MATHEMATICAL ANALYSIS OF THE CHAOTIC BEHAVIOR IN MONETARY POLICY GAMES

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Declaration

I hereby declare that this submission is my own work and that, to the best of my knowledge and belief, it contains no material previously published or written by another person nor material which to a substantial extent has been accepted for the qualification of any other degree or diploma of a university or other institution of higher learning.

Signature of candidate

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Dedication

To AmirSalar

Abstract

This thesis discusses the concept of chaos in monetary policy games, which I believe to be novel. The mathematical framework developed in this thesis addresses two important problems in monetary theory, namely, the time-inconsistency and the complexity in designing, conducting and predicting the impacts of monetary policy on the economy. Considering a noncooperative non-zero-sum differential monetary policy game between the central bank and the public when the coefficients of the system depend on the state and control variables, it is shown that the co-state variables of both players are controllable in three solution concepts. The controllability of the co-state variables means that the monetary policy is time inconsistent even in the open loop Nash game, which is known as a time-consistent policy game in the literature. In other words, the results confirm that the structural time-inconsistency of monetary policy is almost always unavoidable.

To better understand how monetary policy affects the economy, we need to know the response of the public expectations. This can be achieved if the monetary policy behaves in a systematic manner (Walsh, 2003). To this end, this thesis tests the chaotic dynamics of the trajectories of both players. The results reveal that chaotic dynamics is possible in monetary policy games, and it seems that the source of this complexity comes from the chaotic behavior in the public expectations. Chaotic behavior in the strategy of the public sector creates serious difficulties for the policymaker, who wishes to design a policy that controls the business cycles.

Publications

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Chaotic behavior in monetary systems: comparision among different types of Taylor rule (2015)	International Journal of Social, Behavioral, Economic and Management Engineering, 9(8), 2316-2319.
Structural time inconsistency of monetary policy: a differential game theory approach (2018)	Prepared for the 42^{nd} Annual Meeting of the Association for Mathematics Applied to Social and Economics Sciences, Napoli, September 13-15.
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The dynamics of the monetary policy volatility: a spectrum-VAR approach (2019)	International Journal of Economics and Financial Issues, 9(1), 245-252.
Optimal monetary policy games in a growth model: a dynamic structural time inconsistency (2019)	International Journal of Economics and Financial Issues, Accepted for publication.

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Chapter 1

Introduction

Chaotic behavior has captured the attention of many mathematical economists in recent years. Chaos theory presumes that an apparently random system is both nonlinear and deterministic. In other words, the process can be determined if the system has no disturbances (Baba & Nagashima, 1999). However, revolutionary chaos theory has shown that an external disturbance or noise may not be the only source of randomness, and nonlinearity can lead to complexity. In his seminal paper, May (1976) argued that a very simple nonlinear model can exhibit extraordinary rich dynamical behavior. Owing to this behavior, researchers considered limits of predictability on the future behavior of the process from the historical data. In such a system, history can be irrelevant, and our process is time-independent. As a chaotic process is inherently unstable, it cannot be predicted in the long-term. With chaotic dynamics, long-term prediction is impossible even if the structure of the model is completely known (Baumol & Quandt, 1985; Baumol & Benhabib, 1989; DeCoster & Mitchell, 1991). A chaotic process warns the monetary policymaker that "the seemingly random behavior" may not be random at all.

1.1 Problem Statement

Most of the economic models assume that the external noise generates randomness and volatility for the economic system. However, the chaos revolution has revealed another source of this volatile behavior. For instance, in econometric analysis of linear systems, stochastic disturbance at least in some cases is inadequate and nonlinearity can be more appropriate (Moosavi Mohseni & Kilicman, 2013, 2014). Forecasting a chaotic economic system is extremely difficult (Baumol & Quandt, 1985) because the past history is of limited use in predicting the future behavior. Such unpredictability preserves difficulties for both policy designers and economic analysts. Furthermore, in the presence of chaos, anticipating the behaviors of other players is a non-trivial task.

In recent years there has been a growing interest in seeking evidence of nonlinear dynamics, in particular of chaos in economic data (Scheinkman, 1990; DeCoster & Mitchell, 1992; Serletis, 1996; Michener & Ravikumar, 1998; Kaas, 1998; Benhabib, Schmitt-Grohé & Uribe, 2002; Serletis & Shintani, 2003; Shintani & Linton, 2003; Serletis & Shintani, 2006; Kyrtsou & Labys, 2006; Barkoulas, 2008; Barnett & Duzhak, 2008; Grandmont, 2008; Airaudo & Zanna, 2012; Park & Whang, 2012). The dynamics of a chaotic economic system can be predicted only in the short-term; the long-term prediction is difficult even if the correct economic system is known. If the system is actually nonlinearity, the source of this nonlinearity must be identified.

Overall, research in the literature has revealed the existence and importance of nonlinearity and chaos in economic behavior. Therefore further research in this field, especially with regard to policy game outcomes is strongly suggested. The present study is a mathematical attempt to rigorously investigate chaos in monetary policy games.

1.2 Significance of the Study

Searching for chaotic behavior in monetary policy games is important for three reasons. First, we must know whether the trajectory of the evolving strategies of economic players is chaotic or not. Second, if the dynamics is nonlinear and chaotic, it cannot be forecasted by linear stochastic models that are traditionally used in economics. Third, by understanding the behavior of the system we can analyse the interaction between the monetary policy designer and the public and figure out the effectiveness of the monetary policy. To the best of my knowledge, the chaotic dynamics of the monetary policy games has not been discussed in the literature. This study is one attempt to fill this gap.

1.3 Literature Review

This section reviews some of the most important literature related to this thesis. The material preserves at three levels of specification. First, we have a critical overview on the *rule versus discretion* problem. We then discuss some research works related to the *chaos in the monetary systems*. Finally, we discuss a few research papers on the *chaotic games*.

1.3.1 Rule versus Discretion

The rule versus discretion problem in the monetary policy has been studied by many scholars, usually by applying optimization and game theory. The earliest study was due to Simons (1936) who concluded that the monetary policy should obey a definite, stable and legislative rule.

After Simon's seminal work, Friedman advocated a simple money supply rule with no feedback from future, current or past variables. Friedman (1948) recommended a monetary framework that operates under the *rule of law* rather than under the discretionary authority of administrators. He argued that governments must provide this stable framework to eliminate the uncertainty and undesirable political implications of discretionary actions. He posited that lags in policy, cause a large disturbance in any discretionary action. Friedman (1968) discussed the different roles of the monetary policy. He believed that the monetary policy cannot peg the interest rate or unemployment rate.

In the late 1970s, the United States of America and most European economies experienced a high and volatile inflation. The most important puzzle of this decade was the monetary authority allowed such a high inflation. Therefore, from the late 1970s to the early 1980s academic economists tried to explain the discrepancy between the optimal and actual rate of inflation. This problem first was solved by Kydland and Prescott (1977) and then was followed by two brilliant papers Barro and Gordon (1983a, 1983b).

Kydland and Prescott (1977) argued that optimal control theory is a powerful tool for analysing economic systems only if the expectations is invariant to the future of the policy, that is the optimal policy is inconsistent. They concluded that if policymakers would avoid discretion and commit themselves to the rule, they would improve their results. Thus, Kydland and Prescott advocated that to develop the dynamic consistent policy, policymakers should be constrained by suitable rules.

As an economical discription of their theoretical deduction Kydland and Prescott employed the inflation-unemployment (Phillips curve) model. This model is formulated as follows

$$U_t = \lambda(\pi_t^e - \pi_t) + U_t^n; \qquad \lambda > 0, \tag{1.1}$$

where U_t, U_t^n, π_t^e and π_t are current unemployment rate, natural unemployment rate, expected inflation and current inflation, respectively. They argued that this analysis depends mainly on the assumed price expectations. In the conventional approach (adaptive expectation) the optimal path of inflation and unemployment can be determined by the optimal control theory. Sargent and Wallace (1975) who accepted the rational expectation hypothesis suggested that

$$\pi_t^e = E(\pi_t). \tag{1.2}$$

The model is completed by the social objective function, which rationalizes the policy choice

$$S(\pi_t, U_t). \tag{1.3}$$

A consistent policy maximizes (1.3) subject to (1.1). This model can be conceptualized on a diagram. Initially, the expected inflation is zero (point A). At this point, the equilibrium is optimal but time-inconsistent. The policymaker first attempt to reach a position such as B on the highest possible indifference curve. As already mentioned, the private sector is rational, this action is expected by the policymaker. Therefore the expectations shift until the discretionary position reaches the equilibrium point C. Consequently, the utility at point A is higher than that at point C. This means that if the policymaker avoids discretion and commits to the rule, then he would achieve a better result.

Barro and Gordon (1983a) argued that in the natural rate model with the rational expectations, the systematic part of the monetary policy is irrelevant to real economic activities. Therefore, a discretionary policymaker can create surprising inflation which reduces unemployment through the Phillips curve. They believed that the most important distinction between rules and discretion depends on the presence or absence of precommitment. They assumed that the authority controls an instrument that directly effects inflation. They also argued that a positive theory of monetary policy and inflation

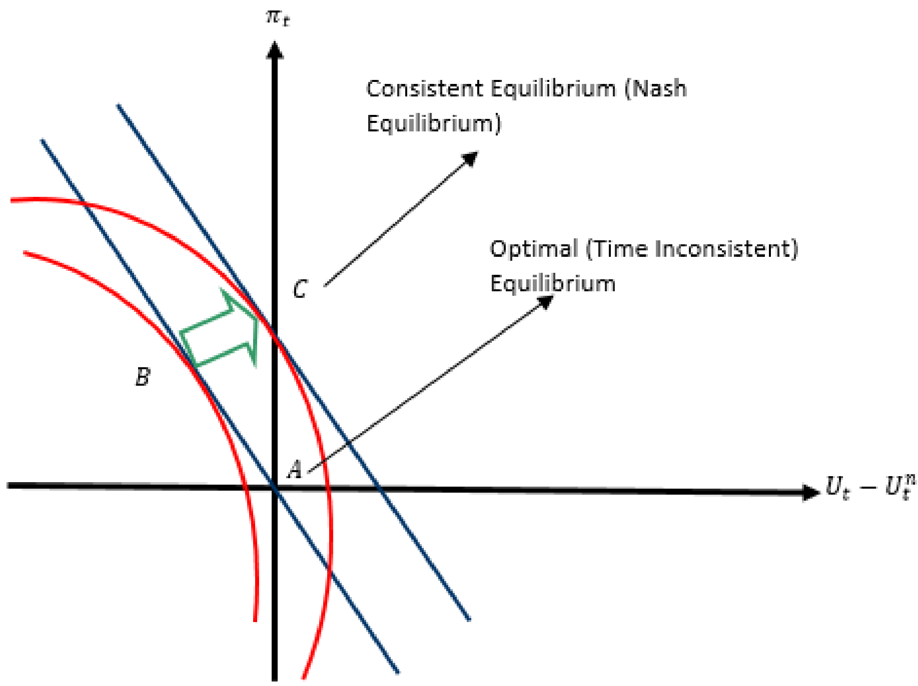

Figure 1.1: Time consistent and inconsistent equilibria

underlies the discretionary solution. In this regime, the inflation rate rises when the policymaker attaches greater benefits to unexpected inflation. The outcomes of this decision increase the natural rate of unemployment.

Barro and Gordon (1983b) included the reputation in their model that had been largely ignored in the previous literature. They employed a very simple model with three policy outcomes: discretion, rule and renege. Considering the time-inconsistency, they argued that a purely active policy leads to higher and stable long-term inflation.

Since the seminal work of Barro and Gordon, a large body of works have examined alternative solutions to the inflationary bias that arises under discretion. Some of the most important works are summarized below.

Canzoneri (1985) argued that Barro and Gordon's model resolved the credibility problem without accounting for private information. He reformulated a model to include the private information (Gray, 1976; Fischer, 1977). This paper provides a new structure for the old controversy over the subject rule versus discretion. Canzoneri concluded the maximizing utility under the private information constraints in the monetary policy game is one of the most important jobs of welfare economists.

According to Rogoff (1985a) the public can rationally accept that central banks have different form of the objective function. He examined various types of intermediate targeting regime in his model and argued that if supply shocks are important, the society prefers that central banks focus on monetary targeting rather than inflation rate targeting. He continued that the best intermediate target is the one that highly correlated with the society's objective function.

Backus and Driffill (1985a) examined the dynamic path of an economy after a change in regime. Their model is a version of Barro and Gordon (1983b) macro policy game. They questioned why government policy has tolerated such a high and persistent inflation over the past decade when a stable price level is desirable. They believed that if the public expects the government inflates then a tight monetary policy aimed at eliminating inflation will reduce the output below the natural rate. On the other hand, since the government policy is dynamically consistent, the government always finds it optimal to stick to the initial plan. In this paper, Backus and Driffill presumed two kinds of governments: wet and hard-nosed. They also considered the Kreps and Wilson (1982b) model of reputation and extended their reputational equilibrium analysis.

Using a new classical model, Andersen (1986) analysed rule versus discretion problem in the environment of asymmetric information between the government and the private sector. He argued that the cooperative solution to the game, which is obtained under constant growth rule is superior to the noncooperative solution. Under asymmetric information when the monetary authority has direct information about a state variable, that is not available to the private agents, the public should determine their expectations. However, the outcomes show that the private sector is never more disadvantaged under discretion than under a rule. This may explain why discretionary powers are admitted by monetary authorities.

Driffill (1988) conducted a survey employing the macro model adopted by Kydland and Prescott and most of the other researchers. He argued that the expectation formation mechanism is the punishment strategy of the private sector. He distinguished two types of modelling process in this research area: separating equilibria and pooling equilibria. A separating equilibrium occurs when the more inflationary government imitates the less inflationary government at the start of the game. He argued that in such an equilibrium the more inflationary government actually disciplines the less inflationary one. This conclusion opposes those of Backus and Driffill (1985a, 1985b) and Barro (1986). A pooling equilibrium occurs when one government is sufficiently less inflationary than the other government.

McCallum (1988) suggested a monetarist's rule for the conduct of monetary policy, which maintains the nominal GNP close to a smooth target growth path. He examined his suggested rules in the United States during the 1954-1985 period. A policy rule should also specify an instrumental variable that the monetary authority can control directly and/or accurately. Supposing that the natural rate hypothesis is valid, thus money should be neutral in the long-term. He proposed a money base rule, which yielded zero inflation in the United State during the study period.

J. B. Taylor (1993) argued that a good policy rule calling for changes in the money supply, monetary base, or short-term interest rate will respond to changes in the price level or real income. Policy rule has empirically significant advantages for policymakers because it is superior to discretion in the time-inconsistent case. The advantage of the rule over discretion resembles the advantage of a cooperative over a non-cooperative solution in game theory. In the Taylor rule, the federal fund rate as a monetary policy instrument depends on the deviations of inflation from its optimal rate and the deviations of output from its target level. Taylor argued that if both inflation and output are on target then the nominal federal fund rate is 4% or 2% in real term.

Clarida, Gali and Gertler (1998) empirically characterized the implementation of monetary policy by European central bank since 1979. They argued that the policy rule is essentially a forward-looking version of the backward-looking approach of the J. B. Taylor (1993). They first noted that the evidence shows many central banks used the short-term interest rate as the main operating instrument of monetary policy. They assumed that central banks set a target for the nominal short-term interest rate based on the structure of the economy. Clarida, Gali and Gertler estimated the parameters of equations using the GMM. Their estimation results provided a guideline for inflation targeting monetary policy with some allowance for output stabilization. Such a rule has proven desirable for G3 (Japan, Germany and the United State) since 1979. On the other hand, a fixed exchange rate mechanism stresses the economy through loss of monetary control. Consequently, it is difficult to build credibility under this scheme.

Ball (1999) defined an efficient rule for monetary policy in a simple IS-Phillips curve model. He aimed to find the best rule that stabilizes the inflation and output. In this paper, a policy rule is set by the interest rate. The class of efficient policies found in this model is equivalent to the class of inflation targeting. In other words, the efficient Taylor rule derives the short-term interest rate that supports inflation targeting with the various speed of adjustment. When inflation expressed as a target, then the policymakers do not need to mention other variables such as output and unemployment. Ball argued the nominal income targeting largely destabilizes both output and inflation in response to aggregate-spending shocks.

J. B. Taylor (1999) argued that the degree of inflation rate fluctuations around its target level is the key variable for evaluating the interest rate rule, but is not the only performance measurement. The real output gap, unemployment gap and unanticipated inflation also can influence the loss function. However, the historical analysis of rule highlights the strong impact of inflation gap on the interest rate.

Over the past two decades, several researchers have empirically examined the performance of simple monetary policy rules. For instance, Clarida, Gali and Gertler (2000) analyzed the conduct of monetary policy pre and post-1970 in the United States, and explored how monetary policy differed before and after Volcker- the former United States Federal Reserve Chairman. They believed that this difference mainly derives changes in the macroeconomic behavior. They then presented a theoretical model that explains the macro performance responses to changes in the policy rule.

Orphanides (2001) compared the real-time monetary accommodation with those obtained from ex-post revised data. Applying the Taylor rule as an example, he showed that estimating the reaction function based on the ex-post revised data rather than on data available in real time can easily overshadow an important fact that forward-looking policy reaction functions appear to provide a more accurate description of the policy than the Taylor type contemporaneous specification.

Kuttner and Posen (2004) analyzed the difficulty of using the Taylor rule in zero interest rate economies such as Japan.

Esanov, Merkl and de Souza (2005) estimated the monetary policy rule in Russia and reviewed the recent conduct of monetary policy in this country. Gerberding, Seitz and Worms (2005) estimated the reaction function and monetary policy implementation in the Bundesbank. Ishak-Kasim and Ahmed (2010) estimated the reaction function of the short-run interest rate in Bank Indonesia, revealing how this bank conducts its monetary policy to meet the inflation target. Finally, Shirazi and Moosavi Mohseni (2015) presented a forward-looking optimum rule in an open economy.

1.3.2 Chaos in Monetary Systems

This section reviews some of the recent studies on chaotic behavior in monetary models.

Using the correlation dimension technique DeCoster and Mitchell (1991) searched for evidence of nonlinearity in various weekly monetary data. The term nonlinearity is used to find the deterministic nonlinear dynamics which shows chaos. The results showed a considerable evidence of nonlinearity in these US monetary data.

