths reaches 200,000 in MC simulations. Such a relative difference is further reduced when the number of paths is increased; demonstrating the convergence of the MC simulations towards our exact solution and hence to a certain extent providing a verification of the correctness of our exact solutions.

Table 4.1: The numerical results of discrete model, continuous model and MC simulations

Sampling Frequency	Monthly(N=12)	Weekly(N=52)	Daily(N=252)
$RV_{d1}(T_s, N, T_e)$	175.00	175.74	175.96
$RV_{d2}(T_s, N, T_e)$	183.91	182.28	181.86
$RV_{c1}(T_s, T_e)$	176.02	176.02	176.02
$RV_{c2}(T_s, T_e)$	181.75	181.75	181.75
$RV_{c3}(T_s, T_e)$	179.76	179.76	179.76
MC for $RV_{d1}(T_s, N, T_e)$	175.1	175.7	176.0
MC for $RV_{d2}(T_s, N, T_e)$	183.9	182.3	181.9

4.3.3 The Effect of Realized-Variance Definitions

As mentioned above, the two definitions, $RV_{d1}(T_s, N, T_e)$ and $RV_{d2}(T_s, N, T_e)$, have been alternatively used as the realized variance in the literature. With the newly found closed-form formulae, Eq. (4.9) and Eq. (4.12), for these two different definitions of realized variance available to us, we can make a comparison of the price difference for two swap contracts being otherwise identical except the payoff involving these two most frequently used definitions of realized variance. Such a comparison should be very interesting, because intuitively the realized variance defined by the actual-return variance, $RV_{d1}(T_s, N, T_e)$, should be a more straightforward definition with a direct financial interpretation than the log-return realized variance, $RV_{d2}(T_s, N, T_e)$. However, the latter seems to be more popular in practice, perhaps due to the mathematical tractability it leads to. One naturally wonders if they would lead to quite different prices if other terms are identical.

Fig. 4.1 displays the variance strike prices computed using the two definitions of realized variance, $RV_{d1}(T_s, N, T_e)$ and $RV_{d2}(T_s, N, T_e)$, as a function of the sampling frequency. The results show that the strike price associated with an actual-return realized $RV_{d1}(T_s, N, T_e)$ is consistently less than that associated with the log-return realized variance variance $RV_{d2}(T_s, N, T_e)$. This finding serves as a confirmation of the conjecture raised by Zhu & Lian (2009*f*) that variance swaps associated with the log-return realized variance should have a higher strike price than those with the actual-return variance realized variance in practice, even though the conjecture then was made under the SV model.

The differences between the two definitions have even been greatly amplified under the SVJJ model, comparing with those under the SV as presented by Zhu & Lian (2009*f*). For the case of weekly sampling ($N = 12$), there is a difference of 3.59% between the strike prices calculated with the two definitions of realized variance. Although the effect of discreteness decreases as sampling

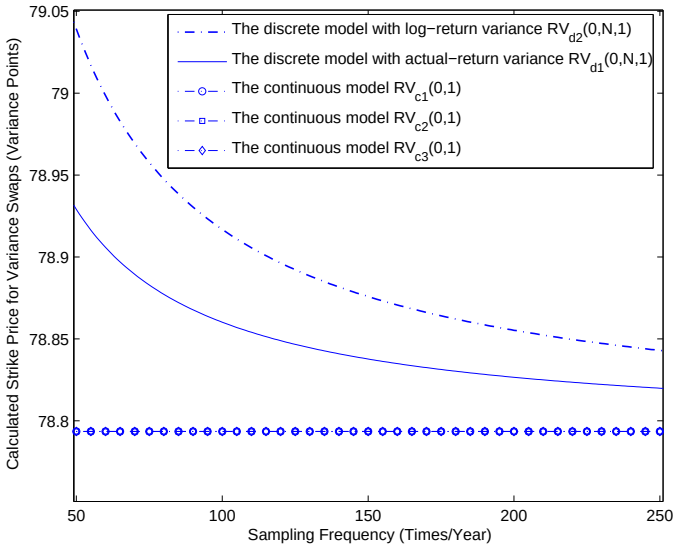

Figure 4.2: Calculated fair strike values in the SV model as a function of the sampling frequency, which ranges from weekly (N=52) to daily (N=252)

frequencies increase, the strike prices obtained with two formulae for discretely-sampled variance swaps do not approach to each other in the limit case. There is a difference of 3.15% even when the sampling frequencies increase to infinity, whereas the two calculated values are the same in SV model.

4.3.4 The Effect of Jump Diffusion

In this section, we first examine the net effect of jumps in the pricing variance swaps, by comparing the strike prices of variance swaps obtained from the SV, SVJ, SVVJ and SVJJ models. For the purpose of comparison, we set the corresponding jump parameters to be zero when there are no jumps assumed for the corresponding processes, and keep other parameters unchanged (i.e., by setting $Z^S = 0$ and $Z^V = 0$ in the SV model, $Z^V = 0$ and $\rho_J = 0$ in the SVJ model, $Z^S = 0$ and $\rho_J = 0$ in the SVVJ model, respectively, with all the rest parameters being those presented in the introduction part of Section 4.3).

In Fig. 4.2, we have shown the strike prices of variance swaps based on the SV model, for the several definitions of realized variance (i.e., the log-return variance

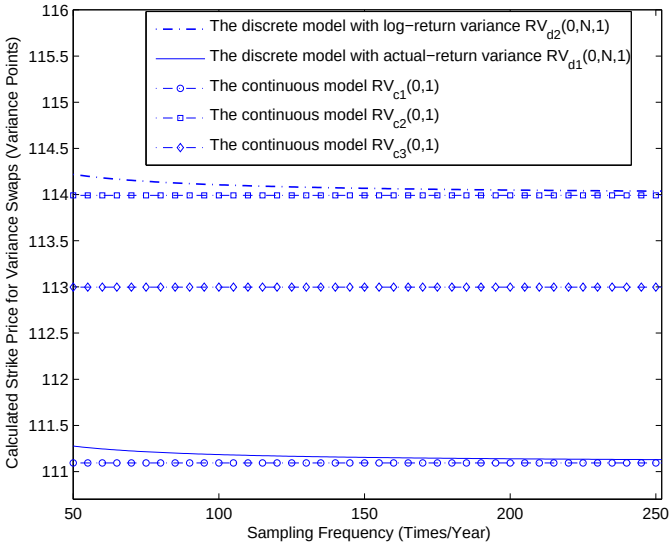

Figure 4.3: Calculated fair strike values in the SVJ model as a function of the sampling frequency, which ranges from weekly (N=52) to daily (N=252)

$RV_{d2}(0, N, 1)$, the actual-return variance $RV_{d1}(0, N, 1)$, the three continuously-sampled variance $RV_{c1}(0, 1)$, $RV_{c2}(0, 1)$, and $RV_{c3}(0, 1)$). It can be observed that the prices of variance in this model have greatly decreased, comparing with their counterparts in the SVJJ model, as presented in Fig. 4.1 and Table 4.1. For example, the price of log-return variance swap with weekly sampling, $RV_{d2}(0, 52, 1)$, is only 79.04 in the SV model, which has decreased as much as 56.6% by comparing with the value of 182.28 in the SVJJ model. It can also be easily noted that the strike prices obtained from the two definitions of discretely-sampled realized variance asymptotically approach to the same price with continuously-sampled realized variance.

Plotted in Fig. 4.3 are the prices of variance swaps based on the SVJ model. In the SVJ model, which allows jumps to occur in the underlying prices, the strike prices of variance swaps are higher than their counterparts in the case of SV model, but lower than those in the case of SVJJ model. For example, the price of log-return variance swap with weekly sampling, $RV_{d2}(0, 52, 1)$, is now 114.21 in the SVJ model, which has increased 44.5% by comparing with the value of 79.04 in the SV model. These comparisons show that inclusion of jumps will

significantly increase the strike prices of variance swaps when other parameters are kept unchanged. This is not surprising at all because random jumps have virtually caused additional uncertainty of the underlying, and hence the realized variance, which is a measure of underlying uncertainty, will naturally increase when jumps are introduced into the process describing the underlying.

On the other hand, it is very interesting to have observed that, different from the jumps effect in the SVJ model, in which jumps are added into the underlying process, the values from the three versions of continuously-sampled variance swaps (i.e., $RV_{c1}(0,1)$, $RV_{c2}(0,1)$, $RV_{c3}(0,1)$) under the SVVJ model, in which jumps are allowed to occur in the volatility process but not in the underlying process, are exactly the same, as shown in Fig. 4.4, whereas those values calculated from the two discretely-sampled variance swaps are still different from each other. This is because when sampling periods approach zero, the additional terms associated with jumps in $RV_{c1}(T_s, T_e)$ and $RV_{c2}(T_s, T_e)$ vanish, resulting in that the definition of continuously-sampled realized variance being identical. On other other hand, similar to the case appeared in the SVJ model, the strike price of a variance swap based on this SVVJ model is higher than the value of an identical contract based on the SV model, but lower than the one in the SVJJ model. For example, the price of a log-return variance swap with weekly sampling, $RV_{d2}(0, 52, 1)$, in the SVVJ model is 127.97, which represents a 61.9% increase to the price of 79.04 calculated from the SV model. This again shows the added jumps into the volatility process can also increase the value of a variance swap, as a result of additional uncertainties associated indirectly through the volatility process rather than the case of a direct impact on the underlying price in the SVJ model.

As for the prices obtained from the two discretely-sampled variance swaps, it is observed that the variance swap prices calculated under the SV, SVJ and SVVJ models all increase as the sampling frequency decreases. It is also observed that with variance swap prices calculated from $RV_{d2}(T_s, N, T_e)$ for all these three models are higher than those calculated from $RV_{d1}(T_s, N, T_e)$ for this particular

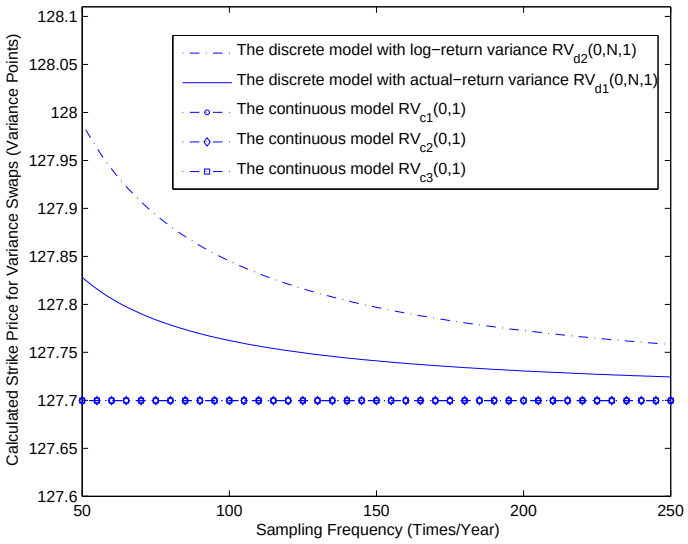

Figure 4.4: Calculated fair strike values in the SVVJ model as a function of the sampling frequency, which ranges from weekly (N=52) to daily (N=252)

set of parameters.

While comparisons along this line give the readers a quantitative sense of the effect of the added jumps, it could be somewhat misleading, as it may give readers a wrong impression that adding jumps would substantially alter the price of a variance swap. It should be pointed out that simply comparing strike prices of variance swaps with or without the jump diffusions while other parameters remaining the same is meaningless in financial practice, since, for the same set of underlying data, one will normally obtain a set of totally different parameters for models with or without jumps during the phase of model calibration. Since the main purpose of this paper is to present analytical formulae to price variance swaps based on the SVJJ model, a financially meaningful examination of jump effect has been left in a future empirical study, utilizing the pricing formulae presented in this paper.

Broadie & Jain (2008*b*) also investigated the effect of ignoring jumps in computing strike price of variance swaps, under the SVJ model, using the value obtained from continuously-sampled log-return realized variance (i.e., Eq. (4.17) in our SVJJ model) as the benchmark. However, a careful study of their paper

shows that their examination of the effect of "ignoring" jumps is different from what we are discussing here in this paper. While we have focused on the effect of "ignoring" jumps starting from the very beginning of the construction of the SVJJ model, their discussion focused on the effect of "ignoring" jumps in the construction of the VIX only. That is, while we have examined both definitions of continuously-sampled actual-return realized variance (i.e., Eq. (4.14)) and continuously-sampled log-return realized variance (i.e., Eq. (4.17)), they queried the effect of "ignoring" jumps in the definition $RV_{c3}(T_s, T_e)$ (i.e., Eq. (4.20)) and added back the jump effect from this point onwards and then compare the results with those obtained from the SVJ model directly. Of course, their approach is based on an unstated assumption that the linear superposition is valid in the addition and deletion of jump components in the adopted stochastic model. With the newly-derived formulae, it is quite easy to follow Broadie & Jain (2008*b*)'s approach to examine the effect of "ignoring" jumps in the definition of VIX based on the more general SVJJ model.

The pricing formula, Eq. (4.20), has ignored the effect of jumps in computing the realized variance. The difference between the variance swap strike price of continuously-sampled actual-return realized variance (i.e., Eq. (4.14)) and the value obtained from replication strategy by ignoring the jumps effect (i.e., Eq. (4.20)) is:

$$\begin{aligned} & E_0^Q[RV_{c1}(T_s, T_e)] - E_0^Q[RV_{c3}(T_s, T_e)] \\ & = \lambda \left(\frac{\exp(2\mu_S + 2\sigma_S^2)}{1 - 2\rho_J \mu_V} - 4\frac{\exp(\mu_S + \frac{\sigma_S^2}{2})}{1 - \rho_J \mu_V} + 2(\mu_S^{\mathbb{Q}} + \rho_J \mu_V) + 3 \right) \end{aligned} \tag{4.23}$$

and the difference between variance swap strike price of continuously-sampled log-return realized variance (i.e., Eq. (4.17)) and the value obtained from replication

strategy by ignoring the jumps effect (i.e., Eq. (4.20)) is:

$$\begin{aligned} &E_0^Q[RV_{c2}(T_s,T_e)] - E_0^Q[RV_{c3}(T_s,T_e)] \\ &= \lambda\left(\sigma_S^2 + \rho_J^2\mu_V^2 + (\mu_S^{\mathbb{Q}} + \rho_J\mu_V + 1)^2 + 1 - 2\frac{\exp\left(\mu_S^{\mathbb{Q}} + \frac{1}{2}\sigma_S^2\right)}{1-\rho_J\mu_V}\right) \end{aligned} \tag{4.24}$$

Eq. (4.23) and Eq. (4.24) indicate that when there is no jump (i.e., $\lambda = 0$) assumed within the model, there will be no difference between the prices obtained from continuously-sampled actual-return realized variance (i.e., Eq. (4.14)), continuously-sampled log-return realized variance (i.e., Eq. (4.17)) and the value obtained from replication strategy by ignoring the jump effect (i.e., Eq. (4.20)). In the case of SVJ and SVJJ models, the three values are different however. For example, using the presented parameters, we can compute the strike prices of variance swaps and obtain that $E_0[RV_{c1}(T_s,T_e)] = 176.02$, $E_0[RV_{c2}(T_s,T_e)] = 181.75$ and $E_0[RV_{c3}(T_s,T_e)] = 179.76$. Thus, by ignoring the jumps, one may over-price the actual-return variance swap by 2.12% and under-price the log-return variance swap by 1.10%. Following the comment made by Little & Pant (2001) that an error level reaching more than 0.5% is "fairly large", the over-estimation of 2.12% or under-estimation of 1.10% is surely unacceptable.

4.3.5 The Effect of Sampling Frequencies

In Fig. 4.1 and Table 4.1, we have also shown a comparison of strike prices of variance swaps with the discrete sampling and the corresponding continuous sampling.

From Fig. 4.1, one can clearly see that the values of the discrete sampling models asymptotically approach those of the continuous approximation counterparts when the sampling frequency increases, i.e., the continuously-sampled realized variance (Eq. (4.14)) is the limit of the actual-return discretely-sampled realized variance, Eq. (3.2), while the continuously-sampled realized variance (Eq. (4.17)) is the limit of the log-return discretely-sampled realized variance,

Eq. (3.3), as $\Delta t \to 0$.

On the other hand, with the daily sampling, there is a relative difference of 0.03% between the results of the actual-return variance model, $RV_{d1}(0, 252, 1)$, and its continuous counterpart, $RV_{c1}(0, 1)$, and a relative difference of 0.06% between the results of the log-return variance model, $RV_{d2}(0, 252, 1)$, and the continuous counterpart, $RV_{c2}(0, 1)$. When the sampling frequency becomes weekly (52 sampling times/years), these corresponding relative differences have increased to 0.16% and 0.29%, respectively. Comparing with the effect of different definitions of realized variance or the effect of including jump diffusion, the errors resulted from the sampling frequency appear to be very small, based on this particular set of model parameters.

Several remarks should be made before leaving this section. Firstly, with the newly-found analytic solutions, all the hedging ratios of a variance swap can also be analytically derived by taking partial derivatives against various parameters in the model. With symbolic calculation packages, such Mathematica or Maple, widely available to researchers and market practitioners, these partial derivatives can be readily calculated and thus omitted here. However, to demonstrate how sensitive the strike price is to the change of the key parameters in the model, we have performed some sensitivity tests for the example presented in this section[†]. Shown in Table 4.2 are the results of the percentage change of the strike price when a model parameter is given a 1% change from its base value used in the example presented in this Section. Clearly, under the SVJJ model, not only can the volatility specification parameters ($\kappa^{\mathbb{Q}}$, $\theta^{\mathbb{Q}}$ and V_0) significantly affect the strike price of a variance swap, the jump diffusion parameters (λ, μ_S, μ_V and ρ_J) paly even more important roles in determining the price of a variance swap. This finding reiterates the importance of investigating the effect of jump diffusion in pricing variance swaps and thereby highlight the significance of this

[†]The sensitivity tests presented here are performed using the pricing formula of actual-return realized variance (Eq. (4.9)). The parameters sensitivities for the case of log-return realized variance (Eq. (4.12)) are very close to the case of actual-return realized variance.

Table 4.2: The sensitivity of the strike price of a variance swap (weekly sampling)

Name	Value	Sensitivity
$\kappa^{\mathbb{Q}}$	3.46	-0.18%
$\theta^{\mathbb{Q}}$	0.008	0.32%
V_0	$(8.7\%)^2$	0.12%
λ	0.47	0.55%
μ_S	-0.087	0.41%
μ_V	0.05	0.38%
ρ_J	-0.38	0.10%
$\sigma_S, \rho, \sigma_V, r$	#	$< 0.01\%$

study. Secondly, due to the notational amount factor L and the size of the contract traded per order, the 1% or 2% relative differences, resulted from adopting different definitions of realized variance, or using the continuous approximation, or even ignoring the jump diffusions, may result in a considerable amount of absolute loss. Combining these points together, it is absolutely preferable to work out the closed-form exact formulae for the variance swaps with popularly-used definitions of discretely-sampled realized variance under the general SVJJ model. The analytical and closed-form properties of those formulae also enables us to efficiently obtain the numerical results. Furthermore, numerical efficiency is also vitally important for any pricing formula; not only producing numerical values of the formula itself requires speedy calculations, calibrating model parameters with financial market data may require thousands, if not millions, of iterations and thus any reduction in computational time per iteration would considerably speed up the calibration process. In this regard, nothing can be better than analytical closed-form exact solutions, as have been shown in this chapter.

4.4 Conclusion

In this chapter, we have applied the Heston stochastic volatility model with random jumps in the underlying return and volatility process (SVJJ model) to describe the underlying asset price and its volatility, and based on this general

SVJJ model, we obtained two closed-form exact solutions for discretely-sampled variance swaps corresponding to two popularly-used definitions of realized variance. Utilizing the newly-found analytical and closed-form solutions for the most general SVJJ model, we have carried out some cross-model and cross-payoff comparisons for discretely-sampled variance swaps. We have also examined some important properties such as the effect of the sampling periods and ultimately the accuracy of the extreme case when the continuously-sampling approximation is adopted as an alternative of pricing discretely-sampled variance swaps. We have found that the price of a variance swap is very sensitive to the parameters of random jumps in the underlying return and volatility process, and the strike price of a variance swap with log-return realized variance is consistently higher than its counterpart associated with actual-return realized variance.

Chapter 5

Pricing Volatility Swaps with Discrete Sampling

5.1 Introduction

In the previous chapters, we have discussed the pricing of variance swaps, based on the definitions of discretely-sampled realized variance. However, despite many common features between volatility swaps and variance swaps, the former is viewed to be more difficult to price analytically than the latter because the pay-off function involves either square root operator or absolute value operator. As a result, quite a few closed-form solution have been discovered for the latter (cf. Zhu & Lian 2009*d*) whereas it is very rare to see a paper discussing closed-form solution for the former, particularly when the sampling is discretely conducted.

The main purpose of this chapter is to present the valuation approach for volatility swaps and we will hereafter focus our main attention on the discussion of volatility swaps. As illustrated in Howison et al. (2004), there are at least two different measurements of realized volatility. The most popularly-used one is defined as the square root of the average of realized variance, and a volatility swap contract based on this definition can be termed as a standard derivation swap. Alternatively, as most of the volatility swaps are traded over-the-counter, it is also

possible to design other measurement of realized volatility and hence construct corresponding volatility swaps, provided that the designed measurement of realized volatility can well capture the historical volatility features of the underlying index. The so-called average of realized volatility is such a measurement which is even more robust than the most popularly-used one, i.e., the square root of the average of realized variance, as pointed out by Howison et al. (2004). They termed a volatility swap based on this definition of average of realized volatility as a volatility-average swap.

However, despite these analytic works have been growing rapidly and enriched the literature of pricing volatility swaps, a common limitation is that the realized variance or realized volatility is defined by a continuously-sampling approximation, whereas in financial practice the realized variance or realized volatility of a swap contract is always discretely sampled. Therefore, these continuously-sampling approximations will surely result in a systematic bias for the actual price of a variance swap or a volatility swap which is discretely sampled.

To properly address this discretely sampling effect, several works have been done very recently. Little & Pant (2001) and Windcliff et al. (2006), respectively, presented numerical algorithms to price discretely-sampled variance swaps under local volatility models. To further extend the research of pricing discretely-sampled variance swaps, some researchers started to explore the possibility of working out analytic closed-form solutions for the price of discretely-sampled variance swaps within the framework of stochastic volatility models. Javaheri et al. (2004) pointed out the importance of investigating discretely-sampled volatility swaps under the GARCH model, but unfortunately they did not present an effective pricing approach. Broadie & Jain (2008*b*) presented a set of closed-form solutions for volatility as well as variance swaps with discrete sampling using stochastic volatility models to price discretely-sampled variance swaps. Alternatively, based on the Heston (1993) stochastic volatility model, Zhu & Lian (2009*d*,*f*) showed a completely different approach, by analytically solving the as-

sociated PDEs, to obtain two closed-form formulae for variance swaps based on two different definitions of discretely-sampled realized variance.

Despite these works in developing more accurate pricing formulae for variance swaps, none has considered the pricing of volatility swaps based on the discretely-sampled realized volatility in the literature. In this chapter, under the Heston stochastic volatility model, we present an approach to price discretely-sampled volatility swaps, and most importantly, a closed-form exact solution for the price of discretely-sampled volatility-average swaps. There are several reasons that we believe this research will benefit the research community as well as market practitioners. Firstly, this study, by working out an exact closed-form solution for the discretely-sampled volatility-average swaps based on the Heston stochastic volatility model, fills a gap that there is no exact pricing formula available for discretely-sampled volatility swaps in the literature. Secondly, this study also demonstrates that our proposed solution approach can be used to work out a lower bound for the standard derivation swap in which the realized volatility is defined as the square root of the average of realized variance. Thirdly, it can be used as a benchmark tool for numerical methods developed to price volatility swaps whose payoff function has made the search of closed-form analytical solution impossible.

The rest of this chapter is organized into four sections. For the easiness of reference, we shall start with a description of our solution approach and our analytical formula for the volatility swaps in Section 5.2. Then, some numerical examples are given in Section 5.3, demonstrating the correctness of our solution from various aspects. In the mean time, we also provide some comparisons to continuous sampling models and discussions on other properties of the volatility swaps. Our conclusions are stated in Section 5.4.

5.2 Our Solution Approach

In this section, we use the Heston (1993) stochastic volatility model to describe the dynamics of the underlying asset. We then present our pricing approach to price discretely-sampled volatility-average swaps and obtain a closed-form exact solution.

In the Heston model, the underlying asset S_t and its stochastic instantaneous variance V_t are modeled by the following diffusion processes, in the risk-neutral probability measure $\mathbb{Q}$:

$$\begin{cases} dS_t = rS_t dt + \sqrt{V_t} S_t d\widetilde{B}_t^S \\ dV_t = \kappa^{\mathbb{Q}}(\theta^{\mathbb{Q}} - V_t)dt + \sigma_V \sqrt{V_t} d\widetilde{B}_t^V \end{cases} \tag{5.1}$$

5.2.1 Volatility Swaps

A volatility swap is a forward contract on realized historical volatility of the specified underlying equity index. In such a contract, the buyer receives a payout at expiry from the counterpart selling the swap if the realized volatility of the stock index over the life of swap contract exceeds the implied volatility swap rate (i,e., the trike price of the forward contract) pre-specified at the inception of the contract. Thus it can be easily used for investors to trade future realized volatility against the implied volatility (the strike price of the volatility swaps), gaining exposure to the so-called volatility risk.

The amount paid at expiration is based on a notional amount times the difference between the realized volatility and implied volatility. More specifically, assuming the current time is 0, the value of a volatility swap at expiry can be written as $(RV(0, N, T) - K_{vol}) \times L$, where the $RV(0, N, T)$ is the annualized realized volatility over the contract life $[0, T]$, K_{vol} is the annualized delivery price for the volatility swap, which is set to make the value of a volatility swap equal to zero for both long and short positions at the time the contract is initially entered. To a certain extent, it reflects market's expectation of the realized volatility in the

future. L is the notional amount of the swap in dollars per annualized volatility point squared. The realized volatility is always discretely sampled over a time period $[0, T]$, with T being referred to as the *total* sampling period, in comparison with the sampling period that is used to define the time span between two sampling points within the total sampling period.

At the beginning of a contract, it is clearly specified the details of how the realized volatility, $RV(0, N, T)$, should be calculated. Important factors contributing to the calculation of the realized volatility include underlying asset(s), the observation frequency of the price of the underlying asset(s), the annualization factor, the contract lifetime, the method of calculating the volatility. Some typical formulae (Howison et al. 2004; Windcliff et al. 2006) for the measure of realized variance are

$$RV_{d1}(0, N, T) = \sqrt{\frac{AF}{N} \sum_{i=1}^{N} \left(\frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}} \right)^2} \times 100 \tag{5.2}$$

or

$$RV_{d2}(0, N, T) = \sqrt{\frac{\pi}{2NT}} \sum_{i=1}^{N} \left| \frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}} \right| \times 100 \tag{5.3}$$

where $t_i, i = 0...N$, is the i-th observation time of the realized variance in the pre-specified time period $[0, T]$, and $t_0 = 0, t_N = T$. S_{t_i} is the closing price of the underlying asset at the i-th observation time t_i, and there are altogether N observations. AF is the annualized factor converting this expression to an annualized variance. For most of the traded variance swaps, or even over-the-counter ones, the sampling period is usually constant to make the calculation of the realized variance easier. Therefore, we assume equally-spaced discrete observations in the period $[0, T]$ in this paper. As a result, the annualized factor is of a simple expression $AF = \frac{1}{\Delta t} = \frac{N}{T}$.

In the literature, these two definitions have been alternatively used to measure the realized volatility (Howison et al. 2004). The definition $RV_{d1}(0, N, T)$

is essentially calculated as the square root of average realized variance. Howison et al. (2004) termed a volatility swap contract using this measurement to calculate realized volatility as a standard derivation swap. On the other hand, the definition $RV_{d2}(0, N, T)$ is just the average of realized volatility, and Howison et al. (2004) termed a volatility swap associated with this definition as a volatility-average swap. Barndorff-Nielsen & Shephard (2003) studied the theoretical properties of the average of realized volatility, Eq. (5.3), and found that definition $RV_{d2}(0, N, T)$ is a more robust measurement of realized volatility. In this paper, we shall first choose the average of realized volatility, Eq. (5.3), as the measurement of realized volatility and present our approach to analytically price volatility swaps based on this measurement of discretely-sampled realized volatility. We shall then discuss the difference between these two definitions $RV_{d1}(0, N, T)$ and $RV_{d2}(0, N, T)$, when we present some numerical examples in Section 5.3.

In the risk-neutral world, the value of a variance swap at time 0 is the expected present value of the future payoff. This should be zero at the beginning of the contract since there is no cost to enter into a swap. Therefore, the fair volatility delivery price can be easily defined as $K_{vol} = E_0^Q[RV(0, N, T)]$, after setting the initial value of volatility swaps to be zero. The volatility swap valuation problem is therefore reduced to calculating the expectation value of the realized volatility in the risk-neutral world.

5.2.2 Pricing Volatility Swaps

We now discuss our analytical solution approach for the determination of the fair price of a volatility swap, under the Heston stochastic volatility model. As discussed above, the fair strike price of a volatility swap can be defined as $K_{vol} = E_0[RV(0, N, T)]$, after specifying the detailed definition of realized volatility, $RV(0, N, T)$. In this paper, we will concentrate on the definition of realized

volatility, $RV_{d2}(0, N, T)$ (Eq. (5.3)), and illustrate our approach to obtain the closed-form analytical solution for fair strike price of a volatility swap.

As illustrated in Eq. (5.3), the expected value of realized volatility in the risk-neutral world is defined as:

$$\begin{aligned} K_{vol} = E_0^{\mathbb{Q}}[RV_{d2}(0, N, T)] &= E_0^{\mathbb{Q}}\left[\sqrt{\frac{\pi}{2NT}}\sum_{i=1}^{N}\left|\frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}}\right|\right] \times 100 \\ &= \sqrt{\frac{\pi}{2NT}}\sum_{i=1}^{N} E_0^{\mathbb{Q}}\left[\left|\frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}}\right|\right] \times 100 \end{aligned} \tag{5.4}$$

where N is a finite number denoting the total sampling times of the swap contract. So the problem of pricing volatility swap is reduced to calculating the N expectations in the form of:

$$E_0^{\mathbb{Q}}\left[\left|\frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}}\right|\right] \tag{5.5}$$

for some fixed equal time interval Δt and N different tenors $t_i = i\Delta t$ ($i = 1, \cdots, N$). Once the details of the volatility swaps are specified (and hence a specific discretization along the time axis $[0, T]$ is made), all the sampling points t_i ($i = 1, \cdots, N$) are fixed points and hence can be regarded as known constants.

As the pricing of variance swaps, the main difficulty associated with this pricing problem is still due to the fact that two stochastic variables, S_{t_i} and $S_{t_{i-1}}$, concurrently exist inside of the expectation operator in Eq. (5.5), as they are the underlying prices at two future sampling points t_i and t_{i-1} for each i ($i = 1, \cdots, N$) (assuming the current time is 0). Following the approach presented in Chapter 3 and 4, we utilize the forward characteristic function presented in Proposition 3 to handle this pricing problem.

Using this forward characteristic function Eq. (3.6), the probability density function, denoted by $p(y_{t_{i-1},t_i})$, of the stochastic variable $y_{t_{i-1},t_i} (= \log S_{t_i} - \log S_{t_{i-1}})$ can be easily obtained by performing the inverse Fourier transform with regard to the forward characteristic function. Furthermore, the probability of the

event $\{y_{t_{i-1},t_i} > 0\}$, denoted by Q_i (i.e., $Q_i = \text{Prob}(y_{t_{i-1},t_i} > 0)$), can also be easily carried out by utilizing the relationship between the characteristic function and the cumulative function in the form of (see Heston 1993; Bakshi et al. 1997)

$$Q_i = \int_0^\infty p(y_{t_{i-1},t_i}) dy_{t_{i-1},t_i} = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \text{Re}\left[\frac{f(\phi j; t_{i-1}, t_i, V_0)}{\phi j}\right] d\phi \tag{5.6}$$

Meanwhile, it can be verified that the function $q(y_{t_{i-1},t_i}) = e^{(y_{t_{i-1},t_i} - r\Delta t)} p(y_{t_{i-1},t_i})$ ($\Delta t = t_i - t_{i-1}$) satisfies the following two properties: (1) $q(y_{t_{i-1},t_i}) \geq 0$; (2) $\int_{-\infty}^{\infty} q(y_{t_{i-1},t_i}) dy_{t_{i-1},t_i} = 1$. Hereby, it can be concluded that the function $q(y_{t_{i-1},t_i}) = e^{(y_{t_{i-1},t_i} - r\Delta t)} p(y_{t_{i-1},t_i})$ is a probability density function of a stochastic variable, whose corresponding characteristic function, denoted by $\widetilde{f}(\phi, t_{i-1}, t_i, V_0)$, can be obtained by performing the Fourier transform with regard to the probability density function $q(y_{t_{i-1},t_i})$, i.e.,

$$\begin{aligned} &\widetilde{f}(\phi, t_{i-1}, t_i, V_0) = \mathcal{F}[e^{(y_{t_{i-1},t_i} - r\Delta t)} p(y_{t_{i-1},t_i})] \\ &= e^{-r\Delta t} \mathcal{F}[e^{y_{t_{i-1},t_i}} p(y_{t_{i-1},t_i})] \\ &= e^{-r\Delta t} f(\phi j + 1; t_{i-1}, t_i, V_0) \end{aligned} \tag{5.7}$$

The last step is followed by noting the relationship $f(\phi j; t_{i-1}, t_i, V_0) = \mathcal{F}[p(y_{t_{i-1},t_i})]$ and the property of Fourier transform, with the Fourier transform being defined as $\mathcal{F}[\psi(x)] = \int_{-\infty}^{\infty} e^{j\phi x} \psi(x) dx$ (see e.g., Poularikas 2000).

