Binary Choice Probabilities on Mixture Sets

CMSS Summer Workshop

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"[O]ne can explain experimental analyses of decision making under risk better (and simpler) as Expected Utility plus noise – rather than through some higher level functional – as long as one specifies the noise appropriately." (Hey, 1995, p.640)

Binary stochastic choice

- Let A be a set of alternatives.
- Let $P : A \times A \rightarrow [0, 1]$ be a **binary choice probability (BCP)**.
- If $a \neq b$ then $P(a, b)$ is the probability of choosing a from $\{a, b\}$. (We leave $P(a, a)$ uninterpreted.)

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- In particular,

$$P(a, a) = \frac{1}{2}$$

for any $a \in A$.

Strong Utility Representation (SUR)

Definition: The BCP P has a **strong utility representation (SUR)** if there exists a utility function $u : A \rightarrow \mathbb{R}$ such that

$$P(a, b) \geq P(c, d) \quad \Leftrightarrow \quad u(a) - u(b) \geq u(c) - u(d)$$

for any $a, b, c, d \in A$.

- This is a standard psychophysical model of choice behaviour: probability of choice depends on the relative stength of stimuli.

Strong Utility Representation (SUR)

What are sufficient conditions (on P) for the existence of a SUR?

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Strong Utility Representation (SUR)

- Debreu showed that the following two conditions suffice for a SUR:

For any $x \in (0,1)$ and any $a, b, c, a', b' \in A$

$$P(a,b) \geq P(a',b') \quad \Leftrightarrow \quad P(a,a') \geq P(b,b') \qquad \textbf{(QC)}$$

$$P(a,b) \geq x \geq P(a,c) \quad \Rightarrow \quad P(a,e) = x \text{ for some } e \in A \qquad \textbf{(S)}$$

Strong Utility Representation (SUR)

The *necessity* of QC is easy to see:

$$P(a,b) \geq P(a',b') \quad \Leftrightarrow \quad P(a,a') \geq P(b,b')$$

$$u(a) - u(b) \geq u(a') - u(b') \quad \Leftrightarrow \quad u(a) - u(a') \geq u(b) - u(b')$$

Strong Utility Representation (SUR)

A weaker (and more intuitive) property than the QC:

Strong Stochastic Transitivity (SST) For all $a, b, c \in A$

$$P(a, b), P(b, c) \geq \frac{1}{2} \quad \Rightarrow \quad P(a, c) \geq \max\{P(a, b), P(b, c)\}$$

Risk and uncertainty

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Risk and uncertainty

- In Dagsvik (2008), A is the unit simplex in $\mathbb{R}^n$ interpreted as lotteries over a fixed set of n possible prizes.

- Dagsvik (2008) builds on Debreu (1958) – he adds two axioms and augments Debreu's proof – to obtain sufficient conditions for a SUR with *linear* utility.

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Risk and uncertainty

Strong Independence (SI) For all $a, b, a', b', c \in A$ and all $\lambda \in (0, 1)$

$$P(a, b) \geq P(a', b') \quad \Rightarrow \quad P(a\lambda c, b\lambda c) \geq P(a'\lambda c, b'\lambda c).$$

Risk and uncertainty

- *Here is an alternative approach, which uses Anscombe and Aumann (1963) rather than Debreu (1958):*

Risk and uncertainty

- Define a binary (preference) relation $\geq^*$ on $A \times A$ as follows:[1]

$$(a, d) \geq^* (b, c) \quad \Leftrightarrow \quad P(a, b) \geq P(c, d)$$

- An ordering on two-state *Anscombe-Aumann (AA) acts*.
  - "Act(ions)" identified with state-contingent consequences.
  - Consequences may be lotteries (objective risk).
- Then P has a SUR iff $\geq^*$ has a *Subjective Expected Utility (SEU)* representation with equi-probable states:

$$(a, d) \geq^* (b, c) \quad \Leftrightarrow \quad P(a, b) \geq P(c, d)$$

$$\tfrac{1}{2}u(a) + \tfrac{1}{2}u(d) \geq \tfrac{1}{2}u(b) + \tfrac{1}{2}u(c) \quad \Leftrightarrow \quad u(a) - u(b) \geq u(c) - u(d)$$

[1] An old idea: see Suppes and Winet (1955, p.261), who credit Donald Davidson.

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Following the lead of Anscombe and Aumann, we obtain sufficient conditions on $\geq^*$ for the existence of a SEU representation with **linear** *utility and* **equi-probable** *states, then translate these conditions into the corresponding restrictions on P.*

New representation theorems

- This proof strategy turns out to be very powerful and very flexible. We can:
  - Replace topological arguments with elementary linear algebra.
  - Strengthen Dagsvik's result by weakening QC to SST.
  - Develop new SUR representation theorems that impose alternative restrictions on u (besides linearity).

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New representation theorems

Definition Given some $M \subseteq A$ we say that $u : A \to \mathbb{R}$ is *M-linear* if

$$u(M) = u(A)$$

and

$$u(\lambda a + (1-\lambda) b) = \lambda u(a) + (1-\lambda) u(b)$$

for any $a \in A$, any $b \in M$ and any $\lambda \in [0,1]$.

New representation theorems

- Several M-linear forms of utility (besides EU) are commonly used to model choice under risk or uncertainty.

- We give a general "recipe" based on a generalisation of the Anscombe-Aumann approach.

- May compare EU with rival (M-linear) utility forms within a random choice framework.

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Given an M-linear class $\mathcal{U}$ of utility functions, what are sufficient conditions for a BCP to possess a SUR with respect to some $u \in \mathcal{U}$?

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Postscript

- Empirical challenges to so-called Fechnerian models (such as the SUR): strength of preference versus ease of comparison (e.g., dominance).