DeCoster and Mitchell (1992) noted that a model with rational expectations will develop nonlinearity and chaotic behavior. After employing a standard version of the Lucas-Sargent rational expectation macro model they observed the time path of the variables in the system reflects the chaotic behavior in a noisy environment. Under the rational expectation hypothesis, the nonlinear dynamic behavior of the system becomes more complex in this model.

Michener and Ravikumar (1998) employed a deterministic version of Lucas Jr and Stokey (1985)'s model to detect chaotic behavior in the cash-in-advance model. The results showed that nonlinearity and chaos in cash-in-advance models can be prevented under certain assumptions on the individual utility function.

Benhabib et al. (2002) analyzed the instability of the Taylor policy rule in a dynamic general equilibrium model. They applied the existence theorem of Yamaguti and Fujii (1979) on chaotic dynamics in a scalar system. They confirmed that the interest rate rules can shape the aperiodic equilibrium cycles and chaos.

Barkoulas (2008) investigated the deterministic chaotic behavior in monetary phenomena using the Lyapunov exponent and correlation dimension approaches. No chaotic dynamics in the different types of monetary series were observed in either approach.

Also using the Lyapunov exponent, Resende and Zeidan (2008) investigated the nonlinearity in the behavior of the exchange rate expectations and reported no evidence of chaotic behavior in this variable.

Yousefpoor, Esfahani and Nojumi (2008) evaluated the behavior of the sample stock returns selected from the Tehran stock exchange. They applied three tests: the BDS, largest Lyapunov exponent, and Kolmogorov entropy tests. The results of all tests confirmed the existence of chaotic behavior in this stock market.

Moosavi Mohseni and Kilicman (2014) checked the Hopf bifurcation in an open monetary economic model. They analyzed and compared the behavior of the economic system under two monetary policy rules namely, Taylor and inflation targeting. The Hopf bifurcation appeared in both economic systems. Comparing the two systems the authors could not identify which rule was most sensitive to the bifurcation coefficient. The results also indicated that openness can change the location of the bifurcation boundaries and largely increase the complexity of the system.

Moosavi Mohseni, Zhang and Cao (2015) investigated the chaotic behavior in monetary systems. They employed three different forms of the Taylor rule: current, backward and forward-looking. The base model employed in this study was based on the modified version of Moosavi Mohseni and Kilicman (2014)'s model for monetary policy analysis in an open economy. Chaotic behavior was found in all three monetary systems. Moreover, inserting the public expectations in the monetary policy rule especially the rational expectation hypothesis increased the complexity of the systems and enhanced the chaotic behavior.

Chaos in monetary systems has also been investigated in Scheinkman (1990), Serletis (1996), Serletis and Shintani (2006), Kyrtsou and Serletis (2006), Barnett and Duzhak (2008), Barnett, Serletis and Serletis (2012), Airaudo and Zanna (2012), Barnett and Eryilmaz (2013), Sanderson (2013) and Moosavi Mohseni and Kilicman (2013), to cite few examples.

1.3.3 Chaotic Games

As mentioned previously, the chaotic dynamics of monetary policy games has not been reported. However, chaotic behavior and strange attractors in dynamic games have been generated by simple difference or differential equations.

Implementing a simple differential evolutionary game introduced by P. D. Taylor and Jonker (1978), Skyrms (1992) presented numerical evidence of chaotic behavior in four strategies. The possibility of complicated behavior in this dynamical game was demonstrated in two examples.

Sato, Akiyama and Farmer (2002) employed a continuous reinforcement dynamic game involving two players in the rock-paper-scissors game. They showed that when the players learn from their strategies, the situation becomes more complicated. The zero-sum dynamic learning game leads to Hamiltonian chaos, meaning that even in a two-player game the learning trajectory can be very complicated with the chaotic strategies in the probability space.

Agiza and Elsadany (2004) investigated the chaotic behavior in a discrete-time duopoly game when the expectations attitude differs between the players. In their paper, the first and second players accepted a rational and an adaptive expectation rule, respectively. In this heterogeneous situation, each player maximizes their payoffs by different strategies. Numerical simulation revealed complex behavior and chaos in this duopoly market.

Károlyi, Neufeld and Scheuring (2005) and Salvetti, Patelli and Nicolo (2007) also revealed that chaos can emerge in the probability space trajectory of a rock-paper-scissors game.

1.4 Research Questions

The main aim of this study is to formulate a non-cooperative two-player (the policymaker and the public) differential monetary policy game model and to analyze the possible chaotic dynamics in both players' trajectories. Specifically, I intend to investigate and solve the following questions

Question 1. How can we analyze the chaotic interactions between the policymaker and the public using the dynamic game theory?

Question 2. Do the evolutionary dynamics of both players' trajectories show chaotic behavior under the Nash open-loop solution concept?

Question 3. Do the evolutionary dynamics of both players' trajectories show chaotic behavior under the Nash feedback solution concept?

Question 4. Do the evolutionary dynamics of both players' trajectories show chaotic behavior under the Stackelberg solution concept?

Question 5. In Question 4, what is the difference between the chaotic behavior of the trajectories if the leader (policymaker) employs time consistent or time-inconsistent policies?

As mentioned previously, the presence of chaos is an important aspect of the decision making, because it implies that one player cannot trivially anticipate the behavior of the other. Therefore the following question is also of interest.

Question 6. What can we conclude by comparing the findings from the above questions?

1.5 Thesis Contributions and Organization

The contributions of this thesis depend on answering the questions in Section 1.4. The remainder of this thesis is organized as follows.

Chapter 2 focuses on the mathematical preliminaries and economic concepts which are required in the rest of the study in order to prepare a model to answer the previous questions. Section 2.2 and 2.3 describe the fundamentals of dynamical systems, control and chaos, differential games, and economic growth theory.

Chapter 3 develops our analytical framework using the well-known neoclassical growth model with money. We briefly discuss the objective functions of the monetary authority and the public sector. The framework is finalized by obtaining the policy-goal relation which is necessary for understanding monetary policy. Finally, in this chapter we present a specific monetary economic model. Parts of the results from **[3]** and the model that I employed in **[4]** and **[6]** are included in this chapter.

Chapter 4 theoretically discusses two solution concepts: Nash and Stackelberg equilibrium. To understand the effectiveness of monetary policy, this chapter distinguishes between simultaneous and hierarchical games. First, we describe the open and feedback Nash solution concepts and analyze the strategies of both players. We then define the Stackelberg solution concept when the central bank has this priority to play as the leader. Parts of the results from **[4]** and **[6]** are include in this chapter.

Chapter 5 applies the specific monetary economic model presented in Chapter 3 to investigate the chaotic behavior in the monetary policy games. The existence of chaos is determined by the largest Lyapunov exponent, the most widely used method for diagnosing chaos in time series data. Parts of the results from **[1]**, **[2]** and **[5]** are include in this chapter.

Chapter 6 presents concluding remarks and proposes some strategies for implementing the study results.

Chapter 2

Preliminaries

2.1 Introduction

This chapter describes some mathematical techniques and economic theories used in the field of dynamical systems and chaos, differential games, optimal control theory and economic growth models. Applying these techniques and theories, we construct our benchmark monetary policy game model and then answer the research questions raised in the Section 1.4.

2.2 Mathematical Preliminaries

This section introduces some mathematical definitions and theorems employed in the subsequent chapters of this thesis.

2.2.1 Dynamic Games

One-shot games or even conventional repeated games are unsuitable for analysing a chaotic game, since no kinematic equations (transition equations) are there to describe how a chaotic game reaches equilibrium. However, dynamic games provide a good benchmark to this problem. Dynamic games include differential games (continuous in time), difference games (discrete in time), and timing games. These types of games were introduced to economics by Roos (1925, 1927), but were neglected until the 1970s. Therefore, they evolved into a standard tool for economic dynamic analysis. At each time period in a dynamic game, which extends over finite or infinite time, the players receive their payoffs. The overall payoff of each player is the sum (in the discrete version) or integral (in the continuous version) of the discounted payoff over the time horizon. This study employs the class of non-cooperative, non-zero-sum differential games.

Now, we consider a differential game with N players over the time horizon $[0,\ T]$ where T is finite or $T = \infty$. The state variables of these players are denoted by $x_1(t), ..., x_m(t)$, where $t \in [0,\ T]$. The ith player's control variable is denoted by $u_i(t)$, where $i = 1, ..., N$ and $t \in [0,\ T]$. Let $x(t) = (x_1(t), ..., x_m(t)) \in X \subseteq \mathbb{R}^m$ and $u(t) = (u_1(t), ..., u_N(t))$. For each $i = 1, ..., N$ we let $u_i(t) \in U^i \subseteq \mathbb{R}$, where U^i is a bounded interval. We call X the state space and and U^i the admissible control of player i, respectively. The state variables and the control variables are related by the following ordinary differential equation

$$x'(t) = f\left(x(t), u(t), t\right); \quad x(0) = x_0, \tag{2.1}$$

where the function f is differentiable. The payoff function of player i is given by

$$J_i(u) = h_i\left(x(T)\right) + \int_0^T g_i\left(x(t), u(t), t\right) dt; \quad \forall i = 1, ..., N. \tag{2.2}$$

where g_i is ith player's utility function and h_i is the terminal payoff function. We assume that g_i and h_i are differentiable and $\lim_{T\to\infty} h_i\left(x(T)\right) = 0 \quad \forall i = 1, ..., N$ and for all $x \in X$.

The above game identifies the optimal strategies u_i^* for the players and analyses the behavior of these trajectories. In particular, in the above N-person differential game, the ith player seeks the solution of the following problem

$$\begin{aligned} \max_{u_i} \quad & J_i(u), \\ \text{subject to} \quad & x'(t) = f\left(x(t), u(t), t\right), \\ & x(0) = x_0. \end{aligned}$$

The necessary conditions for u_i^* to solve the above problem are given by the maximum pricipels that will be explained in the next section. To obtain the conditions we form the Hamiltonian for ith player as

$$H_i\left(x(t), u(t), \lambda, t\right) = g_i\left(x(t), u(t), t\right) + h_i\left(x(T)\right) + \lambda \cdot f\left(x(t), u(t), t\right),$$

where λ is a vector of the costate variables. In such games, the rate of change in the state variables is described by a differential equation, often called the transition, dynamic or kinematic equation of the system. Therefore, the state of the system changes over time at a rate that depends on the control variables of each player. The solution of a dynamic game can be found by open loop Nash solution concept, feedback (Markov-perfect) Nash solution concept, or the Stackelberg solution concept. These three strategies can set different types of dynamic games. A player's open loop strategy is the planned time path of his action. This type of equilibrium concept is time consistent, meaning that along the equilibrium path, no player is incentivized to deviate from his original plan.

Definition 2.2.1. ***Open Loop Nash Strategy*** *(Dockner, Jørgensen, Van Long & Sorger, 2000)*
The N-tuple $(\psi^1, \psi^2, ..., \psi^N)$ of the function $\psi^i : [0, \infty) \to \mathbb{R}$, $i = 1, ..., N$, is called an open loop Nash equilibrium if, for each i, an optimal control path u_i of the above

problem exists and is given by the open loop Nash strategy $u_i = \psi^i$.

In contrast, in a Markov-perfect strategy, the optimal current action should depend on the currently observed state. This technique is better conceptualized, as a player believes that other players change their strategies over time.

Definition 2.2.2. ***Feedback Nash Strategy*** *(Dockner et al., 2000)*
The N-tuple $(\psi^1, \psi^2, ..., \psi^N)$ of the function $\psi^i : X \times [0, \infty) \to \mathbb{R}$, $i = 1, ..., N$, is called a feedback Nash equilibrium if, for each i, an optimal control path u_i of the above problem exists and is given by the feedback strategy $u_i(t) = \psi^i(x, t), \ \forall t \in [0, \infty)$.

The last solution concept is Stackelberg leadership. If one player can commit to a certain strategy before the other player can choose his strategy, the former is called the leader and the latter is called the follower.

Definition 2.2.3. ***Open Loop Stackelberg Equilibrium***[1]
Consider a differential game with two players, L (leader) and F (follower). Let $x \in X$ and $u_i \in U^i, \ \ i = L, \ F$, denote the vector of the state and control variables of each player, respectively. If $H_i(x, u_L, u_F, \lambda, t), \ \ i = L, \ F$, is the Hamiltonian ith player, then

$$R^F(x, u_L, \lambda, t) = \underset{u_F}{\operatorname{argmax}}\, H_F(\cdot),$$

is the follower's best reply. A pair of strategies (u_L^*, R^F) such that for any objective functions $(J_i, \quad i = L, \ F)$

$$\begin{aligned} J_L(u_L^*, R^F) &\geqslant J_L(u_L, R^F), \\ J_F(u_L^*, R^F) &\geqslant J_F(u_L^*, u_F), \end{aligned} \tag{2.3}$$

for all $u_i, \ \ i = L, \ F$, is an open loop stackelberg equilibrium if u_L^*, and u_F^* are open loop strategies, and if (2.3) holds $\forall (u_L, \ u_F)$.

[1]See Simaan and Cruz (1973), Dockner et al. (2000) and Bacchiega, Lambertini and Palestini (2010)

To solve this game, we must adopt the optimal control theory described in the next section.

2.2.2 Optimal Control Theory

In a differential game, each player optimizes his payoff function under a number of dynamical constraints that show the evolution of the state variables. Optimal control theory finds a trajectory of the state variables by choosing a set of control variables. This approach was developed by Pontryagin, Boltyanskii, Gamkrelidz and Mishchenko (1962) and is called Pontryagin maximum principle. We start this part by describing the calculus of variations.

Calculus of Variations

Calculus of Variations which deals with the function of functions derives from a seminal work first discussed by Galileo in 1630 and then by Bernoulli in 1696. The problem was solved by Bernoulli, Newton, and Leibnitz (Takayama, 2006), respectively. The term commonly used to describe a scalar value that depends on a function is commonly called a functional (Bellman & Dreyfus, 2015). Here, we briefly describe a fundamental problem in Calculus of Variations. Consider the following Riemann integral

$$J(x) = \int_a^b f\left(t, x(t), x'(t)\right) dt, \quad x(t) \in \mathbb{R}^m, \ m \geq 1 \tag{2.4}$$

where $x'(t) = \frac{dx}{dt}$ and a and b are constants. Clearly, the above integral depends on the function $x(t)$. Any change in $x(t)$ can change the value of $J(x)$. Suppose that X is a class of differentiable functions defined on the closed interval $[a, b]$. We seek a function $x(t)$ in X such as $J(x)$ is minimized subject to $x(a) = \alpha$ and $x(b) = \beta$. The solution of this problem is obviously a curve (trajectory) joining (a, α) to (b, β).

Lemma 2.2.1. ***Fundamental Lemma of the Calculus of Variations (Gelfand, Silverman et al., 2000)***
Let $F(t)$ be a given continuous function on $[a, b]$. Let X_0 be the set of all continuous function on $[a, b]$ such that $h(t) \in X_0$ implies $h(a) = h(b) = 0$. If $\int_a^b F(t)h(t)dt = 0$ for all $h(t) \in X_0$, then $F(t)$ is identically equal to zero.

Pontryagin Maximum Principle

We desire the optimum trajectory $x(t)$ that maximizes or minimizes a certain objective. Obviously, this trajectory can be controlled by other variables. For instance, in monetary economics, the amount of inflation in time t can be controlled by the money supply and short-term interest rate. We denote these control variables by a vector $u(t)$. To obtain the trajectory of $x(t)$, we choose a function $u(t)$ that optimizes (maximizes or minimizes) a certain objective function. This problem is called the optimal control problem, and the theory of its solution is called optimal control theory. Mathematically, optimal control theory is related to the Calculus of Variations (Chiang, 1992). This problem was solved by the famous Russian mathematician Pontryagin, and as mentioned above the basic result of this problem is called the Pontryagain maximum principle. Analogous to the classical nonlinear Lagrangian maximization method, the Pontryagain maximum principle shows the necessary conditions for optimality.

Consider the following first order differential equations

$$x_i' = f_i\left(t, x(t), u(t)\right), \quad \forall i = 1, 2, ..., n \tag{2.5}$$

where $x(t) = (x_1(t), x_2(t), ..., x_n(t)) \in \mathbb{R}^n$ and $u(t) = (u_1(t), u_2(t), ..., u_m(t)) \in U$ are the state and control variables, respectively. The set $U \subseteq \mathbb{R}^m$ is called the set of admissible controls. We assume that f_i is continuous in each x_i, u_j and t, and has continuous partial derivatives with respect to x_i and t. The initial values of the above differential equations are given by $x_i(0) = x_{i0}$; $\forall i = 1, ..., n$. Finally, the objective function is given as

$$\pi = \sum_{i=1}^{n} c_i x_i(t), \quad t \in (0, T), \tag{2.6}$$

where c_i are constants. Now, The problem now becomes

$$\begin{aligned} &\max_{u(t)\in U} \pi, \\ &subject\ to\ x_i' = f_i\left(t, x(t), u(t)\right), \quad \forall i = 1, 2, ..., n \\ &x_i(0) = x_{i0}. \quad \forall i = 1, 2, ..., n \end{aligned} \tag{2.7}$$

Solving (2.7), we can find $u^*(t) = (u_1^*(t), u_2^*(t), ..., u_m^*(t))$.

Theorem 2.2.1. ***Pontryagin Maximum Principle (Leitmann, 1966)***

Suppose that $u^*(t)$ is a solution of (2.7) with the corresponding state variables $x^*(t)$. There exists a non-zero continuous vector-valued function $p(t) = (p_1(t), p_2(t), ..., p_n(t))$ such that

1. *$p(t)$ together with $u^*(t)$ and $x^*(t)$ solve the following Hamiltonian system for all $i = 1, 2, ..., n$*

$$\begin{aligned} x_i'(t) &= \frac{\partial H^*}{\partial p_i}, \\ p_i'(t) &= -\frac{\partial H^*}{\partial x_i}, \end{aligned}$$

   where H (the Hamiltonian) and H^* are respectively defined by

$$\begin{aligned} H &= \sum_{i=1}^{n} p_i f_i\left(t, x(t), u(t)\right), \\ H^* &= H\left(t, x^*(t), u^*(t)\right). \end{aligned}$$

2. *$H\left(t, x^*(t), u^*(t)\right) \geq H\left(t, x^*(t), u(t)\right)$, for all $u(t) \in U$.*

3. $p_i(t) = c_i$ *for all* $i = 1, 2, ..., n$.