Similarly, the probability, $\widetilde{Q}_i = \int_0^\infty q(y_{t_{i-1},t_i}) dy_{t_{i-1},t_i}$, can be carried out, by utilizing the corresponding characteristic function $\widetilde{f}(\phi; t_{i-1}, t_i, V_0)$, in the form of

$$\widetilde{Q}_i = \int_0^\infty e^{(y_{t_{i-1},t_i} - r\Delta t)} p(y_{t_{i-1},t_i}) dy_{t_{i-1},t_i} = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \text{Re}\left[\frac{e^{-r\Delta t} f(\phi j + 1; t_{i-1}, t_i, V_0)}{\phi j}\right] d\phi \tag{5.8}$$

Using the forward characteristic function Eq. (3.6), and the two expressions of the probabilities Q_i and $\widetilde{Q}_i$, the expectation in Eq. (5.5) can be written in the

form of

$$
\begin{aligned}
&E_0^Q[|\frac{S_{t_i}}{S_{t_{i-1}}}-1|]=E_0^Q[|e^{y_{t_{i-1},t_i}}-1|]=\int_{-\infty}^{\infty}|e^{y_{t_{i-1},t_i}}-1|p(y_{t_{i-1},t_i})dy_{t_{i-1},t_i}\\
&=\int_0^{\infty}(e^{y_{t_{i-1},t_i}}-1)p(y_{t_{i-1},t_i})dy_{t_{i-1},t_i}+\int_{-\infty}^{0}(1-e^{y_{t_{i-1},t_i}})p(y_{t_{i-1},t_i})dy_{t_{i-1},t_i}\\
&=-\int_0^{\infty}p(y_{t_{i-1},t_i})dy_{t_{i-1},t_i}+\int_{-\infty}^{0}p(y_{t_{i-1},t_i}y)dy_{t_{i-1},t_i}\\
&+e^{r\Delta t}\left(\int_0^{\infty}q(y_{t_{i-1},t_i})dy_{t_{i-1},t_i}-\int_{-\infty}^{0}q(y_{t_{i-1},t_i}y)dy_{t_{i-1},t_i}\right)\\
&=1-2Q_i+e^{r\Delta t}(2\widetilde{Q}_i-1)\\
&=\frac{2}{\pi}\int_0^{\infty}\mathrm{Re}\left[\frac{f(\phi j+1;t_{i-1},t_i,V_0)-f(\phi;t_{i-1},t_i,V_0)}{\phi i}\right]d\phi
\end{aligned}
\tag{5.9}
$$

Following this procedure, the summation in Eq. (5.4) can now be carried out all the way with i ranging from 1 to N, consequently leading to the final pricing formula for the volatility swap in the form of:

$$
\begin{aligned}
&K_{vol}=E_0^{\mathbb{Q}}[RV_{d2}(0,N,T)]=\sqrt{\frac{\pi}{2NT}}\sum_{i=1}^{N}\left|\frac{S_{t_i}-S_{t_{i-1}}}{S_{t_{i-1}}}\right|\times 100\\
&=\sqrt{\frac{2}{\pi NT}}\int_0^{\infty}\sum_{i=1}^{N}\mathrm{Re}\left[\frac{f(\phi j+1;t_{i-1},t_i,V_0)-f(\phi j;t_{i-1},t_i,V_0)}{\phi j}\right]d\phi\times 100
\end{aligned}
\tag{5.10}
$$

N is a finite number denoting the total sampling times of the swap contract. The above equation gives a fair strike price for volatility-average swaps, based on the definition of $RV_{d2}(0,N,T)$, in a simple and closed-form solution.

By developing the forward characteristic function and hence integrating the two steps into one proposition in this chapter, the whole derivation procedure of the pricing formula for volatility-average swaps becomes simpler and easier. The great benefit of using this analytic formula for the pricing of volatility-average swaps is illustrated in the next section through some examples.

5.3 Numerical Results and Discussions

In this section, we firstly present some numerical examples of comparing our pricing formula with the Monte Carlo (MC) simulations for illustration purposes. Although theoretically there would be no need to discuss the accuracy of a closed-form exact solution and present numerical results, some comparisons with the MC simulations may give readers a sense of verification for the newly found solution. This is particularly so for some market practitioners who are very used to MC simulations and would not trust analytical solutions that may contain algebraic errors unless they have seen numerical evidence of such a comparison. In addition, comparisons with the continuous sampling model will also help readers to understand the improvement in accuracy with our exact solution of discretely-sampled realized volatility. We shall then discuss the effects of two different measures of realized volatility in volatility swaps, utilizing the newly-found analytical solutions.

To achieve these purposes, we use the following parameters (unless otherwise stated): $V_0 = 0.04$, $\theta^{\mathbb{Q}} = 0.022$, $\kappa^{\mathbb{Q}} = 11.35$, $\rho = -0.64$, $\sigma_V = 0.618$, $r = 0.1$, $T = 1$ in this section. This set of parameters for the square root process was also adopted by Dragulescu & Yakovenko (2002). As for the MC simulations, we took asset price $S_0 = 1$ and the number of the paths $N = 200,000$ for all the simulation results presented here. Following the quotation rules of VIX futures in CBOE, all the numerical values of volatility swaps presented in this section are quoted in terms of volatility points, which are defined as realized volatility multiplied by 100.

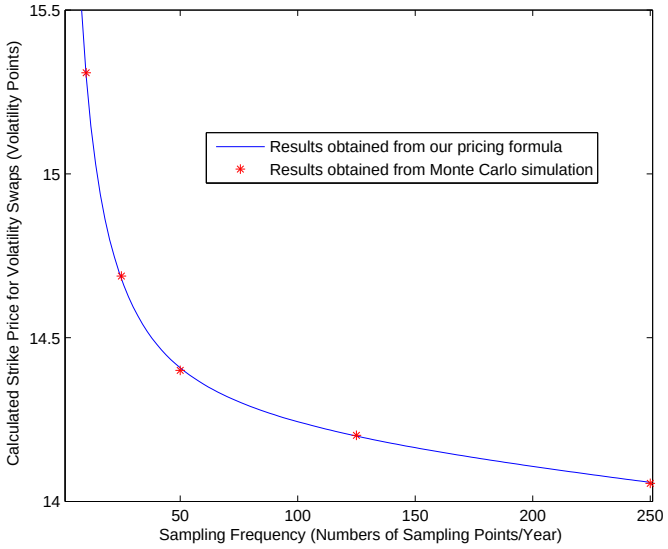

Figure 5.1: A comparison of fair strike prices of volatility swaps based on our explicit pricing formula and the Monte Carlo simulations

5.3.1 Monte Carlo Simulations

Our MC simulations are based on the simple Euler-Maruyama discretization for the Heston model

$$\begin{cases} S_t = S_{t-1} + rS_{t-1}\Delta t + \sqrt{|V_{t-1}|}S_{t-1}\sqrt{\Delta t}W_t^1 \\ V_t = V_{t-1} + \kappa^{\mathbb{Q}}(\theta^{\mathbb{Q}} - V_{t-1})\Delta t + \sigma_V\sqrt{|V_{t-1}|}\sqrt{\Delta t}(\rho W_t^1 + \sqrt{1-\rho^2}W_t^2) \end{cases} \tag{5.11}$$

where W_t^1 and W_t^2 are two independent standard normal random variables.

Shown in Fig. 5.1, as well as in Table 5.1, are two sets of data, for the strike price of volatility swaps obtained with the numerical implementation of Eq. (5.10) and those from MC simulations (5.11). One can clearly observe that the results from our exact solution perfectly match the results from the MC simulations. To make sure that readers have some quantitative concept of how large the difference between the results from our exact solution and those from the MC simulations, we have also tabulated the relative difference of the two as a function of the number of simulation paths, using our exact solution as the reference in the calculation, in Table 5.2. Clearly, when the number of paths

Table 5.1: The numerical results of volatility-average swaps obtained from our analytical pricing formula, MC simulations and continuous sampling approximation

Sampling frequency	Analytical formula	Approximation	MC simulations
Quarterly(N=4)	16.36	14.04	16.33
Monthly(N=12)	15.15	14.04	15.10
Fortnightly(N=26)	14.66	14.04	14.69
Weekly(N=52)	14.39	14.04	14.40
Daily(N=252)	14.13	14.04	14.14

Table 5.2: Relative errors and computational time of MC simulations

Path numbers of the MC	Relative Error %	Computational time(s)
10,000	0.233	6.21
100,000	0.191	61.47
200,000	0.074	254.12
500,000	0.012	1044.23

reaches 200,000 in MC simulations, the relative difference of the two is less than 0.1% already. Such a relative difference is further reduced when the number of paths is increased; demonstrating the convergence of the MC simulations towards our exact solution.

On the other hand, in terms of computational time, the MC simulations take a much longer time than our analytical solution does. To illustrate it clearly, we compare the computational times of implementing Eq. (5.10) and the MC simulations with sampling frequency for the realized variance equalling to 5 times per year. Table 5.2 shows the computational times for different path numbers in the MC simulations. In contrast to a long computational time of 1044.23 seconds using the MC simulations with 500,000 paths, implementing Eq. (5.10) just consumed 0.01 seconds; a roughly 100 thousands folds of reduction in computational time for one data point. The difference is even more significant when the sampling frequency is increased. This is not surprising at all since time-consuming is a well-known drawback of MC simulations.

5.3.2 Other Definition of Realized Volatility

While the focus of the paper is on the volatility swaps with the realized volatility being defined as the average of realized volatility (Eq. (5.3)), one may wonder why volatility swap contracts with the square root of average realized variance (Eq. (5.2)) in their payoff could not be worked out with the same approach. This is primarily due to the following three reasons.

Firstly, the nonlinear nature of the square root operation involved in the measurement, i.e., the square root of average of the discretely-sampled realized variance being outside of the summation operator, has made it extremely difficult to develop explicit pricing formula, because one can no longer exchange the order of these operators.

Secondly, since most of the volatility swaps are traded over the counter, the two participants of a contract can construct their own volatility swap contract to suit their requirements, provided that the required efficient pricing method and effective hedging strategies are available. Consequently, one may wish to choose a contract for which, a closed-form pricing formula can be worked out for its payoff function, in order to take the full advantage offered by an analytical closed-form solution in terms of simplicity in the solution form, significant less computational time and ultimate numerical accuracy.

Thirdly, the definition of the average of realized volatility is also a nice candidate in capturing historical realized volatility, as reported in many previous studies. For example, Andersen & Bollerslev (1997), Andersen & Bollerslev (1998), Granger & Sin (2000) have empirically studied the properties of the average of realized volatility, Eq. (5.3). Barndorff-Nielsen & Shephard (2003) analyzed the theoretical properties of the average of realized volatility, Eq. (5.3), and provided a theory for the use of the average of realized volatility. Davis & Mikosch (1998) even found evidence that if returns do not possess fourth moments then using the average of realized volatility rather than the square root of average

realized variance would be more reliable. Howison et al. (2004) also remarked that the average of realized volatility, Eq. (5.3), is a "more robust" measure of realized volatility. All these previous works seem to imply that the definition of the average of realized volatility may be a better definition than the square root of average realized variance anyway, which has thus motivated us to work out a closed-form exact pricing formula for volatility swaps based on this average of realized volatility definition first.

Of course, searching for a closed-form pricing formula for volatility-average swaps may even facilitate the search for a closed-form pricing formula for standard-deviation volatility swaps based on the square root of the average of realized variance (i.e., Eq. (5.2)) as a natural next step of our study. Besides, the newly-found pricing formula for volatility-average swaps may benefit the pricing of standard-deviation volatility swaps as these two are somewhat related; the former is in fact a lower bound the latter, if all other terms of the contacts being identical except the definitions of the realized variance. This can be easily proved through utilizing the Cauchy inequality (Bronshtein et al. 1997):

$$\sqrt{\frac{1}{NT}}\sum_{i=1}^{N}\left|\frac{S_{t_i}-S_{t_{i-1}}}{S_{t_{i-1}}}\right| \leq \sqrt{\frac{1}{T}\sum_{i=1}^{N}\left(\frac{S_{t_i}-S_{t_{i-1}}}{S_{t_{i-1}}}\right)^2} \tag{5.12}$$

which indicates that $\sqrt{2/\pi}K_{vol}$ is a lower bound for the price of a volatility swap defined in Eq. (5.2), where K_{vol} is given in Eq. (5.10). It should be noted that there have been different lower bounds proposed in the literature for standard-deviation volatility swaps; our newly-found lower bound is obtained and applies for the case of discretely sampling.

In the next section, we shall further compare volatility swaps with the two different definitions of realized volatility through numerical examples.

5.3.3 Continuous Sampling Approximation

In the literature of pricing variance swaps, many researchers (i.e., Swishchuk (2004)) have proposed a continuous sampling approximation for realized variance to price the variance swaps, based on the Heston stochastic volatility model. Some others (e.g., Little & Pant (2001), Broadie & Jain (2008*b*), Zhu & Lian (2009*f*), Zhu & Lian (2009*d*)) however pointed out that under the stochastic volatility model adopting such a continuous sampling approximation for a variance swap with small sampling frequencies or long tenor can result in significant pricing errors, comparing with the exact value of the discretely-sampled variance swap. As for the pricing of volatility swaps, it is also quite interesting to examine the accuracy level that the continuous sampling approximation formula yields as a function of the sampling period. We therefore have worked out the corresponding pricing formula based on the continuous sampling approximation, for the two definitions of realized volatility (i.e., Eq. (5.2) and Eq. (5.3)).

Corresponding to the definition of the square root of average realized variance, $RV_{d1}(0,N,T)$, the continuously-sampled realized volatility is denoted by the $RV_{c1}(0,T)$ and given by:

$$RV_{c1}(0,T) = \lim_{N\to\infty} \sqrt{\frac{AF}{N}\sum_{i=1}^{N}\left(\frac{S_{t_i}-S_{t_{i-1}}}{S_{t_{i-1}}}\right)^2} \times 100 = \sqrt{\frac{1}{T}\int_0^T V_t dt} \times 100 \tag{5.13}$$

where V_t is the spot variance of the underlying price. Under the Heston stochastic volatility mode, Broadie & Jain (2008*b*) showed how to analytically price volatility swaps based on this continuously-sampled realized volatility, Eq. (5.13), and presented the formula as:

$$K_{c1} = E_0^{\mathbb{Q}}[RV_{c1}(t,T)] = \frac{1}{2\sqrt{\pi}}\int_0^\infty \frac{1-E_0^{\mathbb{Q}}[e^{-sRV_{c1}(t,T)}]}{s^{\frac{3}{2}}}ds \tag{5.14}$$

with

$$\begin{aligned}
&E_0^{\mathbb{Q}}[-sRV_{c1}(t,T)] = \exp\left(A(s,T) - B(s,T)V_0\right) \\
&A(s,T) = \frac{2\kappa\theta}{\sigma_V^2}\log\left(\frac{2\gamma(s)\exp\left((\gamma(s)+\kappa)T/2\right)}{(\gamma(s)+\kappa)(\exp\left(\gamma(s)T\right)-1)+2\gamma(s)}\right) \\
&B(s,T) = \frac{2s(\exp\left(T\gamma(s)\right)-1)}{T[(\gamma(s)+\kappa)(\exp\left(\gamma(s)T\right)-1)+2\gamma(s)]} \\
&\gamma(s) = \sqrt{\kappa^2 + 2s\sigma_V^2/T}
\end{aligned} \tag{5.15}$$

Accordingly, the continuously-sampled realized volatility of the average of realized volatility, $RV_{d2}(0,N,T)$, is denoted by the $RV_{c2}(0,T)$ and given by:

$$RV_{c2}(0,T) = \lim_{N\to\infty}\sqrt{\frac{\pi}{2NT}}\sum_{i=1}^{N}\left|\frac{S_{t_i}-S_{t_{i-1}}}{S_{t_{i-1}}}\right| \times 100 = \frac{1}{T}\int_0^T \sqrt{V_t}dt \times 100 \tag{5.16}$$

The expectation of this expression can be carried and hence the approximation pricing formula for a variance swap based on this continuously-sampled realized variance is obtained,

$$K_{c2} = E_0^{\mathbb{Q}}[RV_{c2}(0,T)] = \frac{1}{2T\sqrt{\pi}}\int_0^T\int_0^\infty \frac{1-E_0^{\mathbb{Q}}[e^{-sV_t}]}{s^{\frac{3}{2}}}dsdt \times 100 \tag{5.17}$$

where $E_0^{\mathbb{Q}}[e^{-sV_t}]$ is actually the characteristic function of the stochastic variable V_t and given by $E_0^{\mathbb{Q}}[e^{-sV_t}] = g(-s;t,V_0)$ with $g(\phi;\tau,V)$ being defined in Eq. (3.8).

A question is naturally raised: how close the results of the two approximations and the true values are. One would also like to know when the approximation formulae start to yield large errors when the sampling time is large enough. To address this question, we compare the numerical results obtained from this approximation formulae, the newly-developed analytical formulae for discretely-sampled realized variance and the Monte Carlo simulations.

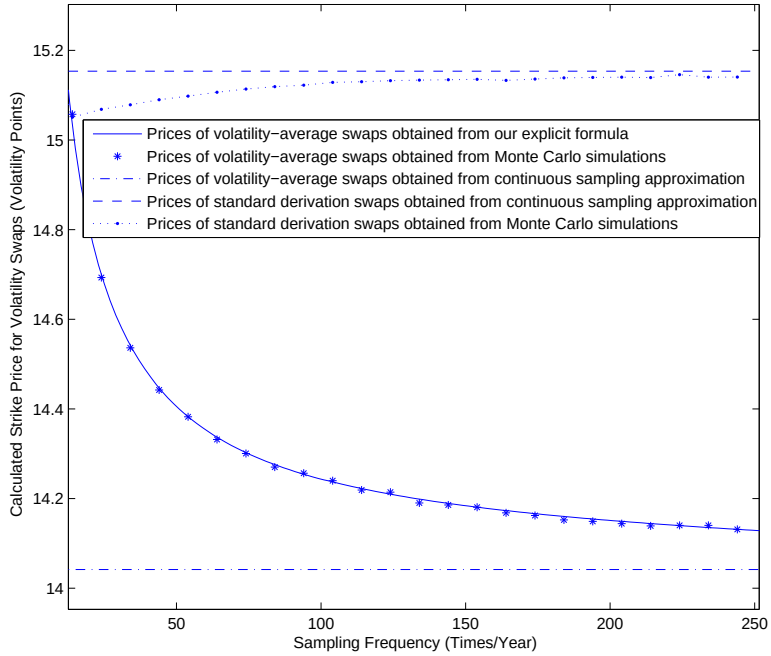

Figure 5.2: A comparison of fair strike prices of volatility swaps based on the two definitions of realized volatility obtained from our explicit pricing formula, the Monte Carlo simulations, and the corresponding continuous sampling approximations.

5.3.4 The Effect of Realized-Variance Definitions

As mentioned above, the two definitions, $RV_{d1}(0, N, T)$ and $RV_{d2}(0, N, T)$, have been alternatively used as the realized volatility in the literature. We now make a comparison of the price difference for two swap contracts being otherwise identical except the payoffs involving these two different definitions of realized volatility. Such a comparison should be very interesting and helpful for us to identify the difference between the two definitions of realized volatility.

Fig. 5.2 displays the strike prices computed using the two definitions of realized volatility, $RV_{d1}(0, N, 1)$ and $RV_{d2}(0, N, 1)$, as a function of the sampling frequency, and their corresponding continuous sampling approximations, Eq. (5.13) and Eq. (5.16), where the numerical results for the standard derivation swaps (based on the definition of $RV_{d1}(0, N, 1)$) are obtained by implementing the MC simulations (5.11), and numerical results for volatility-average swaps (based on

the definition of $RV_{d2}(0, N, 1)$) are obtained by implementing both the MC simulations (5.11) and our closed-form pricing formula (5.10). The results show that the strike price of a volatility-average swap with $RV_{d2}(0, N, 1)$ is consistently less than that of a standard derivation swap with $RV_{d1}(0, N, 1)$. With the increasing of sampling frequency in computing the realized volatility, the difference between the two becomes more significant. It can also be observed that the values of the two discretely-sampled realized volatility, $RV_{d1}(0, N, 1))$ and $RV_{d2}(0, N, 1))$, asymptotically approach the values of their corresponding continuous approximations (Eq. (5.13) and Eq. (5.16)), when the sampling frequency increases.

Several remarks should be made before leaving this section. Firstly, with the newly-found analytical solution, all the hedging ratios of a volatility swap can also be analytically derived by taking partial derivatives against various parameters in the model. With symbolic calculation packages, such Mathematica or Maple, widely available to researchers and market practitioners, these partial derivatives can be readily calculated and thus omitted here. However, to demonstrate how sensitive the strike price is to the change of the key parameters in the model, we performed some sensitivity tests for the example presented in this section. Shown in Table 5.3 are the results of the percentage change of the strike price when a model parameter is given a 1% change from its base value used in the example presented in this Section. Clearly, the strike price of a volatility swap appears to be most sensible to the long-term mean variance, $\theta^{\mathbb{Q}}$, for the case studied. On the other hand, the parameter "*vol of vol*", σ_V, may also have significant influence in terms of the sensitivity of the strike price. We also notice that the strike price of a volatility swap is less sensitive to the spot variance V_0. This finding is surprisingly opposite to the case of a variance swap, which is much more sensitive to the spot variance V_0 but less sensitive to "*vol of vol*" σ_V, as reported in Zhu & Lian (2009*d*). Secondly, due to the notational amount factor L and the size of the contract traded per order, the 1% or 2% relative differences, resulted from adopting different definitions of realized volatility, or using the

Table 5.3: The sensitivity of the strike price of a volatility swap (daily sampling)

Name	Value	Sensitivity
$\kappa^{\mathbb{Q}}$	11.35	-0.044%
$\theta^{\mathbb{Q}}$	0.022	0.50%
σ_V	0.618	-0.16%
V_0	$(20\%)^2$	0.07%

continuous approximations, may result in a considerable amount of absolute loss. Combining these points together, one may realize that it is even more desirable to work out the closed-form exact formula for the discretely-sampled volatility swaps to improve the pricing accuracy. Numerical efficiency is vitally important for any pricing formula; not only producing numerical values of the formula itself requires speedy calculations, calibrating model parameters with financial market data may require thousands, if not millions, of iterations and thus any reduction in computational time per iteration would considerably speed up the calibration process. In this regard, nothing can be better than an analytical closed-form exact solution.

5.4 Conclusion

In this chapter, we have applied the Heston stochastic volatility model to describe the underlying asset price and its volatility, and obtained a closed-form exact solution for discretely-sampled volatility swaps with the realized volatility defined as the average of the absolute percentage increment of the underlying asset price. This can be viewed as a substantial progress made in the field of pricing volatility swaps. Through numerical examples, we have shown that the our discrete model can improve the accuracy in pricing volatility swaps. We have compared the results produced from our new solution with those produced by the MC simulations for the validation purposes and found that our results agree with those from the MC simulations perfectly. This study also demonstrates that

our proposed solution approach can be used to work out a lower bound for the corresponding standard-derivation swap in which the realized volatility is defined as the square root of the average of realized variance. Furthermore, with the newly-found analytical formula, the computational efficiency is enormously enhanced in terms of assisting practitioners to price variance swaps, and thus it can be a very useful tool in trading practice when there is obviously increasing demand of trading variance swaps in financial markets.

Chapter 6

Examining the Accuracy of the Convexity Correction Approximation

6.1 Introduction

In Chapter 5, we presented a closed-form exact solution for discretely-sampled volatility swaps with the realized volatility defined as the average of the absolute percentage increment of the underlying asset price, Eq. (5.3). However, the prices of volatility swap contracts with the square root of average realized variance (Eq. (5.2)) in their payoff could not be worked out with the same approach. In fact, analytically calculating the expectations of these payoff functions containing square root operators can sometimes be very difficult. As a result, the convexity correction approach is used to approximate the square root function, in order to derive analytic approximations to pricing volatility swaps, based on the definition of square root of average realized variance, (see, e.g., Brockhaus & Long 2000; Swishchuk 2004; Javaheri et al. 2004; Elliott et al. 2007; Benth et al. 2007 etc). Lin (2007) also applied a similar analysis to propose an approximation for the strike price of VIX futures. It seems to be quite common in finance practice to

encounter a payoff function of an exotic financial derivative with the square root operator involved. In this chapter, we examine the core issue of the convexity correction approximation (CCA), its accuracy, and the validity condition of this CCA in pricing volatility swaps. For simplicity, our discussion in this chapter is based on the continuously-sampled realized volatility, Eq. (5.13). For the completeness reason, the approximation for VIX futures based on the same CCA technique will also be discussed in this chapter. The detailed discussion about pricing VIX futures is presented in Chapter 7.

In comparison with other solution approaches, analytic approximation formulae developed based on the convexity correction approach certainly have their own advantage in terms of providing simple and speedy pricing formulae for some very complicated pricing problems. However, studies focusing on examining the core issue of the convexity correction approximation (CCA), its accuracy, are very rare in literature, and the validity condition of this CCA remains unclear. The only paper that can be found in the literature is the one by Broadie & Jain (2008*b*), who briefly discussed the convexity correction approximation and concluded that it may not provide a good approximation of fair strikes of volatility swaps in models with jumps in the underlying asset. Our numerical examples presented later in this chapter also show that the CCA, sometimes, is very poor with substantially large pricing errors. Clearly, there is an urgent need to systematically examine the accuracy as well as the reliability of this popularly-used approximation and work out its validity condition.

This chapter addresses these two inter-related and important issues. Particularly, we mainly concern the following two basic questions. First, for a deterministic function $f(x) = \sqrt{x}$, the convergence condition of the Taylor expansion is very clear. It is also a straightforward but important exercise to check whether the point x satisfies the convergence condition before applying the Taylor expansion as an approximation. However, what is the convergence condition of applying the Taylor expansion while the independent variable x is a stochastic

variable, which implies that the realized value of x might be any possible value? More importantly, how to examine whether the convergence condition is satisfied? Second, it can be shown that Brockhaus & Long (2000)'s CCA is essentially the application of the second-order Taylor expansion of the square root function with independent variable being a stochastic variable (e.g., the future realized variance in pricing volatility swaps, the value of future volatility index in pricing VIX futures). As a result, a naturally raised question is: do higher-order Taylor expansions, such as the third order or fourth order, achieve better accuracies to approximate the function $f(x) = \sqrt{x}$ while x is a stochastic variable?

To address these two basic questions, this chapter firstly discusses about the validity condition of this second-order Taylor expansion (i.e., the CCA) from the theoretical analysis aspect, and then presents three specific numerical examples to examine the accuracies of the CCA and its variations (the third- or fourth-order Taylor expansions). The main contribution of this chapter can be summarized in four folds: (1) pointing out the surprisingly large differences in accuracy among approximations in some specific parameters, and further alerting that one should be aware of the inaccuracy of this approximation and be very careful in using it; (2) analyzing the reason why the CCA performs very poor sometimes, and more importantly, proposing a useful mechanism (a test ratio) to detect the possible unacceptable large errors; (3) identifying the pitfall of believing that an further inclusion of higher order terms into the second order expansion will naturally achieve a better accuracy. Our study shows that it is not so at all for most of the cases in approximating square root function involved with stochastic variables; (4) utilizing the proposed test ratio, we propose a more accurate approximation for the pricing of volatility swaps.

The work presented here could not have been carried out without some recently discovered exact solutions for volatility swaps and VIX futures under the Heston model. However, our real goal is to provide a correction formula, accompanying the adoption of the CCA, for the case that there is no closed-form exact

solution and the CCA must be adopted to render a fast and yet accurate enough solution formula.

The remainder of this chapter is organized as follows. In Section 6.2, we show how the CCA can be derived to price volatility swaps and VIX futures, followed by the discussion of the validity condition of the CCA. In Sections 6.3, three specific examples are presented to show the comparison of the CCA and the improved formula in terms of their accuracy. Our conclusion is stated in Section 6.4.

6.2 Convexity Correction and Convergence Analysis

A volatility swap is a forward contract written on the annualized standard deviation of the log asset returns. The payoff at expiry for the long position is equal to the annualized realized volatility over the pre-specified period minus the pre-set delivery price of the contract multiplied by a notional amount of the swap in dollars per annualized volatility point, whereas the short position is just the opposite.

The realized volatility at expiration of a volatility swap on the price of an asset S is commonly calculated as the square root of the realized variance, and the fair strike price of a volatility swap, denoted by K_{vol}, is set at the initiation of the contract so the contract's net present value is equal to zero, i.e.,

$$E_0^{\mathbb{Q}}[(\sqrt{RV(0,T)} - K_{vol}) \times L] = 0 \tag{6.1}$$

where $RV(0,T)$ is the annualized realized variance of the asset S over the contract life $[0,T]$. Therefore, the fair strike price is the expectation value of the realized

volatility in the risk-neutral world, i.e.,

$$K_{vol} = E_0^{\mathbb{Q}}[\sqrt{RV(0,T)}] \tag{6.2}$$

and $RV(0,T)$ is given by

$$RV(0,T) = \lim_{N\to\infty} \frac{1}{N\Delta t} \sum_{i=1}^{N} \log^2 \left(\frac{S_{t_i}}{S_{t_{i-1}}} \right) \times 100^2 \tag{6.3}$$

Due to the nonlinear square root function involved in the expectation in Eq. (6.2), it is difficult to carry out the expectation analytically. On the other hand, Jensen's inequality shows that the fair volatility strike price is upper bounded by the square root of the expectation of the annualized realized variance :

$$K_{vol} = E_0^{\mathbb{Q}}[\sqrt{RV(0,T)}] \leq \sqrt{E_0^{\mathbb{Q}}[RV(0,T)]} \tag{6.4}$$

where $E_0^{\mathbb{Q}}[RV(0,T)]$ is essentially the strike price of a variance swap, and can be relatively more easily computed, as shown by Zhu & Lian (2009*d*), Broadie & Jain (2008*b*), Itkin & Carr (2010), etc.

Obviously, Eq. (6.4) is a very loose upper bound and it may lead to large errors if it is used as a pricing formula. In order to achieve better accuracies, Brockhaus & Long (2000) presented the so-called CCA to approximate the fair volatility strike, using a second-order Taylor expansion of the square root function. Mathematically, their CCA is based on the first three terms of the Taylor expansion of $f(x) = \sqrt{x}$ around the point x_0,

$$\sqrt{x} = \sqrt{x_0} + \frac{(x-x_0)}{2\sqrt{x_0}} - \frac{(x-x_0)^2}{8\sqrt{x_0}^3} + \frac{(x-x_0)^3}{16\sqrt{x_0}^5} - \frac{5(x-x_0)^4}{128\sqrt{x_0}^7} + \mathcal{O}((\frac{x-x_0}{x_0})^5) \tag{6.5}$$

In the specific case of pricing volatility swaps, one just needs to substitute $x = RV(0,T)$ and $x_0 = E_0^{\mathbb{Q}}[RV(0,T)]$ in Eq. (6.5), and then take expectation under the risk-neutral measure on the both sides of Eq. (6.5) to obtain an approximation

formula to price volatility swaps. In this way, Brockhaus & Long (2000)'s CCA can be obtained by taking the second order expansion (i.e., the first three terms) in Eq. (6.5) and ignoring the higher order terms, which results in

$$K_{vol} \approx \sqrt{K_{var}} - \frac{Var_0^{\mathbb{Q}}[RV(0,T)]}{8\sqrt{K_{var}}^3} \tag{6.6}$$

where $K_{var} = E_0^{\mathbb{Q}}[RV(0,T)]$ is the strike price of a variance swap which can be easily determined with the approach recently discussed in Zhu & Lian (2009*d*), Broadie & Jain (2008*b*), Itkin & Carr (2010). Lin (2007) also applied a similar analysis to propose an approximation for the strike price of VIX futures.

For a deterministic function $f(x)$ to be expanded in Taylor series (Eq. (6.5)), it is well known that the convergence condition for the Taylor series expansion is that the x should satisfy the condition $|x - x_0| \leq x_0$. When this condition holds, the Taylor expansion converges very quickly and the higher order terms are negligible compared to the first three terms in the expansion. Hence, if the first three terms on the right hand side of Eq. (6.5) are taken, i.e., with a second-order expansion, we should have a good approximation of $\sqrt{x}$ for all values of x satisfying $|x - x_0| \leq x_0$. Of course, in the convergent radius $|x - x_0| \leq x_0$, a better accuracy can be achieved if higher-order terms are further included.

Intuitively, one may expect a better accuracy can also be achieved by extending Brockhaus & Long (2000)'s second-order CCA to the third order or even fourth order in the Taylor expansion of the square root function as in the deterministic case. In fact, such an extension was indeed attempted. For example, Brenner et al. (2007) proposed the third order Taylor expansion approximation formula for VIX futures, based on the Heston stochastic volatility model (Eq. (9) in their paper). Sepp (2007) presented the fourth order expansion to approximate the expectation of a general smooth nonlinear function of a stochastic variable (cf. Theorem 1.3.2 (Eq. (1.3.6)) of his paper). However, no one has addressed the issue whether a better accuracy can indeed be achieved by including the higher

order terms (the third- or the fourth-order terms) into Brockhaus & Long (2000)'s second-order CCA.

Since the CCA appears to be a natural way to deal with the difficulty whenever there is a presence of a nonlinear operator, such as the square root operator, involved in the payoff function, its convergence in the context of the Taylor series expansion being used in conjunction with a stochastic independent variable needs to be systematically examined. Such an examination will provide a good guidance when the CCA is adopted to derive an approximation formula for pricing any financial derivatives, such as volatility swaps and VIX futures, where there is a square root function $f(x) = \sqrt{x}$ in the payoff with the independent variable x being a stochastic variable (e.g., future realized variances, or future values of VIX^2). We shall show that a higher order expansion does not necessarily achieve a better accuracy in this case. In fact, we found that in most cases, the third order (or the fourth order) expansion performs much worse than the second order one, as will be demonstrated in the numerical examples.

For the CCA, Eq. (6.6), to converge and hence to provide a good approximation of $E_0^{\mathbb{Q}}[\sqrt{RV(0,T)}]$, it is strictly required that the realized variance of the stock price path over the time span $[0,T]$ should satisfy

$$|RV(0,T) - E_0^{\mathbb{Q}}[RV(0,T)]| \leq E_0^{\mathbb{Q}}[RV(0,T)] \tag{6.7}$$

which can be rewritten as

$$0 \leq RV(0,T) \leq 2E_0^{\mathbb{Q}}[RV(0,T)] \tag{6.8}$$

In other words, the CCA would work well if the convergence condition, Eq. (6.8), holds on the stock price path. Since $RV(0,T)$ is a stochastic variable, whose value cannot be determined until a sample path is drawn, we should interpret Eq. (6.8) in the context of probability theory. By defining the excess probability

$p \equiv Prob\big(RV(0,T) \geq 2E_0^{\mathbb{Q}}[RV(0,T)]\big)$, Eq. (6.8) is equivalent in saying that the excess probability p is zero, which is an issue also shown in Broadie & Jain (2008*b*).

Hence $p = 0$ can be viewed as a validity condition for the convexity correction approximation. Our experience is that $p = 0$ holds only for a small number of stochastic processes adopted to price financial derivatives (e.g., pricing volatility swap in the Black-Scholes model). This condition does not hold at all for most of the cases in pricing volatility swaps and VIX futures. Our numeral examples show that when the excess probability p is small (e.g., less than 5%), the second order CCA can still achieve acceptable accuracy, but including higher order (the third- or fourth-order) terms is useless, if they haven't made it worse, in improving accuracies.

It is thereby very important to observe the excess probability in applying the CCA to price volatility swaps. Unfortunately, this excess probability is normally difficult to calculate analytically, which requires the availability of the associate probability density function or characteristic function. For most occasions of using the Taylor expansions, these density functions (or characteristic functions) are not easily obtainable, which is the exact reason one has to use the approximations in the first place. If the density function can be worked out, one can then use it to calculate the expectation of the square root function directly and obtain exact values, instead of using the Taylor expansions.

An alternative way must be sought to avoid this dilemma. We propose a more easily computed ratio which extracts the useful information from the excess probability to serve as an indicator in identifying the relative errors that the CCA may lead to. This useful ratio, denoted by SCV hereafter, is defined as

$$SCV = \frac{Var_0^{\mathbb{Q}}[RV(0,T)]}{\big(E_0^{\mathbb{Q}}[RV(0,T)]\big)^2} \tag{6.9}$$

It can be shown that the SCV ratio is nothing but the square of the coefficient

of variation (CV), which is a normalized measure of dispersion of a probability distribution in statistics and is computed as

$$CV = \frac{Standard \quad Deviation}{Mean} \tag{6.10}$$

Since the CV is a measure of the dispersion of data points around the mean, it should have a close positive correlation with the excess probability $p \equiv Prob(RV(0,T) \geq 2E_0^{\mathbb{Q}}[RV(0,T)])$, which essentially also measures the concentration degree of the stochastic variable $RV(0,T)$ around its mean $E_0^{\mathbb{Q}}[RV(0,T)]$. In simple words, when the SCV (or CV) ratio is low, which means data concentrate more closely around the mean value, the excess probability would also be low, and vice versa.

