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USING COUNTEREXAMPLES, PUZZLES AND PROVOCATIONS FOR ENHANCING TEACHING AND LEARNING OF CALCULUS

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This paper describes author's experiences in both teaching with and research on counterexamples, puzzles and provocations in calculus as a pedagogical strategy. The results of several experimental studies with students and teachers/lecturers of calculus are presented and discussed. Examples of incorrect statements (to be disproved by counterexamples), selected puzzles and provocations offered to the participants of the studies are provided. The overwhelming statistics from those studies showed that both groups – students and lecturers – were very positive about using counterexamples, puzzles and provocations in teaching and learning of calculus. They found the strategy to be effective in the sense that it could help to: deepen conceptual understanding; reduce or eliminate common misconceptions; advance one's mathematical thinking beyond the merely procedural or algorithmic; enhance critical thinking skills; expand a student's 'example space' of functions with interesting properties; make teaching/learning more active and creative.

COUNTEREXAMPLES IN CALCULUS

Dealing with counterexamples for the first time can be challenging for students. When they hear they can disprove a wrong statement by providing one counterexample, many students think they can "prove" a correct statement by showing an example. Even if they know they cannot prove a theorem by providing only examples, it is hard for some students to accept the fact that a single counterexample disproves a statement. Some students believe that a particular counterexample is just an exception to the rule at hand, and that no other 'pathological' cases exist. Selden & Selden (1998) have articulated these ideas in the following way: "Students quite often fail to see a single counterexample as disproving a conjecture. This can happen when a counterexample is perceived as 'the only one that exists', rather than being seen as generic. " A similar observation has been made by Tall (1991): "If an individual works in a restricted context in which all the examples considered have a certain property, then, in the absence of counterexamples, the mind assumes the known properties to be implicit in other contexts".

With experience students understand the role of counterexamples and become interested in creating them. They learn that it is useful to have at their disposal a large assortment of graphs and functions with interesting properties building their 'example space' (Mason & Watson, 2001). Understanding the anomalies and distinguishing features of these functions will provide them a natural starting point for developing their own counterexamples. To construct counterexamples most of the students try to use familiar graphs of polynomial functions, basic trigonometric functions and their inverses, piecewise functions (like step functions), and graphs with sharp corners (like that of the absolute value function $|x|$ or cusps $\sqrt[3]{x^2}$). Search for possible counterexamples to harder examples can add to the student's arsenal graphs of more exotic functions like oscillations with and without damping factors (like $\sin\frac{1}{x}$ and $x\sin\frac{1}{x}$), and even more exotic functions like Dirichlet and Weierstrass functions, which are notable in the history of mathematics. In developing counterexamples students are forced to pay attention to every detail in a statement – the word order, the shape of brackets defining intervals, whether the statement applies to a point or to an interval, and so on. Below are examples of incorrect statements that can be discussed with the students in an introductory calculus course:

Statement 1. If a function $y = f(x)$ is differentiable on (a, b) and its derivative is positive at a point $x = c$ in (a, b), then there is a neighbourhood of the point $x = c$ where the function is increasing.

Statement 2. If a function $y = f(x)$ is differentiable on its domain and its derivative is positive for all x from its domain, then the function is increasing everywhere on its domain.

Statement 3. A function $y = f(x)$ is bounded on $\mathbb{R}$ if for any $x \in \mathbb{R}$ there is $M > 0$ such that $|f(x)| \le M$.

Statement 4. Every continuous and bounded function on $\mathbb{R}$ takes on its extreme values.

Statement 5. If a function is continuous on the interval (a, b) and its graph is a smooth curve (no sharp corners) on that interval then the function is differentiable at every point on (a, b).

Using counterexamples to disprove wrong statements can generate many questions for discussion. What changes will make the statement at hand correct? How can you change a counterexample and have it remain one? Can you think of other statements that your counterexample refutes? Can you find another type of function altogether that will be a counterexample or construct a general class of counterexamples to the statement at hand?

There are different ways of using counterexamples in teaching. One can give students mixtures of correct and incorrect statements; ask students to create their own wrong statements and associated counterexamples; made a deliberate error in a lecture; ask students to spot mistakes in their textbook; give students extra credit for providing counterexamples to challenging statements posed in class; and include on assignments and tests questions that require students to construct counterexamples.

The author conducted two studies that were summarized in (Klymchuk, 2014): on students' attitudes towards the use of counterexamples and their performance. In the first (international) study, that involved 612 students from 10 universities in different countries, 92% of the participants found the use of counterexamples to be very effective. The students reported it helped them to understand concepts better, prevent mistakes, develop logical and critical thinking, and made learning mathematics more challenging, interesting and creative. The second (case) study showed that the use of counterexamples in teaching could improve students' performance on test questions that required conceptual understanding.