2.2.3 Dynamical Systems and Chaos Theory

After solving these games and finding the trajectories by the optimal control theory, we must investigate many relevant issues. The first issue is: Are these trajectories divergent or convergent? We can connect the dynamic games to chaos theory. In other words, if we have an infinitesimally small perturbation in one of the trajectories, will the orbit be periodic or chaotic? As mentioned by May (1976), a very simple but chaotic model can show an extraordinarily complex dynamical behavior. Such an environment presents difficulties to all decision makers (players).

A dynamical system consists of a set of possible states and a rule that determines the present state in terms of the past states (Alligood, Sauer & Yorke, 1996). Each dynamical system under certain conditions can have at least one fixed point (equilibrium). A point p^* is a fixed point of a map f if $f(p^*) = p^*$. Fixed point theorems, especially Brouwer's fixed point theorem, have played a crucial role in equilibrium analysis of dynamical systems.

Theorem 2.2.2. ***Brouwer's Fixed Point Theorem (Border, 1989)***
Let $U \subseteq \mathbb{R}^n$ *be a not empty, compact, convex set. Each continuous map of* U *to itself has at least one fixed point.*

A stable fixed point called a sink is the convergence point of the nearby points as the dynamical system evolves. Obviously, an unstable fixed point (called a source), drives points away as time elapses.

Theorem 2.2.3. ***Stability of Fixed Points (Alligood et al., 1996)***
Let f *be a smooth map on* $\mathbb{R}$ *and assume that* p^* *is a fixed point of* f. *If* $f'(p^*) < 1$, *then* p^* *is a sink (attractor). If* $f'(p^*) > 1$,*then* p^* *is a source (repeller).*

Obviously, fixed points play an important role in the behaviors of orbits. The above theorem implies that if the fixed point is a sink, it provides the final state of the system. The minimum number of iterations in which an orbit returns to its start point is called the period of the orbit.

Definition 2.2.4. ***Periodic Points of Period-k (Verhulst, 2006)***
Let f be a smooth map on $\mathbb{R}$. We call p a periodic point of period- k if k is the smallest positive integer such that $f^k(p) = p$. The orbit with initial point p is called a periodic orbit of period-k.

Clearly the above theorem is useful for investigating the stability of the periodic orbit around a fixed point. Suppose p is a period-k point. The period-k orbit of p is a *period sink* if p is a sink for the map f^k. The orbit of p is a *period source* if p is a source for the map f^k. Let $\{p_1, p_2, \ldots, p_k\}$ denote a period-k orbit of f. Then applying the chain rule in Calculus, we can show that the periodic orbit is a sink if

$$|f'(p_1).f'(p_2).\ldots.f'(p_k)| < 1,$$

and a source if

$$|f'(p_1).f'(p_2).\ldots.f'(p_k)| > 1.$$

In most of economic models, the external noise is considered as the main source of volatile behavior in a dynamical system, but the chaos revolution has revealed another source (Barnett et al., 2012). Barnett and Eryilmaz (2013) argued that economic dynamical systems are subject to the bifurcation, and those bifurcation boundaries can enhance our understanding of the dynamical properties of such systems. Bifurcation theory is the study of points in a mathematical system exhibiting drastic changes in the behavior of the system (Moosavi Mohseni & Kilicman, 2014).

Definition 2.2.5. ***Bifurcation Point (Moosavi Mohseni & Kilicman, 2014)***
In a dynamical system, a bifurcation occurs when a small change in the value of a parameter (the bifurcation parameter) of a system causes a dramatic change in the behavior of the system.

In the general representation, $f(x^*; \tau)$ denotes a strange attractor, whose value depends on the value of the parameter τ. At certain values of τ, called the bifurcation points, the behavior of the system dramatically changes. A bifurcation point can reveal how a mathematical system transmits to chaos.

Chaos can be broadly defined as *stochastic behavior occurring in a deterministic system* (Royal Society, London, 1986). One of the most popular definitions of chaos in mathematical textbooks was proposed by Devaney in 1989. This definition is given below.

Definition 2.2.6. ***Chaos (Devaney, 1989)***
Let V be an interval on $\mathbb{R}$. We can say that $f : V \to V$ is chaotic on V if

i. f is sensitive to the initial conditions,

ii. f is transitive,

iii. Periodic points are dense in V.

Sensitive dependence on the initial condition is a crucial property of a chaotic system. It states that changes in initial measures or calculation errors along the orbit can generally alter the outcome. The existence of chaos as defined by Devaney is commonly detected by the Lyapunov exponents, which measures the average divergence (convergence) between a reference (y_0) and a perturbed trajectory ($y_0 + \Delta y_0$). The separation between two trajectories is an infinitesimally small perturbation Δy_0. Over time, this perturbation from the initial condition can make a new perturbation trajectory Δy that is a function of time and the reference orbit, i.e., $\Delta y(y_0, t)$.

Sensitivity to the initial conditions can be represented as

$$|\Delta y(y_0, t)| \approxeq e^{\lambda t}|\Delta y_0|,$$

where λ denotes the Lyapunov exponents, i.e., the mean rate of separation of trajectories, and $|\quad|$ indicates the absolute value. The Lyapunov exponents for a general orbit are formally defined below

Definition 2.2.7. ***Lyapunov Exponents (Bensaïda, 2014)***
The Lyapunov exponent λ of a dynamical system is defined as follows

$$\lambda = \lim_{t\to\infty} \frac{1}{t} \ln\left(\frac{|\Delta y(y_0, t)|}{|\Delta y_0|}\right). \tag{2.8}$$

In this study we employ the BenSaïda (2012)'s algorithm (Bensaïda & Litimi, 2013; Bensaïda, 2014; BenSaïda, 2015) to numerically estimate the Lyapunov exponents. Properties of the Lyapunov exponents are described by the following theorem.

Theorem 2.2.4. ***(Lynch, 2004)***
If at least one of the average Lyapunov exponents is positive, then the system is chaotic. If the average Lyapunov exponents is negative, then the orbit is periodic. Finally, when the average Lyapunov exponents is zero, the system bifurcates.

A chaotic orbit exhibits unstable behavior at all time. Especially, this orbit near a source is neither fixed nor periodic and is never attracted to a sink (Alligood et al., 1996).

2.3 Economic Preliminaries

This section provides a simple economic growth framework that clarifies an economic concept. This framework prepares the reader for the monetary model that is developed in the next chapter. The starting point of this framework is the so-called the Solow-Swan model[2].

2.3.1 The Economic Environment

To analyze the long-term behavior of the Solow-Swan model, we require the neoclassical aggregate production function, denoted by $F(K, L)$, where K and L denote capital and labor, respectively. The neoclassical production function satisfies the following properties[3].

1. Homogeneity

The function $F(K, L)$ is homogeneous of degree one in K and L. This property is also known as the constant return to scale (CRS).

Definition 2.3.1. ***Homogeneous Function (Acemoglu, 2008)***
The function $f(x, y)$ is called homogeneous of degree m in x and y, where m is a positive integer, if

$$\lambda^m f(x, y) = f(\lambda x, \lambda y); \quad \forall \lambda > 0.$$

This property is useful because of the following theorem.

Theorem 2.3.1. ***Euler's Theorem (Acemoglu, 2008)***
Suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ is continuously differentiable in $x \in \mathbb{R}$ and $y \in \mathbb{R}$ and homogeneous of degree m in x and y, with partial derivatives denoted by f_x and f_y, respectively. Then

$$mf(x, y) = f_x(x, y)x + f_y(x, y)y; \quad \forall\ x,\ y \in \mathbb{R}.$$

Morover, $f_x(x, y)$ and $f_y(x, y)$ are homogeneous of degree $m - 1$ in x and y.

[2]See Solow (1956) and Swan (1956)
[3]See Barro and Sala-i Martin (2004, pp. 27-28)

2. Positive and Diminishing Marginal Product

The production function $F(K, L)$ is twice continuously differentiable in K and L, and satisfies

$$F_K(K,L) > 0, \qquad F_{KK}(K,L) < 0,$$
$$F_L(K,L) > 0, \qquad F_{LL}(K,L) < 0.$$

3. Inada Conditions and Essentiality

The production function $F(K, L)$ satisfies the following conditions[4]

$$\lim_{K\to\infty} F_K(K,L) = 0, \qquad \lim_{K\to 0} F_K(K,L) = \infty,$$
$$\lim_{K\to\infty} F_L(K,L) = 0, \qquad \lim_{K\to 0} F_L(K,L) = \infty.$$

Moreover, each input is essential for production, that is

$$F(0,L) = F(K,0) = 0.$$

In economic growth theory which is concerned with constant returns, we are interested in per capita variables. Hence if we define $\lambda = \frac{1}{L}$, we have

$$Y = F(K,L) = Lf(\frac{K}{L}).$$

Setting $y = \frac{Y}{L}$ and $k = \frac{K}{L}$, then the per capita production function is given by

$$y = f(k). \tag{2.9}$$

[4]Inada (1963)

The remaining equations of the model are as follows

$$K' = I - \Omega K, \qquad 0 < \Omega < 1 \tag{2.10}$$

$$S = sY, \qquad 0 < s < 1 \tag{2.11}$$

$$C = (1 - s)Y, \tag{2.12}$$

$$L = L_0 e^{nt}, \qquad 0 < n < 1 \tag{2.13}$$

$$Y = C + I, \tag{2.14}$$

where S, C and I are saving, private consumption and investment, respectively. Equation (2.10) determines the net increase in capital where Ω is the depreciation rate. Equations (2.11) and (2.12) are private saving and private consumption functions respectively, where s denotes the marginal propensity to save. Equation (2.13) assumes that population grows at a constant rate n. Finally, (2.14) denotes the equilibrium condition in a closed economy with no government sector.

2.3.2 The Solow-Swan Model: Fundamental Law of Motion

To analyze the dynamic behavior of the above economy, we divide both sides of (2.10) by L and exploit the Keynesian closure rule which states that in the long-term $S = I$. This gives

$$\frac{K'}{L} = sf(k) - \Omega k. \tag{2.15}$$

To express the left-hand side of (2.15) in per capita terms, we take the derivative of $k = \frac{K}{L}$ with respect to time. From (2.13), we know that $\frac{L'}{L} = n$. Thus, we have

$$k' = \frac{K'}{L} - nk. \tag{2.16}$$

Substituting the right hand side of (2.15) into (2.16), we get

$$k' = sf(k) - (n + \Omega)k. \tag{2.17}$$

Equation (2.17) describes the fundamental dynamics of the Solow-Swan growth model. In this model, the long-term equilibrium point or the steady state of k, denoted by k^{ss}, corresponds to $k' = 0$. Thus, in the long-term, we have

$$sf(k^{ss}) = (n + \Omega)k^{ss}. \tag{2.18}$$

Using (2.9)-(2.12), we can find the steady-state values of the other endogenous variables of the economy.

The above discussion establishes the following two important propositions

Proposition 2.3.1. ***Steady State of the Solow-Swan Growth Model***

Cosider the above Solow-Swan growth model. There exists a unique steady state point $k^{ss} \in (0, \infty)$ *determined by* (2.18).

Proposition 2.3.2. ***Stability of the Solow-Swan Growth Model***

Consider the above Solow-Swan growth model. Then for any $K(0) > 0$, *the above Solow-Swan growth model is globally asymptotically stable, that is,* $\lim_{t\to\infty} k(t) = k^{ss}$.

Under these two propositions, the Solow-Swan growth model has a stable unique steady state equilibrium point.

Chapter 3

Structure of the Model

This chapter describes the structure of the model. The presented monetary model is a noncooperative, nonzero sum game between the central bank and the public. To derive the dynamics of the system, we first introduce a version of the two-asset economic growth model with the public expectations. We then present the objective functions of both players. The policy-goal relation finalizes the structure of our model. Finally, Section 3.4 presents the framework of our specific model

3.1 Dynamics of the System

The dynamics of the system are derived by two markets: asset and commodity, which are described in Subsections 3.1.2 and 3.1.3, respectively. Appearence of the public expectations in this model is the result of relaxing the usual perfect foresight assumption in growth model. In the following model, the parameters of the system vary as the functions of the variables of the model and this is one of the most important differences between this model and the previous ones in the literature.

3.1.1 Production Function

Following Solow (1956), suppose that a unique final good Y_t is produced by labor L_t and capital K_t.[1] The production function is assumed as a twice continuously differentiable and homogeneous function of degree one in L_t and K_t. Thus, the production function in per capita term can be expressed as $y_t = f(k_t)$, where $y_t = \frac{Y_t}{L_t}$ and $k_t = \frac{K_t}{L_t}$ are the per capita output and capital, respectively.[2] Moreover, we require that f satisfies the Inada conditions: $f(0) = 0$, $f_k > 0$, and $f_{kk} < 0$.

3.1.2 Asset Market

Suppose that there are two main asset categories in our economy: money and capital. The total asset value in this economy is then expressed as

$$A^n = M + P^K K, \tag{3.1}$$

where A^n is the total nominal asset, M and P^K are nominal money balances and capital price, respectively. Assume that the capital price ruling the economy equals to the commodity price, that is, $P^K = P$. Since we are interested in the real per capita variables, we divide both sides of (3.1) by PL, to obtain

$$a = m + k, \tag{3.2}$$

where a and m denote the real per capita assets and the money balances, respectively. Suppose that the real per capita demand for money is $m = \mathcal{L}(y, i, \pi)a$, where i is the nominal interest rate and π is the inflation rate. We know that $\mathcal{L}_y > 0$, $\mathcal{L}_i < 0$ and

[1] This function exhibits a constant return to scale production function.

[2] For simplicity we ignore (t) hereafter.

$\mathcal{L}_\pi < 0$.[3] Substituting a in (3.2) into the real per capita demand for money, we obtain

$$m^d = V(y, i, \pi)k, \tag{3.3}$$

where

$$V(y, i, \pi) = \frac{\mathcal{L}(y, i, \pi)}{1 - \mathcal{L}(y, i, \pi)}.$$

Here, $i = f_k + \pi^e$,[4] where $\pi^e = h(\pi)$ is the expected inflation in terms of π. Moreover, $h_\pi \in [0, 1]$ is the expected inflation rate. Since y is a function of k and $i = f_k + \pi^e$, we can re-write (3.3) in terms of k, π and π^e as

$$m^d = \mathcal{V}(k, \pi, \pi^e)k \tag{3.4}$$

with $\mathcal{V}_k > 0$, $\mathcal{V}_\pi < 0$ and $\mathcal{V}_{\pi^e} < 0$. Equation (3.4) expresses the demand for money as a function of k, π and π^e. We assume that the real per capita supply of money m^s is exogenous and expressed as follows

$$m^s = \frac{M}{PL}. \tag{3.5}$$

Our aim is to derive the fundamental dynamics of inflation from the equilibrium relation of the money market, $m^s = m^d$. Differentiating each term of the log of (3.4) with respect to time, we obtain

$$\frac{m'}{m} = \frac{\mathcal{V}_k(k, \pi, \pi^e)k' + \mathcal{V}_\pi(k, \pi, \pi^e)\pi' + \mathcal{V}_{\pi^e}(k, \pi, \pi^e)\pi^{e\prime}}{\mathcal{V}(k, \pi, \pi^e)} + \frac{k'}{k}.$$

---

[3] A variable with a subscript means the derivative of that variable with respect to the index, i.e., $y_x = \frac{\partial y}{\partial x}$. In addition, a variable with a superscript prime ($\prime$) means the derivative of that variable with respect to time, i.e., $y' = \frac{\partial y}{\partial t}$.

[4] This shows the Fisher equation after Fisher (1896). But under competitive firm profit maximization, we have $r = f_k$, where r is the real interest rate.

After substituting $\pi^{e\prime} = h_\pi \pi'$ and rearranging, we get

$$\frac{m'}{m} = \frac{\mathcal{V}_k(k,\pi,\pi^e)}{\mathcal{V}(k,\pi,\pi^e)}k' + \frac{\mathcal{V}_\pi(k,\pi,\pi^e) + h_\pi \mathcal{V}_{\pi^e}(k,\pi,\pi^e)}{\mathcal{V}(k,\pi,\pi^e)}\pi' + \frac{k'}{k}.$$

The above equation can be simplified as follows

$$\frac{m'}{m} = \left(\frac{1}{k} + \frac{\mathcal{V}_k(k,\pi,\pi^e)}{\mathcal{V}(k,\pi,\pi^e)}\right)k' + \frac{\mathcal{V}_\pi(k,\pi,\pi^e) + h_\pi \mathcal{V}_{\pi^e}(k,\pi,\pi^e)}{\mathcal{V}(k,\pi,\pi^e)}\pi'. \tag{3.6}$$

On the supply side, differentiating each term of the log of (3.5) with respect to time, we have

$$\frac{m'}{m} = \mu - \pi - n, \tag{3.7}$$

where $\mu = \frac{M'}{M}$ is the monetary policy rate parameter and n is the population growth rate.[5] By equating the right hand sides of (3.6) and (3.7), we obtain

$$\begin{aligned}(\mu - \pi - n)\mathcal{V}(k,\pi,\pi^e) = &\left(\frac{\mathcal{V}(k,\pi,\pi^e)}{k} + \mathcal{V}_k(k,\pi,\pi^e)\right)k' \\ &+ \left(\mathcal{V}_\pi(k,\pi,\pi^e) + h_\pi \mathcal{V}_{\pi^e}(k,\pi,\pi^e)\right)\pi'.\end{aligned}$$

It follows that the fundamental dynamics of inflation is given by

$$\pi' = \psi(k,\pi,\pi^e)\left((\mu - \pi - n)\mathcal{V}(k,\pi,\pi^e) - \phi(k,\pi,\pi^e)k'\right), \tag{3.8}$$

where

$$\begin{aligned}\psi(k,\pi,\pi^e) &= \frac{1}{\mathcal{V}_\pi(k,\pi,\pi^e) + h_\pi \mathcal{V}_{\pi^e}(k,\pi,\pi^e)}, \\ \phi(k,\pi,\pi^e) &= \frac{\mathcal{V}(k,\pi,\pi^e)}{k} + \mathcal{V}_k(k,\pi,\pi^e).\end{aligned}$$

[5] Assume that in the long run, full employment always prevails and the labour force grows exponentially at the rate n, i.e., $L(t) = L(0)e^{nt}$.

In the next subsection, we will derive the fundamental dynamics of k.