Our belief of using this SCV ratio as an error indicator of CCA is also supported by another argument. By the Chebyshev's inequality, we have

$$Prob\left(|X - E[X]| \geq \alpha\right) \leq \frac{\sigma^2}{\alpha^2}. \tag{6.11}$$

When α is set to $E[X]$, σ^2/α^2 is just the SCV ratio $Var[X]/E[X]^2$, as defined in Eq. (6.9). Hereby, the SCV ratio is the upper bound of the excess probability. Our numerical examples below will illustrate that the SCV ratio can indeed serve as a good indicator in identifying relative pricing errors resulted from the convexity correction approximation. Our numerical results will also show that the second-order CCA consistently under-estimates the true values. Furthermore, we can identify by using the linear regression that there is a close linear relationship between the relative errors resulted from the second-order CCA and the SCV ratios. Hence, utilizing the fact of the close linear relationship between the SCV ratios and the relative errors, we now can propose an improved approximation as

$$K_{vol} \approx \left(\sqrt{K_{var}} - \frac{Var_0^{\mathbb{Q}}[RV(0,T)]}{8\sqrt{K_{var}}^3}\right) \times (1 + \alpha \cdot SCV) \tag{6.12}$$

α is a correction factor that needs to be determined empirically. Ideally, for a

specific functional form, to which the Taylor expansion is utilized, its value should be within a narrow range and this range may change when another function is to be approximated. Our numerical experience indeed confirms that this factor is around $0.02 - 0.04$ for all the test cases we have so far conducted for the square root function. Therefore, we have set α to 0.03 in all numerical examples presented in the next section, where we demonstrate this improved approximation can substantially reduce the relative errors.

6.3 Illustrations and Discussions

In this section, we shall present some numeral examples, for illustration purpose, to examine the accuracy of the improved approximation formula and to show the robustness of the SCV ratio in identifying the relative errors.

6.3.1 Volatility Swaps in Heston Model

The Heston (1993) model is the most popular stochastic volatility model and has received the most attention, since it can give a satisfactory description of the underlying asset dynamics (Daniel et al. 2005; Silva et al. 2004). Based on the Heston stochastic volatility model, Brockhaus & Long (2000) first proposed the CCA to approximate the value of a volatility swap. A lot of recent studies for the pricing of volatility swaps are also based on the Heston model (see, for example, Elliott et al. 2007; Swishchuk 2004). Hereby, we first examine the accuracy of the CCA in the Heston model.

In the Heston model, the underlying asset S_t is modeled by the following diffusion process with a stochastic instantaneous variance V_t, in the risk-neutral measure $\mathbb{Q}$,

$$\begin{cases} dS_t = rS_t dt + \sqrt{V_t} S_t dB_t^S \\ dV_t = \kappa(\theta - V_t)dt + \sigma_V \sqrt{V_t} dB_t^V \end{cases}$$

where r is the risk-free interest rate, θ is the long-term mean of variance, κ is a

mean-reverting speed parameter of the variance, σ_V is the so-called volatility of volatility. The two Wiener processes dB_t^S and dB_t^V describe the random noise in asset and variance respectively. They are assumed to be correlated with a constant correlation coefficient ρ, that is $(dB_t^S, dB_t^V) = \rho dt$. The stochastic volatility process is the familiar squared-root process. To ensure the variance is always positive, it is required that $2\kappa\theta \geq \sigma^2$ (see Cox et al. 1985; Heston 1993; Zhang & Zhu 2006).

For the volatility swaps based on Heston model, Broadie & Jain (2008*b*) recently proposed an analytical exact solution, as

$$K_{vol} = E_0^{\mathbb{Q}}\left[\sqrt{RV(0,T)}\right] = \frac{1}{2\sqrt{\pi}}\int_0^{\infty}\frac{1 - E[e^{-sRV(0,T)}]}{s^{\frac{3}{2}}}ds \tag{6.13}$$

where $E[e^{-sRV(0,T)}]$ is the Laplace transform of the realized variance, and given by Eq. (D1). It should be noted that there is a typo in the Eq. (A-12) in Broadie & Jain (2008*b*), which has been corrected in Eq. (D1) in this thesis.

We use this formula to obtain exact volatility strike prices as benchmark values to examine the accuracy of the convexity correction approximation (the second order Taylor's expansion) and its higher order extensions (i.e., third order and fourth order expansions). The numerical results of the benchmark values are obtained with the following parameters: $\theta = 0.019$, $\kappa = 6.21$, $\sigma_V = 0.61$, $\sqrt{V_0} = 10.1\%$.

Shown in Fig. 6.1, as well as in Table 6.1, are numerical results of volatility strike prices obtained from the numerical implementation of the exact pricing formula, Eq. (6.13), the second-order, third-order and fourth-order Taylor expansion approximations, and the improved approximation Eq. (6.12), respectively. One can observe that the Brockhaus & Long (2000)'s second order approximation is reasonable for values close to the exact volatility strike for this specific case, with relative error being 2% when time to maturity $T = 0.3$ and less than 0.4% when time to maturity T approaching to 2 years. However, we found that the

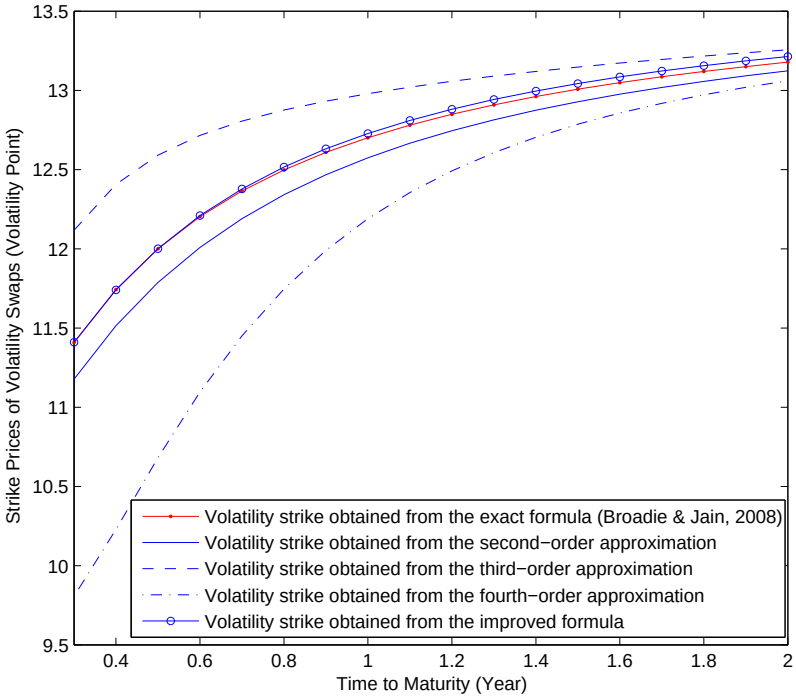

Figure 6.1: A comparison of the exact volatility strike and the approximations based on the Heston model

performance of the third-order and fourth-order Taylor expansions are very poor in approximating volatility strikes, particularly when the time to maturity T decreases. It can also be observed that results from our proposed approximation Eq. (6.12) have the lowest relative pricing errors. In fact, the results obtained from the proposed improved formula match, almost dot-to-dot, with those obtained from the exact solution; this has numerically demonstrated that adopting the proposed improved formula is a far-better choice than adopting higher-order terms in Taylor's expansion to derive a higher-order approximation formula. Of course, the main reason behind this is that higher-order approximations are not necessarily of "higher order" in terms of error reduction anymore when the Taylor expansion is used in the case of expansion of a function of stochastic variables.

To make sure that readers have some quantitative concept of how large the difference between the results from the exact solution and those from the approximations, we have also tabulated the results and the relative differences in

Table 6.1: Strikes of one-year maturity volatility swaps obtained from the exact pricing formula and the approximations in the Heston model

Formulae	Volatility Strikes	Relative Errors	SCV
The exact formula	12.701		
The second-order approximation	12.586	-0.905%	0.407
The third-order approximation	12.979	2.188%	
The fourth-order approximation	12.190	-4.023%	
Our improved approximation	12.728	0.212%	

Table 6.1, for the case of time to maturity $T = 1$. As can be seen, both the third-order and the fourth-order Taylor expansions perform much worse than the second-order expansion. The third-order approximation has over-estimated the true value by 2.188%, which has doubled the relative errors of the second-order approximation of -0.905%. Opposite to third-order approximation, the fourth-order approximation under-estimates the exact volatility strike by -4.023%. And the improved approximation has the lowest pricing error of 0.212%.

To clearly explain the reason why the third-order and fourth-order approximations are even worse than the second-order one, we have run the Monte Carlo simulations to sample the Heston model with $T = 1$ and calculate the realized variance for each sample path of the Heston model. In this way, we can analyze the contribution towards the overall relative errors from three components of the sampled realized variance, $RV(0,1)$, in three intervals $[0, x_0]$, $[x_0, 2x_0]$ and $[2x_0, \infty]$, respectively (Note: $x_0 = E_0^{\mathbb{Q}}[RV(0,T)]$). If the error contribution from the interval $[2x_0, \infty]$, has significantly increased, when the overall relative error becomes high with higher order approximations, it will confirm our hypopiesis that the inclusion of higher order terms does not necessarily improve accuracy, at least based on numerical evidence.

Tabulated in Table 6.2 are the relative error contribution for the three approximations in each of the spacial intervals, respectively. Roughly speaking, the relative errors in three spacial intervals in Table 6.2 contribute to the total relative error of the corresponding approximation listed in Table 6.1, weighted by

Table 6.2: The relative errors of the three approximations in the three intervals

	$[0,\ x_0]$	$[x_0,\ 2x_0]$	$[2x_0,\ \infty\ ]$
Probability	61.0%	31.6%	7.4%
The second-order approximation	1.74%	-0.467%	-10.37%
The third-order approximation	0.74%	0.202%	13.8%
The fourth-order approximation	0.36%	-0.107%	-25.45%

$x_0 = E_0^{\mathbb{Q}}[RV(0,T)]$

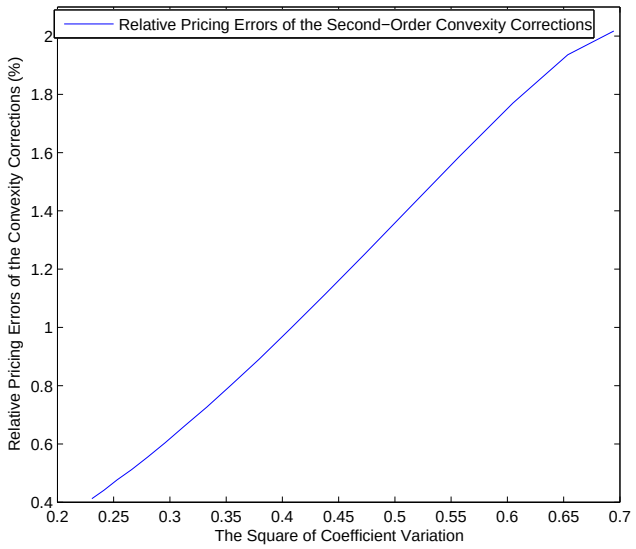

Figure 6.2: Relative pricing errors of the second order approximation as a function of SCV ratio in Heston model

the probability of a sample point appearing in the relevant interval. It can be observed from Table 6.2 that the higher order expansions (the third or fourth order expansions) can indeed reduce the relative errors within the intervals $[0, x_0]$ and $[x_0, 2x_0]$. This is because the Taylor expansion of the square root function, Eq. (6.5), is convergent very well in the interval $[0, 2x_0]$, and hence a higher order approximation should achieve a better accuracy. However, in the interval $[2x_0, \infty]$, the Taylor expansion of the square root function is no longer convergent, and as a result, a higher order expansion will perform substantially worse in this domain. And overall, the higher order approximations results in much larger relative errors, as shown in Table 6.1.

Next, we investigate the relationship between the SCV ratio and the relative

error from the second-order approximation in pricing volatility swaps. Plotted in Fig. 6.2 is the relative error from the second-order approximation as a function of SCV ratio. There seems to be a highly linear relationship between the two variables, as can be observed. To identify the quantitative relationship, we regress the relative error (RE) from the second-order approximation on the SCV ratio and obtain

$$RE = -0.453 + 3.616\text{SCV} + \epsilon \quad R^2 = 99.8\% \tag{6.14}$$

As we know, the coefficient of determination, R^2, is a statistical measure of how well the regression line approximates the real data points and hence gives information about the goodness of fit of a model. With R^2 being almost 1 in our regression, it is therefore shown that the relative pricing errors indeed have a great linear relationship with the SCV. Consequently, it makes sense to use the SCV as an indicator to identify the large pricing errors potentially resulted from adopting the CCA.

6.3.2 Volatility Swaps in GARCH Model

GARCH model is another most widely used model in macroeconomics and finance with many important implications. The variance process in a continuous version of GARCH can be written in the form of

$$dV_t = \kappa(\theta - V_t)dt + \gamma V_t dB_t^V \tag{6.15}$$

Based on this continuous GARCH model, Javaheri et al. (2004) discussed the pricing of volatility swap. They used the flexible PDE approach to determine the first two moments of the realized variance and then obtained the CCA for volatility swaps.

Different from the Heston model, it is very hard to work out the characteristic function for the stochastic variable V_T, conditional on V_0, for this continuous-time

limit GARCH(1,1) model. Although Heston & Nandi (2000) presented a characteristic function based on the discrete GARCH(1,1), their model however differs from the continuous-time limit GARCH(1,1) discussed in this chapter as well as in Javaheri et al. (2004) . Due the lack of any exact formula for pricing volatility swaps under this GARCH model, one has to either resort to some computationally much more expensive numerical methods such as the finite difference or Monte Carlo simulations, or adopt the CCA if one wishes to substantially reduce the computational time. From this viewpoint, it is particularly important to analyze the accuracy of the approximations.

For the purpose of obtaining exact solutions as benchmark values to analyze the accuracy of the approximation, we use the finite difference method. Following Javaheri et al. (2004), it can be shown that the price of a volatility swap, $F(t, V, I)$, satisfies the following PDE

$$\frac{\partial F}{\partial t} + \frac{1}{2}\gamma^2 V^2 \frac{\partial^2 F}{\partial V^2} + \kappa(\theta - V)\frac{\partial F}{\partial V} + V\frac{\partial F}{\partial I} = 0 \tag{6.16}$$

with the payoff function $F(T, V, I) = \sqrt{\frac{I}{T}}$, and the variable I is defined as $I_t = \int_0^t V_s ds$.

We solve this PDE in the region $0 \leq t \leq T$, $V_{min} \leq V \leq V_{max}$, $I_{min} \leq I \leq I_{max}$ with the payoff conditions. Following the studies of Wilmott (2000) and Broadie & Jain (2008*a*), we use the boundary conditions for V and I:

$$\frac{\partial^2 F}{\partial V^2}|_{(V=V_{min},V_{max})} = 0, \quad \frac{\partial^2 F}{\partial I^2}|_{(I=I_{min},I_{max})} = 0 \tag{6.17}$$

We examine the accuracy by taking the set of parameters in Javaheri et al. (2004), i.e., $\theta = 0.0397$, $\kappa = 20.889$, $\gamma = 4.438$ and $V_0 = (19\%)^2$. Shown in Fig. 6.3 are numerical results of volatility strike prices obtained from the numerical implementation of the finite difference method and the second-order Taylor expansion approximation. In this set of specific parameters of the GARCH model,

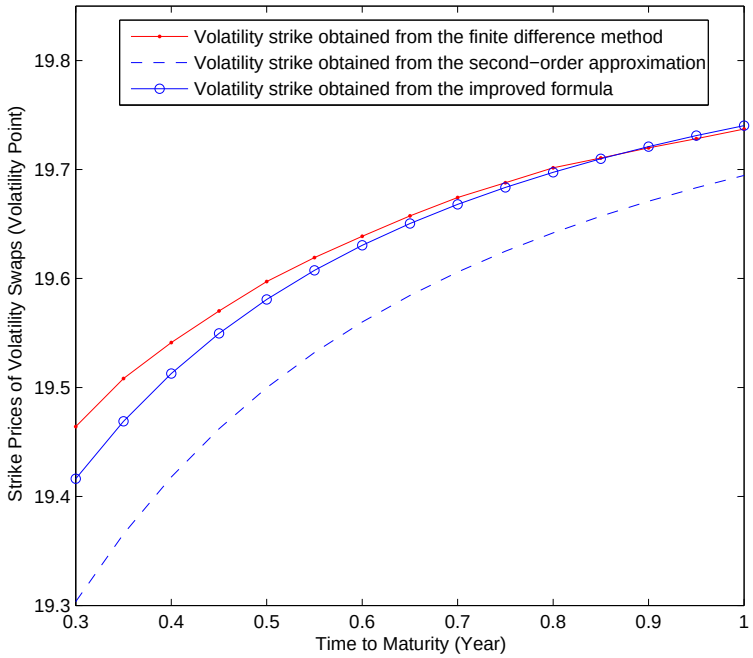

Figure 6.3: A comparison of the volatility strikes from the finite difference and those from approximations in the GARCH model

the Brockhaus & Long (2000)'s second order approximation is very accurate with relative error less than 0.8% when time to maturity $T = 0.3$. While the relative pricing errors resulted from adopting the second-order approximation in this case are lower than those in the Heston model, those resulted from adopting our improved formula (Eq. (6.12)) are even smaller, as shown in Fig. 6.4. Again, it is shown that our improved approximation can further reduce the relative pricing errors. This has demonstrated the consistence of the improved formula across different models.

The regression equation of the relative error of the second-order approximation on the SCV ratio is

$$RE = -0.190 + 5.082\text{SCV} + \epsilon \quad R^2 = 99.4\% \tag{6.18}$$

Once again, these results show that the relative pricing errors are highly linearly related to the SCV ratio, demonstrating the importance of SCV ratio in identi-

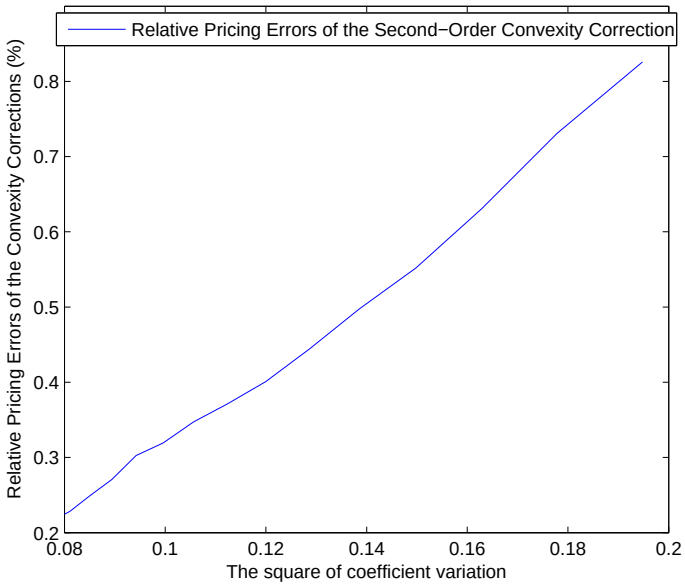

Figure 6.4: Relative pricing errors of the second order approximation as a function of SCV ratio in GARCH model

fying the relative pricing errors while applying the CCA to price volatility swaps in practice.

6.3.3 VIX Futures in SVJJ Model

Now, we examine the accuracy of the convexity correction approximation in pricing VIX futures. The VIX future was introduced by CBOE in 2004, and the interest in trading VIX futures has been growing very quickly. The underlying of VIX futures is the square root of VIX_t^2, which can be computed based on the prices of a portfolio of 30-calendar-day out-of-the-money S&P500 calls and puts with weights being inversely proportional to the squared strike price. The payoff of a VIX future at expiration T is VIX_T, and hence the strike price of a VIX future at time t is

$$F(t,T) = E^{\mathbb{Q}}[\text{VIX}_T|\mathscr{F}_t] = E^{\mathbb{Q}}[\sqrt{\text{VIX}_T^2}|\mathscr{F}_t] \times 100 \tag{6.19}$$

Again, similar to the pricing problem of volatility swaps, the calculation of

VIX future strike also involves an expectation of the square root function. Using the CCA for the square root function, Lin (2007) presented an approximation for the value of the VIX futures under the Heston stochastic volatility models with simultaneous jumps both in the asset price and variance processes (SVJJ model). In particular, Lin (2007)'s analysis was based on the assumption that the dynamics processes of the S&P500 index and its variance under the risk-neutral probability measure $\mathbb{Q}$ follow the processes,

$$\begin{cases} dS_t = S_t r_t dt + S_t \sqrt{V_t} dW_t^S(\mathbb{Q}) + d\left(\sum_{n=1}^{N_t(\mathbb{Q})} S_{\tau_{n-}}[e^{Z_n^S(\mathbb{Q})} - 1]\right) - S_t \overline{\mu}^{\mathbb{Q}} \lambda dt \\ dV_t = \kappa^{\mathbb{Q}}(\theta^{\mathbb{Q}} - V_t)dt + \sigma_V \sqrt{V_t} dW_t^V(\mathbb{Q}) + d\left(\sum_{n=1}^{N_t(\mathbb{Q})} Z_n^V(\mathbb{Q})\right) \end{cases} \tag{6.20}$$

By using the CCA based on this SVJJ model, Lin (2007) presented the VIX futures formula in the form of

$$F(t,T) = E^{\mathbb{Q}}[\mathrm{VIX}_T|\mathscr{F}_t] \approx \sqrt{E_t^{\mathbb{Q}}(\mathrm{VIX}_T^2)} - \frac{var^{\mathbb{Q}}(\mathrm{VIX}_T^2)}{8[E^{\mathbb{Q}}(\mathrm{VIX}_T^2)]^{\frac{3}{2}}} \tag{6.21}$$

where $var^{\mathbb{Q}}(\mathrm{VIX}_T^2)/(8[E^{\mathbb{Q}}(\mathrm{VIX}_T^2)]^{\frac{3}{2}})$ is the convexity adjustment relevant to the VIX futures. Detailed expressions of $E_t^{\mathbb{Q}}(\mathrm{VIX}_T^2)$ and $var^{\mathbb{Q}}(\mathrm{VIX}_T^2)$ are given by Eq. (8) and Eq. (9) in Lin (2007).

For the same problem, Zhu & Lian (2009*a*) managed to obtain a closed-form exact pricing formula for VIX futures in the form of:

$$F(t,T,\mathrm{VIX}_t) = \frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{1 - e^{-sb} f(-sa; t, \tau, \frac{VIX_t^2 - b}{a})}{s^{\frac{3}{2}}} ds \tag{6.22}$$

where $f(\phi; t, \tau, V_t)$ is the moment generating function of the stochastic variable V_T (cf. Eq. (7.8), also Eq. (10) in Zhu & Lian (2009*a*) for the specific form of $f(\phi; t, \tau, V_t)$).

In our examples, we use the parameters (unless otherwise stated) reported in

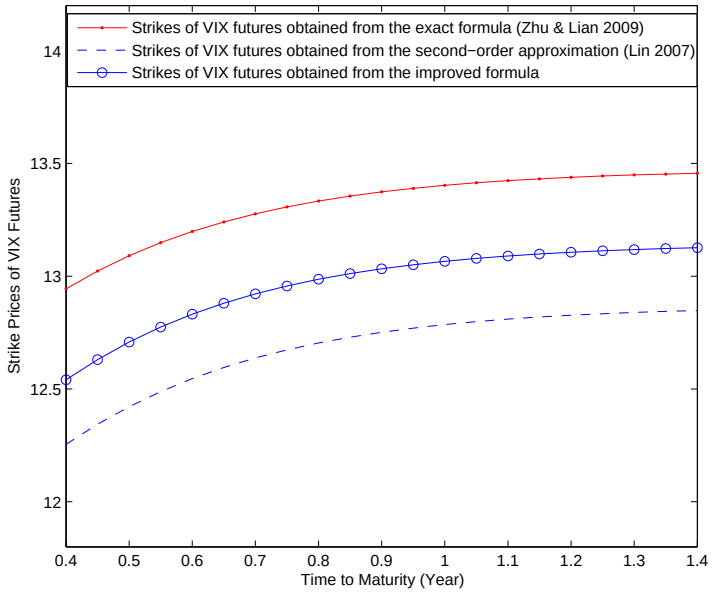

Figure 6.5: A comparison of the VIX futures strikes from the exact formula and those from the convexity correction approximation in the SVJJ model

Duffie et al. (2000) that were founded by minimizing the mean-squared differences between models and the market S&P500 options prices on November 2, 1993. This set of parameters has been adopted by Broadie & Jain (2008*a*) as well. Specifically, these parameters are $\theta = 0.008$, $\kappa = 3.46$, $\sigma_V = 0.14$, $\lambda = 0.47$, $\sigma_S = 0.0001$, $\overline{\mu} = -0.10$, $\mu_V = 0.05$, $\rho_J = -0.38$, $\sqrt{V_0} = 8.7\%$.

In Fig. 6.5, we have plotted the fair price of VIX futures obtained with the numerical implementation of Eq. (6.22), and those obtained from Lin (2007)'s approximation. From this figure, one can clearly see that there are noticeable gaps between numerical results obtained from the exact solution and those from the approximation formula. For a one-year VIX future, the exact solution produces a value of 13.40 while the CCA results in a value of 12.78, exhibiting a relative difference of -4.60%. For example, in the literature of pricing variance swaps, even when the error level reaches more than 0.5%, Little & Pant (2001) already declared that it is "fairly large" so that adopting approximation model to price variance swaps might not be justifiable. With this concept in mind, an error of -4.60% of Lin (2007)'s approximation formula is certainly unacceptable for market

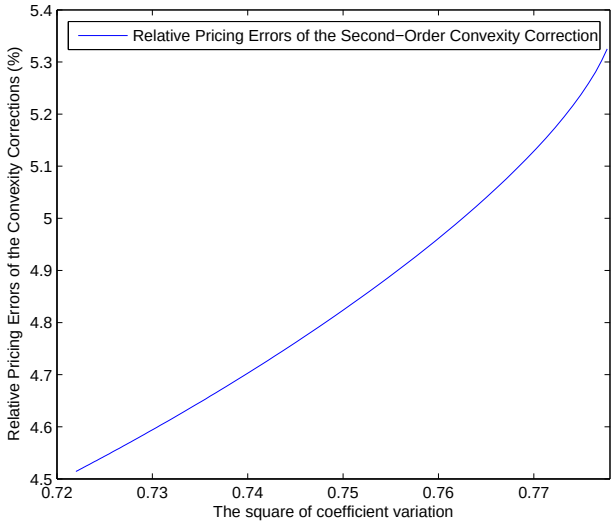

Figure 6.6: Relative pricing errors of the second order approximation in pricing VIX futures as a function of SCV ratio in SVJJ model

traders. At the same time, the price of a VIX future obtained from the improved formula is 13.07, representing a relative error of -2.51%, which is substantially less than the relative error of -4.60% resulted from Lin (2007)'s second-order approximation. When the correction factor in Eq. (6.12) is increased, the resulting relative error of this improved formula will be further reduced.

Of course, when other parameters, such as volatility of volatility, σ_V, are changed, the differences between the exact solution and the Lin (2007)'s second-order approximation might even exponentially grow. When the σ_V reaches 0.5, which is a reasonable and often reported value in the literature of empirical studies (e.g., Zhang & Zhu 2006; Brenner et al. 2007), the relative difference of one-year VIX futures obtained from the two solutions becomes as high as -11.3%!

In this case, there is still a highly linear relationship between the relative error resulting from the Lin (2007)'s approximation and the SCV ratio, as shown in Fig. 6.6. The regression equation of the relative error resulting from the Lin (2007)'s approximation on the SCV ratio is

$$RE = -4.954 + 13.082\text{SCV} + \epsilon \quad R^2 = 98.6\% \tag{6.23}$$

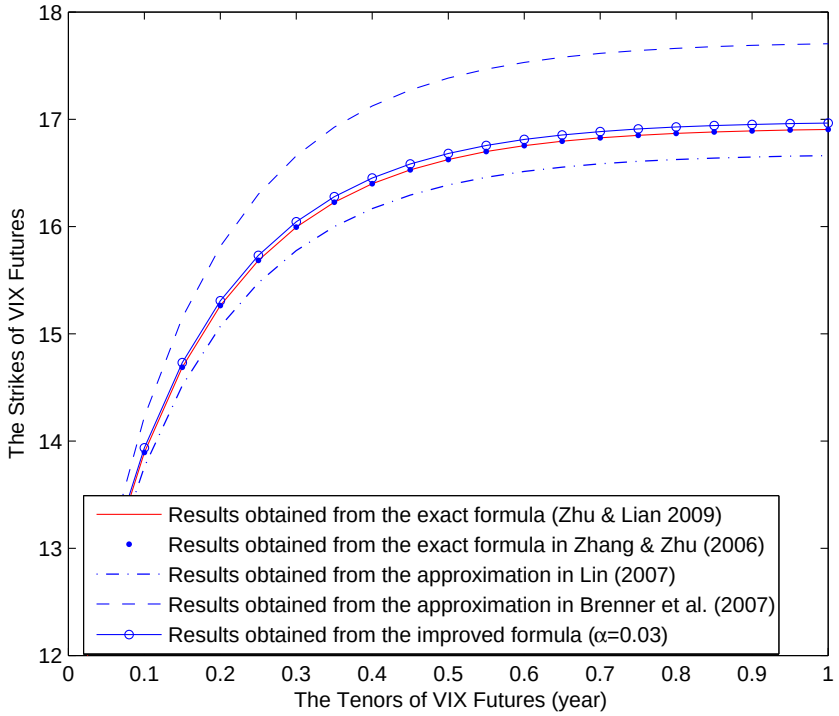

Figure 6.7: A comparison of VIX futures strikes obtained from the exact formula and the second-order and the third-order approximations in the Heston model

The very high value of the coefficient of determination, R^2, again verifies that the SCV ratio is informative in identifying relative errors of the approximation.

Before leaving this section, we want to point out again that the third-order approximation performs even worse in pricing VIX futures. Brenner et al. (2007) explored the third-order approximation already by carrying out the Taylor expansion of the square root function to the third order and obtained an approximation formula for VIX futures, again based on the Heston stochastic volatility model. Plotted in Figure 6.7 displays the comparisons of the results obtained from the exact formula in the special case of SV model, the exact formula presented by Zhang & Zhu (2006), the approximation formula presented by Lin (2007) and the approximation formula presented by Brenner et al. (2007), respectively, with parameters being those presented in Brenner et al. (2007)*. The figure shows

*In this figure, we use the parameters presented in the empirical studies of Brenner et al. (2007) for comparison purpose, i.e., $\kappa = 5.5805$, $\theta = 0.03259$, $\sigma_V = 0.5885$, and $\sqrt{V_0} = 8.7\%$. Since Brenner et al. (2007)'s approximation is obtained based on the Heston stochastic volatility

that the Lin (2007)'s approximation formula always undervalues VIX futures and performs poorly with non-trivial relative pricing errors. For example, for a one-year VIX future, our exact solution produces a value of 16.90 while the second-order CCA results in a value of 16.66, exhibiting a relative difference of -1.8%, which is still quite large and unacceptable for market traders. The Brenner et al. (2007)'s third-order approximation formula works even worse than Lin (2007)'s second-order approximation formula, as can be clearly observed in Figure 6.7. For example, Brenner et al. (2007)'s third-order approximation results in a value of 17.70 for a one-year VIX future, representing a relative difference of 4.73%. The third-order approximation formula has not only reversed the under-pricing characteristics of the second-order approximation formula, but also resulted in more significant over-pricing errors than the second-order approximation. Finally, this figure also shows that our proposed improved approximation can once again substantially reduce the relative errors.

6.4 Conclusion

In this chapter, we have examined the accuracy of the well-known CCA as an approximation to price volatility swaps and VIX futures. In the first part, we point out the reason why the CCA is sometimes inaccurate, and we have demonstrated the validity condition for the application of CCA. Our research shows that the excess probability is vitally important for the application of convexity correction approximation. We then propose a useful ratio to identify the relative errors. With the application of this ratio, we suggest a new approximation to greatly improve the accuracy of the CCA. In the second part, we illustrate our theoretical analysis through three numerical examples: the pricing of volatility swaps in the Heston model; the pricing of volatility swaps in the GARCH model; and the pricing of VIX futures in the SVJJ model. Our study reveals

model, without jump diffusions being considered, other jump diffusion relevant parameters (λ, μ_S, σ_S, μ_V, ρ_J) are set to be zero.

that there are surprisingly large differences in accuracy among the Brockhaus & Long (2000)'s second-order convexity correction approximation in some specific parameters, hence we alert that one should be aware of the inaccuracy of this approximation and be very careful in using it. Furthermore, we have demonstrated that further inclusion of higher order terms into the second-order approximation will normally result in an even worse accuracy in approximating square root function involved with stochastic variables. On the other hand, we recommend that if one aims to reduce the error resulted from adopting a second-order convexity correction approximation when driving a closed-form analytical solution become futile, one is far better off to adopt the newly proposed improved formula than to take higher-order terms in the convexity correction approximation into the consideration, as these the inclusion of these higher-order terms are not necessarily reduce relative errors in most of the cases, as demonstrated in this chapter already.

Chapter 7

Pricing VIX Futures

7.1 Introduction

In the previous chapters, we have demonstrated the approach of analytically pricing variance and volatility swaps, which are historical variance- and volatility-based volatility derivatives. In this chapter, we will address the pricing problem of another important implied-volatility based products - the VIX futures traded in the CBOE. This chapter derives a closed-form exact solution for the fair value of VIX futures under a stochastic volatility model with simultaneous jumps in the asset price and volatility processes. The derivation of this formula for VIX futures with a very general dynamics of VIX represents a substantial progress in identifying and developing more realistic VIX futures models and pricing formulae. With the newly-found pricing formula available, we then conduct empirical studies to examine the performance of four different stochastic volatility models with or without jumps. More importantly, using the Markov chain Monte Carlo (MCMC) method to analyze a set of coupled VIX and S&P500 data, we demonstrate how to estimate model parameters. Our empirical studies show that the Heston stochastic volatility model can well capture the dynamics of S&P500 already and is a good candidate for the pricing of VIX futures. Incorporating jumps into the underlying price can indeed further improve the pricing the VIX futures.

However, jumps added in the volatility process appear to add little improvement for pricing VIX futures.

Since its introduction in 1993 by CBOE (Chicago Board Options Exchange), Volatility Index (VIX) has been considered to be the world's benchmark for stock market volatility. In September 2003, CBOE switched to a new definition of VIX, which is based on a model-free formula and computed from a portfolio of 30-calendar-day out-of-money options written on S&P500 (SPX). This new definition reflects the market's expectation of the 30-day forward S&P500 index volatility and serves as a proxy for investor sentiment, rising when investors are anxious or uncertain about the market and falling during times of confidence. This VIX index, often referred to as the "investor fear gauge", is therefore closely monitored by active traders, financial analysts as well as the media for insight into the financial market.