A collection of 80+ incorrect statements from a first-year introductory calculus course and suggested counterexamples to them illustrated by graphs can be found in (Klymchuk, 2010). The book by Gelbaum & Olmsted (1964) is a classical resource for using counterexamples in *advanced* calculus and mathematical analysis courses.

PUZZLES, PARADOXES AND SOPHISMS IN CALCULUS

A significant number of tertiary engineering and mathematics students drop out from their calculus courses during the first-year not because the courses are too difficult but because, in their words, they 'are too dry and boring'. There are even such special terms as *emotional disengagement* and *academic disinterest* (e.g. Blondal & Adalbjarnardottir, 2012). Interesting puzzles, paradoxes and sophisms can engage students' emotions, creativity and curiosity and also enhance their conceptual understanding, critical thinking skills, problem-solving strategies and lateral thinking "outside the box". By a *puzzle* we mean non-standard, non-routine and unstructured problem presented in an entertaining way. Typically a puzzle appears deceptively simple but has a surprised answer. By a *paradox* we mean a surprising, unexpected, counter-intuitive statement that looks invalid but in fact is true. By a *sophism* we mean intentionally invalid reasoning that looks formally correct, but in fact contains a subtle mistake or flaw. I mention here two famous examples that amaze many students.

Paradox: Torricelli's Trumpet. There is not enough paint in the world to paint the infinite area bounded by the curve $y = \frac{1}{x}$, the x-axis, and the line $x = 1$. However, one can rotate the area around the x-axis and the resulting solid of revolution would have a finite volume of π cubic units. One can fill the solid with π cubic units of paint and thus cover the cross-section area with paint.

The paradox demonstrates a fundamental difference between the 'mathematical' universe and the 'physical' universe that enlighten many students.

Sophism: An Infinitely Fast Fall. Imagine a cat sitting on the top of a ladder learning against a wall. Suppose that the bottom of the ladder of the length l is being pulled away from the wall horizontally at a uniform rate x'. The relationship between the vertical and horizontal distances from the ends of the ladder to the corner at time t is expressed by the Pythagoras Theorem: $y(t) = \sqrt{l^2 - x^2(t)}$. We can 'prove' that the cat speeds up, until eventually falling infinitely fast:

$$\lim_{x \to l} y' = \lim_{x \to l} \left(-\frac{xx'}{\sqrt{l^2 - x^2}} \right) = -\infty$$

Students normally check all calculations and are surprised that they are correct. The sophism illustrates the importance of making correct assumptions when solving application problems.

A collection of paradoxes and sophisms in calculus can be found in (Klymchuk & Staples, 2013).

PROVOCATIONS IN CALCULUS

By a *provocation* we mean a question that looks like a routine question but in fact has a catch. The intention of the study on attention to detail (Klymchuk, 2015) was to draw school mathematics teachers' attention to the importance of checking properties of the functions in the question and *all* conditions of a theorem or a formula and *before* applying it. In other words - checking their 'discipline of noticing' (Mason, 2002).The participants were given a mini-test containing questions that provoked them to use formulae or theorems that were not applicable in those cases. The vast majority of the participants did not notice any catch and failed most of the questions. Examples of calculus questions from the test are below.

Question 1. Show that the equation $\frac{x+\sqrt{x+1}}{x-1}=0$ has a solution on the interval [0, 2].

Question 2. Find the derivative of the function $y=\ln(2\sin(3x)-4)$.

Question 3. Find the integral $\int_{-1}^{1}\frac{1}{x}dx$.

In the questionnaire following the discussion of the results of the test the vast majority of the participants reported that they would definitely make changes in their teaching practice. Some of the comments are below:

> Introduce tricks like this to class to make them think; keep encouraging and creating environment where a deep conceptual knowledge is cultivated; encourage and reward checking of answers; more emphasis on the validity of solutions; teach them to examine the question thoroughly; give students more questions that will force them to think about the conditions surrounding the questions; I would encourage students to think through questions carefully; I try to make my students think more about restricted domains; give them problems occasionally that will 'trip' them up if they have not gone back and re-assessed their solutions; more emphasis on the nature of problem solving; stop answering impulsively, think before respond; I will expose students to such questions to get them to think more deeply about the conditions.

Teachers' comments imply that they would enhance their teaching of calculus by moving from mere procedural teaching to conceptual. This would help smooth the transition from school to university mathematics, in particular calculus.

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