3.1.3 Commodity Market

The equilibrium condition in real terms can be written as

$$Y = \mathcal{C} + I + G, \tag{3.9}$$

where $\mathcal{C} = c(i)Y^D$ is the consumption function with $c_i < 0$, Y^D is the disposable income, I is the investment given by $I = K' - \Omega K$ (Ω is the depreciation rate) and G is the exogenous government expenditure. Defining the real tax as a simple linear function of Y, i.e., $T = \tau Y$, where $0 < \tau < 1$, the disposable income is given by $Y^D = (1-\tau)Y$. It follows that the consumption function is $\mathcal{C} = c(k, \pi^e)(1-\tau)Y$. Substituting $\mathcal{C}$, I and T into (3.9), we get

$$Y = c(k, \pi^e)(1-\tau)Y + K' + \Omega K + \bar{G}.$$

Rearranging the above equation for K' we obtain

$$K' = (1 - c(k, \pi^e)(1-\tau))\, Y + \Omega K - \bar{G}. \tag{3.10}$$

Knowing from (2.16) that

$$\frac{k'}{k} = \frac{K'}{K} - n,$$

and substituting the right-hand side of (3.10) into the above equation and simplifying, we obtain

$$\frac{k'}{k} = \frac{(1 - c(k, \pi^e)(1-\tau))\, Y}{K} - \frac{\bar{G}}{K} + \Omega - n.$$

In terms of per capita variables this becomes

$$\frac{k'}{k} = \frac{\left(1 - c(k, \pi^e)(1 - \tau)\right) \frac{Y}{L}}{\frac{K}{L}} - \frac{\frac{\bar{G}}{L}}{\frac{K}{L}} + \Omega - n.$$

Given that $y = \frac{Y}{L} = f(k)$, $k = \frac{K}{L}$ and $\bar{g} = \frac{\bar{G}}{L}$, we have

$$\frac{k'}{k} = \frac{\left(1 - c(k, \pi^e)(1 - \tau)\right) f(k)}{k} - \frac{\bar{g}}{k} + \Omega - n.$$

Hence, after multiplying both side of the above equation by k, we obtain

$$k' = \left(1 - c(k, \pi^e)(1 - \tau)\right) f(k) - \bar{g} + (\Omega - n)k. \tag{3.11}$$

where $\bar{g}$ is the exogenous per capita government expenditure. Equation (3.11) gives the fundamental dynamics of the commodity market.

3.1.4 Fundamental Dynamics of Model

From (3.11), we know that k' is a function of k and π^e, denoting this function by $\alpha(k, \pi^e)$, it follows that

$$k' = \left(1 - c(k, \pi^e)(1 - \tau)\right) f(k) - \bar{g} + (\Omega - n)k = \alpha(k, \pi^e). \tag{3.12}$$

Now, substituting $k' = \alpha(k, \pi^e)$ into (3.8) and denoting π' as a function of k, π, μ and π^e, namely $\beta(k, \pi, \mu, \pi^e)$, we obtain

$$\begin{aligned} \pi' &= \psi(k, \pi, \pi^e) \left((\mu - \pi - n)\mathcal{V}(k, \pi, \pi^e) - \phi(k, \pi, \pi^e)\alpha(k, \pi^e)\right) \\ &= \beta(k, \pi, \mu, \pi^e). \end{aligned} \tag{3.13}$$

The last two equations describe the fundamental dynamics of the model.

3.2 Objective Functions

To determine policy choice of the central bank, we must specify the preference of the central bank. However, the impact of the monetary policy often depends on the public expectations. Thus, the preferences of both players play important roles in a monetary economic model. The players' preferences are described by the following functions

$$\mathcal{U}^{CB} = \frac{1}{2}\lambda_1 \left(y - y^n\right)^2 - \frac{1}{2}\lambda_2 \left(\pi - \hat{\pi}\right)^2, \tag{3.14a}$$

$$\mathcal{U}^{PS} = \mathcal{U}^{PS}\left(\mathcal{C}(k;\pi^e);\mu\right), \tag{3.14b}$$

where y^n, and $\hat{\pi}$ are the real per capita natural output and the target inflation of the central bank, respectively. The coefficients λ_1 and λ_2 weight the output expansion and inflation stabilization as two main goals of the central bank, respectively. Equation (3.14a) denotes the policy choices of the central bank, as described in Barro and Gordon (1983a, 1983b). Given that $y = f(k)$ and that y^n corresponds to the full employment capital-labor ratio, we can rewrite $\mathcal{U}^{CB}$ as

$$\mathcal{U}^{CB} = \frac{1}{2}\lambda_1 \left(k - k^n\right)^2 + \frac{1}{2}\lambda_2 \left(\pi - \hat{\pi}\right)^2.$$

Equation (3.14b) denotes the money-in-utility function, where ${\mathcal{U}_{\mathcal{C}}}^{PS} > 0$, and $\mathcal{U}_{\mu}^{PS} \lesseqgtr 0$ (Sidrauski, 1967a, 1967b; Walsh, 2003). Following Chang (1998), we employ the following additive utility function for the public sector

$$\mathcal{U}^{PS} = U^{PS1}\left(\mathcal{C}(k;\pi^e)\right) + U^{PS2}\left(\mu\right).$$

Thus, the final objective functions for both players are given by

$$\mathcal{Z}^{CB} = \int_0^\infty \left(\frac{1}{2}\lambda_1 \left(k - k^n\right)^2 + \frac{1}{2}\lambda_2 \left(\pi - \hat{\pi}\right)^2 \right) e^{-\rho t} dt, \tag{3.15a}$$

$$\mathcal{Z}^{PS} = \int_0^\infty \left(U^{PS1} \left(\mathcal{C}(k; \pi^e)\right) + U^{PS2} \left(\mu\right) \right) e^{-\rho t} dt, \tag{3.15b}$$

respectively.

3.3 Policy-Goal Relation

Following Barro and Gordon (1983a), we assume that the instrument wielded by the monetary authority directly affects inflation as the main goal of the central bank. Thus, the remaining part of the model relates inflation to the policy instrument

$$\pi = \pi(\mu); \qquad \pi_\mu \in [0, 1]. \tag{3.16}$$

As the policy instrument makes an indirect impact on k, we should search the transmission of monetary policy.

3.4 A Specific Model

This section describes the framework for our numerical analysis. This monetary economic model[6] is specified by the following relationships

[6]For and exposition of the macroeconomic models, see(Turnovsky, 2000)

$$y = k^{\varrho}; \qquad 0 < \varrho < 1, \tag{3.17a}$$

$$a = m + k, \tag{3.17b}$$

$$m^d = \frac{\eta y}{\iota i + \varpi \pi} a; \qquad \eta, \iota, \varpi > 0, \tag{3.17c}$$

$$m^s = \frac{M}{PL}, \tag{3.17d}$$

$$y = \mathcal{C} + inv + \bar{g}, \tag{3.17e}$$

$$\mathcal{C} = \frac{\theta}{\sigma i} y^D; \qquad \theta, \sigma > 0, \tag{3.17f}$$

$$inv = \frac{K'}{L} + \Omega k; \qquad \Omega > 0, \tag{3.17g}$$

$$t = \tau y; \qquad \tau > 0, \tag{3.17h}$$

$$y^D = y - t, \tag{3.17i}$$

$$\pi = \epsilon \mu; \qquad \epsilon \in [0, 1], \tag{3.17j}$$

where inv denotes the real per capita private investment. Equation (3.17a) specifies the output in terms of a per capita Cobb-Douglas production function. Equations (3.17b) - (3.17d) describe the assets, demand for money and money supply, respectively. The commodity market is described by the (3.17e) - (3.17i). Finally, (3.17j) describes the relationship between goal and policy, which is assumed to be linear.

We are interested in determining the dynamical behavior of the state variables, i.e. k and π. Substituting $i = \varrho k^{\varrho - 1} + \pi^e$ into (3.17c) and (3.17f), we obtain

$$m = \nu(k, \pi; \pi^e) k,$$

$$\mathcal{C} = c(k; \pi^e) k^{\varrho},$$

where

$$\nu(k,\pi;\pi^e) = \frac{\eta k^\varrho}{\iota(\varrho k^{\varrho-1}+\pi^e) - \eta k^\varrho + \varpi\pi},$$
$$c(k;\pi^e) = \frac{\theta(1-\tau)}{\sigma(\varrho k^{\varrho-1}+\pi^e)}.$$

Now, we also have

$$k' = \left(1 - \frac{\theta(1-\tau)}{\sigma(\varrho k^{\varrho-1}+\pi^e)}\right)k^\varrho - \bar{g} + (\Omega - n)k = \alpha(k,\pi^e), \tag{3.18a}$$

$$\pi' = \psi(\cdot)\left((\mu - \pi - n) - \phi(\cdot)\alpha(\cdot)\right) = \beta(k,\pi;\pi^e,\mu) \tag{3.18b}$$

where

$$\psi(k,\pi;\pi^e) = \frac{\left(\iota(\varrho k^{\varrho-1}+\pi^e) - \eta k^\varrho + \varpi\pi\right)^2}{-\left((\iota\delta+\varpi)\beta k^\varrho + \iota\delta\right)},$$
$$\phi(k,\pi;\pi^e) = \frac{\eta k^\varrho}{\iota(\varrho k^{\varrho-1}+\pi^e) - \eta k^\varrho + \varpi\pi}\left(\varrho - \frac{\varrho k^{\varrho-1}\left((\varrho-1)\iota - \varrho\eta k\right)}{\iota(\varrho k^{\varrho-1}+\pi^e) - \eta k^\varrho + \varpi\pi}\right).$$

Equations (3.18a) and (3.18b) together provide the formal behavior of the dynamics of this model. Suppose

$$\mathcal{U}^{PS} = \ln\left(\frac{\theta(1-\tau)}{\sigma(\varrho k^{\varrho-1}+\pi^e)}k^\varrho\right) + \ln(\mu),$$

is an additive utility function for the public sector. Now, the present value of the objective functions of the central bank and the public sector are respectively given by

$$\mathcal{Z}^{CB} = \int_0^\infty \left(\frac{1}{2}\lambda_1\left(k-k^n\right)^2 + \frac{1}{2}\lambda_2\left(\pi-\hat{\pi}\right)^2\right)e^{-\rho t}dt, \tag{3.19a}$$

$$\mathcal{Z}^{PS} = \int_0^\infty \left(\ln\left(\frac{\theta(1-\tau)}{\sigma(\varrho k^{\varrho-1}+\pi^e)}k^\varrho\right) + \ln(\mu)\right)e^{-\rho t}dt. \tag{3.19b}$$

Chapter 4

Analysis of the Solution Concepts

4.1 Introduction

We can now analyze the model introduced in the previous chapter. This chapter applies the Nash and Stackelberg solution concepts described in Chapter 2 to a noncooperative differential monetary policy game with two players.

The next section discusses the Nash equilibrium concepts in the open loop and feedback (Markovian) strategies. Section 4.3 analyses the hierarchical game, which is well-known as the Stackelberg solution concept.

The importance of the time consistency problem is also discussed in this chapter. A policy is said to be time consistent if it includes a period t action that is optimal from the period t point of view (Canzoneri & Henderson, 1991).

Contrary to the previous literature, we demonstrate two types of time consistency problems: behavioral and structural. As its name suggests behavioral time consistency depends on the behavior of a policymaker who reneges on his goals, after being incentive to deviate from the announced policy. Almost all of the literature in this research field focused on this type of time inconsistency. Structural time consistency depends on the economic conditions and causes from the variation of parameters in the economic models (Lucas critique)[1]. Here we argue that structural time inconsistency is unavoidable. This argument explains the rarity of time-consistent policy in practice even when the policymaker is fully committed to the policy.

4.2 Nash Equilibrium

Distinction between *open loop* and *feedback* or *Markov perfect* Nash strategies is useful for understanding the game when one player deliberately deviates from his original plan. This section describes these two different concepts of the Nash equilibrium.

Our optimal control problem is to find a time path for μ and π^e so as to maximize $\mathcal{Z}^{CB}$ and $\mathcal{Z}^{PS}$ subject to $k' = \alpha(k;\pi^e)$ and $\pi' = \beta(k,\pi;\mu,\pi^e)$ and the usual initial condition for the state variables. Define Hamiltonian for both players by

$$\mathcal{H}^{CB} = \frac{1}{2}\lambda_1 (k - k^n)^2 + \frac{1}{2}\lambda_2 (\pi - \hat{\pi})^2 + \omega^{CB}\alpha(k;\pi^e) + \gamma^{CB}\beta(k,\pi;\mu,\pi^e),$$
$$\mathcal{H}^{PS} = U^{PS1}(\mathcal{C}(k;\pi^e)) + U^{PS2}(\mu) + \omega^{PS}\alpha(k;\pi^e) + \gamma^{PS}\beta(k,\pi;\mu,\pi^e).$$

where ω^{CB} and ω^{PS} are the co-state variables of the commodity market for the central bank and the public sector and γ^{CB} and γ^{PS} are the co-state variables of money market for both players, respectively.

4.2.1 Open Loop Nash Strategies

As an open loop Nash strategy, due to its precommitment, it is generally thought to be time consistent but is not robust to perturbations (Dockner et al., 2000; Van Long, 2010). In other words, an open loop Nash strategy is a sequence of precommited actions over the time horizon of the game.

[1]After Lucas (1976) who criticized a range of macroeconometric policy evolutions because they assumed rules are invariant with respect to the law of motion that the public faced (Sargent, 2009). The Lucas critique acknowledged that the model parameters can depend on individual behaviors.

In an open loop Nash strategy, the control variable of one player is an exogenous variable for the others. Mathematically, the control variable of each player depends on time, i.e., $\mu = \mu(t)$ and $\pi^e = \pi^e(t)$[2]. Assuming that the movements of the central bank and the public are synchronous in time, the necessary conditions of the central bank under an open loop Nash solution concept are given by

$$\omega^{CB\prime} - \rho\omega^{CB} = -\lambda_1 (k - k^n) - \omega^{CB}\alpha_k(k;\pi^e) - \gamma^{CB}\beta_k(k,\pi;\mu,\pi^e), \tag{4.1a}$$

$$\gamma^{CB\prime} - \rho\gamma^{CB} = \lambda_2 (\pi - \hat{\pi}) - \gamma^{CB}\beta_\pi(k,\pi;\mu,\pi^e), \tag{4.1b}$$

$$k' = (1 - c(k;\pi^e)(1-\tau)) f(k) - \bar{g} + (\Omega - n)k, \tag{4.1c}$$

$$\pi' = \psi(k,\pi;\pi^e) \left((\mu - \pi - n) - \phi(k,\pi;\pi^e)\alpha(k;\pi^e)\right), \tag{4.1d}$$

$$0 = -\lambda_2\pi_\mu (\pi - \hat{\pi}) + \omega^{CB}\alpha_\mu(k;\pi^e) + \gamma^{CB}\beta_\mu(k,\pi;\mu,\pi^e), \tag{4.1e}$$

$$0 = \lim_{t\to\infty} k(t)\omega^{CB}e^{-\rho t}, \tag{4.1f}$$

$$0 = \lim_{t\to\infty} \pi(t)\gamma^{CB}e^{-\rho t}, \tag{4.1g}$$

$$k(0) = k_0, \quad \pi(0) = \pi_0, \tag{4.1h}$$

where $\alpha_\mu = \frac{\partial\alpha}{\partial k}\frac{\partial k}{\partial\mu}$. Suppose that $\frac{\partial k}{\partial\mu} = 1 - \pi_\mu$, then $\alpha_\mu = \alpha_k(1-\pi_\mu)$. Equation (4.1b) is a first order differential equation, where the solution can be explained as

$$\gamma^{CB} = \gamma^{CB}(k,\pi;\mu,\pi^e). \tag{4.2}$$

Substituting (4.2) into (4.1g), we have

$$\lim_{t\to\infty} \pi(t)\gamma^{CB}(k,\pi;\mu,\pi^e) = 0. \tag{4.3}$$

Hense, the transversality condition (4.1g) holds for γ^{CB} if $\lim_{t\to\infty}\pi(t) = 0$. Now, we

[2] In an open loop Nash game, $\psi(k,\pi;\pi^e) = \frac{\mathcal{V}(k,\pi;\pi^e)}{\mathcal{V}_\pi(k,\pi;\pi^e)}$.

must find ω^{CB}. Substituting (4.2) into (4.1a) and rearranging, we obtain

$$\omega^{CB\prime} - (\rho - \alpha_k)\,\omega^{CB} = -\lambda_1\,(k - k^n) - \beta_\pi \gamma^{CB}(k, \pi; \mu, \pi^e).$$

This is a first order differential equation, where the solution can be expressed as

$$\omega^{CB} = \omega^{CB}(k, \pi; \mu, \pi^e). \tag{4.4}$$

Substituting (4.4) into (4.1f), we obtain

$$\lim_{t\to\infty} k(t)\omega^{CB}(k, \pi; \mu, \pi^e)e^{-\rho t} = 0. \tag{4.5}$$

Since the shadow value of capital at the end of the planning horizon seems to be positive, the transversality condition (4.1f) holds if $\lim_{t\to\infty} k(t) = 0$. This means that the households leave no capital at the end of the planning horizon.