The introduction of VIX has laid a good foundation for constructing tradable volatility products and thus facilitating the hedging against volatility risk and speculating in volatility derivatives. For instance, on March 26, 2004, the CBOE launched a new exchange, the CBOE Futures Exchange (CFE) to start trading VIX futures, which is a type of new futures written on the new definition of VIX. On February 24, 2006, CBOE started the trading of VIX options to enlarge the family of volatility derivatives. Since its inception, the VIX futures market has been rapidly growing. For example, according to the data on the CBOE website, while the actual trading volume was 332 on February 28, 2005, corresponding to US$4 millions, the open interest of VIX futures reached 9240, which corresponds to a market value of US$112 millions. Being warmly welcome by the financial market, these volatility derivatives were awarded the most innovative index derivative products*. In fact, "few proposed types of derivatives securities have attached as much attention and interest as futures and options contracts on volatility" (Grunbichler & Longstaff 1996).

*http://www.cboe.com/AboutCBOE/ShowDocument.aspx?DIR=ACNews&FILE=20061205.doc

Given the growing popularity of trading VIX futures, considerable research interests have been drawn to the development of appropriate pricing models for VIX futures, as discussed in Chapter 1. However, it is worth noting that although the rapid development of those models has indeed greatly enriched the literature in the area of pricing VIX futures, some limitations and weaknesses remain and hence further studies are required. For example, Zhang & Zhu (2006) only considered the model with stochastic volatility characteristic, without paying attention to the importance of possible jumps associated with the underlying S&P500. Besides, their empirical studies show that there is an easily identifiable gap between the prices produced by their model and those observed from the market, indicating that their model and calibration approach may need to be further improved. Grunbichler & Longstaff (1996) and Psychoyios et al. (2007) assumed that the VIX spot evolves independently from the actual evolution of S&P500. As a result, they might have mis-specified the VIX futures and VIX options, especially the volatility-of-volatility risk, as illustrated by Sepp (2008*b*). Sepp (2008*b*) himself considered a model with a jump component in the dynamics that the variance follows, without paying attention to jumps in return distribution of the underlying. Furthermore, the pricing formula in his paper, which involves complex-value integration and recursive computation, seems to be too complicate to be used in terms of price calculation and model calibration. As for the studies by Lin (2007), although both jumps in asset price and variance process have been taken into consideration, a major problem is that their VIX futures pricing formula is based on the convex adjustment approximation, which is not justifiable for models with stochastic volatility or jumps, as shall be shown later.

With the steadily increasing demand of trading VIX futures, there is an apparent need to conduct further studies for the theoretical as well as practical purpose. Hence this study is well motivated. In particular, we will complete three main tasks in this chapter: (1) deriving an efficient exact pricing formula

for VIX futures under a general framework of stochastic models with jumps being incorporated in both the underlying and the variance; (2) analyzing the accuracy of the well-known convexity correction approximation in pricing VIX futures; and (3) estimating corresponding model parameters from joint VIX and S&P500 market data and empirically examining the performance of alternative models in terms of pricing VIX futures.

We firstly develop a closed-form and exact pricing formula to evaluate the VIX futures in a general framework that allows for stochastic volatility, random jumps in return distribution, and random jumps in variance process. Such a model will be refereed to as the SVJJ model hereafter. This specification is general enough to cover most of the already-known alternative models as its special cases, including (i) the Heston's (1993) stochastic volatility (SV) model, (ii) the stochastic volatility with jumps in asset return (SVJ) model, (iii) the stochastic volatility model with jumps in variance process (SVVJ) model, and (iv) the stochastic volatility, random jumps in both return distribution and variance process (SVJJ) model. The Heston (1993)'s SV model has the advantage of non-negative variance, easily capturing volatility smile as well as the mean-reverting feature observed in options market. Bates (1996) and Bakshi et al. (1997) extended the SV model to the SVJ model, which was found to be extremely useful in improving the performance of pricing short-term options. However, researchers found strong evidence for model misspecification in the SVJ model framework, and hence called for further extension models, such as adding jumps in the variance process. The further inclusion of jumps in the variance process leads to the so-called SVVJ model and the SVJJ model (e.g., Duffie et al. 2000; Pan 2002; Eraker 2004).

The former three models are special cases of the SVJJ. Consequently, we concentrate our effort on the SVJJ model when obtaining the analytical formula for VIX futures. As shall be shown later, the characteristic function is found by analytically solving the associated governing partial integro-differential equation (PIDE) in the SVJJ model. Contrast to the research conducted by Lin (2007),

who proposed an approximate formula for VIX futures in the general SVJJ model using the convexity adjustment approximation, we found an exact formula for VIX futures by inverting the characteristic function. With this newly-found formula, the numerical computation for the price of a VIX futures contract can be efficiently carried out. The numerical comparison shows that the results from our exact solution perfectly match with the those obtained from the Monte Carlo simulations. Further comparison indicates there is nontrivial differences between our results and those from the Lin (2007)'s approximation solution. We also find that the three-order approximation proposed in Brenner et al. (2007) performs even worse in the SVJJ model than Lin (2007)'s approximation formula. Naturally, the advantage of using our exact solution over Lin (2007)'s approximation is clearly demonstrated.

With the pricing formulae now available for the four models (SV, SVJ, SVVJ and SVJJ), one of the natural questions is naturally raised; which one of them is the most suitable in terms of pricing VIX futures in practice, i.e., which one results in the lowest pricing errors. Although there exist many studies in the literature discussing the effects of model specification in pricing and hedging options (e.g., Bakshi et al. 1997; Pan 2002; Eraker 2004; Broadie et al. 2007), there are very few papers empirically examining the model specification in pricing and hedging VIX futures. Lin (2007) first conducted the empirical studies to investigate the pricing and hedging performances of several dynamics specification for VIX and VIX futures, and found that the SVJ model outperformed for the short-dated futures, and adding volatility jumps (SVJJ model) can overall enhance hedging performance.

In this chapter, we re-examine the effects of adding jumps in underlying and the volatility processes, taking advantage of our newly-developed exact pricing formula. Using the joint time series data of S&P500 and VIX, we demonstrate the determination of model parameters with the MCMC approach. With these parameters extracted from the market data, we then empirically examined the

pricing performance of four models (SV, SVJ, SVVJ and SVJJ). Our empirical studies show that the Heston stochastic volatility model (SV) can well capture the dynamics of S&P500 already and is a good candidate for the pricing of VIX futures. Incorporating jumps into the underlying price can further improve the pricing the VIX futures. However, jumps added in the volatility process appear to be of little improvement in pricing VIX futures.

This chapter is organized into four sections. In Section 7.2, a detailed description of the general S&P500 specification is first provided, followed by the explanation of VIX definition and then our analytical formula for VIX futures. Some numerical examples are also provided to illustrate some fundamental properties of our newly-derived VIX futures pricing formula. In Section 7.3, we show how model parameters can be determined from the coupled data of S&P500 and VIX, using the MCMC approach. The pricing performance of each of these alternative models is empirically examined and compared, taking advantaging of our explicit VIX futures pricing formula. In Section 7.4, a brief summary is provided.

7.2 VIX Futures Models

The purpose of this section is to derive a closed-form formula for VIX futures, in the framework of stochastic volatility with jump-diffusion characteristics observed in the voluminous time-series literature. This general pricing framework includes all those models (SV, SVJ and SVVJ) to be studied in the empirical studies as special cases. For the purpose of verifying the newly-developed formula, some comparisons with the Monte Carlo simulation are presented. A closed-form exact solution in such a general framework also enables us to closely scrutinize the accuracy of some approximation formulae in the literature (e.g., Lin 2007; Brenner et al. 2007).

7.2.1 Volatility Index

The VIX was introduced in 1993 by the Chicago Board Options Exchange (CBOE) and switched to a new methodology in 2003. The new VIX is calculated in a model-free manner as a weighted sum of out-of-money option prices across all available strikes on the S&P500 index. As detailed in the CBOE white paper[†], the new VIX, which is the underlying of VIX futures and options, is defined by means of VIX_t^2,

$$\text{VIX}_t^2 = (\frac{2}{\overline{\tau}} \sum_i \frac{\Delta K_i}{K_i^2} e^{r\overline{\tau}} Q(K_i) - \frac{1}{\overline{\tau}} [\frac{F}{K_0} - 1]^2) \times 100 \tag{7.1}$$

where $\overline{\tau} = \frac{30}{365}$, K_i is the strike price of the i-th out-of-the-money option in the calculation, F is the time-t forward index level, $Q(K_i)$ denotes the time-t midquote price of the out-of-the-money option at strike K_i, K_0 is the first strike below the forward index level, r denotes the time-t risk-free rate with maturity $\overline{\tau}$.

For a better understanding of the financial interpretation, this expression of the VIX squared can be presented in terms of the risk-neutral expectation of the log contract, with mathematical simplification (see Lin 2007; Duan & Yeh 2007 for more details),

$$\text{VIX}_t^2 = -\frac{2}{\overline{\tau}} E^{\mathbb{Q}}[\ln(\frac{S_{t+\overline{\tau}}}{F}) | \mathscr{F}_t] \times 100^2 \tag{7.2}$$

where $\mathbb{Q}$ is the risk-neutral probability measure, $F = S_t e^{r\overline{\tau}}$ denotes the 30-day forward price of the underlying S&P500 with a risk-free interest rate r under the risk-neutral probability, and $\mathscr{F}_t$ is the filtration up to time t. Under the assumption that the S&P500 index does not jump, Carr & Wu (2006) have further shown that the VIX squared is just the conditional risk-neutral expectation of the annualized realized variance of the S&P500 return over the next 30 calendar days, which means VIX squared can be viewed as an approximation of the one-month

[†]see the white paper of VIX, available at http://www.cboe.com/micro/vix/vixwhite.pdf

variance swap rate up to discretization error (from using a finite number of options in the VIX definition). However, when jumps are taken into consideration, the VIX squared differs from the one-month realized variance of underlying S&P500. Broadie & Jain (2008*b*) analyzed the difference between the VIX squared and the one-month continuous realized variance and concluded that the mean value of the jump size strongly influences whether the portfolio of options (or the VIX index) under- or over-approximates the realized variance.

It is worth noting that although the construction of VIX squared is model-free, which can be replicated by a portfolio of out-of-money options written on S&P500 as illustrated in Equation (7.1), the VIX itself cannot be replicated by a portfolio of options. The main difficulty in replicating the VIX is that the computation of the VIX involves a square root operator against the price of the portfolio of options. Carr & Wu (2006) pointed that although the at-the-money implied volatility is a good approximation of the volatility swap rate, the payoff on a volatility swap (which is essentially the VIX) is notoriously difficult to replicate. As a result, any pricing formula for the fair value of VIX futures should be model-dependent. This kind of issues about the pricing of VIX derivatives has been shown in literature (e.g., Lin 2007; Sepp 2008*b*). With the purpose of obtaining the exact pricing formula in a general framework, we demonstrate our general model and pricing approach in the following two sections.

7.2.2 Affine Model Specification

Due to the fact that the VIX_t^2 is the risk-neutral expectation of the log contract of S&P500, one natural method to model the VIX_t^2 and VIX_t is to model the dynamics of S&P500. In literature, there have been elaborate efforts by researchers to build models that admit the "volatility smile" in the implied volatility extracted from the cross-section option prices, or "fat tails" in return distributions. Previous research has mainly focused on two approaches: (1) developing stochas-

tic volatility models that allow for "leverage effect", and (2) developing models that incorporate discontinuous jumps in the asset price or the stochastic volatility process.

Our general analysis model in this chapter incorporates stochastic volatility characteristic and simultaneous jumps in asset price and volatility process. This general model was initially proposed by Duffie et al. (2000). Under the physical probability measure $\mathbb{P}$, the S&P500 index, denoted by S_t, is assumed to follow

$$\begin{cases} dS_t = S_t(r_t + \gamma_t)dt + S_t\sqrt{V_t}dW_t^S + d\left(\sum_{n=1}^{N_t} S_{\tau_{n-}}[e^{Z_n^S} - 1]\right) - S_t\overline{\mu}\lambda dt \\ dV_t = \kappa(\theta - V_t)dt + \sigma_V\sqrt{V_t}dW_t^V + d\left(\sum_{n=1}^{N_t} Z_n^V\right) \end{cases} \tag{7.3}$$

where:

r_t *is the constant spot interest rate;*

V *is the diffusion component of the variance of the underlying asset dynamics (conditional on no jumps occurring);*

dW_t^S *and* dW_t^V *are two standard Brownian motions correlated with* $E[dW_t^S, dW_t^V] = \rho dt$*;*

κ, θ *and* σ_V *are respectively the mean-reverting speed parameter, long-term mean, and variance coefficient of the diffusion* V_t*;*

N_t *is the independent Poisson process with intensity* λ*, that is,* $Pr\{N_{t+dt} - N_t = 1\} = \lambda dt$ *and* $Pr\{N_{t+dt} - N_t = 0\} = 1 - \lambda dt$*. The jumps happen simultaneously in underlying dynamics* S_t *and variance process* V_t*;*

The jump sizes are assumed to be $Z_n^V \sim exp(\mu_V)$*, and* $Z_n^S | Z_n^V \sim N(\mu_S + \rho_J Z_n^V, \sigma_S^2)$*;*

$\overline{\mu} = e^{\mu_S + \frac{1}{2}\sigma_S^2}/(1 - \rho_J\mu_V) - 1$ *is the risk premium of the jump term in the process to compensate the jump component, and* γ_t *is the total equity premium.*

One of the reasons that we spent some efforts to study this rather general model and obtain a closed-form solution is that this general model, combining

both the stochastic volatility and jump diffusions characteristics, takes the following four models as special cases according to the specification of jump components in Equation (7.3).

The first special case is that Eq. (7.3) reduces to the Heston (1993) stochastic volatility (SV) model, when the jumps are set to zero, i.e., $\lambda = 0$, $Z_t^S = Z_t^V = 0$. The Heston (1993) model contains some very unique features and has become a widely accepted model in option pricing theory. For example, the parameter σ_V is commonly referred to as the "volatility-of-volatility". Higher value of σ_V results in "fatter tails" in the return distribution of the underlying. The correlation parameter, ρ, is typically found to be negative, implying that a fall in price of the underlying usually will be accompanied by an increase in volatility. This effect is sometimes referred to as the "leverage effect" (Black 1976).

The stochastic volatility with jumps (SVJ) in return model is an extension to the SV model that allows random jumps to occur in the underlying prices. Again, the SVJ model can be regarded as a special case of the general dynamics Equation (7.3) with $Z_t^V = 0$ and Z^S being a jump size process, typically specified to be normal distribution as $Z_t^S \sim N(\mu_S, \sigma_S^2)$. This extension can be easily justified as it reflects an important assumption that the discrete and unexpected arrival of new information has resulted in an instantaneous revision of underlying prices. Adding the jump component in the stock returns distribution should improve fitness of model to the observed stock return in financial market, since the jump component adds mass to the tails of the returns distribution. Bakshi et al. (1997) used this model with stochastic interest rate as their general specification to test the source of model misspecification in the option pricing and found that adding jump feature to the SV model can greatly improve the performance in pricing and hedging options, especially in pricing short-term options.

The third special case of the Equation (7.3) is the case in which jumps are allowed appearing in the variance, V_t, process but no jumps in the underlying prices. Differing from the SVJ model, this special case, abbreviated by SVVJ

hereafter, is nested by setting $Z_t^S = 0$ and $Z_t^V \sim exp(\mu_V)$. Recently, this specification was employed by Sepp (2008*b*) to price the VIX options as well as VIX futures. One of the motivations of this model is the added volatility jump component will typically add right skewness in the distribution of volatility, and hence, overall fatten the tails of the returns distribution. Another motivation is that model with jumps in volatility may generate the level of skewness implied by the volatility observed smirk in market data, as illustrated in Bakshi et al. (1997) and Bates (1996).

Finally, the general model (Eq. (7.3)) itself is a combination of SVJ and SVVJ model, which is labeled stochastic volatility with simultaneous jumps in underlying and volatility processes (SVJJ). In this model, jumps in volatility and prices are driven by the same Poisson process, meaning that price jumps will simultaneously impact both prices and volatility. The jump arrival intensity λ is assumed to be constant in this chapter. This assumption is supported by Chernov et al. (2003) and Andersen et al. (2002), who found that there is no evidence for a time-varying intensity based on their time-series-based analyses. Bates (2000) also found that strong evidence for misspecification in models with state-dependent intensities. This assumption of constant jump arrival intensity has also been adopted by Broadie et al. (2007). While we have assumed that jumps would occur at the same time, the jump sizes do not have to be the same; jump sizes are assumed to be $Z_n^V \sim \exp(\mu_V)$, and $Z_n^S | Z_n^V \sim N(\mu_S + \rho_J Z_n^V, \sigma_S^2)$ with correlation ρ_J. One may note that in the SVJ model only price movements resulting from Brownian shocks will have an impact on volatility while price moves stemming from jumps have no impact on volatility. By introducing simultaneous jumps in both returns and volatility processes, this shortcoming is corrected in the general model, which is one of the major advantages over the SVJ mode. Hence, this specification has received considerable attention in recent literature. For examples, Duffie et al. (2000), Pan (2002), Eraker et al. (2003), Eraker (2004), Lin (2007), Broadie et al. (2007) employed this model in their theoretical or

empirical analysis. Pan (2002) argued that the addition of a volatility jump component might explain her findings of a severely pronounced increase in the volatility smile for short maturity, far in the money put options. It is also possible for skewness to be added into the conditional returns distribution through the parameter ρ_J in the SVJJ model. The term $\lambda\overline{\mu} = E^{\mathbb{P}}[Z_t^S dN]$ compensates the jump component in return.

In the literature of pricing VIX futures, Zhang & Zhu (2006) and Brenner et al. (2007) used the SV model. Zhu & Zhang (2007) employed the SV model with parameters assumed to be time-varying. Sepp (2008*b*)'s study is based on the SVVJ model. Lin (2007) and Sepp (2008*a*) respectively used the general SVJJ model to study the VIX futures and VIX options[‡]. In this study, we managed to obtain a closed-form and exact pricing formula for VIX futures based on the general SVJJ model and this solution is presented in the next section. The pricing performance of the various SV, SVJ, SVVJ and SVJJ models in pricing VIX futures will be demonstrated in the empirical study section.

7.2.3 Pricing VIX Futures

In this section, we discuss our analytical solution approach for the determination of the fair price of a VIX future contract. As we shall show later, the associated PDE is analytically solved and an explicit closed-form solution is obtained.

We firstly present the dynamics processes of the S&P500 index and its variance under the risk-neutral probability measure $\mathbb{Q}$, following the standard analysis in

[‡]In Zhu & Zhang (2007) and Sepp (2008*b*), some volatility structural parameters are assumed to be time-varying. In Lin (2007), the jump density λ is assumed to be a linear specification $\lambda_0 + \lambda_1 V_t$, for some nonnegative constants λ_0 and λ_1.

literature (i.e., Duffie et al. 2000; Pan 2002; Eraker 2004; Broadie et al. 2007),

$$\begin{cases} dS_t = S_t r_t dt + S_t\sqrt{V_t}dW_t^S(\mathbb{Q}) + d\left(\sum_{n=1}^{N_t(\mathbb{Q})} S_{\tau_{n-}}[e^{Z_n^S(\mathbb{Q})}-1]\right) - S_t\overline{\mu}^{\mathbb{Q}}\lambda dt \\ dV_t = \kappa^{\mathbb{Q}}(\theta^{\mathbb{Q}} - V_t)dt + \sigma_V\sqrt{V_t}dW_t^V(\mathbb{Q}) + d\left(\sum_{n=1}^{N_t(\mathbb{Q})} Z_n^V(\mathbb{Q})\right) \end{cases} \tag{7.4}$$

where $\overline{\mu}^{\mathbb{Q}} = e^{\mu_S^{\mathbb{Q}}+\frac{1}{2}\sigma_S^2}/(1-\rho_J\mu_V) - 1$ and $\mu_S^{\mathbb{Q}}$ is the corresponding risk-neutral parameter of μ_S. Consistent with the specification considered in Pan (2002) or Eraker (2004), the risk premium parameters in our study are specified as: diffusive volatility risk premium $\eta_V = \kappa^{\mathbb{Q}} - \kappa$ and jump risk premium $\eta_J = \mu_S^{\mathbb{Q}} - \mu_S$. One can notice that the σ_V, ρ, $\kappa\theta$, λ and other jumps parameters are the same under both the physical probability measure $\mathbb{P}$ and the risk-neutral probability measure $\mathbb{Q}$. The specification for diffusive volatility risk premium η_V is standard in literature, whereas there are various ways of specifying the measure changes (jump risk premium) for the jump processes. Broadie et al. (2007) considered a more general specification for the measure changes for the jump processes by allowing the jump intensity and all the jump parameters to change across measures $\mathbb{P}$ and $\mathbb{Q}$.

As shown in Eq. (7.2), VIX squared is virtually just the conditional risk-neutral expectation of the log contract of the S&P500 over the next 30 calendar days. Under the general specification Eq. (7.4), this expectation can be carried out explicitly in the form of,

$$\text{VIX}_t^2 = (aV_t + b) \times 100^2 \tag{7.5}$$

where

$$\begin{cases} a = \dfrac{1-e^{-\kappa^{\mathbb{Q}}\overline{\tau}}}{\kappa^{\mathbb{Q}}\overline{\tau}}, \quad and \quad \overline{\tau} = 30/365 \\ b = (\theta^{\mathbb{Q}} + \dfrac{\lambda\mu_V}{\kappa^{\mathbb{Q}}})(1-a) + \lambda c \\ c = 2[\overline{\mu}^{\mathbb{Q}} - (\mu_S^{\mathbb{Q}} + \rho_J\mu_V)] \end{cases}$$

as shown in Lin (2007), Broadie & Jain (2007) and Duan & Yeh (2007).

The VIX squared is thus a linear function of the instantaneous variance, V_t. One can take advantage of this linear relationship to calculate the instantaneous variance, V_t, of the S&P500, once the VIX value is given. This has considerably facilitated the calculation involved in our newly-developed formula which explicitly relates the price of a VIX future with the instantaneous variance, V_t. It can be clearly observed that the VIX is lower bounded by $\sqrt{b}$. This theoretical positive lower bound can easily explain the fact that the lowest value of actual VIX value data in CBOE was 9.31 volatility points during January 2nd, 1990 to August 29th, 2008[§]. Differing from our VIX process modeling approach by starting from the process of S&P500 and deriving the VIX according its definition Eq. (7.1), Grunbichler & Longstaff (1996) and Psychoyios et al. (2007) directly modeled VIX process by a mean-reverting squared-root process, assuming the VIX spot evolves separating from the actual evolution of S&P500. As a result, the VIX value in the their models may theoretically approach to zero or even negative, which is inconsistent with the VIX value data. In this perspective, our approach by taking advantage of the relationship to model VIX process seems to be more realistic in describing the VIX process.

Carr & Wu (2006) illustrated that under the assumption of no-arbitrage and continuous marking to market, the VIX futures price, $F(t,T)$, is a martingale under the risk-neutral probability measure $\mathbb{Q}$. Lin (2007) and Zhang & Zhu (2006) also concluded that the futures price is a martingale. Hence the futures price is

$$F(t,T) = E^Q[\text{VIX}_T|\mathscr{F}_t] = E^Q[\sqrt{aV_T+b}|\mathscr{F}_t] \times 100 \tag{7.6}$$

where $F(t,T)$ is the value of the VIX futures at time t with settlement at time T.

A couple of more points should be remarked before proceeding to present our exact and closed-form solution for the VIX futures.

[§]9.31 is the lowest daily closing value of VIX. The lowest intraday value of VIX is 8.63

Firstly, one may wonder now that if the VIX squared can be replicated by a portfolio of options, why a similar approach cannot be adopted to construct a portfolio of the S&P500 options to replicating the VIX itself and hence pricing the VIX futures with the value of the portfolio. This is actually due to the fact that VIX, which is the underlying value of the VIX futures, involves a square root operator against the price of the portfolio of options. The nonlinear nature of the square root operation has prevented the replication approach being applied to construct a portfolio of options for the VIX. Some researchers have tried and concluded that the construction of a portfolio of options for the VIX is extremely difficult. For example, Carr & Wu (2006) pointed out the replicate strategy for the square-root nonlinear function is "notoriously difficult to construct". Lin (2007) also illustrated that one cannot use the replicate approach for VIX futures and as a result the pricing formula for the fair value of VIX futures should be model-dependent.

Secondly, due to the fact that VIX is not a tradable asset, there is no cost-of-carry relationship between VIX futures and their underlying value, VIX, as that of the stock futures and underlying stock price. As a result, the drift term of the VIX process under the risk neutral probability measure is not the risk-free interest rate any more, and the classic stock futures pricing approach is inappropriate to price the VIX futures. In other words, the price of a VIX future contract (at time t with settlement at time T), $F(t,T)$, cannot be simply calculated by $F(t,T) = \text{VIX}_t e^{r(T-t)}$, a formula that would be normally used to calculate the price of a future contract written on stocks.

In order to find a closed-form formula for the exact price of a VIX future contract, we must proceed to carry out the expectation in Eq. (7.6) by explicitly working out the conditional probability density function $p^{\mathbb{Q}}(V_T|V_t)$. With the instantaneous variance following the stochastic differential equation (SDE) in Eq. (7.4), the corresponding risk-neutral probability density function can be determined by inverting the associated characteristic function.

We consider the moment generating function, $f(\phi; t, \tau, V_t)$, of the stochastic variable V_T, conditional on the filtration $\mathscr{F}_t$, with time to expiration $\tau = T - t$.

$$f(\phi; t, \tau, V_t) = E^{\mathbb{Q}}[e^{\phi V_T} | \mathscr{F}_t] \tag{7.7}$$

Accordingly, the characteristic function is just $f(\phi i; t, \tau, V_t)$. The moment generating function can be interpreted as a contingent claim whose payoff at expiry T is $e^{\phi V_T}$ with interest rate being 0. Feynman-Kac theorem implies that $f(\phi, \tau)$ must satisfy the following backward PIDE

$$\begin{cases} -f_\tau + \kappa^{\mathbb{Q}}(\theta^{\mathbb{Q}} - V) f_V + \frac{1}{2}\sigma^2 V f_{VV} + \lambda E^{\mathbb{Q}}[f(V + Z^V) - f(V) | \mathscr{F}_t] = 0 \\ f(\phi; t + \tau, 0, V) = e^{\phi V} \end{cases}$$

Following the solution procedure in literature (Heston (1993), Duffie et al. (2000), and among many others), the moment generating function for the variance process V_t in Eq. (7.4) has the following exponential affine form : $f(\phi; t, \tau, V_t) = e^{C(\phi,\tau) + D(\phi,\tau)V_t + A(\phi,\tau)}$, under some technical regularity conditions. The coefficients $C(\phi, \tau)$, $D(\phi, \tau)$, and $A(\phi, \tau)$ are obtained through solving a set of ordinary differential equations (ODE). In Appendix E, it is shown that the solution of PIDE (7.8) is

$$f(\phi; t, \tau, V_t) = e^{C(\phi,\tau) + D(\phi,\tau)V_t + A(\phi,\tau)} \tag{7.8}$$

where

$$\begin{cases} A(\phi, \tau) = \dfrac{2\mu_V \lambda}{2\mu_V \kappa^{\mathbb{Q}} - \sigma_V^2} \ln\left(1 + \dfrac{\phi(\sigma_V^2 - 2\mu_V \kappa^{\mathbb{Q}})}{2\kappa^{\mathbb{Q}}(1 - \mu_V \phi)}(e^{-\kappa^{\mathbb{Q}}\tau} - 1)\right) \\ C(\phi, \tau) = \dfrac{-2\kappa\theta}{\sigma_V^2} \ln\left(1 + \dfrac{\sigma_V^2 \phi}{2\kappa^{\mathbb{Q}}}(e^{-\kappa^{\mathbb{Q}}\tau} - 1)\right) \\ D(\phi, \tau) = \frac{2\kappa^{\mathbb{Q}}\phi}{\sigma_V^2 \phi + (2\kappa^{\mathbb{Q}} - \sigma_V^2 \phi)e^{\kappa^{\mathbb{Q}}\tau}} \end{cases}$$

Assuming we stand at time t, the Fourier inversion of the characteristic func-

tion $f(\phi i; t, \tau, V_t)$ provides the required conditional density function $p^{\mathbb{Q}}(V_T|V_t)$

$$p^{\mathbb{Q}}(V_T|V_t) = \frac{1}{\pi}\int_0^\infty Re[e^{-i\phi V_T} f(i\phi; t, \tau, V_t)]d\phi \tag{7.9}$$

The price of a VIX future contract at time t is thus expressed in the form of

$$F(t,T) = E^{\mathbb{Q}}[\text{VIX}_T|\mathscr{F}_t] = \int_0^\infty p^{\mathbb{Q}}(V_T|V_t)\sqrt{aV_T + b}dV_T \times 100 \tag{7.10}$$

Under the Heston stochastic volatility framework, which is a special case covered by the general dynamics presented in this chapter, Zhu & Zhang (2007) proposed their VIX futures pricing formula. Similarly in form to the pricing formula shown in Eq. (7.10), their pricing formula for the VIX futures is expressed in the form of a two-dimensional integral with one of the integrands being the complex function, i.e., the Fourier inverse transform, in order to obtain the probability density function from the characteristic function. We initially decided to leave our final solution in this two-dimensional integral form too. However, after a careful examination of the properties of the integrand, we realized that the integration could be further simplified, by utilizing a mathematical identity to avoid the complicate Fourier inverse transform and obtain a closed-form solution as our final solution for the price of VIX futures. This had significantly simplified the calculation time and made it possible for us to adopt our formula in the empirical analyses later.

Schürger (2002) has shown that, after interchanging the expectation and integral using Fubini's theorem, the expectation of square root function can be expressed as,

$$E[\sqrt{x}] = \frac{1}{2\sqrt{\pi}}\int_0^\infty \frac{1 - E[e^{-sx}]}{s^{\frac{3}{2}}}ds \tag{7.11}$$

Invoking this identity, Formula (7.10) can be simplified as

$$F(t,T;\Phi) = \frac{1}{2\sqrt{\pi}}\int_0^\infty \frac{1 - e^{-sb} f(-sa; t, \tau, V_t)}{s^{\frac{3}{2}}}ds \tag{7.12}$$

where $f(\phi; t, \tau, V_t)$ is the moment generating function shown in Eq. (7.8). Noticeably, this pricing formula for VIX futures under the general SVJJ model has a parameter vector $\Phi = \{\kappa, \theta, \sigma_V, \eta_V, \lambda, \mu_S, \sigma_S, \mu_V, \rho_J, \eta_V\}$. And the price of VIX futures in Eq. (7.12) is a function of the instantaneous variance, V_t, which can be easily calculated from a given VIX value through Eq. (7.5). Therefore, as a conclusion of our formula derivation, we have successfully obtained a one-to-one function between the VIX futures price and the VIX itself, as stated in the following proposition.

Proposition 5 *If S&P500 index follows the general dynamics given by Equation (7.4), the conditional probability density function of VIX_T, denoted by $p^{\mathbb{Q}}(VIX_T|VIX_t)$, is given by*

$$p^{\mathbb{Q}}(VIX_T|VIX_t) = \frac{2VIX_T}{a\pi}\int_0^\infty Re\left[e^{-i\phi(\frac{VIX_T^2-b}{a})}f\left(i\phi; t, \tau, (VIX_t^2-b)/a\right)\right]d\phi \tag{7.13}$$

and the price of a VIX future at time t with maturity T is given by the following formula:

$$F(t, T, VIX_t) = \frac{1}{2\sqrt{\pi}}\int_0^\infty \frac{1-e^{-sb}f(-sa; t, \tau, \frac{VIX_t^2-b}{a})}{\sqrt{s}^3}ds \tag{7.14}$$

where $f(\phi; t, \tau, V_t)$ is the moment generating function of the stochastic variable V_T, and given by Eq. (7.8).

The pricing formula has several distinctive features. Firstly, this pricing formula is so far more general than any closed-form exact solutions reported in the literature; it applies to economies with stochastic volatility, jump risk in the price process, and jump in the variance process, taking the existing SV, SVJ and SVVJ models as special cases. For example, the pricing formula proposed by Zhang & Zhu (2006) in the Heston model (SV) can be obtained by setting $\lambda = 0$ in Eq. (7.10). This has considerably reduced the effort of deriving closed-form pricing formulae for the SV, SVJ, SVVJ and SVJJ models individually. With this most

general formula in hands, all one needs is to decide what model would be the most appropriate dynamics to describe the underlying first and then take the appropriate special case as needed.

Secondly, the pricing formula (7.12) for VIX futures involves only one dimensional integral with its integrand being a well-defined and smooth real function, since it has completely avoided numerically performing the complex-valued Fourier inverse transform. However, Zhu & Zhang (2007) had left their final VIX futures pricing formula in the form of two-dimensional integral without being able to carry out the complex-valued Fourier inverse transform. Although the parameters in their discussions were assumed to be time-varying in the framework of the Heston SV model, we find that our approach presented in this chapter can also be applied to simplify their final solution and avoid the complex-valued Fourier inverse transform. The main disadvantage of a solution being left in terms of complex-valued integrals is that the numerical calculation of these integrals has to be handled very carefully as the integrands are multi-valued complex functions, which may cause some problems when one needs to decide which root is the correct one to take. There have been examples reported in the literature (e.g., Kahl & Jackel 2005) for the wrong numerical integration when a Fourier inversion is performed. In comparison with those complicated integral calculations, the numerical advantage of our compact solution (7.12) is obvious. Such advantage has also been clearly demonstrated by Zhu & Lian (2009*d*) when they developed their variance swaps pricing model.

Thirdly, under the general dynamics as specified in Eq. (7.3), Lin (2007) employed the so-called convexity correction approximation (Brockhaus & Long 2000; Bates 2006), which is essentially the second-order Taylor expansion of the square root function, for the square root of latent affine stochastic process to calculate the expectation in Eq. (7.6) and hence obtained an approximation formula for VIX futures. By using the convexity correction approximation, he

was able to present the VIX futures formula in the form of

$$F(t,T) = E^{\mathbb{Q}}[\text{VIX}_T|\mathscr{F}_t] \approx \sqrt{E_t^{\mathbb{Q}}(\text{VIX}_T^2)} - \frac{var^{\mathbb{Q}}(\text{VIX}_T^2)}{8[E^{\mathbb{Q}}(\text{VIX}_T^2)]^{\frac{3}{2}}} \tag{7.15}$$

where $var^{\mathbb{Q}}(\text{VIX}_T^2)/\{8[E^{\mathbb{Q}}(\text{VIX}_T^2)]^{\frac{3}{2}}\}$ is the convexity adjustment relevant to the VIX futures. However, Broadie & Jain (2008*b*) pointed out that the convexity correction is not justifiable at all for the Heston model, or Merton jump diffusion model. They also have shown some numerical evidence that the convexity correction performs poorly in the Heston stochastic volatility model and even worse in models with jumps, such as Merton jump diffusion model or Bates model (SVJ). Even though their analysis aimed at examining the performance of convexity correction approximation for volatility swaps, we have every reason to doubt that Lin (2007)'s approximation-based VIX futures formula doesn't work well either. Therefore, this adds another motivation for us to find out an exact formula for VIX futures as presented in this chapter. The numerical comparisons between Lin (2007)'s approximation formula (7.15) and our exact formula (7.12) confirm that the convexity correction indeed doesn't work well for some parameters, as was also confirmed by Broadie & Jain (2008*b*)'s findings. Furthermore, our numerical comparisons reveal that the third-order Taylor expansion approximation performs even worse than seconde-order approximation, which is the totally at odds with the conclusion of Brenner et al. (2007) who explored the third-order Taylor expansion to give an approximate formula for VIX futures prices and claimed that it was "very accurate" for reasonable set of parameter values. All the numerical analyses will be shown in the next section.

Fourthly, this pricing formula (7.12) for VIX futures inherently possesses a number of interesting properties, consistent with many reported properties about volatility futures in the literature (c.f., Grunbichler & Longstaff 1996; Psychoyios

et al. 2007). For example,

$$\lim_{(T-t)\to 0} F(t,T) = \text{VIX}_t \tag{7.16}$$

which is the standard convergence property of futures prices to the underlying spot value at maturity, as a necessary condition for any futures contract to be correctly priced. When the time-to-maturity increases, however, the VIX futures prices have distinct property, in that the futures prices are becoming less sensitive to the spot VIX value and fail to capture the evolution of the VIX as time-to-maturity increases. In the limiting case, the futures prices approach to a constant that is independent of the VIX value, i.e.,

$$\lim_{(T-t)\to \infty} F(t,T) = Constant \tag{7.17}$$

As will be shown in the empirical studies later, this term structure of VIX futures prices is indeed consistent with the actually traded prices in CBOE. This feature is quite unique itself, in contrast to those of futures contracts written on commodities or equities; the latter always move in an one-to-one fashion with the underlying spot price, even with very large time to expiring.

7.2.4 Numerical Examples

In this section, we show some numerical results to illustrate the properties of our newly-found VIX futures pricing formula. We firstly compare the results obtained from the implementation of Eq. (7.12) with those from Monte Carlo simulations to verify the correctness of our newly-found formula. Although theoretically there would be no need to discuss the accuracy of a closed-form exact solution and present numerical results, some comparisons may give readers a sense of verification for the newly-found analytical solution. We then present some numerical comparisons with the results obtained from our exact solution Eq. (7.12)

and those from the convexity correction approximations (e.g., Lin 2007; Brenner et al. 2007). These comparisons will help readers understand the improvement in accuracy of our exact solution.

In our examples, we use the parameters (unless otherwise stated) reported in Duffie et al. (2000) that were founded by minimizing the mean-squared differences between models and the market S&P500 options prices on November 2, 1993. Listed in Table 7.1 are these parameters, which are the same set of parameters adopted by Broadie & Jain (2008*a*) as well.

Table 7.1: Parameters for SV, SVJ and SVJJ models

Parameters	SV model	SVJ model	SVJJ model
θ	0.019	0.014	0.008
κ	6.21	3.99	3.46
σ_V	0.61	0.27	0.14
λ	N/A	0.11	0.47
σ_S	N/A	0.15	0.0001
$\overline{\mu}$	N/A	-0.12	-0.10
μ_V	N/A	N/A	0.05
ρ_J	N/A	N/A	-0.38
$\sqrt{V_0}$	10.1%	9.4%	8.7%

To verify the correctness of our solution, we have used Monte Carlo method to simulate the underlying process (7.3) and calculate VIX futures prices according to Eq. (7.6). We took $200,000$ paths for all the simulation results presented here. It should be remarked that a nice simulation of the CIR variance process is anything but straightforward. For simplicity, we have employed the simple Euler-Maruyama discretization for the variance dynamics:

$$v_t = v_{t-1} + \kappa^{\mathbb{Q}}(\theta^{\mathbb{Q}} - v_{t-1})\Delta t + \sigma\sqrt{|v_{t-1}|}\sqrt{\Delta t}W_t + \sum_{n=1}^{N_t} Z_n^V \tag{7.18}$$

where W_t is a standard normal random variables, $Z_n^V \sim \exp(\mu_V)$, and N_t is the independent Poisson process with intensity $\lambda\Delta t$.

Plotted in Fig. 7.1 are three sets of data, the fair price of VIX futures ob-

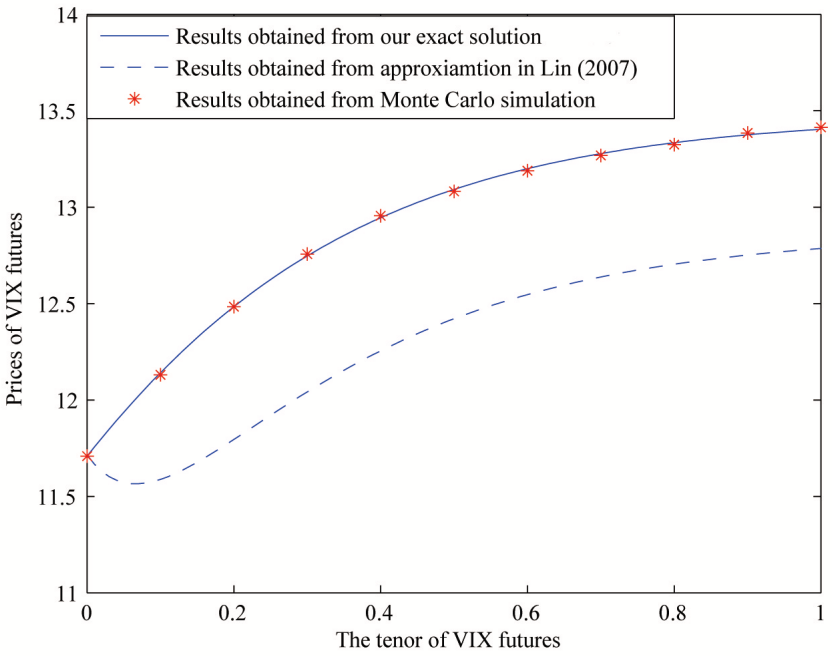

Figure 7.1: A comparison of VIX futures strikes obtained from our exact formula, the MC simulations and Lin (2007)'s approximation, as a function of tenor, based on the SVJJ model

tained with the numerical implementation of Eq. (7.12), those obtained from the approximation solution (7.15) and the numerical results obtained from the Monte Carlo simulations (8.23). All these numerical results are obtained using the SVJJ model with the input parameters listed in Table 7.1.

One can clearly observe that the results from our exact solution perfectly match with the results from the Monte Carlo simulations. The value of the relative difference between of our results and those of the Monte Carlo simulations is less than 0.16% already when the number of paths reaches 200,000 in the Monte Carlo simulations. Such a relative difference is further reduced when the number of paths is increased; demonstrating the convergence of the Monte Carlo simulations towards our exact solution. On the other hand, in terms of computational time, the Monte Carlo simulations take a much longer time than our analytical solution does. In contrast to a formidable computational time of 273.219 seconds for one data point using the Monte Carlo simulations with 200,000 paths, im-

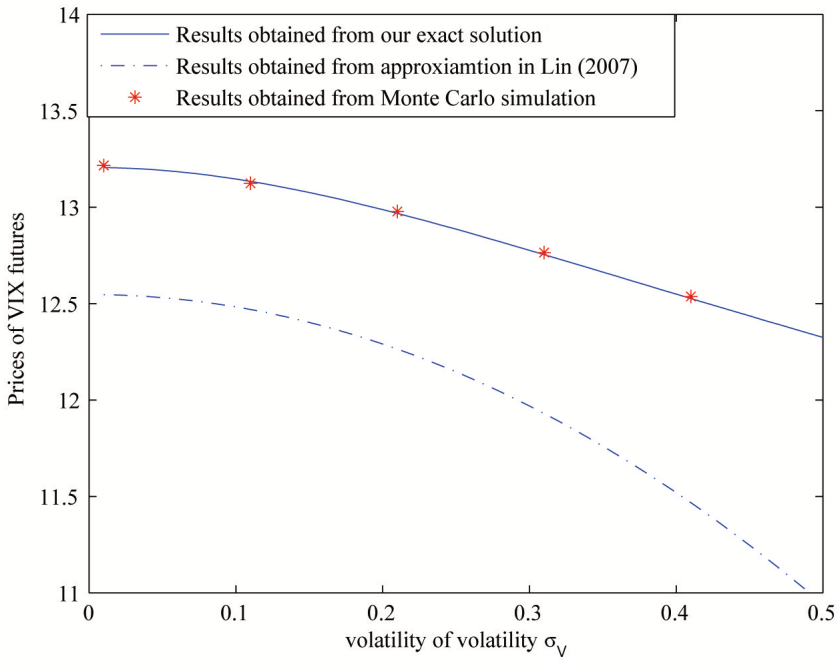

Figure 7.2: A comparison of VIX futures strikes obtained from our exact formula, the MC simulations and Lin (2007)'s approximation, as a function of "vol of vol", based on the SVJJ model

plementing Formula (7.12) just consumed 0.07 seconds; a roughly 4,000 folds of reduction in computational time for one data point. This is not surprising at all since time-consuming is a well-known drawback of Monte Carlo simulations. Of course, some variance reduction techniques (e.g., Glasserman 2004) may be used to enhance the computational efficiency of the Monte Carlo simulations. Since our aim in this chapter is primarily to obtain values from the Monte Carlo simulations for the comparison and verification purpose, we did not focus our attention on improving the numerical efficiency of the Monte Carlo method. However, from our previous experience, it is unlikely that any improved Monte Carlo simulation would have an efficiency exceeding that of an analytical solution no matter what reduction techniques one may adopt.

In Fig. 7.1, we have also plotted the numerical results of Lin (2007)'s approximation solution which is obtained using the convexity correction approach. From this figure, one can clearly see that there are non-trivial gaps between numeri-

cal results obtained from our exact solution and those from the approximation formula. For a one-year VIX future, our exact solution produces a value of 13.4 while the convexity approximation results in a value of 12.8, exhibiting a relative difference of -4.47%, which is quite large and unacceptable for market traders. For example, in the literature of pricing variance swaps, even when the error level reaches more than 0.5%, Little & Pant (2001) already declared that it is "fairly large" so that adopting approximation model to price variance swaps might not be justifiable. With this concept in mind, an error of -4.47% of Lin (2007)'s approximation formula is certainly unacceptable. This finding is consistent with the conclusion by Broadie & Jain (2008*b*), who concluded that the convexity correction formula "performs poorly in the Heston stochastic volatility model and even worse in models with jumps".

Of course, when other parameters, such as volatility of volatility, σ_V, are changed, the differences between our exact solution and the approximation solution might even exponentially grow. Plotted in Fig. 7.2 is the effect of changing parameter σ_V while the other parameters are held the same. As one can see, both solutions indicate that the VIX futures prices decrease when σ_V increases. However, the values produced by the approximation solution decrease much faster than those produced from our exact solution. When the σ_V reaches 0.5, which is a reasonable and often reported value in the literature of empirical studies (e.g., Zhang & Zhu 2006; Brenner et al. 2007), the relative difference between the results of the two solutions becomes as high as -11.3%! In Fig. 7.2, one can also observe that our solution matches up the results from Monte Carlo simulations, once again verifying the correctness of our exact solution.

Lin (2007)'s convexity correction approximation is essentially a Taylor-series expansion of the square root function to the second order. One may wonder if a better accuracy can be achieved by extending the convexity correction approximation to the third order in the Taylor expansion of the square root function. Brenner et al. (2007) explored such an extension already by carrying out the

Taylor expansion of the square root function to the third order and obtained an approximation formula for VIX futures, based on the Heston stochastic volatility model as well. Brenner et al. (2007) claimed that the third-order Taylor expansion-based approximation formula "is very accurate for reasonable set of parameter values" for the SV model. It is thereby quite interesting to examine how their third-order approximation formula has improved the accuracy.

We tried to replicate Brenner et al. (2007)'s word in order to examine the degree of improvement of their formula over Lin (2007)'s second-order approximation. In Brenner et al. (2007)'s study, they did not illustrate the specific definition of reasonable set of parameter values, nor did they give any examples of reasonable sets of parameter values. For the purpose of examining the accuracy of Brenner et al. (2007)'s approximation formula, we adopted the parameters presented in their empirical studies to do numerical comparisons, $\kappa = 5.5805$, $\theta = 0.03259$ and $\sigma_V = 0.5885$. This set of parameters was obtained by using the calibration method, i.e., minimizing the sum of square differences between VIX futures market prices and the approximation formula based-theoretical prices, and hence it should be a "reasonable set of parameter values". Plotted in Fig. 7.3 and Fig. 7.4 display the comparisons of the results obtained from our exact formula in the special case of SV model, the exact formula presented by Zhang & Zhu (2006), the approximation formula presented by Lin (2007) and the approximation formula presented by Brenner et al. (2007), respectively, with parameters being those presented by Brenner et al. (2007). As can be seen in the both figures, results of our exact formula match up with those from Zhang & Zhu (2006)'s exact formula for VIX futures, once again verifying the correctness of our exact formula. It should also be noted that Zhang & Zhu (2006)'s exact formula for VIX futures is based on the Heston stochastic volatility (SV) framework, which is a special case covered by our general model (SVJJ). The two figures also show that the Lin (2007)'s approximation formula always undervalues VIX futures and performs poorly. For examples, the relative error is -1.8% for

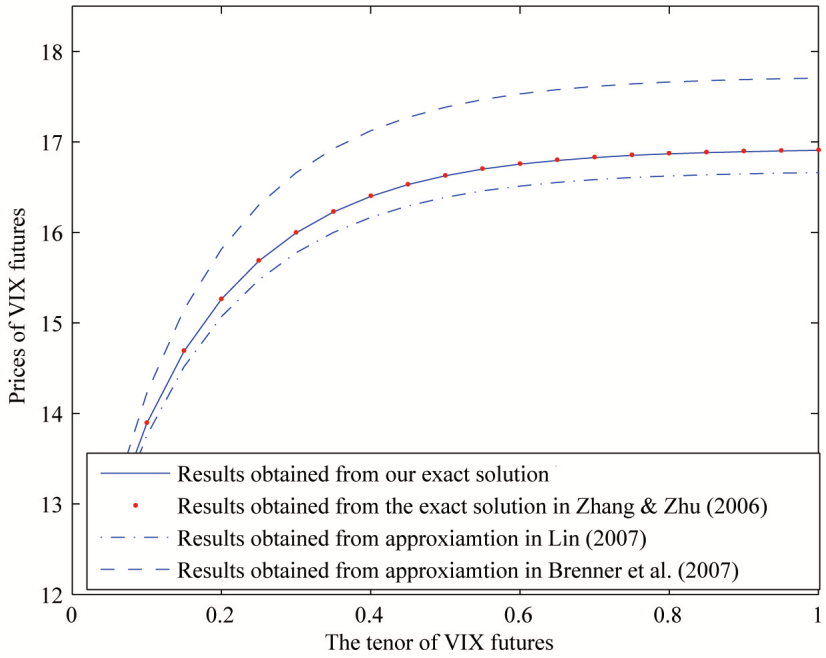

Figure 7.3: A comparison of VIX futures strikes obtained from our exact formula and the approximations in literature, as a function of tenor, based on the Heston model

a one-year VIX futures contract, which is the largest relative difference between the solid line and the dash-dotted line displayed in Fig. 7.3. The Brenner et al. (2007)'s third-order approximation formula works even worse than Lin (2007)'s second-order approximation formula. As can be clearly observed in Fig. 7.3, the third-order approximation formula has not only reversed the under-pricing characteristics of the second-order approximation formula, but also resulted in significant over-pricing errors in comparison with the prices obtained with our new exact solution.

The consistent over-pricing from the third-order approximation, as opposed to the consistent under-pricing from the second-order approximation, can be exhibited more clearly when we plot the VIX futures prices against the volatility of volatility, σ_V, in Fig. 7.4. From this diagram, we can also conclude that the convexity correction approximation works well when σ_V is sufficiently small. However, when σ_V has passed certain threshold (it is roughly 0.5 in this particular

example), the deviation resulted from the convexity correction approximation, no matter if it is from the second-order or the third-order approximation, will become unacceptably too large. In this particular case, we can see that Brenner et al. (2007)'s third-order approximation formula performs far worse than the Lin (2007)'s second-order approximation. Theoretically, if the Taylor expansion series converges well, formulae derived with a third-order expansion should exhibit better accuracy than those derived with a second-third order expansion. If there is no sign of improvement in accuracy when a higher-order expansion is used, or the third-order expansion performs even worse than the second-order expansion as shown in Fig. 7.4, the series may not even converge and neither of them can really be used as a reliable approximation. Of course, it is quite possible that under some other sets of parameters, the two approximations may work well and the third-order approximation formula may indeed achieve higher accuracies than the second-order approximation formula. The fact that the accuracies of the both Lin (2007)'s and Brenner et al. (2007)'s approximation formulae are sensitive to the volatility of volatility, σ_V, suggests that adopting the convexity correction approximation based on a Taylor series expansion of square root function is not suitable at all; this further reinforces the case that exact solutions needs to be derived as we present in this chapter.

7.3 Empirical Studies

Like any other pricing formulae, such as the Black-Scholes formula, to apply our newly-developed general formula to price VIX futures in practice, one needs to know what parameters to use. The determination of the model-needed parameters in a proper and sensible way can itself be a difficult problem. Furthermore, just as a question raised by Bakshi et al. (1997), one now naturally has to choose, according to some criteria, the most suitable one to price VIX futures, among the four available models (SV, SVJ, SVVJ and SVJJ models). The most commonly

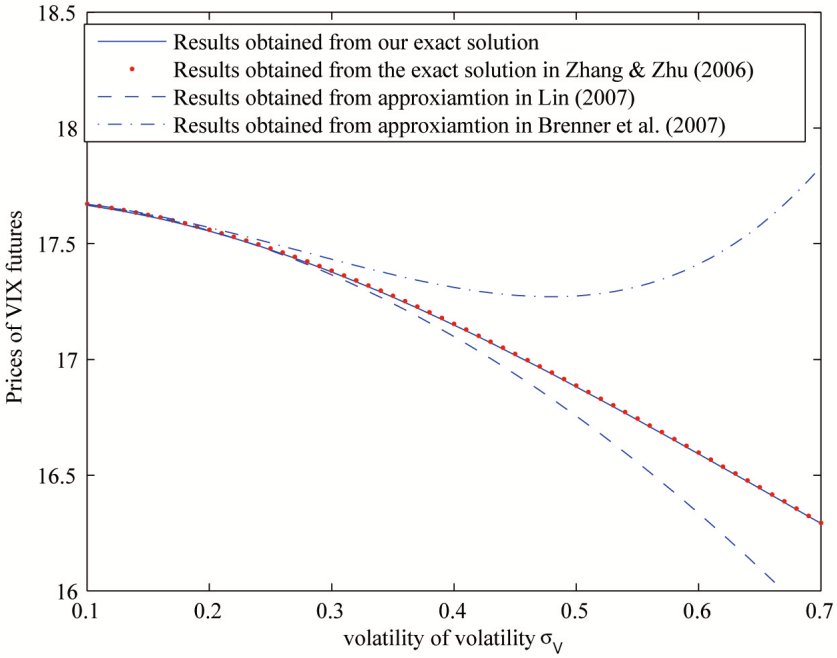

Figure 7.4: A comparison of VIX futures strikes obtained from our exact formula and the approximations in literature, as a function of "vol of vol", based on the Heston model

adopted criteria are i) by the least "pricing errors" between the model-predicted values and the set of market data chosen for the empirical study under some appropriately designed norms to measure the "pricing errors"; ii) by the best "hedging performance" in the sense that the chosen model can indeed render various hedging (such as Delta hedge) against risks specified within the model; iii) by the best fit of the model-implied parameters, which are determined from the derivative prices obtained from the model and market data in an "indirect" and "implied" sense, and those determined directly from analyzing the time series of the underlying such as the S&P500 for the case of pricing VIX options or futures. However, implementing any of these criteria usually means that one faces a highly computationally intensive task as any routine required to carry out the computational task usually involves millions, if not billions, of iterations. Now, with our newly-found closed-form pricing formula that covers four different models, the computation involved in the parameter determination will be substantially

reduced, thus allowing us to compare which model is the most suitable one to price VIX futures. In this section, we present such an empirical study conducted to test the pricing performance of the four models (SV, SVJ, SVVJ and SVJJ).

7.3.1 The Econometric Methodology

"In applying the option pricing models, one always encounters the difficulties that the spot volatility and the structural parameters are unobservable" (Bakshi et al. 1997). To address these difficulties, a number of methods have recently been proposed to estimate the uncertain structural parameters in the latent volatility dynamics as well as the jump diffusion, including the generalized method of moments (GMM) (Singleton 2001; Pan 2002), the efficient method of moments (EMM) (Gallant et al. 1997; Andersen et al. 1999; Andersen et al. 2002; Ortelli & Trojani 2005), the maximum likelihood estimation (MLE) (Bates 2006; Ait-Sahalia & Kimmel 2007), the quasi-maximum likelihood estimation (QMLE) (Ruiz 1994; Sandmann & Koopman 1998), the Markov chain Monte Carlo (MCMC) (Eraker et al. 2003; Eraker 2004; Johannes & Polson 2002; Jacquier et al. 2004; Forbes 2007; Yu & Meyer 2006), and the calibration method (Bakshi et al. 1997; Duffie et al. 2000; Broadie et al. 2007). Zhou (2000) performed a Monte Carlo study on EMM, GMM, QMLE, and MLE for the Heston square-root stochastic volatility model. Although there are pros and cons of each these estimation methods, as pointed out by Andersen et al. (2002) and Bates (2006) in their brief review of these methods, the MCMC method appears to be a robust and popular method.

There are three main reasons why we chose the MCMC method in this study. Firstly, in order to estimate the parameters needed to price VIX futures, we initially chose the calibration method to infer the parameters by minimizing the squared differences between theoretical values calculated from any VIX futures model and those observed in the market, as Bakshi et al. (1997), Zhang & Lim (2006), Broadie et al. (2007) did. Our experience is that some very minor dis-

turbance in the initial value for the optimization results in huge changes in the optimized solution obtained from the optimization algorithm. In other words, the calibration approach appears to be unstable. The instability may result from the highly nonlinear inherence in the object function itself, as has been reported by Zhang & Lim (2006). Secondly, it has been well reported in the literature that the MCMC method has sampling properties superior to other methods. For example, Jacquier et al. (1994) found that the MCMC method outperforms GMM and QMLE in parameter estimation of stochastic volatility models. Andersen et al. (1999) found that the MCMC method also outperforms EMM. Some other advantages (such as computational efficiency, accounting for estimation risk and providing estimations of the latent volatility as well as jumps parameters) are also reported (Eraker et al. 2003). Finally, as commented by Broadie et al. (2007), an efficient estimation procedure should utilize not only the information stored in the underlying that varies as a function of time over the period of study but also the cross-sectional information stored in the derivatives prices over the same period of time. This is also a view shared by other (e.g., Pan 2002; Jones 2003; Eraker 2004). Incorporating all these three features, especially the last one that the joint data of underlying and the cross-sectional derivatives prices were used to estimate the model parameters, the MCMC method naturally became our selected method to conduct the empirical study, in which three sets of market data (S&P500, VIX values and VIX futures prices) were available to us; simultaneously utilizing these sets of data would allow the extracted parameters to ultimately reflect the most unbias information contained in each individual set.

The MCMC method has been used in analyzing time series for a long time. Eraker et al. (2003) adopted this approach to estimate stochastic volatility models with jumps in returns and volatility. Jacquier et al. (2004) comprehensively discussed its application with a number of examples. This method was then extended by Eraker (2004) using not only the underlying time-series data but also the options prices as well. In our study, we employ the MCMC method to estimate

model parameters and analyze model performance. These MCMC analyses are implemented by using the software package WinBUGS, which provides an easy and efficient implementation of the Gibbs sampler, and has been successfully applied for a variety of statistic models such as random effects, generalized linear, proportional hazards, latent variables, and even state space models (Yu & Meyer 2006). Quite a few papers have been proposed in estimating stochastic volatility models using the WinBUGS (for example, Meyer & Yu 2000; Berg et al. 2004; Yu 2005; Yu & Meyer 2006). Readers are referred to Meyer & Yu (2000) for a comprehensive introduction on using WinBUGS to determine the parameters used in stochastic volatility models.

In order to use the MCMC method to estimate the structural parameters and the latent stochastic volatility in our VIX futures pricing model, we construct a time-discretization of Eq. (7.3).

$$\begin{cases} Y_t = \mu + \sqrt{V_{t-1}}\varepsilon_t^S + Z_t^S dq \\ V_t = V_{t-1} + \kappa(\theta - V_{t-1}) + \sigma_V \sqrt{V_{t-1}}\varepsilon_t^V + Z_t^V dq \\ \text{VIX}_t^2 = (aV_t + b) \times 100^2 + \varepsilon_t^{VIX} \end{cases} \tag{7.19}$$

where dq =1 indicates a jump arrival, ε_t^S and ε_t^V are standard normal random variables with correlation ρ, Y_t are continuous daily returns, e.g., $Y_t = \ln(S_t/S_{t-1})$. All the parameters are quoted using a daily time interval following the convention in the time-series literature.

Before continuing with the algorithm, it is important to note the following remarks. Firstly, one may note that there should be a variance risk premium in the return drift, $\mu + \beta V_{t-1}$. The term βV_{t-1} has been ignored from our analysis since the resulted bias is insignificant in daily-interval discretization, consistent with the similar conclusions drawn by Andersen et al. (2002), Pan (2002) and Eraker et al. (2003). Secondly, provided that the Feller condition holds, the V_t process will have a positive solution (Johannes & Polson 2002). Thirdly, an additional

term that represents the difference between the model-predicted value and the recorded market value, or the so-called "pricing errors", ε_t^{VIX}, is introduced in Eq. (7.19). The main reason to introduce such a "pricing error" term is to deal with the stochastic singularity, or what one would call an over-determination problem in solving a system of mathematical equations. In finance practice, there are always more observed market values than the number of parameters that are used in a pricing model, which means that no model is capable of simultaneously fitting all of the recorded market values tick by tick. The implied volatility smile is a typical example of this type of over-determination problems. With the introduction of a "pricing error" term, we are then able to use the MCMC method to overcome the difficulties involved in this over-determination problem in a statistical manner.

For pricing errors, Eraker (2004) adopted a serial dependent AR(1) model, which is equivalent to assuming that pricing errors follow the independent Ornstein-Uhlenbeck processes, based on the prior belief that if an asset is mispriced at time t, it is also likely to be mispriced at time $t+1$. In our study, we follow Johannes & Polson (2002), assuming that ε_t^{VIX} at different t is independent and normally distributed with the zero mean and a known variance, σ_U^2. We have also adopted the prior distributions suggested by Eraker et al. (2003) and Eraker (2004) for the unknown parameters, to implement the MCMC inference model.

7.3.2 Data Description

The daily VIX index value and VIX futures prices can be obtained directly from the CBOE. The VIX index data, including the daily open, high, low and close, are available from the January 2, 1990 to the present. And the VIX futures prices, including open, high, low and close and settle prices, as well as the trading volume together with the open interest, are downloadable from the CBOE from March 26, 2004 to the present. In our studies, we use the VIX daily close levels and VIX

futures daily settle prices over the period from March 26, 2004 to July 11, 2008.

Several exclusion filters were applied to the raw data to construct the VIX futures prices data that are eventually used in our analysis. Firstly, VIX futures that are less than 5 days to maturity were taken out of the raw sample to avoid any liquidity-related bias. This is because there are cases in the last few days before expiration when the VIX futures prices move in the opposite direction to that of the underlying VIX movement. This filter principle was also used by Bakshi et al. (1997) and Zhang & Lim (2006). Secondly, VIX futures data with the associated open interest less than 200 contracts were excluded from the sample to avoid any liquidity-related bias. Lastly, futures prices that are less than 0.5 were not used to mitigate the impact of prices discreteness because of the tick size of 0.01. This is because most option pricing models assume continuous price movements, whereas in the real world the price moves in ticks. Nandi (1996), Bakshi et al. (1997) and Zhang & Lim (2006) used this filter rule. In our studies, the minimum futures price in the raw data is 9.95 anyway and so no sample data has been filtered out by this rule. Based on the criterion, we have 6433 VIX futures prices. Because the VIX futures price is independent with the risk-free interest rate, we do not need to use any interest-rate proxy, such as the LIBOR rate.

Prior to March 26, 2007, the underlying value of VIX futures contract is VIX times 10 under the symbol "VXB", i.e., VXB=VIX×10. And the VIX futures contract size is \$100 times VXB. For example, with a VIX value of 17.33 on March 26, 2004, the VXB would be 173.3 and the contrac size would be \$17,330. In order to bring the traded futures contract prices in line with the underlying VIX index, CBOE Futures Exchange (CFE) rescaled the VIX futures contracts, effective on March 26, 2007, by using the VIX index level as the underlying instead of the VXB. At the same time, CFE increased the previous multiplier for the VIX futures contracts from \$100 to \$1,000. As a result, the traded futures price were reduced by a factor of ten and the minimum tick was reduced from \$0.10 to \$0.01

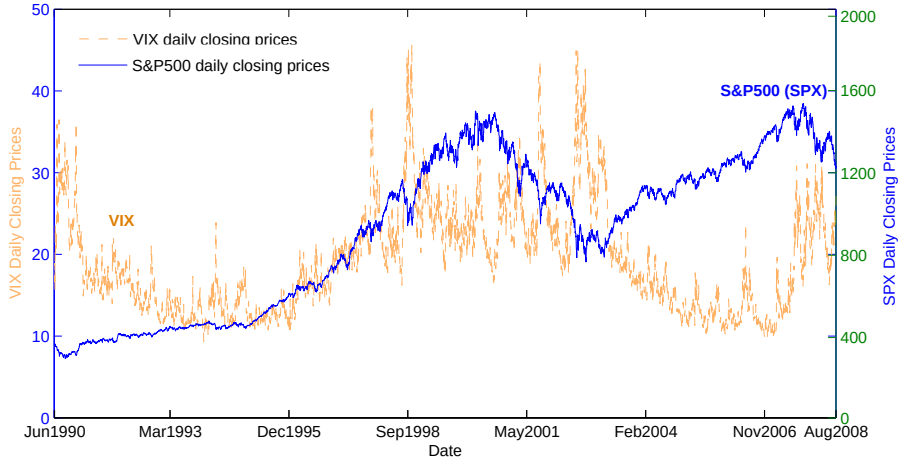

Figure 7.5: The historical data of VIX index and S&P500 index from Jun. 1990 to Aug. 2008

index point, but the dollar value of both remained the same. Thus the rescaling did not change the dollar value of VIX futures contracts. The settlement date is usually the Wednesday prior to the third Friday of the expiration month. In our studies, we rescale the VIX futures price among the period from March 26, 2004 to March 25, 2007 by dividing the contract prices by 10, as guided by the CFE rescaling method[¶].

To illustrate, Figure 7.5 plots the time series of S&P500 and VIX index. As can be immediately observed from the figure, the VIX index has a mean-reverting behavior and has a high volatile behavior.

Table 7.2 provides some basic statistic properties of the S&P500, VIX index and VIX futures. The futures data are divided into 3 categories according to the term to expiration as (i) short-term (< 60 days); (ii) medium-term (60-180 days); and (iii) long-term (> 180 days). This classification was also used by Lin (2007) in pricing VIX futures and Bakshi et al. (1997) in analyzing S&P500 options.

[¶]http://cfe.cboe.com/Data/HistoricalData.aS&P500

Table 7.2: Descriptive statistics of VIX and daily settlement prices of the VIX futures across maturities

			Daily settlement prices of VIX futures			
	S&P500 return	VIX value	All	<60 days	60-180 days	>180 days
Obs. #	4537	1081	6433	2479	1868	2086
Mean	0.000268	15.63	17.89	17.03	18.53	18.34
Median	0.000432	14.02	16.31	15.13	16.50	16.73
Std	0.01012	4.74	4.16	4.56	4.18	3.39
Minimum	-0.07113	9.89	10.37	10.37	12.53	13.52
Maximum	0.05574	32.24	30.61	30.61	27.24	26.26
Skewness	-0.11414	1.22	0.60	0.82	0.44	0.77
Kurtosis	6.61850	3.63	2.08	2.37	1.64	2.09

7.3.3 Empirical Results

By implementing the above MCMC procedure in the software package WinBUGS, we obtained the volatility and jumps parameters, using the joint data of VIX value and S&P500 as inputs to estimate the parameters Φ. This estimation was separately done for each of the four models. Table 7.3 reports the mean and standard deviations of each estimated parameters in the four models. Following the convention in the literature (Eraker 2004), all the parameters are quoted using a daily time interval, which can be annualized to be comparable to the typical results in the literature (e.g., Pan 2002; Lin 2007).

These reported parameters are quite informative. Table 7.3 shows that θ values are 1.761, 1.684, 1.624, and 1.541 respectively for the SV, SVJ, SVVJ and SVJJ models, which correspond to the annualized long-term volatilities of 21.1%, 20.6%, 20.2%, 19.7%. These estimations are slightly higher than the unconditionally sampled standard deviation of S&P500 return data, which corresponds an annualized value of 16.1% (see Table 7.2)[‖]. These discrepancies indicate that the sample period for our VIX futures (2004-2008) may be a relatively higher volatile period than that of the S&P500 (1990-2008). Our estimations for θ are slightly smaller than those reported in Lin (2007), Eraker (2004), Zhang & Zhu

[‖] $16.1\% = 0.01012\sqrt{252}$

Table 7.3: The parameters of the SV, SVJ, SVCJ, and SVSCJ models estimated from the MCMC method

Parameters	SV	SVJ	SVVJ	SVJJ
θ	1.761	1.684	1.624	1.541
	(0.283)	(0.303)	(0.280)	(0.205)
$\kappa^{\mathbb{Q}}$	0.009	0.009	0.007	0.008
	(0.001)	(0.000)	(0.001)	(0.000)
σ_V	0.153	0.120	0.136	0.045
	(0.020)	(0.055)	(0.055)	(0.010)
η_V	-0.008	-0.010	-0.007	-0.007
	(0.002)	(0.002)	(0.001)	(0.001)
ρ	-0.753	-0.668	-0.766	-0.577
	(0.023)	(0.034)	(0.038)	(0.081)
λ		0.002	0.001	0.0007
		(0.000)	(0.000)	(0.000)
$\mu_S^{\mathbb{Q}}$		-0.510		-0.736
		(0.061)		(0.070)
σ_S		2.007		2.305
		(0.722)		(0.922)
μ_V			2.044	0.374
			(1.020)	(0.047)
η_J		-0.101		-0.218
		(0.043)		(0.037)
ρ_J				0.422
				(0.034)

Note. This table reports the means and standard deviations (within parentheses) of each estimated parameters in the four models, using the joint data of VIX value and S&P500. Following the convention in the literature (Eraker 2004), all the parameters are quoted using a daily time interval, which can be annualized to be comparable to the typical results in the literature (e.g., Pan 2002; Lin 2007).