Proposition 4.2.1. *In an open loop Nash solution concept, the dynamic path of the monetary policy depends on the dynamic behavior of the public expectations, that is,*

$$\mu' = \Upsilon^{11}(k, \pi; \mu; \pi^e) + \Upsilon^{12}(k, \pi; \mu; \pi^e)\pi^{e\prime}, \tag{4.6}$$

where

$$\Upsilon^{11}(k, \pi; \mu; \pi^e) = -\frac{\alpha_\mu \omega^{CB\prime} + \beta_\mu \gamma^{CB\prime} + \left(\omega^{CB}\alpha_{k\mu} + \gamma^{CB}\beta_{k\mu}\right)\alpha + \omega^{CB}\beta_{\pi\mu}\beta}{\lambda_2\pi_\mu^2 + \omega^{CB}\alpha_{k\mu}(1-\pi_\mu) + \gamma^{CB}\beta_{\mu\mu}},$$
$$\Upsilon^{12}(k, \pi; \mu; \pi^e) = -\frac{\omega^{CB}\alpha_{\pi^e\mu} + \gamma^{CB}\beta_{\pi^e\mu}}{\lambda_2\pi_\mu^2 + \omega^{CB}\alpha_{k\mu}(1-\pi_\mu) + \gamma^{CB}\beta_{\mu\mu}}.$$

Proof. Differentiating (4.1e) with respect to time, we have

$$-\lambda_2\pi_\mu\pi' + \alpha_\mu\omega^{CB\prime} + \omega^{CB}\alpha_\mu{}' + \beta_\mu\gamma^{CB\prime} + \gamma^{CB}\beta_\mu{}' = 0. \tag{4.7}$$

From the policy relation, we have $\pi' = \pi_\mu \mu'$. Differentiating $\alpha_\mu = \alpha_\mu(k; \pi^e)$ and $\beta_\mu = \beta_\mu(k, \pi; \mu, \pi^e)$ with respect to time, we also obtain

$$\alpha_\mu{}' = \alpha_{k\mu} k' + \alpha_{\pi^e\mu} \pi^{e\prime} + \alpha_{k\mu} \frac{\partial k}{\partial \mu} \mu',$$
$$\beta_\mu{}' = \beta_{k\mu} k' + \beta_{\pi\mu} \pi' + \beta_{\pi^e\mu} \pi^{e\prime} + \beta_{\mu\mu} \mu'.$$

By knowing that $k' = \alpha$, $\pi' = \beta$ and $\frac{\partial k}{\partial \mu} = 1 - \pi_\mu$ and substituting them into the above equations, we have

$$\alpha_\mu{}' = \alpha_{k\mu} \alpha + \alpha_{\pi^e\mu} \pi^{e\prime} + \alpha_{k\mu}(1 - \pi_\mu) \mu', \tag{4.8a}$$
$$\beta_\mu{}' = \beta_{k\mu} \alpha + \beta_{\pi\mu} \beta + \beta_{\pi^e\mu} \pi^{e\prime} + \beta_{\mu\mu} \mu'. \tag{4.8b}$$

Substituting (4.8a) and (4.8b) into (4.7) and rearranging, we obtain

$$\begin{aligned} \mu' = &- \frac{\alpha_\mu \omega^{CB\prime} + \beta_\mu \gamma^{CB\prime} + \left(\omega^{CB} \alpha_{k\mu} + \gamma^{CB} \beta_{k\mu}\right) \alpha + \omega^{CB} \beta_{\pi\mu} \beta}{\lambda_2 \pi_\mu^2 + \omega^{CB} \alpha_{k\mu}(1 - \pi_\mu) + \gamma^{CB} \beta_{\mu\mu}} \\ &- \frac{\omega^{CB} \alpha_{\pi^e\mu} + \gamma^{CB} \beta_{\pi^e\mu}}{\lambda_2 \pi_\mu^2 + \omega^{CB} \alpha_{k\mu}(1 - \pi_\mu) + \gamma^{CB} \beta_{\mu\mu}} \pi^{e\prime}, \end{aligned} \tag{4.9}$$

which means that the strategy trajectory of the central bank (control variables) depends on the dynamics of the public expectations, i.e., π^e. □

As seen in (4.2) and (4.4), the co-state variables of the central bank are functions of the control variable of the public sector. Hence, the co-state variables of the central bank are controlled by the public. The controllability of the co-state variables is now defined as follows.[3]

Definition 4.2.1. ***(Controllability).*** *The co-state variable of a player is said to be controllable if it depends on the control variables of the other players.*

[3](Dockner et al., 2000, p. 116).

Proposition 4.2.2. *In an open loop Nash game with the variation of the parameters, the co-state variables of the central bank are controllable.*

In this situation, manoeuvring the public expectation changes the co-state variables of the central bank, leading to the time consistency problems[4].

Definition 4.2.2. ***(Time Consistency).*** *A policy is time consistent if an action planned at time t for time $t+i$ remains optimal to be implemented when time $t+i$ actually arrives*[5].

We must now find the trajectory of the public expectations. From (3.15b), the first order conditions for the public sector are given by

$$\omega^{PS\prime} - \rho\omega^{PS} = -\mathcal{U}_k^{PS} - \omega^{PS}\alpha_k(k;\pi^e) - \gamma^{PS}\beta_k(k,\pi;\mu,\pi^e), \tag{4.10a}$$

$$\gamma^{PS\prime} - \rho\gamma^{PS} = -\gamma^{PS}\beta_\pi(k,\pi;\mu,\pi^e), \tag{4.10b}$$

$$k' = (1 - c(k;\pi^e)(1-\tau))\, f(k) - \bar{g} + (\Omega - n)k, \tag{4.10c}$$

$$\pi' = \psi(k,\pi;\pi^e)\left((\mu - \pi - n) - \phi(k,\pi;\pi^e)\alpha(k;\pi^e)\right), \tag{4.10d}$$

$$0 = \mathcal{U}_{\pi^e}^{PS} + \omega^{PS}\alpha_{\pi^e}(k;\pi^e) + \gamma^{PS}\beta_{\pi^e}(k,\pi;\mu,\pi^e), \tag{4.10e}$$

$$0 = \lim_{t\to\infty} k(t)\omega^{PS}e^{-\rho t}, \tag{4.10f}$$

$$0 = \lim_{t\to\infty} \pi(t)\gamma^{PS}e^{-\rho t}, \tag{4.10g}$$

$$k(0) = k_0, \quad \pi(0) = \pi_0, \tag{4.10h}$$

where $\mathcal{U}_k^{PS} = \mathcal{U}_{\mathcal{C}}^{PS1}\frac{\partial \mathcal{C}}{\partial k}$, and $\mathcal{U}_{\pi^e}^{PS} = \mathcal{U}_{\mathcal{C}}^{PS1}\frac{\partial \mathcal{C}}{\partial \pi^e}$. The solution of (4.10b) can be represented as

$$\gamma^{PS} = \gamma^{PS}(k,\pi;\mu,\pi^e). \tag{4.11}$$

[4]Xie (1997), Van Long (2010), and Bacchiega et al. (2010) showed that controllability in the Stackelberg solution concept leads to time inconsistency.

[5]Walsh (2003).

Substituting (4.11) into (4.10g), we obtain

$$\lim_{t\to\infty} \pi(t)\gamma^{PS}(k,\pi;\mu,\pi^e) = 0,$$

which means the transversality condition (4.10g) satisfies if $\lim_{t\to\infty}\pi(t) = 0$.

Substituting (4.11) into (4.10a) and rearranging, we obtain

$$\omega^{PS\prime} - (\rho - \alpha_k)\,\omega^{PS} = -\mathcal{U}_k^{PS} - \beta_k\gamma^{PS}(k,\pi;\mu,\pi^e).$$

Again this is a first order non-homogeneous differential equation. Its solution can be expressed as

$$\omega^{PS} = \omega^{PS}(k,\pi;\mu,\pi^e). \tag{4.12}$$

Substituting (4.12) into (4.10f), we obtain

$$\lim_{t\to\infty} k(t)\omega^{PS}(k,\pi;\mu,\pi^e) = 0.$$

The above transversality condition satisfis if $\lim_{t\to\infty} k(t) = 0$. That is, the households will leave no capital as an asset at the end of the planning horizon.

Proposition 4.2.3. *In an open loop Nash solution concept with the variation of parameters, the co-state variables of the public sectors are controllable.*

Proof. Examining (4.11) and (4.12), we find that the co-state variables of the public sector are functions of the control variable of the central bank. Hence, the co-state variables of the public sector are controlled by the central bank. □

This means that central bank can change π^e by manoeuvring the policy instrument.

Proposition 4.2.4. *In an open loop Nash solution concept, the dynamic path of the* *public expectations depends on the dynamic behavior of the policy instrument, that is*

$$\pi^{e\prime} = \Upsilon^{21}(k,\pi;\mu;\pi^e) + \Upsilon^{22}(k,\pi;\mu;\pi^e)\mu', \tag{4.13}$$

where

$$\Upsilon^{21} = -\frac{\Upsilon_k\alpha + \Upsilon_\pi\beta}{\Upsilon_{\pi^e}},$$
$$\Upsilon^{22} = -\frac{\Upsilon_\mu}{\Upsilon_{\pi^e}}.$$

Proof. Substituting γ^{PS} and ω^{PS} into (4.10e), we obtain

$$\mathcal{U}^{PS}_{\pi^e} + \omega^{PS}(k,\pi;\mu;\pi^e)\alpha_{\pi^e}(k,\pi;\mu;\pi^e) + \gamma^{PS}(k,\pi;\mu;\pi^e)\beta_{\pi^e}(k,\pi;\mu;\pi^e) = 0. \tag{4.14}$$

Obviously, (4.14) depends on the state and control variables. For simplicity, we rewrite the above equation as follows

$$\Upsilon(k,\pi;\mu;\pi^e) = 0. \tag{4.15}$$

Now differentiating (4.15) with respect to time, we have

$$\Upsilon_k k' + \Upsilon_\pi\pi' + \Upsilon_\mu\mu' + \Upsilon_{\pi^e}\pi^{e\prime} = 0,$$

Substituting $k' = \alpha$ and $\pi' = \beta$, into the above expression and rearranging, we obtain

$$\pi^{e\prime} = -\frac{\Upsilon_k\alpha + \Upsilon_\pi\beta}{\Upsilon_{\pi^e}} - \frac{\Upsilon_\mu}{\Upsilon_{\pi^e}}\mu'. \tag{4.16}$$

□

Proposition 4.2.5. *In an open loop Nash game, assuming* $\Upsilon^{12}\Upsilon^{22} \neq 1$*, the strategies of* *the central bank and the public sector are given by*

$$\mu' = \frac{1}{1 - \Upsilon^{12}\Upsilon^{22}} \left(\Upsilon^{11} + \Upsilon^{12}\Upsilon^{21}\right), \tag{4.17a}$$

$$\pi^{e\prime} = \frac{1}{1 - \Upsilon^{12}\Upsilon^{22}} \left(\Upsilon^{22}\Upsilon^{11} + \Upsilon^{21}\right), \tag{4.17b}$$

respectively.

Proof. To find the trajectories of both players, we rewrite (4.6) and (4.13) as the following matrix notation

$$\begin{pmatrix} 1 & -\Upsilon^{12} \\ -\Upsilon^{22} & 1 \end{pmatrix} \begin{pmatrix} \mu' \\ \pi^{e\prime} \end{pmatrix} = \begin{pmatrix} \Upsilon^{11} \\ \Upsilon^{21} \end{pmatrix}. \tag{4.18}$$

Solving the system (4.18), for μ' and π', we obtain (4.17a) and (4.17b), respectively. □

As emphasized in Propositions 3.2. and 3.6., the reaction of the public sector on the policymaker and vice versa cannot be ignored.

Proposition 4.2.6. *In an open loop Nash solution concept, when the coefficients vary with the control and state variables of the system, the trajectories of both players depend on the variables of the system. In other words, the optimal monetary policy in an open loop Nash solution concept is structurally time inconsistent.*

Proof. From (4.17a) and (4.17b), we respectively have

$$\mu' = \mu^{OLE}(k, \pi; \mu, \pi^e), \tag{4.19a}$$

$$\pi^{e\prime} = \pi^{OLE}(k, \pi; \mu, \pi^e), \tag{4.19b}$$

that is, the optimal monetary policy is a function of the state of the system at each instant of the time horizon. □

Corollary 4.2.1. *The open loop Nash solution concept is time consistent if all coefficients of the system are invariant with respect to the control and state variables.*

Two dynamics of the system, i.e., $k' = \alpha(k; \pi^e)$, and $\pi' = \beta(k, \pi; \mu, \pi^e)$, together with (4.19a) and (4.19b) determine the optimal monetary policy under an open loop Nash solution concept.

Proposition 4.2.7. *The open loop Nash solution concept for the optimal monetary policy in a neoclassical growth model is given as follows*

$$k' = \alpha(k; \pi^e), \tag{4.20a}$$

$$\pi' = \beta(k, \pi; \mu, \pi^e), \tag{4.20b}$$

$$\mu' = \mu^{OLE}(k, \pi; \mu, \pi^e), \tag{4.20c}$$

$$\pi^{e\prime} = \pi^{OLE}(k, \pi; \mu, \pi^e), \tag{4.20d}$$

4.2.2 Feedback Nash Strategies

In the feedback Nash solution concept, one player believes that the other player's action in each period will accord with the observed level of at least one state variable. Mathematically, the control variables of the game between the central bank and the public sector can be described as follows

$$\pi^e = \pi^e(\pi) = \delta\pi; \qquad \delta \in [0, 1], \tag{4.21a}$$

$$\mu = \mu(\pi - \hat{\pi}); \qquad \mu_\pi < 0. \tag{4.21b}$$

This solution concept can be the preferred choice if we believe the players manipulate. The necessary conditions of the central bank in this solution concept are given by

$$\omega^{CB\prime} - \rho\omega^{CB} = -\lambda_1\left(k - k^n\right) - \omega^{CB}\alpha_k(k;\pi^e) - \gamma^{CB}\beta_k(k,\pi;\mu,\pi^e), \tag{4.22a}$$

$$\gamma^{CB\prime} - \rho\gamma^{CB} = \lambda_2\left(\pi - \hat{\pi}\right) - \omega^{CB}\alpha_\pi(k;\pi^e) - \gamma^{CB}\beta_\pi(k,\pi;\mu,\pi^e), \tag{4.22b}$$

$$k' = \left(1 - c(k;\pi^e)(1-\tau)\right)f(k) - \bar{g} + (\Omega - n)k, \tag{4.22c}$$

$$\pi' = \psi(k,\pi;\pi^e)\left((\mu - \pi - n) - \phi(k,\pi;\pi^e)\alpha(k;\pi^e)\right), \tag{4.22d}$$

$$0 = -\lambda_2\pi_\mu\left(\pi - \hat{\pi}\right) + \omega^{CB}\alpha_\mu(k;\pi^e) + \gamma^{CB}\beta_\mu(k,\pi;\mu,\pi^e), \tag{4.22e}$$

$$0 = \lim_{t\to\infty} k(t)\omega^{CB}e^{-\rho t}, \tag{4.22f}$$

$$0 = \lim_{t\to\infty} \pi(t)\gamma^{CB}e^{-\rho t}, \tag{4.22g}$$

$$k(0) = k_0, \quad \pi(0) = \pi_0, \tag{4.22h}$$

where

$$\alpha_\pi(k;\pi^e) = \alpha_{\pi^e}(k;\pi^e)\frac{\partial\pi^e}{\partial\pi},$$
$$\alpha_\mu(k;\pi^e) = \alpha_k\frac{\partial k}{\partial\mu} + \alpha_{\pi^e}(k;\pi^e)\frac{\partial\pi^e}{\partial\pi}\frac{\partial\pi}{\partial\mu}.$$

From (4.22e), we have

$$\gamma^{CB} = \frac{\lambda_2\pi_\mu\left(\pi - \hat{\pi}\right)}{\beta_\mu} - \frac{\alpha_\mu}{\beta_\mu}\omega^{CB}. \tag{4.23}$$

Substituting (4.23) into (4.22a), we obtain

$$\omega^{CB\prime} - \left(\rho + \alpha_\mu\frac{\beta_k}{\beta_\mu} - \alpha_k\right)\omega^{CB} = -\lambda_1(k - k^n) + \lambda_2\pi_\mu(\pi - \hat{\pi})\frac{\beta_k}{\beta_\mu}. \tag{4.24}$$

This is a first order differential equation. The solution can be expressed as

$$\omega^{CB} = \omega^{CB}(k,\pi;\mu,\pi^e), \tag{4.25}$$

Substituting (4.25) into (4.22g), we obtain

$$\lim_{t\to\infty} k(t)\omega^{CB}(k,\pi;\mu,\pi^e) = 0.$$

The transversality condition (4.22g) holds if $\lim_{t\to\infty} k(t) = 0$.

Now, we must find γ^{CB}. Substitute (4.25) into (4.22b) and rearrangeing, we obtain a first order differential equation

$$\gamma^{CB\prime} - (\rho - \beta_\pi)\gamma^{CB} = \lambda_2(\pi - \hat{\pi}) - \alpha_\pi \omega^{CB}(k,\pi;\mu,\pi^e). \tag{4.26}$$

The solution of (4.26) can be expressed as

$$\gamma^{CB} = \gamma^{CB}(k,\pi;\mu,\pi^e). \tag{4.27}$$

Substituting (4.27) into (4.22f), we have

$$\lim_{t\to\infty} \pi(t)\gamma^{CB}(k,\pi;\mu,\pi^e) = 0.$$

The transversality condition (4.22f) holds if $\lim_{t\to\infty} \pi(t) = 0$.

Proposition 4.2.8. *In a feedback Nash solution concept with variable parameters, the co-state variables of the central bank are controllable.*

Substituting (4.25) and (4.27) into (4.22e), we have

$$-\lambda_2 \pi_\mu (\pi - \hat{\pi}) + \alpha_\mu \omega^{CB} + \beta_\mu \gamma^{CB} = 0.$$

For convinience, we rewrite the above equation as

$$\Gamma(k,\pi;\mu,\pi^e) = 0.$$

Differentiation this expression with respect to time gives

$$\Gamma_k k' + \Gamma_\pi \pi' + \Gamma_\mu \mu' + \Gamma_{\pi^e} \pi^{e\prime} = 0.$$

We know that $k' = \alpha$ and $\pi' = \beta$. Morover, differentiating (4.21a) with respect to time gives $\pi^{e\prime} = \delta\beta$. Substituting these terms in the above equation and rearanging, we have

$$\mu' = -\frac{\Gamma_k \alpha}{\Gamma_\mu} - \frac{\beta\left(\Gamma_\pi + \Gamma_{\pi^e}\delta\right)}{\Gamma_\mu}. \tag{4.28}$$

The above equation describes the trajectory of the monetary policy instrument. Obviously, the dynamics of μ are independent of the dynamics of the public expectations.