(2006), however very close to the impled estimation in Bakshi et al. (1997). Theoretically, the effective long-term mean variance is θ for the SV and SVJ models,

and $\theta + \frac{\lambda\mu_V}{\kappa}$ for SVVJ and SVJJ models. For the SVVJ and SVJJ models, the estimated values of θ are smaller than those in SV and SVJ models, suggesting that the jump components in volatility processes have indeed captured a portion of the unconditional return variance. This feature is indeed in line with those reported in the literature (e.g., Pan 2002; Lin 2007).

Our estimates of volatility of volatility σ_V are slightly larger than those reported by Eraker et al. (2003) obtained from time-series analysis on long-time S&P500 return, while smaller than those estimated by Eraker (2004) using joint data of return and option prices. These estimates in our study are a little smaller than those in the literature of VIX futures studies, such as Zhang & Zhu (2006) and Lin (2007). As pointed out by Eraker (2004), there is a certain disagreement whether estimates obtained previously are reasonable.

Our estimates of the "leverage effect", ρ, range from -0.577 to -0.766 in the four models. The absolute values of these estimates are slightly larger than those documented in the literature, for example, ρ=-0.39 in Jacquier et al. (2004), -0.40 in Eraker et al. (2003), -0.58 for SVJJ in Eraker (2004). Interestingly, Bakshi et al. (1997) obtained estimates of -0.64, -0.76 and -0.70 for ρ in the SV model, using the data of all options, short-term options and at-the-money options respectively. Lin (2007) presented an estimate of -0.6936 for SVJ model. This disagreement indicates the estimate of ρ is still inconclusive. For the purpose of pricing VIX future, the estimate of ρ is not so important because the VIX and VIX futures are independent of this parameter.

7.3.4 Comparative Studies of Pricing Performance

In this section, we discuss the empirical performance of the four models in this chapter in fitting the historical VIX futures prices. By following the studies in Lin (2007), we employ three measures of "goodness of fitting" (the root mean squared error (RMSE), the mean percentage error (MPE) and the mean absolute

error (MAE)) to assess the pricing performance for each of the four VIX futures pricing models. For this purpose, we firstly compute the model-determined future price using parameters reported in Table 7.3, then subtract it from its observed market counterpart, to obtain the squared pricing error, percentage pricing error, and absolute pricing error. This procedure is repeated for every future and each day in the sample to eventually obtain the mean values of the three tests.

The RMSE, MPE and MAE values for the short-term, mid-term and long-term futures contracts are tabulated in Table 7.4. Firstly, the RMSE and MAE are the lowest (except the short-term futures contracts) for the SVJJ model, ranking SVJJ model the best. This suggests that the specification benefits are indeed generated by introducing simultaneous jumps in return and volatility processes. On the other hand, from the panel of MPE values, in contrast to the above conclusion, SV model generally outperforms the other three models. Secondly, we find very few benefits generated by adding jump in return, as the SVJ model outperforms the SV model only marginally, or even performs worse than SV model according to the MPE test. Comparing with the SVVJ model, the SVJJ model, which is constructed by adding jump in underlying into the SVVJ model, only yields marginal improvement, except for the long-term futures contracts. Therefore, it may not be worthwhile for the effort spent on adding up jumps in volatility process. Thirdly, it is shown that SVVJ model performs very well for the short-term and medium-term futures. However, it significantly overprices the long-term futures with MPE as high as 10.790%. Lastly, all the three tests show that the four models perform better for short-term futures than for long-term contracts. For example, the MPE is 3.303% for short-term futures in SVJJ model, whereas it increases to 8.942% for long-term contracts, which is more than doubled. This is also true for other test measures or other models.

To illustrate the pricing performance of the various models more clearly, we examine the performance of models in fitting the VIX futures term structure. Following the basic idea of VIX futures term structure proposed by Brenner et al.

Table 7.4: The test of pricing performance of the four models

Pricing Errors	Models	Time to Expiration			
		All Futures	<=60	60-180	>=180
RMSE	SV	2.668	1.782	2.940	3.230
	SVJ	2.615	1.731	2.856	3.198
	SVVJ	2.578	1.633	2.718	3.271
	SVJJ	2.485	1.625	2.657	3.095
MPE(%)	SV	5.399	2.880	5.112	8.651
	SVJ	5.624	3.174	5.340	8.790
	SVVJ	6.184	2.556	5.855	10.790
	SVJJ	5.774	3.303	5.514	8.942
MAE	SV	2.343	1.479	2.713	3.037
	SVJ	2.296	1.443	2.635	3.006
	SVVJ	2.237	1.335	2.505	3.068
	SVJJ	2.174	1.351	2.449	2.907

Note. For a given model, we compute the price of each VIX future using the previously estimated parameters reported in Table 7.3, the current day's VIX and the maturity of the VIX future, then subtract it from its observed market counterpart, to obtain the squared pricing error, percentage pricing error, and absolute pricing error. This procedure is repeated for every future and each day in the sample to eventually obtain the mean values of the three tests.

(2007), we sort all the observed futures prices according to the expiration and group these futures by every 30 day to expiration, and then compute the average prices of each group. In this procedure, we obtain an empirical term structure of VIX futures, as plotted in Fig. 7.6. Then, we compute futures values as a function of expiration, using our empirically obtained parameters in Table 7.3 with the VIX value in VIX futures pricing formula (7.12) being the mean value of VIX 15.63 (see Table 7.2). Hence we figure out four VIX futures term structure curves, corresponding to the four models SV, SVJ, SVVJ, and SVJJ.

It can be observed that the empirical term structure of VIX futures price as well as the model-based theoretical term structures is of upward sloping, indicating the short-term mean level of volatility is relatively low compared with the long-term mean level and that the volatility is increasing to the long-term high level. It can be also easily observed that all term structure curves are concave,

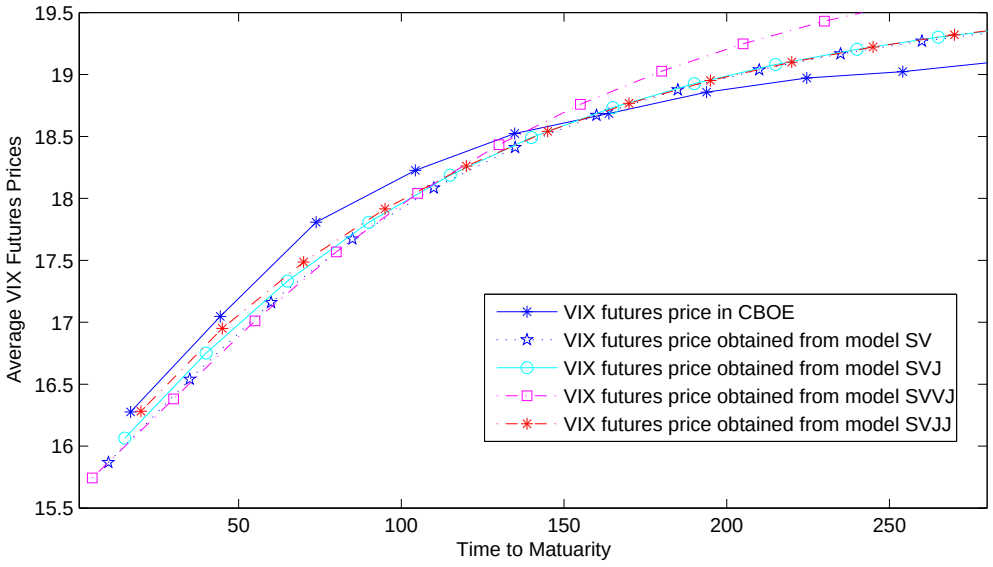

Figure 7.6: A comparison of the term structures of average VIX futures prices obtained from empirical market data and the four models

and asymptotically approach their upper bounds, indicating that futures prices become less sensitive to time-to-maturities when time-to-maturities increase, and eventually independent of time-to-maturities when time-to-maturities are large enough. This interesting property, observed from the empirical data of VIX futures, is consistent with our theoretical analysis, Eq. (7.17). As shown in Figure 7.6, all four models can overall capture the term structure of the VIX futures very well. In particular, SVJJ model performs the best, and SVVJ model the worst, as SVVJ model performs poorly in fitting the long-term contracts.

In Figure 7.7, the model implied density distribution for the VIX is compared with the empirical frequency of the VIX, which is calculated from VIX closing levels observed in CBOE from March 26, 2004 to July 11, 2008. The model implied density is computed based on the Eq. (7.9) and the relationship between VIX and V_t as shown in Eq. (7.5), using the parameters in Table 7.3. It should be noted Eq. (7.9) is the conditional transitional probability density, and empirical VIX frequency is a steady-rate one. To reduce the effect of the spot VIX_t, we choose VIX_t to be the mean of VIX value, 15.63, and $T-t = 10$ years in computing the model implied density. It can be observed in Figure 7.7 that none of the four

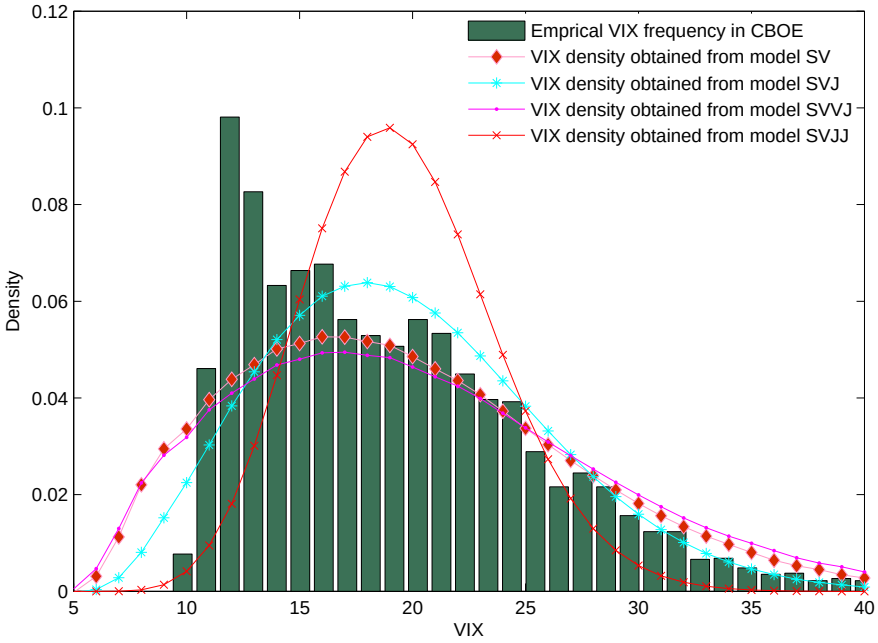

Figure 7.7: A comparison of the steady-rate VIX density functions obtained from empirical market data and the four models

models can capture the "right tail" of the VIX as observed in empirical data. However, relatively, SVJJ model is better than the other three. SVVJ is again the worst. In the related literature, only Sepp (2008*b*) and Sepp (2008*a*) discussed this issue. By calibrating the model to the VIX options data observed on July 25, 2007, Sepp (2008*b*) obtained his parameters for the model and worked out the VIX density. Unfortunately, his model-implied density cannot capture the right tail feature of the VIX empirical frequency either. Sepp (2008*a*) estimated the model parameters by minimizing the squared difference between the model and empirical quantiles. In this way, he found the model-implied density fits the empirical counterpart very well. On the contrary, we found models based on this set of parameters in Sepp (2008*a*) cannot capture the VIX futures term-structure as shown in Figure 7.6. The calculated performance tests (RMSE, MPE and MAE) based on Sepp (2008*a*)'s parameters are also significantly larger than those in Table 7.4. It appears to be a dilemma that is difficult in fitting the VIX futures and VIX values well at the same time. This is actually an essential

question in the empirical study literature, as addressed by Bakshi et al. (1997) and Eraker (2004). Just similar to the well-known question raised by Bakshi et al. (1997) and Bates (1996) in the empirical studies of options pricing, the implied structural volatility parameters that well fit the derivatives market prices (such as S&P500 options or VIX futures) cannot capture the corresponding underlying processes (S&P500, VIX). Although Eraker (2004) found reconciling evidences from spot and option prices by using the MCMC method to infer the related model parameters, we are so far convinced that it is still a very difficult task, trying to obtain the parameters that can well capture the VIX and VIX futures both at the same time.

7.4 Conclusion

In this chapter, we have presented a newly-found closed-form exact solution for VIX futures. The analytic pricing formula has some very unique features. First of all, it is an "umbrella" solution that covers four different stochastic volatility models with or without jumps in underlying and volatility processes to describe the S&P500. Or it is an amazingly "four-in-one" closed-form pricing formula for VIX futures. Secondly, this formula can be efficiently numerically evaluated since it involves a single integral with a real integrand. With this high computational computational efficiency, not only is a much shorter computational time needed to compute the price of a VIX futures contract in comparison with the Monte Carlo simulations, it also greatly facilitates the determination of model parameters, needed when a model is used in practice. Finally, while we have demonstrated that our new formula takes some previously derived formula(e.g., SV) as a special case, it has filled up a gap that there is no closed-form exact solution available in the literature for some other cases (SVJ, SVVJ and SVJJ). Consequently, we were able to use the new formula to examine the accuracy of the analytic approximations previously available for the SVJ, SVVJ and SVJJ cases.

We were also able to use these formulae to conduct empirical studies. Using the joint time series data of S&P500 and VIX, we have demonstrated the determination of model parameters with the MCMC approach thorough a numerical example. With these parameters extracted from the market data, we then empirically examined the pricing performance of four models (SV, SVJ, SVVJ and SVJJ), taking advantage of our newly-found explicit pricing formula. Our empirical studies show that the Heston stochastic volatility model (SV) can well capture the dynamics of S&P500 already and is a good candidate for the pricing of VIX futures. Incorporating jumps into the underlying price can further improve the pricing the VIX futures. However, jumps added in the volatility process appear to add little improvement for pricing VIX futures.

Chapter 8

Pricing VIX Options

8.1 Introduction

In Chapter 7, we demonstrated the derivation of a closed-form exact solution for the fair value of VIX futures under stochastic volatility model with simultaneous jumps in the asset price and volatility processes (SVJJ). We also showed how to estimate model parameters and compare the pricing performance of four models, using the Markov chain Monte Carlo (MCMC) method to analyze a set of coupled VIX and S&P500 data.

In this chapter, we present an analytical exact solution for the price of VIX options under stochastic volatility model with simultaneous jumps in the asset price and volatility processes. We shall demonstrate that our new pricing formula can be used to efficiently compute the numerical values of an VIX option. While we also show that the numerical results obtained from our formula consistently match up with those obtained from Monte Carlo simulation perfectly as a verification of the correctness of our formula, numerical evidence is offered to illustrate that the correctness of the formula proposed in Lin & Chang (2009) is in serious doubt. Moreover, some important and distinct properties of the VIX options (e.g., put-call parity, hedging ratios) are also examined and discussed.

Trading volatility is nothing new for option traders. Most option traders

rely heavily on volatility information to choose their trades. For this reason, the Chicago Board Options Exchange (CBOE) Volatility Index, more commonly known by its ticker symbol VIX, has been a popular trading tool for option and equity traders since its introduction in 1993. Until recently, traders used regular equity or index options to trade volatility, but many quickly realized that this was not the best method. On February 24, 2006, the CBOE started trading options on the VIX, giving investors a direct and effective way to use volatility. The VIX option contracts are the first products on market volatility listed on an SEC-regulated securities exchange*. As a natural extension of the successful introduction of VIX Futures launched on March 26, 2004 on the CBOE Futures Exchange (CFE), the introduction of VIX options have greatly facilitated the hedging against market volatility and consequently allow traders to better manage their portfolio. All VIX futures and options listed on CBOE have well-defined expiration dates; the Wednesday that is thirty days prior to the third Friday of the calendar month immediately following the expiring month. Investors and traders don't have to establish expensive long straddles and strangles or short butterflies and condors to make a volatility play; if they expect increasing market volatility, they can use a long call option on the VIX to attempt to capitalize on their forecast. Similarly, they can replace negative volatility strategies like short straddles and strangles or long butterflies and condors with a long put option on the VIX. Needlessly to say, VIX options are very powerful risk management tools.

The special feathers of VIX options create all sorts of potential opportunities that were previously unavailable for traders and risk managers, and the trading popularity of VIX options has been growing very quickly since their introduction. According to the CBOE Futures Exchange press release on Jul. 11, 2007, in June 2007 the average daily volume of VIX option was 95,283 contracts, making the VIX the second most actively traded index and the fifth most actively traded

*http://www.cboe.com/micro/vix/vixoptions.aspx

product on the CBOE. On July 11, open interest in VIX options stood a 1,845,820 contracts (1,324,775 calls and 521,045 puts). In the same month, the VIX futures totalled 78,578 contracts traded with open interest at 49,894 contracts at the end of June.

Given the rapidly growing popularity of trading VIX options and futures, as well as the unique and interesting features of VIX options and futures, considerable research attention has been drawn to the development of appropriate pricing models for VIX options and futures, as discussed in Chapter 1. However, research on the valuation of VIX options is far from concluded. Very recently, Lin & Chang (2009) presented a closed-form pricing formula for VIX options that reconcile the most general price processes of the S&P500 in the literature: stochastic volatility, price jumps, and volatility jumps. Their solution was obtained by analytically working out the characteristic function of the $\log(\text{VIX}^2)$, through solving the associated PDE. Utilizing this closed-form pricing formula for VIX options, they empirically investigated how much each generalization of the S&P500 price dynamics improves VIX option pricing, and concluded that a model with stochastic volatility and state-dependent correlated jumps in S&P500 returns and volatility (i.e., Duffie et al. 2000) is a better alternative to the others in terms of pricing VIX options. By applying the exactly same pricing formula for VIX options shown in Lin & Chang (2009), Lin & Chang (2010) further studied the relationships among stylized features on S&P 500, VIX and options on VIX, and examined how jump factors impact VIX option pricing and hedging.

Unfortunately, a careful scrutinization of Lin & Chang (2009)'s formula reveals that there is an error contained in their derivation for the characteristic function $\log(\text{VIX}^2)$; our numerical test results obtained from their pricing formula for VIX options substantially differ from those obtained from the Monte Carlo simulations. This error, which does not seem to be a typo, is fatal and uncorrectable, unless one starts to reconstruct the exact closed-form solution using a different approach.

In this chapter, following the approach shown in Zhu & Lian (2009*a*,*d*), we

demonstrate our approach to obtain a correct formula for the price of VIX options. Although we adopt the same general specification of the S&P500 process as Lin & Chang (2009) did, our solution approach is totally different from theirs. Of course, the closed-form pricing formula for VIX options we have derived is different from theirs as well. In order to support our arguments with convincing evidence that our solution is correct, we provide numerical simulation results, which clearly demonstrate that the numerical values of the option price obtained from our newly-derived formula match perfectly with those produced with the Monte Carlo simulations, whereas the results obtained from Lin & Chang (2009)'s pricing formula substantially deviate from those of the MC simulations. The contribution of this study to the existing literature is therefore obvious: (1) by pointing out the incorrectness of Lin & Chang (2009)'s pricing formula, we alert that it should no longer be used; (2) more importantly, a new closed-form pricing formula for VIX has been presented, together with its great efficiency in computing the numerical values of VIX options being clearly demonstrated; (3) some important and distinct properties of the VIX options (e.g., put-call parity, hedging ratios) have also been discussed.

The rest of the chapter is organized as follows. In Section 8.2, based on the general SVJJ model, we present our closed-form pricing formula for VIX options. In Section 8.3, some numerical examples are provided to examine the correctness of our formula and the incorrectness of Lin & Chang (2009)'s formula. Some other important properties of the VIX options (e.g., put-call parity, the hedging ratios) are also discussed. In Section 8.4, a brief conclusion is provided.

8.2 VIX Options

The stochastic volatility model with simultaneous jumps in both asset price and volatility processes (referred to as SVJJ hereafter) is the most general process used for the equity derivatives in literature (see, Andersen et al. 2002 Duffie et al.

2000; Eraker et al. 2003; Eraker 2004; Broadie et al. 2007; Lin & Chang 2009, etc). Like Lin & Chang (2009)'s starting point, our analysis is also based on this general SVJJ model[†].

In the risk-neutral probability measure $\mathbb{Q}$, we assume the dynamics of S&P500 index, denoted by S_t, follows the SVJJ model, i.e., form of

$$\begin{cases} dS_t = S_t r_t dt + S_t \sqrt{V_t} dW_t^S(\mathbb{Q}) + d\left(\sum_{n=1}^{N_t(\mathbb{Q})} S_{\tau_n -}[e^{Z_n^S(\mathbb{Q})} - 1]\right) - S_t \overline{\mu}^{\mathbb{Q}} \lambda dt \\ dV_t = \kappa^{\mathbb{Q}}(\theta^{\mathbb{Q}} - V_t)dt + \sigma_V \sqrt{V_t} dW_t^V(\mathbb{Q}) + d\left(\sum_{n=1}^{N_t(\mathbb{Q})} Z_n^V(\mathbb{Q})\right) \end{cases} \tag{8.1}$$

where:

V is the diffusion component of the variance of the underlying asset dynamics (conditional on no jumps occurring);

dW_t^S and dW_t^V are two standard Brownian motions correlated with $E[dW_t^S, dW_t^V] = \rho dt$;

κ, θ and σ_V are respectively the mean-reverting speed parameter, long-term mean, and variance coefficient of the diffusion V_t;

N_t is the independent Poisson process with intensity λ, that is, $Pr\{N_{t+dt} - N_t = 1\} = \lambda dt$ and $Pr\{N_{t+dt} - N_t = 0\} = 1 - \lambda dt$. The jumps happen simultaneously in underlying dynamics S_t and variance process V_t;

The jump sizes are assumed to be $Z_n^V \sim exp(\mu_V)$, and $Z_n^S | Z_n^V \sim N(\mu_S + \rho_J Z_n^V, \sigma_S^2)$, $e^{Z_n^S(\mathbb{Q})} - 1$ is the percentage price jump size with mean $\overline{\mu}$;

$\overline{\mu} = e^{\mu_S + \frac{1}{2}\sigma_S^2}/(1 - \rho_J \mu_V) - 1$ is the risk premium of the jump term in the process to compensate the jump component, and γ_t is the total equity premium.

As discussed in Chapter 7, the square of VIX (denoted by VIX^2) defined in

[†]In Lin & Chang (2009), the jump density λ is assumed to be a linear specification $\lambda_0 + \lambda_1 V_t$, for some nonnegative constants λ_0 and λ_1. In this study, the λ_1 is set to be zero for simplicity. Our approach presented in this paper can also be applied to Lin & Chang (2009)'s model with hardly any additional effort.

the CBOE white paper[‡] can be interpreted, with mathematical simplification, as the risk-neutral expectation of the log contract (see Lin 2007; Duan & Yeh 2007; Zhu & Lian 2009*a* for more details),

$$\text{VIX}_t^2 = -\frac{2}{\overline{\tau}} E^{\mathbb{Q}}[\ln(\frac{S_{t+\overline{\tau}}}{F})|\mathscr{F}_t] \times 100^2 \tag{8.2}$$

where $\mathbb{Q}$ is the risk-neutral probability measure, $F = S_t e^{r\overline{\tau}}$ denotes the 30-day forward price of the underlying S&P500 with a risk-free interest rate r under the risk-neutral probability, and $\mathscr{F}_t$ is the filtration up to time t.

Under the general specification Eq. (8.1), the expectation in Eq. (8.2) can be carried out explicitly in the form of,

$$\text{VIX}_t^2 = (aV_t + b) \times 100^2 \tag{8.3}$$

where

$$\begin{cases} a = \dfrac{1 - e^{-\kappa^{\mathbb{Q}}\overline{\tau}}}{\kappa^{\mathbb{Q}}\overline{\tau}}, \quad \text{and} \quad \overline{\tau} = 30/365 \\ b = (\theta^{\mathbb{Q}} + \dfrac{\lambda\mu_V}{\kappa^{\mathbb{Q}}})(1-a) + \lambda c \\ c = 2[\overline{\mu}^{\mathbb{Q}} - (\mu_S^{\mathbb{Q}} + \rho_J\mu_V)] \end{cases}$$

Since the underlying for VIX options is the expected, or forward, value of VIX at the expiry, rather than the current, or spot VIX value, we can know that the price of a European VIX call option, $C(t,T)$ at time t with time-to-maturity τ (or expiring at date $T = \tau + t$) and strike K is given by

$$C(t,T) = e^{-t(T-t)} E^{\mathbb{Q}}[\max\{F(T,T,\text{VIX}_T) - K, 0\}|\mathscr{F}_t] \tag{8.4}$$

where $F(t,T,\text{VIX}_t)$ is the VIX future price at the maturity date T. Given that the maturity date T is the same for both the VIX future and the VIX option, and also that the VIX future price coincides with the VIX index at this date, this

[‡]see the white paper of VIX, available at http://www.cboe.com/micro/vix/vixwhite.pdf

pricing formula can be rewritten as

$$C(t,T) = e^{-t(T-t)} E^{\mathbb{Q}}[\max\{\text{VIX}_T - K, 0\}|\mathscr{F}_t] \tag{8.5}$$

A major objective of this paper is to show how this expectation can be analytically worked out to price VIX call options, utilizing the characteristic function of the stochastic variable V_T, as demonstrated in the next section.

8.2.1 Our Formula

As stated in the Introduction, we believe that Lin & Chang (2009)'s approach contains fundamental errors and thus a completely different way of finding the exact solution is necessary. Clearly, in order to obtain a closed-form formula for the price of a VIX call option[§], one needs to work out the transitional probability density function of stochastic variable VIX_T to calculate the expectation in Eq. (8.5). This required transitional probability density function actually has already been presented in Proposition 5 in Chapter 7, in the form of

$$p^{\mathbb{Q}}(\text{VIX}_T|\text{VIX}_t) = \frac{2\text{VIX}_T}{a\pi} \int_0^\infty Re\left[e^{-i\phi(\frac{VIX_T^2 - b}{a})} f\left(i\phi; t, \tau, (\text{VIX}_t^2 - b)/a\right)\right] d\phi \tag{8.6}$$

where $\tau = T - t$ and $f(\phi; t, \tau, V_t)$ is the moment generating function of the stochastic variable V_T, given by

$$f(\phi; t, \tau, V_t) = e^{C(\phi,\tau) + D(\phi,\tau)V_t + A(\phi,\tau)} \tag{8.7}$$

[§]For VIX put options, our approach can be adopted as well. Alternatively, one can use the put-call parity, which will be discussed in the next section of this chapter.

with

$$\begin{cases} A(\phi,\tau) = \dfrac{2\mu_V\lambda}{2\mu_V\kappa^{\mathbb{Q}} - \sigma_V^2}\ln\left(1 + \dfrac{\phi(\sigma_V^2 - 2\mu_V\kappa^{\mathbb{Q}})}{2\kappa^{\mathbb{Q}}(1-\mu_V\phi)}(e^{-\kappa^{\mathbb{Q}}\tau} - 1)\right) \\ C(\phi,\tau) = \dfrac{-2\kappa\theta}{\sigma_V^2}\ln\left(1 + \dfrac{\sigma_V^2\phi}{2\kappa^{\mathbb{Q}}}(e^{-\kappa^{\mathbb{Q}}\tau} - 1)\right) \\ D(\phi,\tau) = \frac{2\kappa^{\mathbb{Q}}\phi}{\sigma_V^2\phi + (2\kappa^{\mathbb{Q}} - \sigma_V^2\phi)e^{\kappa^{\mathbb{Q}}\tau}} \end{cases}$$

Thus, the price of a VIX call option at time t is expressed in the form of

$$C(t,T) = e^{-r(T-t)}\int_0^\infty p^{\mathbb{Q}}(\text{VIX}_T|\text{VIX}_t)(\text{VIX}_T - K)^+ d\text{VIX}_T \tag{8.8}$$

This formula, however, is computational expensive, due to the existence of the double integral with one of the integrands being the complex inverse Fourier transform to obtain the probability density function from the characteristic function. A key to the success of our study hinges on whether or not we can reduce the dimensionality of the integral involved and thus improve the computation efficiency, in order to derive a pricing formula that can achieve the same goal as that of Lin & Chang (2009) but in the mean time is error-free. To improve the computational efficiency, we realized that this formula could be substantially simplified, by interchanging the order of the two integral calculations and utilizing the generalized Fourier transform (Lewis 2000; Poularikas 2000; Sepp 2007):

$$\int_0^\infty e^{-\phi y}(\sqrt{y} - K)^+ dy = \frac{\sqrt{\pi}}{2}\frac{1 - \text{erf}(K\sqrt{\phi})}{\sqrt{\phi}^3} \tag{8.9}$$

where ϕ is the complex Fourier transform variable with $Re[\phi] > 0$, and erf(Z) is the complex error function defined by

$$\text{erf}(Z) = \frac{2}{\sqrt{\pi}}\int_0^Z e^{-s^2} ds \tag{8.10}$$

Since the algorithm for the numerical calculation of the error function is standard and very efficient, we can in this way obtain a simplified pricing formula and

hence substantially improve the computational efficiency. In fact, this approach can also be applied to obtain a pricing formula for the strike price of a VIX future, by interchanging the order of integral and utilizing the following Fourier transform

$$\int_0^\infty e^{-\phi y}\sqrt{y}dy = \frac{\sqrt{\pi}}{2}\frac{1}{\sqrt{\phi}^3} \tag{8.11}$$

where ϕ is the complex Fourier transform variable with $Re[\phi] > 0$. The both pricing formulae are stated in the following proposition

Proposition 6 *If S&P500 index follows the general dynamics given by Equation (8.1), the conditional probability density function of VIX_T, denoted by $p^{\mathbb{Q}}(VIX_T|VIX_t)$, is given by*

$$p^{\mathbb{Q}}(VIX_T|VIX_t) = \frac{2\,VIX_T}{a\pi}\int_0^\infty Re\left[e^{-i\phi(\frac{VIX_T^2-b}{a})}f\left(i\phi;t,\tau,(VIX_t^2-b)/a\right)\right]d\phi \tag{8.12}$$

and the price of a VIX call option, $C(t,T,VIX_t)$, at time t with maturity T is given by the following formula:

$$C(t,T,VIX_t) = \frac{e^{-r(T-t)}}{2a\sqrt{\pi}}\int_0^\infty Re\left[e^{i\phi b/a}f\left(\phi i;t,\tau,(VIX_t^2-b)/a\right)\frac{1-erf(K\sqrt{\phi i/a})}{\sqrt{\phi i/a}^3}\right]d\phi \tag{8.13}$$

and the strike price of a VIX future, $F(t,T,VIX_t)$, at time t with maturity T is

$$F(t,T,VIX_t) = \frac{1}{2a\sqrt{\pi}}\int_0^\infty Re\left[e^{i\phi b/a}f\left(\phi i;t,\tau,(VIX_t^2-b)/a\right)/\sqrt{\phi i/a}^3\right]d\phi \tag{8.14}$$

where $f(\phi;t,\tau,V_t)$ is the moment generating function of the stochastic variable V_T, and given by Eq. (8.7).

As shown in Cox et al. (1985), for the simple Heston model, the transitional probability density function of the square root process of instantaneous variance

V_T in the risk-neutral probability measure $\mathbb{Q}$ is given in the form of

$$g^{\mathbb{Q}}(V_T|V_t) = ce^{-W-v}(\frac{v}{W})^{q/2}I_q(2\sqrt{Wv}) \tag{8.15}$$

with

$$c = \frac{2\kappa^{\mathbb{Q}}}{\sigma_V^2(1-e^{-\kappa^{\mathbb{Q}}(T-t)})}, \qquad W = cV_te^{-\kappa^{\mathbb{Q}}(T-t)}, \qquad v = cV_T, \qquad q = \frac{2\kappa^{\mathbb{Q}}\theta^{\mathbb{Q}}}{\sigma_V^2} - 1 \tag{8.16}$$

and $I_q(\cdot)$ is the modified Bessel function of the first kind of order q. The distribution function is the noncentral chi-square, $\chi^2(2v; 2q+2, 2W)$, with $2q+2$ degrees of freedom and parameters of noncentrality $2W$ proportional to the current variance, V_t. Utilising this explicit form of transitional probability density function for the Heston stochastic volatility model, we can obtain the following proposition.

Proposition 7 *If S&P500 index follows the Heston (1993) stochastic volatility model, the conditional probability density function of* VIX_T, $p^{\mathbb{Q}}(VIX_T|VIX_t)$, *is given by*

$$g^{\mathbb{Q}}(VIX_T|VIX_t) = \frac{2\,VIX_T}{a}f^{\mathbb{Q}}(\frac{VIX_T^2-b}{a}|\frac{VIX_t^2-b}{a}) \tag{8.17}$$

where $g^{\mathbb{Q}}(V_T|V_t)$ *is the transitional probability density function of the instantaneous variance* V_T *in the Heson model and is given by Eq. (8.15). The price of VIX futures with maturity* T *is given by following formula (Zhang & Zhu (2006)'s formula)*

$$F(t,T,VIX_t) = E^{\mathbb{Q}}[VIX_T] = \int_0^\infty VIX_T g^{\mathbb{Q}}(VIX_T|VIX_t)d\,VIX_T \tag{8.18}$$

and the price of a VIX call option, $C(t,T,VIX_t)$, *at time* t *with maturity* T *is given by*

$$C(t,T,VIX_t) = e^{-r(T-t)}\int_0^\infty (VIX_T-K)^+ g^{\mathbb{Q}}(VIX_T|VIX_t)d\,VIX_T \tag{8.19}$$

It should be remarked that both Eq. (8.13), which is for the price of a VIX call option based on the general SVJJ model, and Eq. (8.19), which is for the price of a VIX call option based on the simple Heston stochastic volatility model, involve the quadrature of a single integral only and can thus be extremely efficiently computed in order to obtain numerical values. We have carried out some numerical validation, which shall be presented in the next section, to demonstrate the correctness of our newly-derived pricing formulae for VIX options.

8.3 Numerical Results and Discussions

For the purpose of demonstrating the correctness of our formula and the incorrectness of Lin & Chang (2009), we present some numerical examples in this section. We compare the results obtained from our formula, those from Lin & Chang (2009)'s formula, and those from Monte Carlo simulation. We then discuss the put-call parity, risk management ratios, and some other important properties of VIX options.

8.3.1 Lin & Chang (2009)'s Formula

For the time-t price, $C(t,T)$, of a European call option written on VIX with strike price K and expiry at time T (or time-to-maturity τ), Lin & Chang (2009) have shown that $C(t,T)$ can be obtained by solving the following partial differential equation (PDE) (the Eq. (5) in their paper)[¶]

$$\begin{aligned}&\frac{1}{2}V\frac{\partial^2 C}{\partial L^2}+\left[r-\lambda_0\kappa^{\mathbb{Q}}-(\lambda_1\kappa^{\mathbb{Q}}+\frac{1}{2})V\right]\frac{\partial C}{\partial L}+\rho\sigma_V V\frac{\partial^2 C}{\partial L\partial V}+\frac{1}{2}\sigma_V V\frac{\partial^2 C}{\partial V^2}\\&+E_t^{\mathbb{Q}}\{[\lambda_0+\lambda_1(V+Z_V)]C(t,\tau;L+Z_S,V+Z_V)-(\lambda_0+\lambda_1 V)C(t,\tau;L,V)\}\\&+\kappa^{\mathbb{Q}}(\theta^{\mathbb{Q}}-V)\frac{\partial C}{\partial V}-\frac{\partial C}{\partial \tau}-rC=0\end{aligned} \tag{8.20}$$

where $L=\ln S$, and the terminal condition is $C(T,T)=\max(\text{VIX}_T-K,0)$.

[¶]The notations in their paper have been converted into our notations.

Then, following a similar solution procedure presented by Heston (1993), they worked out a characteristic function, $f_2(t,\tau;i\phi)$ (the equation in Appendix A in their paper), of the stochastic variable $\ln(\text{VIX}_T^2)$, (i.e., $f_2(t,\tau;i\phi) = E_t^{\mathbb{Q}}[\exp(i\phi \ln(\text{VIX}_T^2))]$) and presented the price of a VIX call option in the form of (Eq. (6) in their paper)

$$C(t,T) = e^{-r(T-t)}[F(t,T,\text{VIX})\Pi_1 - K\Pi_2] \tag{8.21}$$

where the risk-adjusted probabilities, Π_1 and Π_2, are recovered from inverting the characteristic functions:

$$\begin{aligned}\Pi_1 &= \frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re\left[\frac{e^{-i\phi\ln(K^2)}f_2(t,\tau;i\phi+1/2)}{i\phi f_2(t,\tau;1/2)}\right]d\phi \\ \Pi_2 &= \frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re\left[\frac{e^{-i\phi\ln(K^2)}f_2(t,\tau;i\phi)}{i\phi}\right]d\phi\end{aligned} \tag{8.22}$$

They also commented that the fair value of the VIX futures can be obtained by setting $i\phi = 1/2$ in $f_2(t,\tau;i\phi)$, i.e., $F(t,T,\text{VIX}) = f_2(t,\tau;1/2)$.

However, a mis-match of the results obtained from Eq. (8.21) and those obtained from a Monte Carlo simulation has alerted us for a possible incorrectness of Lin & Chang (2009)'s characteristic functions $f_2(t,\tau;i\phi)$ of $\ln(\text{VIX}_T^2)$. We initially thought that there might be a typo in Lin & Chang (2009)'s formula. However, after carefully examining their solution procedure of the characteristic function of $\ln(\text{VIX}^2)$[‖], we started to realize that the fatal mistake actually stems from the fact that they tried to follow the same solution procedure described in Heston (1993), by assuming the solution of the PDE (Eq. (B6) in their paper) has a specific form as $\exp[C(T-t)+J(T-t)+D(T-t)\ln(\text{VIX}_t^2)]$. This specific representation of solution implies that all the functions $C(T-t)$, $D(T-t)$ and $J(T-t)$ are independent of the variable $\ln(\text{VIX}_t^2)$. However, the form of $f_2(t,\tau;i\phi)$ in the eventually obtained solution (Eq. (B16)) is clearly at odds with this

[‖]Although there is no detailed derivation of this characteristic function in the journal paper Lin & Chang (2009), a very detailed mathematical derivation of this characteristic function can be easily found in their working paper downloadable from http://www.fma.org/Texas/Papers/vixopt_FMA2008-fullpaper.pdf

assumption! We also considered the possibility that Lin & Chang (2009) might have actually assumed $C(T-t)$, $D(T-t)$ and $J(T-t)$ were dependent on the variables $T-t$ and $\ln(\text{VIX}_t^2)$ but have intentionally omitted $\ln(\text{VIX}_t^2)$ in presenting Eq. (B6) for the sake of simplicity. Unfortunately, it does not work either as it is then impossible to transform the PDE (Eq. (B6)) to the ODEs (Eq. (B15)), when $C(T-t)$, $D(T-t)$ and $J(T-t)$ are assumed to be functions of both variables $T-t$ and $\ln(\text{VIX}_t^2)$. The conflict between the assumed functional form and the obtained solution in the solution process for $f_2(t,\tau;i\phi)$ clearly suggests that there is an inherent flaw in the solution process itself and thus the obtained $f_2(t,\tau;i\phi)$ is a wrong solution to the PDE. Consequently, we believe that both the formulae for VIX options and VIX futures (i.e., $F(t,T,\text{VIX}) = f_2(t,\tau;1/2)$) in Lin & Chang (2009) and Lin & Chang (2010) are wrong and our numerical experiments further confirm our conjecture, as shall be demonstrated below.

8.3.2 Monte Carlo Simulations

For simplicity, we have employed the simple Euler-Maruyama discretization for the variance dynamics:

$$V_t = V_{t-1} + \kappa^{\mathbb{Q}}(\theta^{\mathbb{Q}} - V_{t-1})\Delta t + \sigma\sqrt{|V_{t-1}|}\sqrt{\Delta t}W_t + \sum_{n=1}^{N_t} Z_n^V \tag{8.23}$$

where W_t is a standard normal random variables, $Z_n^V \sim \exp(\mu_V)$, and N_t is the independent Poisson process with intensity $\lambda\Delta t$. With the sampled path of V, we calculate the sampling VIX path according to Eq. (8.3), and obtain the prices of VIX call options and futures based on this simulation procedure.

8.3.3 Numerical Results

For the purpose of examining the correctness of Lin & Chang (2009)'s formula, our numerical examples presented here are based on the Heston stochastic volatil-

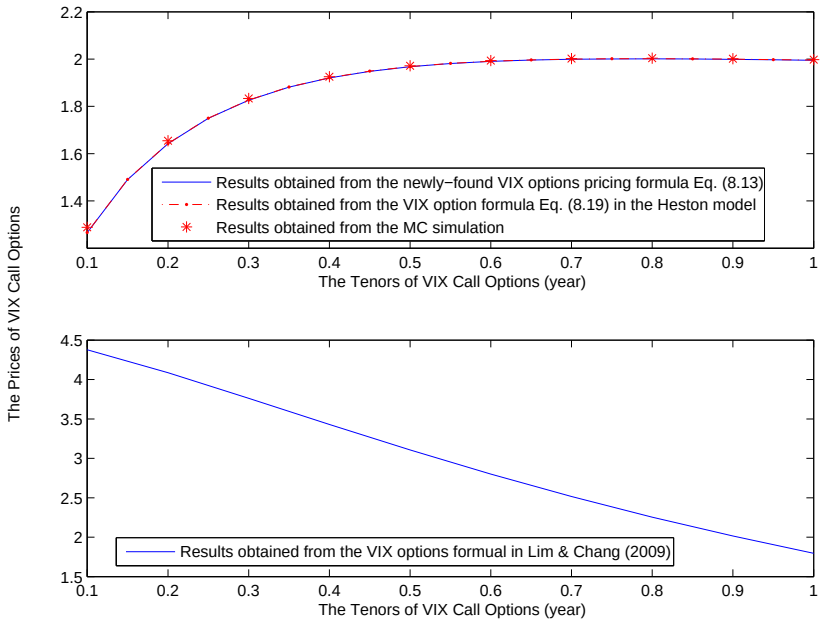

Figure 8.1: A Comparison of the Prices of VIX Options Obtained from Our Exact Formula and the Formula in Lin & Chang (2009), as A Function of Tenor, based on the Heston Model ($K = 13$)

ity model, which is a special case covered by our general SVJJ model and by Lin & Chang (2009)'s model. We show that even for this simple and special case of the general SVJJ model, for which Lin & Chang (2009)'s formula has covered, results obtained from Lin & Chang (2009)'s formula significantly differ from those obtained from our formula, while the latter match up perfectly with those obtained from the MC simulations. Listed in the second column of Table 7.1 are the parameters used in the numerical examples.

Plotted in Fig. 8.1 are four sets of data: the prices obtained with the numerical implementation of Eq. (8.13), those obtained from the newly-derived formula Eq. (8.19), those obtained from the Lin & Chang (2009)'s formula and the numerical results obtained from the Monte Carlo simulations (8.23).

One can clearly observe that the results from our exact solutions Eq. (8.13) and Eq. (8.19) perfectly match with the results from the Monte Carlo simulations; the relative difference between of our results and those of the Monte Carlo simulations is less than 0.10% already when the number of paths reaches 200,000

in the Monte Carlo simulations. This has demonstrated the correctness of our exact solutions from a different angle. On the other hand, one can observe that the results obtained from Lin & Chang (2009)'s formula significantly differ from those obtained from our solutions, and from the Monte Carlo simulations.

This significant discrepancy exists not only in the pricing of VIX options, but also in the pricing of VIX futures. Lin & Chang (2009) showed that the fair strike price of a VIX future is $F(t, T, \text{VIX}) = f_2(t, \tau; 1/2)$. By comparing the numerical results from this formula and our VIX pricing formulae Eq. (8.14) and Eq. (8.18), the formula in Zhu & Lian (2009*a*), as well as the results obtained from Monte Carlo simulation, we found that there is also a great difference between the results obtained from Lin & Chang (2009)'s formula and those from the other methods, as illustrated in Figure 8.2.

For exactly the same pricing model and the same parameters, we should have every reason, theoretically, to believe that the pricing formulae should yield the same numerical values in terms of pricing VIX options and futures, although they may appear in different analytical forms. However, this is not the case as clearly exhibited in Figure 8.2. After carefully checking our computational code and with the fact that the results from our solution match with those obtained from the Monte Carlo simulations so well, we had to rule out the possibility that there might be a typo in Lin & Chang (2009)'s formula. This then led us to carefully examine their solution procedure of the characteristic function of $\ln(\text{VIX}^2)$ and eventually found the reason why their pricing formulae for both VIX futures and options obtained are wrong, as discussed in Section 8.3.1. Of course, this also led us to search for a correct way of finding the exact solution Eq. (8.13) as presented in Section 8.2.1.

8.3.4 Properties of VIX Options

Now, we discuss some important and distinct properties of VIX options.

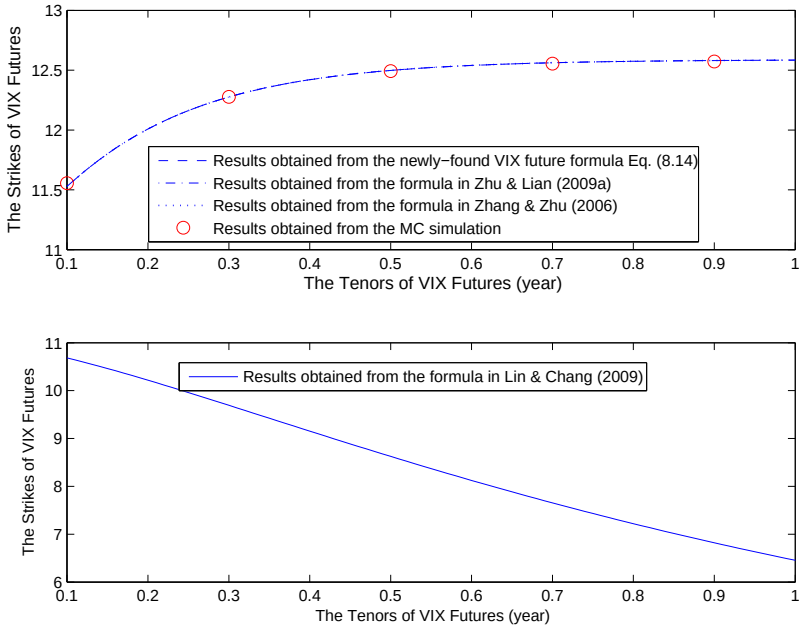

Figure 8.2: A Comparison of VIX Futures Strikes Obtained from Our Exact Formula and the Formulae in Literature, as A Function of Tenor, based on the Heston Model

Firstly, VIX options have one major difference from most other options; i.e., their "underlying instrument". For example, when pricing an equity option, such as options written on the General Electric (GE), the "underlying price" is clearly the price of GE stocks. Similarly, for options on November soybean futures, the underlying price is the price of the November futures contract. For VIX options, however, the underlying price is the price of the corresponding VIX futures contract rather than the spot VIX index, which is calculated, starting from year 2003, based on a set of S&P500 options. Although as time passes the estimated VIX forward prices from the S&P500 options gradually converge the spot VIX and eventually equal to the spot VIX at expiry, resulting in a seemingly-the-same payoff value (see Eq. (8.4) and (8.5)) at expiry, the fact that their underlying is the corresponding VIX futures price, which is not correlated to VIX spot price by a simple cost-of-carry relationship has made the pricing of these options different from equity options written on the tradable asset.

Secondly, because VIX itself is not a tradable asset, there is no cost-of-carry

relationship between VIX futures and the spot VIX values, i.e., $F(t,T,\text{VIX}_t) \neq \text{VIX}_t e^{r\tau}$, as illustrated in Zhang & Zhu (2006), Zhu & Lian (2009*a*) and Zhang et al. (2010). As a result, the put-call parity for VIX options is different from that for options written on stocks. On Feb. 19, 2010, for example, the VIX index closed at 20.02, and the May 2010 VIX 20-strike call and put closed at 5.05 and 0.87, respectively. At first glance, it might seem that the traditional put-call parity has been violated and an arbitrage opportunity existed. But, the pitfall is because the traditional put-call parity relation does not hold for these options any more, as explained by Grunbichler & Longstaff (1996) who showed that since volatility is not a traded asset, the price of a volatility call can be below their intrinsic value. A better way to show the correct put-call parity through the following mathematical deduction:

$$\begin{aligned} & C(t,T,\text{VIX}_t) - P(t,T,\text{VIX}_t) \\ &= e^{-r\tau} E^{\mathbb{Q}}[(\text{VIX}_T - K)^+ - (K - \text{VIX}_T)^+ | \mathscr{F}_t] \\ &= e^{-r\tau} E^{\mathbb{Q}}[\text{VIX}_T - K | \mathscr{F}_t] \\ &= e^{-r\tau} F(t,T,\text{VIX}_t) - K e^{-t\tau} \end{aligned} \tag{8.24}$$

Clearly, this has demonstrated that all one needs to so is to replace the underlying price in the traditional put-call parity by the "discounted" forward volatility, which is reflected in a tradable asset, the VIX future. In other words, for VIX options, the put-call parity should be interpreted as an equality holding among a VIX call and a VIX put and their "true" underlying, the VIX futures price. With this in mind, the closing prices listed in the example given above would be almost perfectly in line with the put-call parity, Eq. (8.24), if one notices that the May 2010 VIX futures actually closed at 24.10 on Feb. 19, 2010.

Thirdly, with the availability of the exact and analytical pricing formulae Eq. (8.13) and (8.19), all the hedging ratios of a VIX call option can be easily deduced analytically. Here, to demonstrate the sensitivity of a VIX call to the parameters, we present some numerical examples of the Δ, based on the SVJJ model with

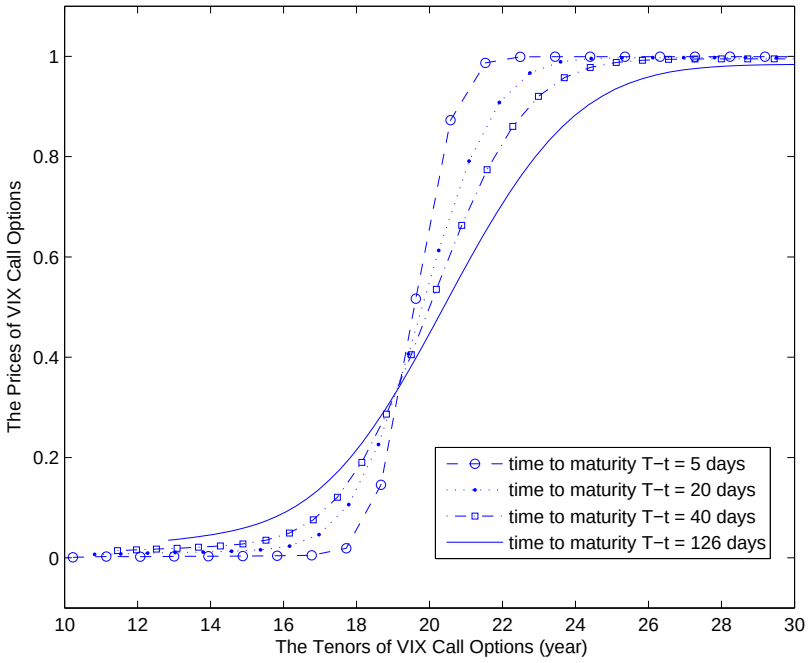

Figure 8.3: The Delta of VIX Options with different maturities: = 5, 20, 40 and 128 days, based on the SVJJ Model.

parameters being those in the fourth column of Table 7.1.

Δ is the sensitivity of the call price with respect to the forward VIX values (i.e., the VIX futures prices). The justification for using VIX futures prices as the underlying of VIX call options is that VIX options are priced based on VIX futures rather than on the spot VIX. The magnitude of Δ is related to volatility call option hedging effectiveness. The highest the Δ, the more sensitive to volatility changes is the volatility call. Figure 8.3 shows the Δ of the VIX calls as a function of the underlying (i.e., VIX futures). The figure is drawn for r = 3.19%, and K=20. Other model parameters are those in the fourth column of Table 7.1. We can easily observe that that Δ is always positive. The magnitude of Δ depends on the level of volatility and the time-to-maturity. It can also be observed that as time-to-maturity $T - t$ increases, the values of the Δ decreases and flattens out. This implies that the sensitivity of the VIX call option price to the underlying decreases as time-to-maturity increases. In other words, as time to maturity increases, the VIX call option loses its hedging effectiveness. The

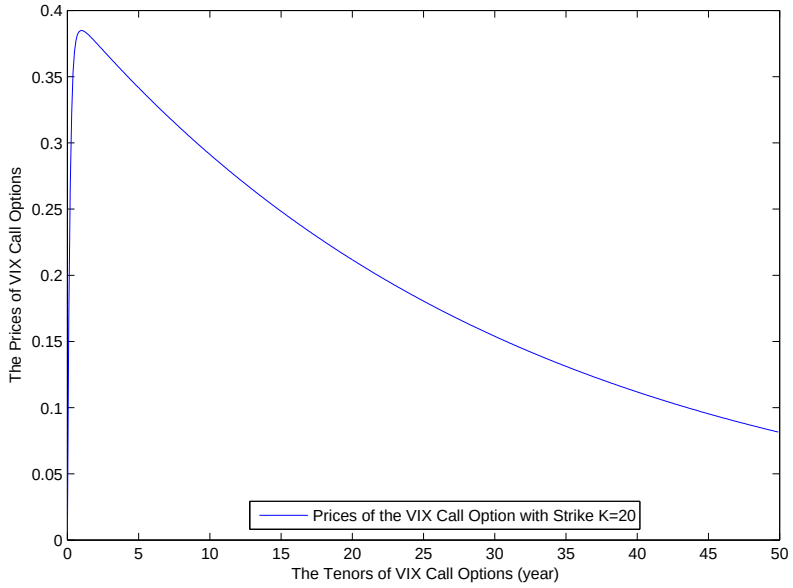

Figure 8.4: The Prices of VIX Options, as A Function of the Time to Maturity, based on the SVJJ Model.

important implication of the result is that long maturity volatility calls are not effective for hedging or trading volatility purposes

Finally, it can be shown that the VIX call pricing formula (8.13) has the following limiting property:

$$\lim_{T-t\to\infty} C(t,T,\text{VIX}_t) = 0 \tag{8.25}$$

This equation shows that for very long maturities the VIX call option, in contrast to the standard equity call options, is going to be worthless, just as it was the case in the models of Detemple & Osakwe (2000), and Grunbichler and Longstaff (1996). This is due to the mean reverting nature of volatility. In the long-run, volatility will revert to it's long run mean value. Figure 8.4 shows the value of the VIX call option as a function of the time-to-maturity. We can see that the call options, in contrast to standard options, are concave functions of volatility; the value of the volatility call initially increases and then flattens out.

8.4 Conclusion

In this chapter, we have derived an analytical exact solution for the price of VIX options under stochastic volatility model with simultaneous jumps in the asset price and volatility processes. The approach presented in this study is totally different from the approach presented by Lin & Chang (2009) in obtaining a closed-form pricing formula for VIX options. We then showed some numerical examples to demonstrate that the results obtained from our formula perfectly match up with those obtained from the Monte Carlo simulations as a verification of the correctness of our formula, whereas the results obtained from Lin & Chang (2009)'s pricing formula significantly differ from those from Monte Carlo simulations, confirming our doubt that their pricing formula is not correct at all. It was shown that our pricing formula is very efficient in computing the numerical prices of VIX options. Some important and distinct properties of the VIX options (e.g., put-call parity, the hedging ratios) have also been discussed. Clearly, our formula can be a very useful tool in trading practice when there is obviously increasing demand of trading VIX options in financial markets.

Chapter 9

Concluding Remarks

In this thesis, we develop some highly efficient approaches to analytically price volatility derivatives. In particular, using our approaches, we present a set of closed-form exact pricing formulae for discretely-sampled variance swaps, forward-start variance swaps, volatility swaps, VIX futures and options.

We first discuss the pricing of variance swaps. We present an approach to solve the partial differential equation (PDE), based on the Heston (1993) two-factor stochastic volatility, to obtain closed-form exact solutions to price variance swaps with discrete sampling times. We then extend our approach to price forward-start variance swaps to obtain closed-form exact solutions. Finally, our approach is extended to price discretely-sampled variance by further including random jumps in the return and volatility processes. We show that our solutions can substantially improve the pricing accuracy in comparison with those approximations in literature. Our approach is also very versatile in terms of treating the pricing problem of variance swaps with different definitions of discretely-sampled realized variance in a highly unified way.

Following the study of pricing variance swaps, we discuss the pricing of another important volatility derivatives, i.e., volatility swaps. Papers focusing on analytically pricing discretely-sampled volatility swaps are rare in literature, mainly due to the inherent difficulty associated with the nonlinearity in the pay-off

function. We present a closed-form exact solution for the pricing of discretely-sampled volatility swaps, under the framework of Heston (1993) stochastic volatility model, based on the definition of the so-called average of realized volatility. Our closed-form exact solution for discretely-sampled volatility swaps can significantly reduce the computational time in obtaining numerical values for the discretely-sampled volatility swaps, and substantially improve the computational accuracy of discretely-sampled volatility swaps, comparing with the continuous sampling approximation. We also investigate the accuracy of the well-known convexity correction approximation in pricing volatility swaps. Through both theoretical analysis and numerical examples, we show that the convexity correction approximation would result in significantly large errors on some specifical parameters. The validity condition of the convexity correction approximation and a new improved approximation are also presented.

Finally, we study the pricing of VIX futures and options. We derive closed-form exact solutions for the fair value of VIX futures and VIX options, under stochastic volatility model with simultaneous jumps in the asset price and volatility processes. As for the pricing of VIX futures, we show that our exact solution can substantially improve the pricing accuracy in comparison with the approximation in literature. We then demonstrate how to estimate model parameters, using the Markov Chain Monte Carlo (MCMC) method to analyze a set of coupled VIX and S&P500 data, and further empirically examine the performance of four different stochastic volatility models with or without jumps. Our empirical studies show that the Heston stochastic volatility model can well capture the dynamics of S&P500 already and is a good candidate for the pricing of VIX futures. Incorporating jumps into the underlying price can indeed further improve the pricing the VIX futures. However, jumps added in the volatility process appear to add little improvement for pricing VIX futures. As for the pricing of VIX options, we point out that Lin & Chang (2009)'s pricing formula for VIX options is incorrect at all. More importantly, we present a totally different closed-form

exact pricing formula for VIX options. It is shown that our pricing formula for VIX options is very efficient in computing the numerical prices of VIX options. The numerical examples show that the results obtained from our formula consistently match up with those obtained from Monte Carlo simulation perfectly, verifying the correctness of our formula. However the results obtained from Lin & Chang (2009)'s pricing formula significantly differ from those from Monte Carlo simulation, confirming our doubt that their pricing formula is incorrect. Some important and distinct properties of the VIX options (e.g., put-call parity, the hedging ratios) have also been discussed in this thesis.

Several directions may be worthy of further pursuing. The approaches presented in this thesis can be extended to price some other even complicated volatility derivatives, for example, the "options on variance", gamma swaps etc. In literature, Sepp (2008*a*) and Carr & Lee (2005) respectively discussed the pricing of options on variance, based on the continuous sampled realized variance. But no one has addressed the issue of pricing these options based on the discretely-sampled realized variance. Some studies on this problem are certainly meaningful for the academic as well as practical purpose. In terms of pricing VIX derivatives, empirically, it is still not known whether and by how much each generalization (e.g., inclusion random jumps) of S&P500 price dynamics improves VIX option pricing. And more importantly, it remains to be an open question why the model parameters in the same model empirically estimated from the market data with different approaches are significantly different and what is the most reliable way to estimate the parameters in the asset and volatility process. Hereby, future study on the model calibration is another extremely important issue.

Appendix A

A Sample Term Sheet of A Variance Swap

S&P 500 INDEX VARIANCE SWAP
March 4, 2005

Transaction Summary:	Buyer enters into a variance swap, under which it • Receives a Payoff at Maturity if the Volatility, as calculated on the Valuation Date, is greater than the Strike. • Pays a Payoff at Maturity if the Volatility, as calculated on the Valuation Date, is lower than the Strike.
General:	
Seller:	
Buyer:	Bear Stearns International, Limited ("BSIL")
Trade Date:	March 4, 2005
Valuation Date:	April 15, 2005
Payment Date:	3 Exchange Business Days following the Valuation Date
Volatility Units (USD per point):	USD 100,000
Variance Units (USD per point):	4,385.9649 (Equal to {Volatility Units/ (2*Strike) }
Underlying Index:	S&P 500 Index ("SPX")
Strike:	**11.4%**
Payoff at Maturity:	Variance Units * [$(\text{Volatility})^2$ - $(\text{Strike})^2$] If such amount is a positive number, then Seller shall make a payment to Buyer If such amount is a negative number, then Buyer shall make a payment equal to the absolute amount to Seller
Volatility:	$100 * \sqrt{\frac{252 x \sum_{i=1}^{n} (Return(i))^2}{n}}$, where: $\text{Return}(i) = \ln\left[\frac{(Index_i)}{(Index_{i-1})}\right]$ 252 = Annualization Factor n = Number of observations excluding the initial observation on Trade Date, but including the Valuation Date Index_i = The closing level of the Underlying Index i business days from the Trade Date except for Index_0 which shall equal the closing level on the Trade Date and Index_n which shall equal the special quotation of the Underlying Index on the Valuation Date.
Documentation:	As per ISDA
Currency:	USD
Exchange:	New York Stock Exchange
Market Disruption:	Postponement
Collateral:	Upfront: Subject to portfolio margin calculation agreed upon Mark-to-Market: As per ISDA

Figure A.1: A sample term sheet of a variance swap written on the variance of S&P500. Source: Bear Stearns Equity Derivatives Strategy, Bloomberg.

Appendix B

Proofs for Chapter 2

B.1 Proof of Proposition 1

We now present a brief proof of Proposition 1.

The PDE system is

$$\begin{cases} \dfrac{\partial U}{\partial t} + \dfrac{1}{2}vS^2\dfrac{\partial U^2}{\partial S^2} + \rho\sigma_V vS\dfrac{\partial U^2}{\partial S\partial v} + \dfrac{1}{2}\sigma_V^2 v\dfrac{\partial U^2}{\partial v^2} + rS\dfrac{\partial U}{\partial S} + [\kappa^{\mathbb{Q}}(\theta^{\mathbb{Q}} - v)]\dfrac{\partial U}{\partial v} - rU = 0 \\ U(S, v, T) = H(S) \end{cases} \tag{B1}$$

Firstly, we do the transform by letting

$$\begin{cases} \tau = T - t \\ x = \ln S \end{cases} \tag{B2}$$

After the transformation, the PDE system is converted to

$$\begin{cases} \dfrac{\partial U}{\partial \tau} = \dfrac{1}{2}v\dfrac{\partial U^2}{\partial x^2} + \rho\sigma_V v\dfrac{\partial U^2}{\partial x\partial v} + \dfrac{1}{2}\sigma_V^2 v\dfrac{\partial U^2}{\partial v^2} + (r - \dfrac{1}{2}v)\dfrac{\partial U}{\partial x} + [\kappa^{\mathbb{Q}}(\theta^{\mathbb{Q}} - v)]\dfrac{\partial U}{\partial v} - rU = 0 \\ U(x, v, 0) = H(e^x) \end{cases} \tag{B3}$$

Solution for this PDE system can be obtained through generalized Fourier transform with respect to x. More details about the generalized Fourier transform, one can refer to Lewis (2000) and Poularikas (2000). Based on the generalized Fourier transform, we can do the transformation

$$\mathcal{F}[e^{j\alpha t}] = 2\pi\delta_\alpha(\omega) \tag{B4}$$

where $j = \sqrt{-1}$ and $\delta_\alpha(\omega)$ is the generalized delta function satisfying

$$\int_{-\infty}^{\infty} \delta_\alpha(t)\Phi(t)dt = \Phi(\alpha) \tag{B5}$$

with α being any complex number.

Applying the transform to the PDE with respect to the variable x, we obtain the following problem for $\widetilde{U}(\omega, v, \tau) = \mathcal{F}[U(x, v, \tau)]$

$$\begin{cases} \dfrac{\partial \widetilde{U}}{\partial \tau} = \dfrac{1}{2}\sigma_V^2 v \dfrac{\partial \widetilde{U}^2}{\partial v^2} + [\kappa^{\mathbb{Q}}\theta^{\mathbb{Q}} + (\rho\sigma_V\omega j - \kappa^{\mathbb{Q}})v]\dfrac{\partial \widetilde{U}}{\partial v} + [(r\omega j - r) - \dfrac{1}{2}(\omega j + \omega^2)v]\widetilde{U} \\ \widetilde{U}(\omega, v, 0) = \mathcal{F}[H(e^x)] \end{cases} \tag{B6}$$

Following Heston's (1993) solution procedure, the solution of the above PDE system can be assumed of the form:

$$\widetilde{U}(\omega, v, \tau) = e^{C(\omega,\tau)+D(\omega,\tau)v}\widetilde{U}(\omega, v, 0) \tag{B7}$$

One can then substitute this function into the PDE to reduce it to two ordinary differential equations,

$$\begin{cases} \dfrac{dD}{d\tau} = \dfrac{1}{2}\sigma_V^2 D^2 + (\rho\omega\sigma_V j - \kappa^{\mathbb{Q}})D - \dfrac{1}{2}(\omega^2 + \omega j) \\ \dfrac{dC}{d\tau} = \kappa^{\mathbb{Q}}\theta^{\mathbb{Q}}D + r(\omega j - 1) \end{cases} \tag{B8}$$

with the initial conditions

$$C(\omega, 0) = 0, \quad D(\omega, 0) = 0 \tag{B9}$$

The solutions of these equations can be easily found as

$$\begin{cases} C(\omega, \tau) = r(\omega j - 1)\tau + \dfrac{\kappa^{\mathbb{Q}}\theta^{\mathbb{Q}}}{\sigma_V^2}[(a + b)\tau - 2\ln(\dfrac{1 - ge^{b\tau}}{1 - g})] \\ D(\omega, \tau) = \dfrac{a + b}{\sigma_V^2}\dfrac{1 - e^{b\tau}}{1 - ge^{b\tau}} \end{cases} \tag{B10}$$

where

$$a = \kappa - \rho\sigma_V\omega j, \quad b = \sqrt{a^2 + \sigma_V^2(\omega^2 + \omega j)}, \quad g = \frac{a + b}{a - b} \tag{B11}$$

One should note that the Fourier transform variable ω appears as a parameter in function C and D.

Therefore, the solution of the original PDE can be obtained after the inverse Fourier transform in form as

$$U(x,v,\tau) = \mathcal{F}^{-1}[\widetilde{U}(\omega,v,\tau)] \tag{B12}$$

$$= \mathcal{F}^{-1}[e^{C(\omega,T-t)+D(\omega,T-t)v}\mathcal{F}[H(e^x)]] \tag{B13}$$

B.2 The Derivation of Eq. (2.32)

If setting stochastic variable $\chi_t^2 = 2cV_t$, then χ_t^2 is subject to noncentral chi-squared distribution, $\chi^2(2v; 2q+2, 2W)$, with probability density function denoted by $p_{\chi_t^2}(x)$. We can easily verify that $p(V_T|V_t) = 2cp_{\chi^2_{T-t}}(2cV_T)$. c, W, q and $p(V_T|V_t)$ are given in Eq. (2.28) and Eq. (2.25).

Hence,

$$E_0^Q[(\frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}})^2] = \int_0^\infty e^{r\Delta t} f(v_{t_{i-1}}) p(v_{t_{i-1}}|v_0) dv_{t_{i-1}} \tag{B14}$$

$$= e^{r\Delta t} E_0^Q[e^{\widetilde{C}(\Delta t)+\widetilde{D}(\Delta t)v_{t_{i-1}}} + e^{-r\Delta t} - 2] \tag{B15}$$

$$= e^{r\Delta t}(e^{\widetilde{C}(\Delta t)} E_0^Q[e^{\widetilde{D}(\Delta t)v_{t_{i-1}}}] + e^{-r\Delta t} - 2) \tag{B16}$$

$$= e^{r\Delta t}(e^{\widetilde{C}(\Delta t)} E_0^Q[e^{\frac{\widetilde{D}(\Delta t)}{2c}\chi^2_{t_{i-1}}}] + e^{-r\Delta t} - 2) \tag{B17}$$

$$= e^{r\Delta t}(e^{\widetilde{C}(\Delta t)}(1-2\Phi)^{-(q+1)} e^{\frac{2W\Phi}{1-2\Phi}}|_{\Phi=\frac{\widetilde{D}(\Delta t)}{2c}} + e^{-r\Delta t} - 2) \tag{B18}$$

$$= e^{r\Delta t}(e^{\widetilde{C}(\Delta t)+\frac{W\widetilde{D}(\Delta t)}{c-\widetilde{D}(\Delta t)}}(\frac{c}{c-\widetilde{D}(\Delta t)})^{\frac{2\kappa^{\mathbb{Q}}\theta^{\mathbb{Q}}}{\sigma_V^2}} + e^{-r\Delta t} - 2) \tag{B19}$$

It should be noted the parameters c, W are determined by the time t_{i-1} in Eq. (2.28) with $T = t_{i-1}$ and $t = 0$.

$$f_i(V_0) = e^{\widetilde{C}(\Delta t)+\frac{c_i e^{-\kappa^{\mathbb{Q}} t_{i-1}}}{c_i-\widetilde{D}(\Delta t)}\widetilde{D}(\Delta t)V_0}(\frac{c_i}{c_i-\widetilde{D}(\Delta t)})^{\frac{2\kappa^{\mathbb{Q}}\theta^{\mathbb{Q}}}{\sigma_V^2}} + e^{-r\Delta t} - 2 \tag{B20}$$

where $c_i = \frac{2\kappa^{\mathbb{Q}}}{\sigma_V^2(1-e^{-\kappa^{\mathbb{Q}} t_{i-1}})}$.