Proposition 4.2.9. *In a feedback Nash solution concept, the time path of the policy instrument depends on the control and state variables of the system. In other words, the optimal monetary policy is structurally time inconsistent.*

We now need to derive the trajectory of the public expectations under the feedback strategy. The first order conditions for the public sector are given by

$$\omega^{PS\prime} - \rho\omega^{PS} = -\mathcal{U}_k^{PS} - \omega^{PS}\alpha_k(k;\pi^e) - \gamma^{PS}\beta_k(k,\pi;\mu,\pi^e), \tag{4.29a}$$

$$\gamma^{PS\prime} - \rho\gamma^{PS} = -\mathcal{U}_\pi^{PS} - \omega^{PS}\alpha_\pi(k;\pi^e) - \gamma^{PS}\beta_\pi(k,\pi;\mu,\pi^e), \tag{4.29b}$$

$$k' = (1 - c(k;\pi^e)(1-\tau))\, f(k) - \bar{g} + (\Omega - n)k, \tag{4.29c}$$

$$\pi' = \psi(k,\pi;\pi^e)\left((\mu - \pi - n) - \phi(k,\pi;\pi^e)\alpha(k;\pi^e)\right), \tag{4.29d}$$

$$0 = \mathcal{U}_{\pi^e}^{PS} + \omega^{PS}\alpha_{\pi^e}(k;\pi^e) + \gamma^{PS}\beta_{\pi^e}(k,\pi;\mu,\pi^e), \tag{4.29e}$$

$$0 = \lim_{t\to\infty} k(t)\omega^{PS}e^{-\rho t}, \tag{4.29f}$$

$$0 = \lim_{t\to\infty} \pi(t)\gamma^{PS}e^{-\rho t}, \tag{4.29g}$$

$$k(0) = k_0, \quad \pi(0) = \pi_0. \tag{4.29h}$$

where

$$\mathcal{U}_k^{PS} = \frac{\partial \mathcal{U}^{PS1}}{\partial c}\frac{\partial c}{\partial k},$$
$$\mathcal{U}_\pi^{PS} = \frac{\partial \mathcal{U}^{PS1}}{\partial c}\frac{\partial c}{\partial \pi^e}\frac{\partial \pi^e}{\partial \pi},$$
$$\mathcal{U}_{\pi^e}^{PS} = \frac{\partial \mathcal{U}^{PS1}}{\partial c}\frac{\partial c}{\partial \pi^e},$$
$$\alpha_\pi = \frac{\partial \alpha}{\partial \pi^e}\frac{\partial \pi^e}{\partial \pi}.$$

Solving (4.29e) for ω^{PS}, we obtain $\omega^{PS} = -\frac{\mathcal{U}_{\pi^e}^{PS}}{\alpha_{\pi^e}} - \frac{\beta_{\pi^e}}{\alpha_{\pi^e}}\gamma^{PS}$. Substitute this result into (4.29b), gives the first order differential equation

$$\gamma^{PS\prime} - (\rho - \beta_\pi + \delta\beta_{\pi^e})\,\gamma^{PS} = 0,$$

which the solution can be expressed as

$$\gamma^{PS} = \gamma^{PS}(k, \pi; \mu, \pi^e). \tag{4.30}$$

Substituting (4.30) into (4.29g), we have

$$\lim_{t\to\infty} \pi(t)\gamma^{PS}(k, \pi; \mu, \pi^e) = 0,$$

hence the transversality condition (4.29g) holds if $\lim_{t\to\infty} \pi(t) = 0$. Substituting (4.30) into (4.29a) and rearranging, we obtain ω^{PS} as

$$\omega^{PS\prime} - (\rho - \alpha_k)\,\omega^{PS} = -\mathcal{U}_k^{PS} - \beta_k\gamma^{PS}(k, \pi; \mu, \pi^e), \tag{4.31}$$

which is a first order differential equation with the following implicit solution

$$\omega^{PS} = \omega^{PS}(k, \pi; \mu, \pi^e). \tag{4.32}$$

Knowing that

$$\lim_{t\to\infty} k(t)\omega^{PS}(k,\pi;\mu,\pi^e) = 0,$$

The above transversality condition holds if $\lim_{t\to\infty} k(t) = 0$.

Proposition 4.2.10. *In a feedback Nash solution concept, with varying parameters, the co-state variables of the public sector are controllable.*

Substituting (4.30) and (4.32) into (4.29e) the trajectory of the public expectations under the feedback solution concept is obtained as

$$\mathcal{U}^{PS}_{\pi^e} + \alpha_{\pi^e}\omega^{PS}(k,\pi;\mu,\pi^e) + \beta_{\pi^e}\gamma^{PS}(k,\pi;\mu,\pi^e) = 0. \tag{4.33}$$

For simplicity, we rewrite (4.33) as $\Theta(k,\pi;\mu,\pi^e) = 0$. Differentiation this expression-with respect to time, we obtain

$$\Theta_k k' + \Theta_\pi \pi' + \Theta_\mu \mu' + \Theta^e_\pi \pi^{e\prime} = 0, \tag{4.34}$$

with $k' = \alpha$ and $\pi' = \beta$ and differentiating (4.21b) with respect to time, we obtain $\mu' = \mu_\pi \beta$. Substituting this result into (4.34) the dynamics of the public expectations are described by

$$\pi^{e\prime} = -\frac{\Theta_k\alpha + \Theta_\pi\beta + \Theta_\mu\mu_\pi\beta}{\Theta_{\pi^e}}. \tag{4.35}$$

The above equation denotes the trajectory of the public expectation under the feedback Nash solution concept.

Proposition 4.2.11. *In a feedback Nash solution concept, the time path of the public expectations depends on the control and state variables of the system. In other words, the optimal monetary policy is structurally time inconsistent.*

Proposition 4.2.12. *The feedback Nash solution concept for the optimal monetary policy in a neoclassical growth model is given by*

$$k' = \alpha(k; \pi^e), \tag{4.36a}$$

$$\pi' = \beta(k, \pi; \mu, \pi^e), \tag{4.36b}$$

$$\mu' = \Gamma^{FBE}(k, \pi; \mu, \pi^e), \tag{4.36c}$$

$$\pi^{e\prime} = \Theta^{FBE}(k, \pi; \mu, \pi^e). \tag{4.36d}$$

4.3 Stackelberg Equlibrium

Suppose that the players are not required to start the game simultaneously. In this situation, the first player (the leader) is granted priority to choose his strategy. This type of game is a game with hierarchical play, and its equilibrium is the well-known Stackelberg solution concept. If the leader (hereafter the central bank) commits to the time path of its policy instrument and knows the best public response along any given of time path, then the best strategy of the central bank depends on the best reply of the follower (hereafter the public sector). Using this information the central bank can choose its strategy before the public. When the central bank knows the reaction of the public sector (here as an open loop player), it can generate a time path of the monetary policy instrument by employing the first order condition of the open loop Nash game for the public. In this situation, the central bank constrained by four differential equations (Starr & Ho, 1969; Chen & Cruz, 1972; Simaan & Cruz, 1973; Van Long, 2010; Dockner et al., 2000). These two additional constraints include two state variables γ^{PS} and ω^{PS} arising from the optimal control solution of the public in the open loop Nash solution concept. Hence we have

$$\omega^{PS\prime} - \rho\omega^{PS} = -\mathcal{U}_k^{PS} - \omega^{PS}\alpha_k(k;\pi^e) - \gamma^{PS}\beta_k(k,\pi;\mu,\pi^e), \tag{4.37a}$$

$$\gamma^{PS\prime} - \rho\gamma^{PS} = -\gamma^{PS}\beta_\pi(k,\pi;\mu,\pi^e), \tag{4.37b}$$

$$0 = \mathcal{U}_{\pi^e}^{PS} + \omega^{PS}\alpha_{\pi^e}(k;\pi^e) + \gamma^{PS}\beta_{\pi^e}(k,\pi;\mu,\pi^e). \tag{4.37c}$$

Rearranging (4.37c) as $\gamma^{PS} = -\frac{\mathcal{U}_{\pi^e}^{PS}}{\beta_{\pi^e}} - \frac{\alpha_{\pi^e}}{\beta_{\pi^e}}\omega^{PS}$ and substituting it into (4.37a), we obtain the third constraint on the central bank (the first two constraints are (3.12) and (3.13)). The final constraint comes from rearranging (4.37b). These two constraints are respectively given as follows

$$\omega^{PS\prime} = \left(\rho - \alpha_k - \beta_k\frac{\alpha_{\pi^e}}{\beta_{\pi^e}}\right)\omega^{PS} + \frac{\beta_k}{\beta_{\pi^e}}\mathcal{U}_{\pi^e}^{PS} - \mathcal{U}_k^{PS} = \Delta(k,\pi,\omega^{PS};\mu,\pi^e), \tag{4.38a}$$

$$\gamma^{PS\prime} = (\rho - \beta_\pi)\,\gamma^{PS} = \Lambda(k,\pi,\gamma^{PS};\mu,\pi^e), \tag{4.38b}$$

where $\omega^{PS}(0)$ and $\gamma^{PS}(0)$ can be freely chosen by the central bank (Xie, 1997; Dockner et al., 2000; Van Long, 2010).

Proposition 4.3.1. *In an open loop Stackelberg solution concept with the variation of parameters, the co-state variables of the public are controllable.*

The Hamiltonian of the central bank in the Stackelberg solution concept is given by

$$\begin{aligned}\mathcal{H}^{CB} = {} & \frac{1}{2}\lambda_1\left(k - k^n\right)^2 + \frac{1}{2}\lambda_2\left(\pi - \hat{\pi}\right)^2 + \omega^{CB}\alpha(k;\pi^e) + \gamma^{CB}\beta(k,\pi;\mu,\pi^e)\\ & + \xi^{CB}\Delta(k,\pi,\omega^{PS};\mu,\pi^e) + \zeta^{CB}\Lambda(k,\pi,\gamma^{PS};\mu,\pi^e),\end{aligned} \tag{4.39}$$

where ω^{CB}, γ^{CB}, ξ^{CB}, and ζ^{CB} are the co-state variables of the system. To find the solution of this game, we must solve the following Pontryagin maximum principle

$$-\frac{\partial \mathcal{H}^{CB}}{\partial k} = \omega^{CB\prime} - \rho\omega^{CB}, \qquad -\frac{\partial \mathcal{H}^{CB}}{\partial \pi} = \gamma^{CB\prime} - \rho\gamma^{CB},$$
$$-\frac{\partial \mathcal{H}^{CB}}{\partial \omega^{PS}} = \xi^{CB\prime} - \rho\xi^{CB}, \qquad -\frac{\partial \mathcal{H}^{CB}}{\partial \gamma^{PS}} = \zeta^{CB\prime} - \rho\zeta^{CB},$$
$$k' = \alpha(k;\pi^e), \qquad \pi' = \beta(k,\pi;\mu,\pi^e),$$
$$\omega^{PS\prime} = \Delta(k,\pi,\omega^{PS};\mu,\pi^e), \qquad \gamma^{PS\prime} = \Lambda(k,\pi,\gamma^{PS};\mu,\pi^e),$$
$$\frac{\partial \mathcal{H}^{CB}}{\partial \mu} = 0,$$

given $k(0) = k_0$ and $\pi(0) = \pi_0$, where $\omega^{PS}(0)$, and $\gamma^{PS}(0)$ are both free and are chosen by the central bank . Finally, the transversality conditions of this optimal control problem are given by[6]

$$\lim_{t\to\infty} k(t)\omega^{CB}e^{-\rho t} = 0, \qquad \lim_{t\to\infty} \pi(t)\gamma^{CB}e^{-\rho t} = 0,$$
$$\lim_{t\to\infty} \omega^{PS}(t)\xi^{CB}e^{-\rho t} = 0, \qquad \lim_{t\to\infty} \gamma^{PS}(t)\zeta^{CB}e^{-\rho t} = 0.$$

From the results of the above system of equations, the central bank suggests two time paths for ω^{PS} and γ^{PS} that satisfy the transversality conditions. After the public sector substitutes these two time paths into (4.37c) and solves to obtain the trajectory of the public expectations the optimization problem is completed.

In the open loop Stackelberg solution concept, both players make their decisions in a hierarchical manner. Similar to the open loop Nash solution concept, the time path actions are planned over the time horizon. Therefore in both of this games $\mu = \mu(t)$ and $\pi^e = \pi^e(t)$ are still valid.

[6] Turnovsky and Brock (1980); Xie (1997); Dockner et al. (2000).

Meanwhile, the first order conditions for the central bank are given by

$$\omega^{CB\prime} - \rho\omega^{CB} = \lambda_1 (k - k^n) - \omega^{CB}\alpha_k - \gamma^{CB}\beta_k - \xi^{CB}\Delta_k - \zeta^{CB}\Lambda_k, \tag{4.40a}$$

$$\gamma^{CB\prime} - \rho\gamma^{CB} = -\lambda_2 (\pi - \hat{\pi}) - \gamma^{CB}\beta_\pi - \xi^{CB}\Delta_\pi - \zeta^{CB}\Lambda_\pi, \tag{4.40b}$$

$$\xi^{CB\prime} - \rho\xi^{CB} = -\xi^{CB}\left(\rho - \alpha_k - \beta_k \frac{\alpha_{\pi^e}}{\beta_{\pi^e}}\right), \tag{4.40c}$$

$$\zeta^{CB\prime} - \rho\zeta^{CB} = -\zeta^{CB}(\rho - \beta_\pi), \tag{4.40d}$$

$$k' = (1 - c(1-\tau)) f(k) - \bar{g} + (\Omega - n)k, \tag{4.40e}$$

$$\pi' = \psi((\mu - \pi - n) - \phi\alpha), \tag{4.40f}$$

$$\omega^{PS\prime} = \left(\rho - \alpha_k - \beta_k \frac{\alpha_{\pi^e}}{\beta_{\pi^e}}\right)\omega^{PS} + \frac{\beta_k}{\beta_{\pi^e}}\mathcal{U}^{PS}_{\pi^e} - \mathcal{U}^{PS}_k, \tag{4.40g}$$

$$\gamma^{PS\prime} = (\rho - \beta_\pi)\gamma^{PS}, \tag{4.40h}$$

$$0 = \lambda_2 \pi_\mu (\pi - \hat{\pi}) + \gamma^{CB}\beta_\mu + \xi^{CB}\Delta_\mu + \zeta^{CB}\Lambda_\mu, \tag{4.40i}$$

$$0 = \lim_{t\to\infty} k(t)\omega^{CB}e^{-\rho t}, \tag{4.40j}$$

$$0 = \lim_{t\to\infty} \pi(t)\gamma^{CB}e^{-\rho t}, \tag{4.40k}$$

$$0 = \lim_{t\to\infty} \omega^{PS}(t)\xi^{CB}e^{-\rho t}, \tag{4.40l}$$

$$0 = \lim_{t\to\infty} \gamma^{PS}(t)\zeta^{CB}e^{-\rho t}, \tag{4.40m}$$

$$k(0) = k_0, \;\; \pi(0) = \pi_0, \;\; \omega^{PS}(0); free, \;\; \gamma^{PS}(0); free. \tag{4.40n}$$

Rearranging (4.40c) and (4.40d), we obtain the following pair of homogeneous differential equations

$$\xi^{CB\prime} - \left(\alpha_k + \beta_k \frac{\alpha_{\pi^e}}{\beta_{\pi^e}}\right)\xi^{CB} = 0,$$

$$\zeta^{CB\prime} - \beta_\pi \zeta^{CB} = 0,$$

the solution of the above equations can be expressed as

$$\xi^{CB} = \xi^{CB}(k, \pi; \mu, \pi^e), \tag{4.41a}$$

$$\zeta^{CB} = \zeta^{CB}(k, \pi; \mu, \pi^e). \tag{4.41b}$$

Both of the above solutions should satisfy the corresponding transversality conditions

$$\lim_{t \to \infty} \omega^{PS}(t) \xi^{CB}(k, \pi; \mu, \pi^e) = 0,$$

$$\lim_{t \to \infty} \gamma^{PS}(t) \zeta^{CB}(k, \pi; \mu, \pi^e) = 0.$$

To find γ^{CB}, we substitute (4.41a) and (4.41b) into (4.40i) and obtain

$$\gamma^{CB} = -\frac{\lambda_2 \pi_\mu (\pi - \hat{\pi})}{\beta_\mu} - \frac{\Delta_\mu}{\beta_\mu} \xi^{CB}(k, \pi; \mu, \pi^e) - \frac{\Lambda_\mu}{\beta_\mu} \zeta^{CB}(k, \pi; \mu, \pi^e). \tag{4.42}$$

Now, substitute (4.41a), (4.41b) and (4.42) into (4.40a) and rearranging, we get

$$\begin{aligned} \omega^{CB\prime} - (\rho - \alpha_k)\, \omega^{CB} = \lambda_1 (k - k^n) + \lambda_2 \pi_\mu \frac{\beta_k}{\beta_\mu} (\pi - \hat{\pi}) + \left(\Delta_\mu \frac{\beta_k}{\beta_\mu} - \Delta_k \right) \xi \\ + \left(\Lambda_\mu \frac{\beta_k}{\beta_\mu} - \Lambda_k \right) \zeta. \end{aligned} \tag{4.43}$$

Solution to (4.43) can be expressed as

$$\omega^{CB} = \omega^{CB}(k, \pi, \omega^{PS}, \gamma^{PS}; \mu, \pi^e). \tag{4.44}$$

Proposition 4.3.2. *In an open loop Stackelberg solution concept, with the variation of parameters, the co-state variables of the central bank are controllable.*

For simplicity, (4.40i) can be written as follows

$$\Xi^1(k, \pi, \gamma^{PS}, \omega^{PS}; \mu, \pi^e) = 0 \tag{4.45}$$

To find the dynamics of μ, we differentiate (4.45) with respect to time

$$\Xi^1_k k' + \Xi^1_\pi \pi' + \Xi^1_{\gamma^{PS}} \gamma^{PS\prime} + \Xi^1_{\omega^{PS}} \omega^{PS\prime} + \Xi^1_\mu \mu' + \Xi^1_{\pi^e} \pi^{e\prime} = 0,$$

where $\gamma^{PS\prime} = \Lambda$, $\omega^{PS\prime} = \Delta$, $k' = \alpha$ and $\pi' = \beta$. Substituting these terms into the above equation and rearranging, we derive the dynamics of the monetary polict rate as follows

$$\mu' = -\frac{\Xi^1_k \alpha + \Xi^1_\pi \beta + \Xi^1_{\gamma^{PS}} \Lambda + \Xi^1_{\omega^{PS}} \Delta}{\Xi^1_\mu} - \frac{\Xi^1_{\pi^e}}{\Xi^1_\mu} \pi^{e\prime}. \tag{4.46}$$

Therefore, the dynamics of the policy instrument depends on the dynamics of the public expectations and more importantly is the variation of coefficients depends on the states and control variables of the system. For simplicity, we rewrite (4.46) as

$$\mu' = \Xi^{11} + \Xi^{12} \pi^{e\prime}, \tag{4.47}$$

where

$$\begin{aligned}
\Xi^{11} &= -\frac{\Xi^1_k \alpha + \Xi^1_\pi \beta + \Xi^1_{\gamma^{PS}} \Lambda + \Xi^1_{\omega^{PS}} \Delta}{\Xi^1_\mu}, \\
\Xi^{12} &= -\frac{\Xi^1_{\pi^e}}{\Xi^1_\mu}.
\end{aligned}$$

Proposition 4.3.3. *In an open loop Stackelberg solution concept, the dynamic path of the monetary policy instrument depends on the dynamic behavior of the public expectations.*

The dynamics of the public expectations can be found from equation (4.37c). For convinience, we rewrite this equation as $\Xi^2(k, \pi, \gamma^{PS}, \omega^{PS}; \mu, \pi^e) = 0$. Differentiating it with respect to time, we obtain

$$\Xi^2_k k' + \Xi^2_\pi \pi' + \Xi^2_{\gamma^{PS}} \gamma^{PS\prime} + \Xi^2_{\omega^{PS}} \omega^{PS\prime} + \Xi^2_\mu \mu' + \Xi^2_{\pi^e} \pi^{e\prime} = 0,$$

Substituting $k' = \alpha$, $\pi' = \beta$, $\omega^{PS\prime} = \Delta$ and $\gamma^{PS\prime} = \Lambda$ into the above equation and rearranging, we get

$$\pi^{e\prime} = -\frac{\Xi^2_k \alpha + \Xi^2_\pi \beta + \Xi^2_{\gamma^{PS}} \Lambda + \Xi^2_{\omega^{PS}} \Delta}{\Xi^2_{\pi^e}} - \frac{\Xi^2_\mu}{\Xi^2_{\pi^e}} \mu', \tag{4.48}$$

which means that the dynamics of the public expectations also depends on the dynamics of the monetary policy instrument. Again, the trajectory coefficients are not invariant and depend on the states and control variables of the system. For simplicity, we rewrite (4.48) as follows

$$\pi^{e\prime} = \Xi^{21} + \Xi^{22} \mu', \tag{4.49}$$

where

$$\begin{aligned} \Xi^{21} &= -\frac{\Xi^2_k \alpha + \Xi^2_\pi \beta + \Xi^2_{\gamma^{PS}} \Lambda + \Xi^2_{\omega^{PS}} \Delta}{\Xi^2_{\pi^e}}, \\ \Xi^{12} &= -\frac{\Xi^2_\mu}{\Xi^2_{\pi^e}}. \end{aligned}$$

Proposition 4.3.4. *In an open loop Stackelberg solution concept, the dynamic path of the public expectations depends on the dynamic behavior of the policy instrument.*

To find the dynamics of the policy instrument and the public expectations, we simultaneously solve (4.47) and (4.49) as follows

$$\begin{pmatrix} \mu' \\ \pi^{e\prime} \end{pmatrix} = \begin{pmatrix} 1 & -\Xi^{12} \\ -\Xi^{22} & 1 \end{pmatrix}^{-1} \begin{pmatrix} \Xi^{11} \\ \Xi^{21} \end{pmatrix}. \tag{4.50}$$

Suppose that $1 - \Xi^{12}\Xi^{22} \neq 0$, we conclude

$$\mu' = \mu^{SOE}(k, \pi, \gamma^{PS}, \omega^{PS}; \mu, \pi^e), \tag{4.51a}$$

$$\pi^{e\prime} = \pi^{SOE}(k, \pi, \gamma^{PS}, \omega^{PS}; \mu, \pi^e). \tag{4.51b}$$

Proposition 4.3.5. *In an open loop Stackelberg solution concept, when the coefficients depend on the control and state variables of the system, the trajectories of both players depend on the variables of the system. In other words, the optimal monetary policy under an open loop Stackelberg solution concept is structurally time inconsistent.*

The open loop Stackelberg solution concept is described by the following proposition.

Proposition 4.3.6. *An open loop Stackelberg solution concept for the optimal monetary policy in a neoclassical growth model is conceptualized as follow*

$$k' = \alpha(k; \pi^e), \tag{4.52a}$$

$$\pi' = \beta(k, \pi; \mu, \pi^e), \tag{4.52b}$$

$$\omega^{PS\prime} = \Delta(k, \pi, \omega^{PS}; \mu, \pi^e), \tag{4.52c}$$

$$\gamma^{PS\prime} = \Lambda(k, \pi, \gamma^{PS}; \mu, \pi^e), \tag{4.52d}$$

$$\mu' = \mu^{SOE}(k, \pi, \gamma^{PS}, \omega^{PS}; \mu, \pi^e), \tag{4.52e}$$

$$\pi^{e\prime} = \pi^{SOE}(k, \pi, \gamma^{PS}, \omega^{PS}; \mu, \pi^e), \tag{4.52f}$$

Chapter 5

Chaos: A Numerical Investigation

5.1 Introduction

This chapter investigates the chaos in the monetary policy game between the central bank and the public. The analysis is performed in a specific neoclassical growth framework which is presented in Chapter 3. The results can stimulate improvements in both the theory and applications of monetary policy chaotic games.

First, we discuss the mathematical solutions of this model in different solution concepts, then present the numerical solutions that are more suited to economic applications. Because the solutions in each case were analyzed by the same method, we can compare the results of the different games.

The central bank and the public sector are required to maximize (3.19a) and (3.19b) subject to (3.18a) and (3.18b) respectively, given the usual initial conditions of the state variables. The Hamiltonian of the players in this specific monetary policy game are easly introduced as

$$\begin{aligned}\mathcal{H}^{CB}(k,\pi;\pi^e,\mu;\omega^{CB},\gamma^{CB}) = \frac{1}{2}\lambda_1\left(k-k^n\right)^2+\frac{1}{2}\lambda_2\left(\pi-\hat{\pi}\right)^2 \\ +\,\omega^{CB}\alpha(k,\pi^e)+\gamma^{CB}\beta(k,\pi,\mu,\pi^e),\end{aligned} \tag{5.1a}$$

$$\begin{aligned}\mathcal{H}^{PS}(k,\pi;\pi^e,\mu;\omega^{PS},\gamma^{PS}) = \ln\left(\frac{\theta(1-\tau)}{\sigma(\varrho k^{\varrho-1}+\pi^e)}k^{\varrho}\right)+\ln\left(\mu\right) \\ +\,\omega^{PS}\alpha(k,\pi^e)+\gamma^{PS}\beta(k,\pi,\mu,\pi^e).\end{aligned} \tag{5.1b}$$

5.2 Solution Concepts of the Specific Model

This section is mainly devoted to the equilibrium of our three solution concepts introduced in Chapter 3. The trajectories of both players are found by the approaches developed in the previous chapter.

5.2.1 Nash Equilibrium: Open Loop Strategy

In the open loop strategy, each player should commit to his planned action over the time horizon. Therefore, the control variable of each player does not depend on the states of the system and we can simply write

$$\pi^e = \pi^e(t), \tag{5.2}$$

$$\mu = \mu(t). \tag{5.3}$$

Suppose that the central bank and the private sector move simultaneously at each instance in time. Using the Hamiltonian of the players given in the previous section, i.e., (5.1a) and (5.1b) the optimum path of both players can be determined by the Pontryagin maximum principle. Hence, the necessary conditions for the central bank are given as follows

$$\omega^{CB\prime} - \rho\omega^{CB} = -\lambda_1(k - k^n) - \omega^{CB}\alpha_k(k;\pi^e) - \gamma^{CB}\beta_k(k,\pi;\mu,\pi^e), \tag{5.4a}$$

$$\gamma^{CB\prime} - \rho\gamma^{CB} = \lambda_2(\pi - \hat{\pi}) - \gamma^{CB}\beta_\pi(k,\pi;\mu,\pi^e), \tag{5.4b}$$

$$k' = \left(1 - \frac{\theta(1-\tau)}{\sigma(\varrho k^{\varrho-1} + \pi^e)}\right)k^\varrho - \bar{g} + (\Omega - n)k, \tag{5.4c}$$

$$\pi' = \psi(k,\pi;\pi^e)\left((\mu - \pi - n) - \phi(k,\pi;\pi^e)\alpha(k;\pi^e)\right), \tag{5.4d}$$

$$0 = -\lambda_2\epsilon(\pi - \hat{\pi}) + \omega^{CB}\alpha_\mu(k;\pi^e) + \gamma^{CB}\beta_\mu(k,\pi;\mu,\pi^e), \tag{5.4e}$$

$$0 = \lim_{t\to\infty} k(t)\omega^{CB}e^{-\rho t}, \tag{5.4f}$$

$$0 = \lim_{t\to\infty} \pi(t)\gamma^{CB}e^{-\rho t}, \tag{5.4g}$$

$$k(0) = k_0, \quad \pi(0) = \pi_0, \tag{5.4h}$$

where $\alpha_\mu(k;\pi^e) = \alpha_k(1-\epsilon)$ and $\beta_\mu(k,\pi;\pi^e) = \psi\left((1-\epsilon) - \phi\alpha_\mu\right)$. Solving (5.4b), we obtain the following first order differential equation

$$\gamma^{CB\prime} - (\rho - \beta_\pi)\gamma^{CB} = \lambda_2(\pi - \hat{\pi}),$$

where $\beta_\pi = -\psi$. Solution to the above differential equation can be expressed as

$$\gamma^{CB} = \gamma^{CB}(k,\pi;\mu,\pi^e). \tag{5.5}$$

Substituting (5.5) into (5.4a), we obtain the following first order differential equation

$$\omega^{CB\prime} - (\rho - \alpha_k)\omega^{CB} = -\lambda_1(k - k^n) - \beta_k\gamma^{CB}(k,\pi;\mu,\pi^e), \tag{5.6}$$

where

$$\alpha_k = \varrho\left(1 - \frac{\theta(1-\tau)}{\sigma(\varrho k^{\varrho-1} + \pi^e)}\right)k^{(\varrho-1)} + \frac{\theta(1-\tau)\varrho(\varrho-1)}{\sigma(\varrho k^{\varrho-1} + \pi^e)^2}k^{(\varrho-2)} + \Omega - n.$$

and

$$\beta_k = -\psi\phi\alpha_k.$$

Solution to (5.6) can be expressed as

$$\omega^{CB} = \omega^{CB}(k, \pi; \mu, \pi^e) \tag{5.7}$$

To find the trajectory of the central bank decision variable, i.e., μ, we differentiate (5.4e) with respect to time, thereby obtain

$$-\lambda_2\epsilon\pi' + \alpha_\mu\omega^{CB\prime} + \alpha'_\mu\omega^{CB} + \beta_\mu\gamma^{CB\prime} + \beta'_\mu\gamma^{CB} = 0. \tag{5.8}$$

We have already found ω^{CB} and γ^{CB}, given that

$$\alpha_\mu(k; \pi^e) = \alpha_k(1-\epsilon),$$
$$\beta_\mu(k, \pi; \pi^e) = \psi\left((1-\epsilon) - \phi\alpha_\mu\right),$$

we obtain

$$\alpha'_\mu = (1-\epsilon)\left(\alpha_{kk}k' + \alpha_{k\pi^e}\pi^{e\prime} + \alpha_{kk}(1-\epsilon)\mu'\right). \tag{5.9}$$

$$\beta'_\mu = \beta_{\mu k}k' + \beta_{\mu\pi}\pi' + \beta_{\mu\pi^e}\pi^{e\prime}. \tag{5.10}$$

We also have $k' = \alpha$ and $\pi' = \beta$. Substituting (5.9) and (5.10) into (5.8) and rearranging for μ, we get

$$\mu' = \Upsilon^{11} + \Upsilon^{12}\pi^{e\prime}, \tag{5.11}$$

where

$$\Upsilon^{11} = \frac{-\lambda_2\epsilon\pi' + \alpha_\mu\omega^{CB\prime} + \beta_\mu\gamma^{CB\prime} + \omega^{CB}(1-\epsilon)\alpha_{kk}\alpha + \gamma^{CB}\left(\beta_{\mu k}\alpha + \beta_{\mu\pi}\beta\right)}{-\omega^{CB}(1-\epsilon)^2\alpha_{kk}},$$
$$\Upsilon^{12} = \frac{\omega^{CB}(1-\epsilon)\alpha_{k\pi^e} + \gamma^{CB}\beta_{\mu\pi^e}}{-\omega^{CB}(1-\epsilon)^2\alpha_{kk}}.$$

Equation (5.11) describes the trajectory of the monetary policy rate.

We must now find the trajectory of the public sector. To this end, we solve the following Pontryagin maximum principle

$$\omega^{PS\prime} - \rho\omega^{PS} = -\mathcal{U}_k^{PS} - \omega^{PS}\alpha_k(k;\pi^e) - \gamma^{PS}\beta_k(k,\pi;\mu,\pi^e), \tag{5.12a}$$

$$\gamma^{PS\prime} - \rho\gamma^{PS} = -\gamma^{PS}\beta_\pi(k,\pi;\mu,\pi^e), \tag{5.12b}$$

$$k' = \left(1 - \frac{\theta(1-\tau)}{\sigma(\varrho k^{\varrho-1} + \pi^e)}\right)k^\varrho - \bar{g} + (\Omega - n)k, \tag{5.12c}$$

$$\pi' = \psi(k,\pi;\pi^e)\left((\mu - \pi - n) - \phi(k,\pi;\pi^e)\alpha(k;\pi^e)\right), \tag{5.12d}$$

$$0 = \mathcal{U}_{\pi^e}^{PS} + \omega^{PS}\alpha_{\pi^e}(k;\pi^e) + \gamma^{PS}\beta_{\pi^e}(k,\pi;\mu,\pi^e), \tag{5.12e}$$

$$0 = \lim_{t\to\infty} k(t)\omega^{PS}e^{-\rho t}, \tag{5.12f}$$

$$0 = \lim_{t\to\infty} \pi(t)\gamma^{PS}e^{-\rho t}, \tag{5.12g}$$

$$k(0) = k_0, \quad \pi(0) = \pi_0, \tag{5.12h}$$

where

$$\mathcal{U}_k^{PS} = \mathcal{U}_{\mathcal{C}}^{PS}\frac{\partial\mathcal{C}}{\partial k} = \frac{\varrho(k^{\varrho-1} + \pi^e)}{(\varrho k^{\varrho-1} + \pi^e)k},$$
$$\mathcal{U}_{\pi^e}^{PS} = \mathcal{U}_{\mathcal{C}}^{PS}\frac{\partial\mathcal{C}}{\partial\pi^e} = \frac{-1}{(\varrho k^{\varrho-1} + \pi^e)}.$$

From (5.12b), we have

$$\gamma^{PS\prime} - (\rho - \beta_\pi)\gamma^{PS} = 0, \tag{5.13}$$

which is a linear homogeneous first order differential equation. The solution to (5.13) can be represented as

$$\gamma^{PS} = \gamma^{PS}(k, \pi; \mu, \pi^e). \tag{5.14}$$

To find ω^{PS}, we substitute (5.14) into (5.12a). The result is another first order differential equation

$$\omega^{PS\prime} - (\rho - \alpha_k)\omega^{PS} = -\mathcal{U}_k^{PS} - \beta_k \gamma^{PS}(k, \pi; \mu, \pi^e). \tag{5.15}$$

Solution to (5.15) can be expressed as

$$\omega^{PS} = \omega^{PS}(k, \pi; \mu, \pi^e). \tag{5.16}$$

Substituting the known values of γ^{PS} and ω^{PS} into (5.12e), we can find the dynamics of the public expectation. First, we differentiate (5.12e) with respect to time

$$\mathcal{U}_{\pi^e k}^{PS} k' + \mathcal{U}_{\pi^e \pi^e}^{PS} \pi^{e\prime} + \omega^{PS\prime} \alpha_{\pi^e} + \omega^{PS} \alpha'_{\pi^e} + \gamma^{PS\prime} \beta_{\pi^e} + \gamma^{PS} \beta'_{\pi^e} = 0, \tag{5.17}$$

where $\beta_{\pi^e} = -\psi\phi\alpha_{\pi^e}$, $\alpha_{\pi^e} = \delta\alpha_\pi$ and

$$\begin{aligned}
\mathcal{U}_{\pi^e k}^{PS} &= \frac{\varrho(\varrho - 1)k^{\varrho - 2}}{(\varrho k^{\varrho - 1} + \pi^e)^2}, \\
\mathcal{U}_{\pi^e \pi^e}^{PS} &= \frac{1}{(\varrho k^{\varrho - 1} + \pi^e)^2}, \\
\omega^{PS\prime} &= \frac{\partial \omega^{PS}}{\partial k} k' + \frac{\partial \omega^{PS}}{\partial \pi} \pi' + \frac{\partial \omega^{PS}}{\partial \mu} \mu' + \frac{\partial \omega^{PS}}{\partial \pi^e} \pi^{e\prime}, \\
\gamma^{PS\prime} &= \frac{\partial \gamma^{PS}}{\partial k} k' + \frac{\partial \gamma^{PS}}{\partial \pi} \pi' + \frac{\partial \gamma^{PS}}{\partial \pi^e} \pi^{e\prime}, \\
\alpha'_{\pi^e} &= \frac{\partial \alpha_{\pi^e}}{\partial k} k' + \frac{\partial \alpha_{\pi^e}}{\partial \pi^e} \pi^{e\prime}, \\
\beta'_{\pi^e} &= \frac{\partial \beta_{\pi^e}}{\partial k} k' + \frac{\partial \beta_{\pi^e}}{\partial \pi} \pi' + \frac{\partial \beta_{\pi^e}}{\partial \mu} \mu' + \frac{\partial \beta_{\pi^e}}{\partial \pi^e} \pi^{e\prime}.
\end{aligned}$$