Hence,

$$E_0^Q[(\frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}})^2] = e^{r\Delta t} f_i(v_0) \tag{B21}$$

B.3 The Derivation of Eq. (2.55)

Now, we prove Eq. (2.55). Using l'Hôpital's rule, one can easily verify that

$$\lim_{\Delta t\to 0}\widetilde{C}(\Delta t)=0 \qquad \lim_{\Delta t\to 0}\widetilde{D}(\Delta t)=0 \tag{B22}$$

and

$$\lim_{\Delta t\to 0} e^{\widetilde{C}(\Delta t)+\widetilde{D}(\Delta t)v_0}+e^{-r\Delta t}-2=0 \tag{B23}$$

$$\lim_{\Delta t\to 0}\frac{e^{\widetilde{C}(\Delta t)+\widetilde{D}(\Delta t)v_0}+e^{-r\Delta t}-2}{\Delta t}=v_0 \tag{B24}$$

$$\lim_{\Delta t\to 0}\frac{f_i(v_0)}{\Delta t}=v_0e^{-\kappa^{\mathbb{Q}}(i-1)\Delta t}+\theta^{\mathbb{Q}}(1-e^{-\kappa^{\mathbb{Q}}(i-1)\Delta t}) \tag{B25}$$

Therefore,

$$\lim_{\Delta t\to 0}\frac{e^{r\Delta t}}{T}[f(v_0)+\sum_{i=2}^{N}f_i(v_0)]=\frac{1}{T}\lim_{\Delta t\to 0}\sum_{i=2}^{N}\Delta t(v_0+\frac{f_i(v_0)}{\Delta t}) \tag{B26}$$

$$=\frac{1}{T}\lim_{\Delta t\to 0}\sum_{i=1}^{N}\Delta t[v_0e^{-\kappa^{\mathbb{Q}}(i-1)\Delta t}+\theta^{*}(1-e^{-\kappa^{*}(i-1)\Delta t})] \tag{B27}$$

$$=\frac{1}{T}\int_0^T[v_0e^{-\kappa^{\mathbb{Q}}t}+\theta^{\mathbb{Q}}(1-e^{-\kappa^{\mathbb{Q}}t})]dt \tag{B28}$$

$$=v_0\frac{1-e^{-\kappa^{\mathbb{Q}}T}}{\kappa^{\mathbb{Q}}T}+\theta^{\mathbb{Q}}(1-\frac{1-e^{-\kappa^{\mathbb{Q}}T}}{\kappa^{\mathbb{Q}}T}) \tag{B29}$$

Appendix C

Proof for Chapter 3 and 4

we now give a brief proof for Proposition 3 and 4. Assuming the current time is 0, we let $y_{t,T} = \log S_T - \log S_t$ $(t < T)$, where S_t is the underlying price following the SVJJ model (i.e., Eq. (4.1)). The forward characteristic function $f(\phi; t, T, V_0)$ of the stochastic variable $y_{t,T}$ is defined as

$$f(\phi; t, T, V_0) = E^{\mathbb{Q}}[e^{\phi y_{t,T}} | y_0, V_0], \quad t < T \tag{C1}$$

This expectation can be analytically carried out by solving two PDE successively, due to the tower rule of expectation, i.e.,

$$f(\phi; t, T, V_0) = E^{\mathbb{Q}}[e^{\phi y_{t,T}} | y_0, V_0] = E^{\mathbb{Q}}\left[E^{\mathbb{Q}}[e^{\phi y_{t,T}} | y_t, V_t] \mid y_0, V_0\right] \tag{C2}$$

The inner expectation, $U(\phi; t, T, X, V) = E^{\mathbb{Q}}[e^{\phi y_{t,T}} | y_t, V_t]$, can be carried out by solving the following PIDE, utilizing the Feynman-Kac theorem:

$$\begin{cases} \frac{\partial U}{\partial t} + (r - \lambda\overline{\mu} - \frac{1}{2}V)\frac{\partial U}{\partial X} + [\kappa^{\mathbb{Q}}(\theta^{\mathbb{Q}} - V_t)]\frac{\partial U}{\partial V} + \frac{1}{2}V\frac{\partial U^2}{\partial X^2} + \rho\sigma_V V\frac{\partial U^2}{\partial XV} \\ + \frac{1}{2}\sigma_V^2 V\frac{\partial U^2}{\partial V^2} + \lambda E^Q[U(X + J^X, V + J^V) - U(X, V)|\mathscr{F}_t] = 0 \\ U(\phi; t = T, T, X, V) = e^{\phi X} \end{cases} \tag{C3}$$

where $X = \log S$. Following the solution procedure used by Heston (1993), Bakshi et al. (1997) and Duffie et al. (2000), the solution of the above PIDE can be assumed of the form:

$$U(\phi; t, T, X, V_t) = e^{C(\phi, T-t) + D(\phi, T-t)V_t + \phi X + A(\phi, T-t)} \tag{C4}$$

One can then substitute this function into the PDE to reduce it to the following three ordinary differential equations (ODE)

$$\begin{cases} -\dfrac{\partial C}{\partial t} = (r-\overline{\mu})\phi + \kappa^{\mathbb{Q}}\theta^{\mathbb{Q}}D \\ -\dfrac{\partial D}{\partial t} = \dfrac{1}{2}(\phi-1)\phi + (\rho\sigma_V\phi - \kappa^{\mathbb{Q}})D + \dfrac{1}{2}\sigma^2D^2 \\ -\dfrac{\partial A}{\partial t} = \lambda E^{Q}[e^{\phi Z_n^S(\mathbb{Q})+Z_n^V(\mathbb{Q})D} - 1|\mathscr{F}_t] \end{cases} \tag{C5}$$

with the initial conditions

$$C(\phi,0)=0, \quad D(\phi,0)=0, \quad A(\phi,0)=0 \tag{C6}$$

The solutions of these equations can be easily found as

$$\begin{cases} C(\phi,\tau) = (r-\overline{\mu})\phi\tau + \dfrac{\kappa^{\mathbb{Q}}\theta^{\mathbb{Q}}}{\sigma_V^2}[(a+b)\tau - 2\log(\dfrac{1-ge^{b\tau}}{1-g})] \\ D(\phi,\tau) = \dfrac{a+b}{\sigma_V^2}\dfrac{1-e^{b\tau}}{1-ge^{b\tau}} \\ A(\phi,\tau) = \lambda\left(\exp\left(\mu_S\phi + \dfrac{1}{2}\sigma_S^2\phi^2\right)\right)\left(\dfrac{(a+b)\tau}{c(a+b)+\mu_V\widetilde{\phi}} + \dfrac{2\mu_V\widetilde{\phi}}{(ac+\mu_V\widetilde{\phi})^2-(bc)^2}\log B\right) \\ \qquad -\lambda\tau \\ B = 1 + \dfrac{c(b-a)-\mu_V\widetilde{\phi}}{2bc}(e^{-b\tau}-1) \\ a = \kappa^{\mathbb{Q}} - \rho\sigma_V\phi, \quad b = \sqrt{a^2+\sigma_V^2\widetilde{\phi}}, \quad g = \dfrac{a+b}{a-b}, \quad c = 1-\rho_J\mu_V\phi, \quad \widetilde{\phi} = \phi(1-\phi) \\ \bar{\mu} = \lambda(\dfrac{\exp(\mu_S+\frac{1}{2}\sigma_S^2)}{1-\rho_J\mu_V} - 1) \end{cases} \tag{C7}$$

This affine-form solution obtained from the calculation of the inner expectation facilitates the calculation of the exterior expectation, which is

$$E^{\mathbb{Q}}\left[E^{\mathbb{Q}}[e^{\phi y_{t,T}}|y_t,V_t] \mid y_0,V_0\right] = E^{\mathbb{Q}}\left[e^{C(\phi,T-t)+D(\phi,T-t)V_t+\phi X+A(\phi,T-t)} \mid y_0,V_0\right] \tag{C8}$$

This expectation can be carried out by using the characteristic function, $g(\phi;T-t,V_t)$, of the stochastic variable V_T. Utilizing the Feynman-Kac theorem, the function $g(\phi;T-t,V_t)$, which is defined as $g(\phi;T-t,V_t)=E^{\mathbb{Q}}[e^{\phi V_T}|y_t,V_t]$, should

satisfy the following PDE

$$\begin{cases} \dfrac{\partial g}{\partial t} + \kappa^{\mathbb{Q}}(\theta^{\mathbb{Q}} - V)\dfrac{\partial g}{\partial V} + \dfrac{1}{2}\sigma^2 V \dfrac{\partial^2 g}{\partial V^2} + \lambda E^{\mathbb{Q}}[g(V + Z^V) - g(V)|\mathscr{F}_t] = 0 \\ g(\phi; 0, V) = e^{\phi V} \end{cases} \tag{C9}$$

Again, following the same solution procedure, we can solve this PIDE in closed-form by using a guess of the affine-form solution as,

$$g(\phi; T-t, V_t) = e^{E(\phi,T-t)+F(\phi,T-t)+G(\phi,T-t)V_t} \tag{C10}$$

One can then substitute this function into the PIDE to reduce it to three ordinary differential equations,

$$\begin{cases} -\dfrac{\partial G}{\partial t} = -\kappa^{\mathbb{Q}} G + \dfrac{1}{2}\sigma_V^2 G^2 \\ -\dfrac{\partial F}{\partial t} = \kappa^{\mathbb{Q}}\theta^{\mathbb{Q}} G \\ -\dfrac{\partial E}{\partial t} = \lambda E^{\mathbb{Q}}[e^{GZ_t^V} - 1|\mathscr{F}_t] \end{cases}$$

with the initial conditions

$$E(\phi, 0) = 0, \quad F(\phi, 0) = 0, \quad G(\phi, 0) = \phi \tag{C11}$$

The solutions to these ODEs are

$$\begin{cases} E(\phi,\tau) = \dfrac{2\mu_V\lambda}{2\mu_V\kappa^{\mathbb{Q}} - \sigma_V^2}\log\left(1 + \dfrac{\phi(\sigma_V^2 - 2\mu_V\kappa^{\mathbb{Q}})}{2\kappa^{\mathbb{Q}}(1-\mu_V\phi)}(e^{-\kappa^{\mathbb{Q}}\tau} - 1)\right) \\ F(\phi,\tau) = \dfrac{-2\kappa\theta}{\sigma_V^2}\log\left(1 + \dfrac{\sigma_V^2\phi}{2\kappa^{\mathbb{Q}}}(e^{-\kappa^{\mathbb{Q}}\tau} - 1)\right) \\ G(\phi,\tau) = \frac{2\kappa^{\mathbb{Q}}\phi}{\sigma_V^2\phi + (2\kappa^{\mathbb{Q}} - \sigma_V^2\phi)e^{\kappa^{\mathbb{Q}}\tau}} \end{cases}$$

where $\tau = T - t$.

Summarizing above discussion, we can obtain the forward characteristic function, $f(\phi; t, T, V_0)$, of stochastic variable, $y_{t,T}$, in the form of

$$\begin{aligned} &f(\phi; t, T, V_0) = E^{\mathbb{Q}}[e^{\phi y_{t,T}}|y_0, V_0] = E^{\mathbb{Q}}\left[E^{\mathbb{Q}}[e^{\phi y_{t,T}}|y_t, V_t] \mid y_0, V_0\right] \\ &= E^{\mathbb{Q}}[e^{C(\phi,T-t)+D(\phi,T-t)V_t+A(\phi,T-t)}|y_0, V_0] = e^{C(\phi,T-t)+A(\phi,T-t)}E^{\mathbb{Q}}[e^{D(\phi,T-t)V_t}|y_0, V_0] \\ &= e^{C(\phi,T-t)+A(\phi,T-t)}g(D(\phi, T-t); t, V_0) \end{aligned} \tag{C12}$$

Appendix D

The Laplace Transform of the Realized Variance in Chapter 6

Appendix The Laplace transform of the realized variance $RV(0,T)$ in the Heston stochastic volatility model is given by,

$$E_0^Q[e^{-sRV(0,T)}] = \exp\left[A(T,s) - B(T,s)V_0\right] \tag{D1}$$

where

$$\begin{cases} A(T,s) = \dfrac{2\kappa\theta}{\sigma_V^2}\log\left(\dfrac{2\gamma(s)e^{\frac{(\gamma(s)+\kappa)T}{2}}}{(\gamma(s)+\kappa)(e^{\gamma(s)T}-1)+2\gamma(s)}\right) \\ B(T,s) = \dfrac{2s(e^{\gamma(s)T}-1)}{T\left[(\gamma(s)+\kappa)(e^{\gamma(s)T}-1)+2\gamma(s)\right]} \\ \gamma(s) = \sqrt{\kappa^2 + 2\dfrac{s\sigma_V^2}{T}} \end{cases}$$

Appendix E

Proof for Chapter 7

For the SVJJ model, Feynman-Kac theorem implies that $f(\phi; t, \tau, V_t)$ satisfies

$$\begin{cases} -f_\tau + \kappa^{\mathbb{Q}}(\theta^{\mathbb{Q}} - V)f_V + \frac{1}{2}\sigma^2 V f_{VV} + \lambda E^{\mathbb{Q}}[f(V + Z^V) - f(V)|\mathscr{F}_t] = 0 \\ f(\phi; t + \tau, 0, V) = e^{\phi V} \end{cases}$$

Following the solution procedure used by Heston (1993), Bakshi et al. (1997), Duffie et al. (2000) and many others, we can solve this PIDE in closed-form by using a guess of the affine-form solution as,

$$f(\phi; t, \tau, V_t) = e^{C(\phi,\tau)+D(\phi,\tau)V_t+A(\phi,\tau)} \tag{E1}$$

One can then substitute this function into the PIDE to reduce it to three ordinary differential equations,

$$\begin{cases} D_\tau = -\kappa^{\mathbb{Q}} D + \frac{1}{2}\sigma_V^2 D^2 \\ C_\tau = \kappa^{\mathbb{Q}}\theta^{\mathbb{Q}} D \\ A_\tau = \lambda E^{\mathbb{Q}}[e^{DZ_t^V} - 1|\mathscr{F}_t] \end{cases}$$

with the initial conditions

$$C(\phi, 0) = 0, \quad D(\phi, 0) = \phi, \quad A(\phi, 0) = 0 \tag{E2}$$

The solutions to these ODEs are

$$\begin{cases} A(\phi,\tau) = \dfrac{2\mu_V\lambda}{2\mu_V\kappa^{\mathbb{Q}} - \sigma_V^2}\ln\left(1 + \dfrac{\phi(\sigma_V^2 - 2\mu_V\kappa^{\mathbb{Q}})}{2\kappa^{\mathbb{Q}}(1-\mu_V\phi)}(e^{-\kappa^{\mathbb{Q}}\tau} - 1)\right) \\ C(\phi,\tau) = \dfrac{-2\kappa\theta}{\sigma_V^2}\ln\left(1 + \dfrac{\sigma_V^2\phi}{2\kappa^{\mathbb{Q}}}(e^{-\kappa^{\mathbb{Q}}\tau} - 1)\right) \\ D(\phi,\tau) = \frac{2\kappa^{\mathbb{Q}}\phi}{\sigma_V^2\phi + (2\kappa^{\mathbb{Q}} - \sigma_V^2\phi)e^{\kappa^{\mathbb{Q}}\tau}} \end{cases}$$

Bibliography

Ait-Sahalia, Y. & Kimmel, R. (2007), 'Maximum likelihood estimation of stochastic volatility models', *Journal of Financial Economics* **83**(2), 413–452.

Andersen, T., Benzoni, L. & Lund, J. (2002), 'An empirical investigation for continuous-time equity return models', *Journal of Finance* **57**, 1239–1284.

Andersen, T. & Bollerslev, T. (1997), 'Intraday periodicity and volatility persistence in financial markets', *Journal of empirical finance* **4**(2-3), 115–158.

Andersen, T. & Bollerslev, T. (1998), 'Deutsche mark-dollar volatility: Intraday activity patterns, macroeconomic announcements, and longer run dependencies', *the Journal of Finance* **53**(1), 219–265.

Andersen, T., Chung, H. & Sørensen, B. (1999), 'Efficient method of moments estimation of a stochastic volatility model: A Monte Carlo study', *Journal of Econometrics* **91**(1), 61–87.

Andreasen, J. (2006), 'Stochastic Volatility for Real', *Bank of America Fixed Income Quantitative Research, available at papers. ssrn. com* .

Bakshi, G., Cao, C. & Chen, Z. (1997), 'Empirical Performance of Alternative Option Pricing Models', *The Journal of Finance* **52**(5), 2003–2049.

Barndorff-Nielsen, O. & Shephard, N. (2003), 'Realized power variation and stochastic volatility models', *Bernoulli* **9**(2), 243–265.

Bates, D. (1996), 'Jumps and Stochastic Volatility: Exchange Rate Processes Implicit in Deutsche Mark Options', *Review of Financial Studies* pp. 69–107.

Bates, D. (2000), 'Post-'87 crash fears in the S&P 500 futures option market', *Journal of Econometrics* **94**(1-2), 181–238.

Bates, D. (2006), 'Maximum Likelihood Estimation of Latent Affine Processes', *Review of Financial Studies* **19**(3), 909–965.

Benth, F., Groth, M. & Kufakunesu, R. (2007), 'Valuing volatility and variance swaps for a non-Gaussian Ornstein-Uhlenbeck stochastic volatility model', *Applied Mathematical Finance* **14**(4), 347.

Berg, A., Meyer, R. & Yu, J. (2004), 'DIC as a model comparison criterion for stochastic volatility models', *Journal of Business and Economic Statistics* **22**, 107–120.

Black, F. (1976), Studies of stock price volatility changes, *in* 'Proceedings of the 1976 Meetings of the Business and Economic Statistics Section, American Statistical Association', Vol. 177, p. 81.

Black, F. & Scholes, M. (1973), 'The pricing of options and corporate liabilities', *J. of Political Economy* **81**, 637–654.

Breeden, D. (1979), 'An Intertemporal Asset Pricing Model with Stochastic Investment and Consumption Opportunities', *Journal of Financial Economics* **7**, 265–296.

Brenner, M., Shu, J. & Zhang, J. (2007), 'The Market for Volatility Trading; VIX Futures', *working paper, available at http://archive.nyu.edu/handle/2451/26289* .

Broadie, M., Chernov, M. & Johannes, M. (2007), 'Model Specification and Risk Premia: Evidence from Futures Options', *The Journal of Finance* **62**(3), 1453–1490.

Broadie, M. & Jain, A. (2007), 'VIX and VIX futures', *Working paper* .

Broadie, M. & Jain, A. (2008*a*), 'Pricing and Hedging Volatility Derivatives', *The Journal of Derivatives* **15**(3), 7–24.

Broadie, M. & Jain, A. (2008*b*), 'The Effect Of Jumps And Discrete Sampling On Volatility And Variance Swaps', *International Journal of Theoretical and Applied Finance* **11**(8), 761–797.

Broadie, M. & Kaya, Ö. (2006), 'Exact simulation of stochastic volatility and other affine jump diffusion processes', *Operations Research* **54**(2), 217–231.

Brockhaus, O. & Long, D. (2000), 'Volatility swaps made simple', *Risk* **19**(1), 92–95.

Bronshtein, I., Semendyayev, K., Musiol, G. & Muehlig, H. (1997), *Handbook of mathematics*, Springer Berlin.

Carr, P. & Corso, A. (2001), 'Commodity Covariance Contracting', *Energy and Power Risk Management* .

Carr, P. & Lee, R. (2005), 'Robust replication of volatility derivatives', *Working paper: Courant Institute, NYU http://math.nyu.edu/research/carrp/research.html* .

Carr, P. & Lee, R. (2009), 'Volatility Derivatives', *Annual Review of Financial Economics* **1**, 319–339.

Carr, P. & Madan, D. (1998), 'Towards a Theory of Volatility Trading', *Volatility, Risk Publications, R. Jarrow, ed* pp. 417–427.

Carr, P. & Wu, L. (2006), 'A Tale of Two Indices ', *Journal of Derivatives* **13**(3), 13–29.

Chernov, M., Ronald Gallant, A., Ghysels, E. & Tauchen, G. (2003), 'Alternative models for stock price dynamics', *Journal of Econometrics* **116**(1-2), 225–257.

Cont, R. & Tankov, P. (2004), *Financial modelling with jump processes*, Chapman & Hall/CRC.

Cox, J., Ingersoll Jr, J. & Ross, S. (1985), 'A Theory of the Term Structure of Interest Rates', *Econometrica* **53**(2), 385–407.

Daniel, G., Joseph, N. & Brée, D. (2005), 'Stochastic volatility and the goodness-of-fit of the Heston model', *Quantitative Finance* **5**(2), 199–211.

Davis, R. & Mikosch, T. (1998), 'Limit theory for the sample ACF of stationary process with heavy tails with applications to ARCH', *Annals of Statistics* **26**, 2049–80.

Delbaen, F. & Schachermayer, W. (1994), 'A general version of the fundamental theorem of asset pricing', *Mathematische Annalen* **300**(1), 463–520.

Demeterfi, K., Derman, E., Kamal, M. & Zou, J. (1999), 'More than you ever wanted to know about volatility swaps', *Goldman Sachs Quantitative Strategies Research Notes* .

Derman, E. & Kani, I. (1994), 'Riding on a smile', *Risk* **7**(2), 32–39.

Detemple, J. & Osakwe, C. (2000), 'The valuation of volatility options', *Review of Finance* **4**(1), 21.

Dragulescu, A. & Yakovenko, V. (2002), 'Probability distribution of returns in the Heston model with stochastic volatility', *Quantitative Finance* **2**(6), 443–453.

Duan, J. & Yeh, C. (2007), 'Jump and Volatility Risk Premiums Implied by VIX'.

Duffie, D., Pan, J. & Singleton, K. (2000), 'Transform Analysis and Asset Pricing for Affine Jump-diffusions', *Econometrica* **68**(6), 1343–1376.

Dupire, B. (1994), 'Pricing with a smile', *Risk* **7**(1), 18–20.

Dupire, B. (2005), 'Exploring Volatility Derivatives: New Advances in Modelling', *presentation at NYU* .

Elliott, R., Siu, T. & Chan, L. (2007), 'Pricing Volatility Swaps Under Heston's Stochastic Volatility Model with Regime Switching', *Applied Mathematical Finance* **14**(1), 41–62.

Eraker, B. (2004), 'Do Stock Prices and Volatility Jump? Reconciling Evidence from Spot and Option Prices', *The Journal of Finance* **59**(3), 1367–1404.

Eraker, B., Johannes, M. & Polson, N. (2003), 'The Impact of Jumps in Volatility and Returns', *The Journal of Finance* **58**(3), 1269–1300.

Forbes, C. (2007), 'Inference for a Class of Stochastic Volatility Models Using Option and Spot Prices: Application of a Bivariate Kalman Filter', *Econometric Reviews* **26**(2), 387–418.

Fouque, J., Sircar, K. & Papanicolaou, G. (2000), *Derivatives in Financial Markets with Stochastic Volatility*, Cambridge University Press.

Friz, P. & Gatheral, J. (2005), 'Valuation of volatility derivatives as an inverse problem', *Quantitative Finance* **5**(6), 531–542.

Gallant, A. R., Hsie, D. & Tauchen, G. (1997), 'Estimation of stochastic volatility models with diagnostics', *Journal of Econometrics* **81**, 159–192.

Garman, M. (1977), 'A general theory of asset valuation under diffusion state processes'.

Glasserman, P. (2003), *Monte Carlo Methods in Financial Engineering*, New York, Springer.

Glasserman, P. (2004), *Monte Carlo methods in financial engineering*, Springer.

Granger, C. & Sin, C. (2000), 'Modelling the absolute returns of different stock indices: exploring the forecastability of an alternative measure of risk', *Journal of Forecasting* **19**(4), 277–298.

Grunbichler, A. & Longstaff, F. (1996), 'Valuing Futures and Options on Volatility', *Journal of Banking and Finance* **20**(6), 985–1001.

Harrison, J. & Kreps, D. (1979), 'Martingales and arbitrage in multiperiod securities markets', *Journal of economic theory* **20**(3), 381–408.

Harrison, J. & Pliska, S. (1981), Martingales and stochastic integrals in the theory of continuous trading, *in* 'Proc. Appl', Vol. 11, pp. 215–260.

Heston, S. (1993), 'A closed-form solution for options with stochastic volatility with applications to bond and currency options', *Review of Financial Studies* **6**(2), 327–343.

Heston, S. (2000), 'Derivatives on Volatility: Some Simple Solutions Based on Observables'.

Heston, S. & Nandi, S. (2000), 'A closed-form GARCH option valuation model', *Review of Financial Studies* **13**(3), 585.

Higham, D. & Mao, X. (2005), 'Convergence of Monte Carlo simulations involving the meanreverting square root process', *Journal of Computational Finance* **8**(3), 35–62.

Howison, S., Rafailidis, A. & Rasmussen, H. (2004), 'On the pricing and hedging of volatility derivatives', *Applied Mathematical Finance* **11**, 317–346.

Hull, J. & White, A. (1987), 'The pricing of options on assets with stochastic volatilities', *The Journal of Finance* **42**(2), 281–300.

Itkin, A. & Carr, P. (2010), 'Pricing swaps and options on quadratic variation under stochastic time change modelsdiscrete observations case', *Review of Derivatives Research* .

Jacod, J. & Protter, P. (1998), 'Asymptotic error distributions for the Euler method for stochastic differential equations', *Annals of Probability* **26**(1), 267–307.

Jacquier, E., Polson, N. & Rossi, P. (1994), 'Bayesian Analysis of Stochastic Volatility Models', *Journal of Business and Economic Statistics* **12**(4), 69–87.

Jacquier, E., Polson, N. & Rossi, P. (2004), 'Bayesian analysis of stochastic volatility models with fat-tails and correlated errors', *Journal of Econometrics* **122**(1), 185–212.

Javaheri, A., Wilmott, P. & Haug, E. (2004), 'GARCH and Volatility swaps', *Quantitative Finance* **4**(5), 589–595.

Johannes, M. & Polson, N. (2002), *MCMC Methods for Continuous-Time Financial Econometrics*, Handbook of Financial Econometrics (Elsevier, New York).

Johnson, N., Kotz, S. & Balakrishnan, N. (1970), *Continuous univariate distributions.*, New York: Wiley.

Jones, C. (2003), 'The dynamics of stochastic volatility: evidence from underlying and options markets', *Journal of Econometrics* **116**(1-2), 181–224.

Jung, J. (2006), 'Viva la variance: Despite large losses from simple variance swap trades in mid-2006, dealers are pitching more exotic instruments as a means of getting a cheaper, more tailored exposure to volatility', *Risk-London-Risk Magazine Limited-* **19**(10), 43.

Kahl, C. & Jackel, P. (2005), 'Not-so-complex logarithms in the Heston model', *Wilmott Magazine* pp. 94–103.

Karatzas, I., Lehoczky, J. & Shreve, S. (1991), 'Equilibrium models with singular asset prices', *Mathematical Finance* **1**(3), 11–29.

Lewis, A. (2000), *Option valuation under stochastic volatility*, Finance Press Newport Beach, CA.

Lin, Y. (2007), 'Pricing VIX futures: Evidence from integrated physical and risk-neutral probability measures', *Journal of Futures Markets* **27**(12), 1175.

Lin, Y. & Chang, C. (2009), 'VIX option pricing', *Journal of Futures Markets* **29**(6).

Lin, Y. & Chang, C. (2010), 'Consistent modeling of S&P 500 and VIX derivatives', *Journal of Economic Dynamics and Control* .

Little, T. & Pant, V. (2001), 'A finite-difference method for the valuation of variance swaps', *Journal of Computational Finance* **5**(1), 81–101.

Lu, Z. & Zhu, Y. (2009), 'Volatility components: The term structure dynamics of VIX futures', *Journal of Futures Markets* **30**(3), 230–256.

Madan, D., Carr, P. & Chang, E. (1998), 'The variance gamma process and option pricing', *Review of Finance* **2**(1), 79.

Merton, R. (1973), 'The theory of rational option pricing', *Bell Journal of Economics and Management Science* **1**, 141–183.

Merton, R. (1976), 'Option pricing when underlying stock returns are discontinuous', *Journal of Financial Economics* **3**(1-2), 125–144.

Meyer, R. & Yu, J. (2000), 'BUGS for a Bayesian analysis of stochastic volatility models', *The Econometrics Journal* **3**(2), 198–215.

Ortelli, C. & Trojani, F. (2005), 'Robust efficient method of moments', *Journal of Econometrics* **128**(1), 69–97.

Pan, J. (2002), 'The jump-risk premia implicit in options: evidence from an integrated time-series study', *Journal of Financial Economics* **63**(1), 3–50.

Poularikas, A. (2000), *The transforms and applications handbook*, CRC Press.

Psychoyios, D., Dotsis, G. & Markellos, R. (2007), 'A Jump Diffusion Model for VIX Volatility Options and Futures', *Working paper, availble at http://www.aislab.aueb.gr/accfin/DownLoads/seminars/VIX%20Options.pdf* .

Ruiz, E. (1994), 'Quasi-maximum likelihood estimation of stochastic volatility models', *Journal of Econometrics* **63**(1), 289–306.

Sandmann, G. & Koopman, S. (1998), 'Estimation of stochastic volatility models via Monte Carlo maximum likelihood', *Journal of Econometrics* **87**(2), 271–301.

Schöbel, R. & Zhu, J. (1999), 'Stochastic Volatility With an Ornstein–Uhlenbeck Process: An Extension', *European Finance Review* **3**(1), 23–46.

Schürger, K. (2002), 'Laplace transforms and suprema of stochastic processes', *Advances in Finance and Stochastics: Essays in Honour of Dieter Sondermann* .

Scott, L. (1997), 'Pricing Stock Options in a JumpDiffusion Model with Stochastic Volatility and Interest Rates: Applications of Fourier Inversion Methods', *Mathematical Finance* **7**(4), 413–426.

Sepp, A. (2007), 'Affine models in mathematical finance: an analytical approach'.

Sepp, A. (2008*a*), ' pricing options on realized variance in the heston model with jumps in returns and volatility', *Journal of Computational Finance* **11**(4), 33–70.

Sepp, A. (2008*b*), 'Option pricing: Vix option pricing in a jump-diffusion model', *RISK-LONDON-RISK MAGAZINE LIMITED-* **21**(4), 84.

Silva, A., Prange, R. & Yakovenko, V. (2004), 'Exponential distribution of financial returns at mesoscopic time lags: a new stylized fact', *Physica A: Statistical Mechanics and its Applications* **344**(1-2), 227–235.

Singleton, K. (2001), 'Estimation of affine asset pricing models using the empirical characteristic function', *Journal of Econometrics* **102**(1), 111–141.

Stein, E. & Stein, J. (1991), 'Stock price distributions with stochastic volatility: an analytic approach', *Review of Financial Studies* **4**(4), 727–752.

Swishchuk, A. (2004), 'Modeling of variance and volatility swaps for financial markets with stochastic volatilities', *Wilmott magazine, September Issue, Technical Article* pp. 64–72.

Voit, J. (2005), *The Statistical Mechanics of Financial Markets*, Springer.

Wiggins, J. (1987), 'Option values under stochastic volatility: Theory and empirical estimates', *Journal of Financial Economics* **19**(2), 351–372.

Wilmott, P. (1998), *Derivatives: The Theory and Practice of Financial Engineering*, J. Wiley.

Wilmott, P. (2000), *Paul Wilmott on Quantitative Finance*, London; New York: Springer, 2002.

Wilmott, P., Dewynne, J. & Howison, S. (1993), *Option Pricing: Mathematical Models and Computation*, Oxford Financial Press.

Windcliff, H., Forsyth, P. & Vetzal, K. (2006), 'Pricing methods and hedging strategies for volatility derivatives', *Journal of Banking & Finance* **30**(2), 409–431.

Yu, J. (2005), 'On leverage in a stochastic volatility model', *Journal of Econometrics* **127**(2), 165–178.

Yu, J. & Meyer, R. (2006), 'Multivariate Stochastic Volatility Models: Bayesian Estimation and Model Comparison', *Econometric Reviews* **25**(2), 361–384.

Zhang, J. E. & Huang, Y. (2010), 'The CBOE S&P 500 three-month variance futures', *Journal of Futures Markets,* **30**, 48–70.

Zhang, J., Shu, J. & Brenner, M. (2010), 'The new market for volatility trading', *Journal of Futures Markets, (forthcoming, articles online in advance of print)* .

Zhang, J. & Zhu, Y. (2006), 'VIX Futures', *Journal of Futures Markets* **26**(5), 521–531.

Zhang, Z. & Lim, K. (2006), 'An Non-Lattice Pricing Model Of American Options Under Stochastic Volatility', *Journal of Futures Markets* **26**(5), 417–448.

Zhou, H. (2000), 'A Study of the Finite Sample Properties of EMM, GMM, QMLE, and MLE for a Square-Root Interest Rate Diffusion Model', *Working paper, availble ar SSRN* .

Zhu, S. & Lian, G. (2009*a*), 'An analytical pricing formula for vix futures and its applications in empirical study', *Journal of Futures Markets, (Submitted)* .

Zhu, S. & Lian, G. (2009*b*), 'Analytically pricing variance swaps with stochastic volatility and random jumps', *Journal of Economic Dynamics and Control, (Submitted)* .

Zhu, S. & Lian, G. (2009*c*), 'Analytically pricing volatility swaps with stochastic volatility', *Quantitative Finance, (Submitted)* .

Zhu, S. & Lian, G. (2009*d*), 'A closed-form exact solution for pricing variance swaps with stochastic volatility', *Mathematical Finance, (Accepted)* .

Zhu, S. & Lian, G. (2009*e*), 'On the valuation of variance swaps with stochastic volatility', *Applied Mathematical Finance, (Submitted)* .

Zhu, S. & Lian, G. (2009*f*), 'Pricing forward-start variance swaps with stochastic volatility', *The IMA Journal of Applied Mathematics, (Submitted)* .

Zhu, S. & Lian, G. (2010*a*), 'On the convexity correction approximation in pricing volatility swaps and vix futures', *Review of Derivatives Research, (Submitted)* .

Zhu, S. & Lian, G. (2010*b*), 'Pricing vix options with stochastic volatility and random jumps', *SIAM Journal on Financial Mathematics, Submitted (Submitted)* .

Zhu, Y. & Zhang, J. (2007), 'Variance Term Structure and VIX Futures Pricing', *International Journal of Theoretical and Applied Finance* **10**, 111–127.

Publication List of the Author

Zhu, S. & Lian, G. (2009*a*), 'An analytical pricing formula for vix futures and its applications in empirical study', *Journal of Futures Markets, (Submitted)* .

Zhu, S. & Lian, G. (2009*b*), 'Analytically pricing variance swaps with stochastic volatility and random jumps', *Journal of Economic Dynamics and Control, (Submitted)* .

Zhu, S. & Lian, G. (2009*c*), 'Analytically pricing volatility swaps with stochastic volatility', *Quantitative Finance, (Submitted)* .

Zhu, S. & Lian, G. (2009*d*), 'A closed-form exact solution for pricing variance swaps with stochastic volatility', *Mathematical Finance, (Accepted).*

Zhu, S. & Lian, G. (2010*e*), 'On the convexity correction approximation in pricing volatility swaps and vix futures', *Applied Stochastic Models in Business and Industry, (Submitted)* .

Zhu, S. & Lian, G. (2009*f*), 'On the valuation of variance swaps with stochastic volatility', *Applied Mathematical Finance, (Submitted).*

Zhu, S. & Lian, G. (2009*g*), 'Pricing forward-start variance swaps with stochastic volatility', *The IMA Journal of Applied Mathematics, (Submitted)* .

Zhu, S. & Lian, G. (July 2009*h*), 'Pricing variance swaps with stochastic

volatility', *Proceedings of 2009 International Conference of Financial Engineering, London* **Vol. II**, 1359–1364.

Lian, G. & Zhu, S. (2010*h*), 'Pricing VIX Options with Stochastic Volatility and Random Jumps', , *SIAM Journal on Financial Mathematics, (Submitted)* .