Substitute the above equations in (5.17) and rearranging for $\pi^{e\prime}$, we obtain

$$\pi^{e\prime} = \Upsilon^{21} + \Upsilon^{22}\mu', \tag{5.18}$$

where

$$\begin{aligned}
\Upsilon^{21} &= -\frac{\alpha\left(\frac{\varrho(\varrho-1)k^{\varrho-2}}{(\varrho k^{\varrho-1}+\pi^e)^2} + \frac{\partial\omega^{PS}}{\partial k}\alpha_{\pi^e} + \frac{\partial\alpha^{\pi^e}}{\partial k}\omega_{PS} + \frac{\partial\gamma^{PS}}{\partial k}\beta_{\pi^e} + \frac{\beta_{\pi^e}}{\partial k}\gamma^{PS}\right)}{\frac{1}{(\varrho k^{\varrho-1}+\pi^e)^2} + \frac{\partial\omega^{PS}}{\partial\pi^e}\alpha_{\pi^e} + \frac{\partial\alpha_{\pi^e}}{\partial\pi^e}\omega^{PS} + \frac{\partial\gamma^{PS}}{\partial\pi^e}\beta_{\pi^e} + \frac{\partial\beta_{\pi^e}}{\partial\pi^e}\gamma^{PS}} \\
&\quad - \frac{\beta\left(\frac{\partial\omega^{PS}}{\partial\pi}\alpha_{\pi^e} + \frac{\partial\gamma^{PS}}{\partial\pi}\beta_{\pi^e} + \frac{\beta_{\pi^e}}{\partial\pi}\gamma^{PS}\right)}{\frac{1}{(\varrho k^{\varrho-1}+\pi^e)^2} + \frac{\partial\omega^{PS}}{\partial\pi^e}\alpha_{\pi^e} + \frac{\partial\alpha_{\pi^e}}{\partial\pi^e}\omega^{PS} + \frac{\partial\gamma^{PS}}{\partial\pi^e}\beta_{\pi^e} + \frac{\partial\beta_{\pi^e}}{\partial\pi^e}\gamma^{PS})}, \\
\Upsilon^{22} &= -\frac{\frac{\partial\omega^{PS}}{\partial\mu}\alpha_{\pi^e} + \frac{\partial\beta_{\pi^e}}{\partial\mu}\gamma^{PS}}{\frac{1}{(\varrho k^{\varrho-1}+\pi^e)^2} + \frac{\partial\omega^{PS}}{\partial\pi^e}\alpha_{\pi^e} + \frac{\partial\alpha_{\pi^e}}{\partial\pi^e}\omega^{PS} + \frac{\partial\gamma^{PS}}{\partial\pi^e}\beta_{\pi^e} + \frac{\partial\beta_{\pi^e}}{\partial\pi^e}\gamma^{PS})}.
\end{aligned}$$

To find the the players' strategies, we need to solve (5.11) and (5.18) simultaneously. The result is

$$\mu' = \frac{1}{1-\Upsilon^{12}\Upsilon^{22}}\left(\Upsilon^{11} + \Upsilon^{12}\Upsilon^{21}\right), \tag{5.19}$$

$$\pi^{e\prime} = \frac{1}{1-\Upsilon^{12}\Upsilon^{22}}\left(\Upsilon^{22} + \Upsilon^{11}\Upsilon^{21}\right). \tag{5.20}$$

Thus the solution of the open loop Nash strategy is as follows

$$k' = \left(1 - \frac{\theta(1-\tau)}{\sigma\left(\varrho k^{\varrho-1} + \pi^e\right)}\right)k^{\varrho} - \bar{g} + (\Omega - n)k, \tag{5.21a}$$

$$\pi' = \quad (k, \pi; \pi^e)\left((\mu - \pi - n) - \phi(k, \pi; \pi^e)\alpha(k; \pi^e)\right), \tag{5.21b}$$

$$\mu' = \frac{1}{1-\Upsilon^{12}\Upsilon^{22}}\left(\Upsilon^{11} + \Upsilon^{12}\Upsilon^{21}\right), \tag{5.21c}$$

$$\pi^{e\prime} = \frac{1}{1-\Upsilon^{12}\Upsilon^{22}}\left(\Upsilon^{22} + \Upsilon^{11}\Upsilon^{21}\right). \tag{5.21d}$$

5.2.2 Nash Equilibrium: Feedback Strategy

In the previous chapter, we explained in the feedback Nash strategy the players observe the states of the system to choose their next movement accordingly. For instance, consider the following reaction functions of the central bank and the public

$$\mu = \mu(\pi - \hat{\pi}); \qquad \mu_\pi < 0, \tag{5.22a}$$

$$\pi^e = \pi^e(\pi) = \delta\pi; \qquad \delta \in [0,1]. \tag{5.22b}$$

In this solution concept, each player solves the optimal control problem given the opponent's strategy.

Consider the Pontryagin maximum principle described in Theorem 2.2.1. Substituting (5.22a) and (5.22b) into the respective Hamiltonian of the players, we can find the feedback Nash equilibria which differ from the equilibria in the previous section. The necessary conditions for the central bank are given by

$$\omega^{CB\prime} - \rho\omega^{CB} = -\lambda_1 (k - k^n) - \omega^{CB}\alpha_k(k;\pi^e) - \gamma^{CB}\beta_k(k,\pi;\mu,\pi^e), \tag{5.23a}$$

$$\gamma^{CB\prime} - \rho\gamma^{CB} = \lambda_2 (\pi - \hat{\pi}) - \omega^{CB}\alpha_\pi(k;\pi^e) - \gamma^{CB}\beta_\pi(k,\pi;\mu,\pi^e), \tag{5.23b}$$

$$k' = \left(1 - \frac{\theta(1-\tau)}{\sigma(\varrho k^{\varrho-1} + \pi^e)}\right)k^\varrho - \bar{g} + (\Omega - n)k, \tag{5.23c}$$

$$\pi' = \psi(k,\pi;\pi^e)\left((\mu - \pi - n) - \phi(k,\pi;\pi^e)\alpha(k;\pi^e)\right), \tag{5.23d}$$

$$0 = -\lambda_2\epsilon(\pi - \hat{\pi}) + \omega^{CB}\alpha_\mu(k;\pi^e) + \gamma^{CB}\beta_\mu(k,\pi;\mu,\pi^e), \tag{5.23e}$$

$$0 = \lim_{t\to\infty} k(t)\omega^{CB}e^{-\rho t}, \tag{5.23f}$$

$$0 = \lim_{t\to\infty} \pi(t)\gamma^{CB}e^{-\rho t}, \tag{5.23g}$$

$$k(0) = k_0, \quad \pi(0) = \pi_0, \tag{5.23h}$$

where

$$\begin{aligned}
\alpha_\pi(k;\pi^e) &= \frac{\partial \pi^e}{\partial \pi}\alpha_{\pi^e}(k;\pi^e) = \delta\alpha_{\pi^e}(k;\pi^e),\\
\alpha_{\pi^e}(k;\pi^e) &= \frac{k^\varrho}{\sigma^2(\varrho k^{\varrho-1}+\pi^e)^2},\\
\alpha_\mu(k;\pi^e) &= \alpha_k\frac{\partial k}{\partial \mu} + \alpha_{\pi^e}(k;\pi^e)\frac{\partial \pi^e}{\partial \pi}\frac{\partial \pi}{\partial \mu} = (1-\epsilon)\alpha_k + \delta\epsilon\alpha_{\pi^e},\\
\beta_\mu(k,\pi;\mu,\pi^e) &= \psi\left((1-\epsilon)-\phi\alpha_\mu\right).
\end{aligned}$$

Rearranging (5.23e), we get

$$\gamma^{CB} = \frac{\lambda_2\epsilon\left(\pi-\hat{\pi}\right)}{\beta_\mu} - \frac{\alpha_\mu}{\beta_\mu}\omega^{CB}.$$

Substituting the above equation into (5.23a) and rearranging, we obtain

$$\omega^{CB\prime} - \left(\rho - \alpha_k + \alpha_\mu\frac{\beta_k}{\beta_\mu}\right)\omega^{CB} = -\lambda_1\left(k-k^n\right) - \lambda_2\epsilon\left(\pi-\hat{\pi}\right)\frac{\beta_k}{\beta_\mu}. \tag{5.24}$$

Equation (5.24) is a first order differential equation, where solution to this equation can be expressed as

$$\omega^{CB} = \omega^{CB}(k,\pi;\mu,\pi^e). \tag{5.25}$$

Substituting (5.25) into (5.23b) and rearranging, we obtain another first order differential equation

$$\gamma^{CB\prime} - (\rho-\beta_\pi)\gamma^{CB} = \lambda_2\left(\pi-\hat{\pi}\right) - \alpha_\pi\omega^{CB}(k,\pi;\mu,\pi^e). \tag{5.26}$$

Solution to (5.26) can be represented as

$$\gamma^{CB} = \gamma^{CB}(k,\pi;\mu,\pi^e). \tag{5.27}$$

Inserting ω^{CB} and γ^{CB} calculated by (5.25) and (5.27) respectively into (5.23e), we obtain the dynamics of the central bank's strategy. Differentiating (5.23e) with respect to time, we get

$$-\lambda_2 \epsilon \pi' + \omega^{CB} \alpha'_\mu + \omega^{CB\prime} \alpha_\mu + \gamma^{CB} \beta'_\mu + \gamma^{CB\prime} \beta_\mu = 0, \tag{5.28}$$

where

$$\begin{aligned}
\omega^{CB\prime}(k, \pi; \mu, \pi^e) &= \frac{\partial \omega^{CB}}{\partial k} k' + \left(\frac{\omega^{CB}}{\partial \pi} + \delta \frac{\partial \omega^{CB}}{\partial \pi^e} \right) \pi' + \frac{\partial \omega^{CB}}{\partial \mu} \mu', \\
\gamma^{CB\prime}(k, \pi; \mu, \pi^e) &= \frac{\partial \gamma^{CB}}{\partial k} k' + \left(\frac{\partial \gamma^{CB}}{\partial \pi} + \delta \frac{\partial \gamma^{CB}}{\partial \pi^e} \right) \pi' + \frac{\partial \gamma^{CB}}{\partial \mu} \mu', \\
\alpha'_\mu(k; \pi^e) &= \frac{\partial \alpha_\mu}{\partial k} k' + \delta \frac{\partial \alpha_\mu}{\partial \pi^e} \pi', \\
\beta'_\mu(k, \pi; \mu, \pi^e) &= \frac{\partial \beta_\mu}{\partial k} k' + \left(\frac{\partial \beta_\mu}{\partial \pi} + \delta \frac{\partial \beta_\mu}{\partial \pi^e} \right) \pi' + \frac{\partial \beta_\mu}{\partial \mu} \mu'.
\end{aligned}$$

Given $k' = \alpha$ and $\pi' = \beta$ the above equations are substituted into (5.28) which is then rearranged for the μ'

$$\begin{aligned}
\mu' = -\alpha & \frac{\alpha_\mu \frac{\partial \omega^{CB}}{\partial k} + \beta_\mu \frac{\partial \gamma^{CB}}{\partial k} + \omega^{CB} \frac{\partial \alpha_\mu}{\partial k} + \gamma^{CB} \frac{\partial \beta_\mu}{\partial k}}{\alpha_\mu \frac{\partial \omega^{CB}}{\partial \mu} + \beta_\mu \frac{\partial \gamma^{CB}}{\partial \mu} + \gamma^{CB} \frac{\partial \beta_\mu}{\partial \mu}} \\
- \beta & \frac{-\lambda_2 \epsilon + \alpha_\mu \left(\frac{\omega^{CB}}{\partial \pi} + \delta \frac{\partial \omega^{CB}}{\partial \pi^e} \right) + \beta_\mu \left(\frac{\partial \gamma^{CB}}{\partial \pi} + \delta \frac{\partial \gamma^{CB}}{\partial \pi^e} \right) + \omega^{CB} \delta \frac{\partial \alpha_\mu}{\partial \pi^e} + \gamma^{CB} \left(\frac{\partial \beta_\mu}{\partial \pi} + \delta \frac{\partial \beta_\mu}{\partial \pi^e} \right)}{\alpha_\mu \frac{\partial \omega^{CB}}{\partial \mu} + \beta_\mu \frac{\partial \gamma^{CB}}{\partial \mu} + \gamma^{CB} \frac{\partial \beta_\mu}{\partial \mu}}.
\end{aligned}$$

The above equation describes the trajectory of the central bank's control variable. For simplicity, we rewrite this equation as

$$\mu' = \Psi^\mu(k, \pi; \mu, \pi^e). \tag{5.29}$$

Next, we derive the trajectory of the public expectations. The first order conditions for the public sector are given as follows

$$\omega^{PS\prime} - \rho\omega^{PS} = -\mathcal{U}_k^{PS} - \omega^{PS}\alpha_k(k;\pi^e) - \gamma^{PS}\beta_k(k,\pi;\mu,\pi^e), \tag{5.30a}$$

$$\gamma^{PS\prime} - \rho\gamma^{PS} = -\mathcal{U}_\pi^{PS} - \omega^{PS}\alpha_\pi(k;\pi^e) - \gamma^{PS}\beta_\pi(k,\pi;\mu,\pi^e), \tag{5.30b}$$

$$k' = \left(1 - \frac{\theta(1-\tau)}{\sigma(\varrho k^{\varrho-1} + \pi^e)}\right)k^\varrho - \bar{g} + (\Omega - n)k, \tag{5.30c}$$

$$\pi' = \psi(k,\pi;\pi^e)\left((\mu - \pi - n) - \phi(k,\pi;\pi^e)\alpha(k;\pi^e)\right), \tag{5.30d}$$

$$0 = \mathcal{U}_{\pi^e}^{PS} + \omega^{PS}\alpha_{\pi^e}(k;\pi^e) + \gamma^{PS}\beta_{\pi^e}(k,\pi;\mu,\pi^e), \tag{5.30e}$$

$$0 = \lim_{t\to\infty} k(t)\omega^{PS}e^{-\rho t}, \tag{5.30f}$$

$$0 = \lim_{t\to\infty} \pi(t)\gamma^{PS}e^{-\rho t}, \tag{5.30g}$$

$$k(0) = k_0, \quad \pi(0) = \pi_0, \tag{5.30h}$$

where $\mathcal{U}_\pi^{PS} = \mathcal{U}_{\pi^e}^{PS}\frac{\partial\pi^e}{\partial\pi} = \frac{-\delta}{(\varrho k^{\varrho-1}+\pi^e)}$. Rearranging (5.30e), we obtain

$$\omega^{PS} = -\frac{\mathcal{U}_{\pi^e}^{PS}}{\alpha_{\pi^e}} - \frac{\beta_{\pi^e}}{\alpha_{\pi^e}}\gamma^{PS}. \tag{5.31}$$

Substituting (5.31) into (5.30b) and rearranging, we get the first order differential equation

$$\gamma^{PS\prime} - (\rho - \beta_\pi + \delta\beta_{\pi^e})\gamma^{PS} = 0. \tag{5.32}$$

Solution to (5.32) can be represented as

$$\gamma^{PS} = \gamma^{PS}(k,\pi;\mu,\pi^e). \tag{5.33}$$

We must now find ω^{PS}. Subdtituting (5.33) into (5.30a) and rearranging, we obtain the following first order differential equation

$$\omega^{PS\prime} - (\rho - \alpha_k)\omega^{PS} = -\mathcal{U}_k^{PS} - \beta_k\gamma^{PS}(k,\pi;\mu,\pi^e). \tag{5.34}$$

The solution to the above equation can be expressed as

$$\omega^{PS} = \omega^{PS}(k, \pi; \mu, \pi^e). \tag{5.35}$$

Substituting (5.33) and (5.35) into (5.30e) and differentiating with respect to time, we obtain the dynamics of the public expectations as follows

$$\mathcal{U}^{PS}_{\pi^e k} k' + \mathcal{U}^{PS}_{\pi^e \pi^e} \pi^{e\prime} + \omega^{PS\prime} \alpha_{\pi^e} + \omega^{PS} \alpha'_{\pi^e} + \gamma^{PS\prime} \beta_{\pi^e} + \gamma^{PS} \beta'_{\pi^e} = 0, \tag{5.36}$$

where

$$\begin{aligned}
\mathcal{U}^{PS}_{\pi^e k} &= \frac{\varrho(\varrho - 1)k^{\varrho-2}}{(\varrho k^{\varrho-1} + \pi^e)^2}, \\
\mathcal{U}^{PS}_{\pi^e \pi^e} &= \frac{1}{(\varrho k^{\varrho-1} + \pi^e)^2}, \\
\alpha'_{\pi^e} &= \frac{\partial \alpha_{\pi^e}}{\partial k} k' + \frac{\partial \alpha_{\pi^e}}{\partial \pi^e} \pi^{e\prime}, \\
\beta'_{\pi^e} &= \frac{\partial \beta_{\pi^e}}{\partial k} k' + \frac{\partial \beta_{\pi^e}}{\partial \pi} \pi' + \frac{\partial \beta_{\pi^e}}{\partial \pi^e} \pi^{e\prime} + \frac{\partial \beta_{\pi^e}}{\partial \mu} \mu', \\
\omega^{PS\prime} &= \frac{\partial \omega^{PS}}{\partial k} k' + \frac{\partial \omega^{PS}}{\partial \pi} \pi' + \frac{\partial \omega^{PS}}{\partial \pi^e} \pi^{e\prime} + \frac{\partial \omega^{PS}}{\partial \mu} \mu', \\
\gamma^{PS\prime} &= \frac{\partial \gamma^{PS}}{\partial k} k' + \frac{\partial \gamma^{PS}}{\partial \pi} \pi' + \frac{\partial \gamma^{PS}}{\partial \pi^e} \pi^{e\prime} + \frac{\partial \gamma^{PS}}{\partial \mu} \mu'.
\end{aligned}$$

Substituting the known terms $k' = \alpha$, $\pi' = \beta$ and $\mu' = \mu_\pi \pi'$ into (5.36) and rearranging for $\pi^{e\prime}$, we obtain

$$\begin{aligned}
\pi^{e\prime} = &-\alpha \frac{\frac{\varrho(\varrho-1)k^{\varrho-2}}{(\varrho k^{\varrho-1}+\pi^e)^2} + \omega^{PS} \frac{\partial \alpha_{\pi^e}}{\partial k} + \gamma^{PS} \frac{\partial \beta_{\pi^e}}{\partial k} + \alpha_{\pi^e} \frac{\partial \omega^{PS}}{\partial k} + \beta_{\pi^e} \frac{\partial \gamma^{PS}}{\partial k}}{\frac{1}{(\varrho k^{\varrho-1}+\pi^e)^2} + \omega^{PS} \frac{\partial \alpha_{\pi^e}}{\partial \pi^e} + \gamma^{PS} \frac{\partial \beta_{\pi^e}}{\partial \pi^e} + \beta_{\pi^e} \frac{\partial \gamma^{PS}}{\partial \pi^e} + \alpha_{\pi^e} \frac{\partial \omega^{PS}}{\partial \pi^e}} \\
&- \beta \frac{\alpha_{\pi^e} \left(\frac{\partial \omega^{PS}}{\partial \pi} + \mu_\pi \frac{\partial \omega^{PS}}{\partial \mu} \right) + \beta_{\pi^e} \left(\frac{\partial \gamma^{PS}}{\partial \pi} + \mu_\pi \frac{\partial \gamma^{PS}}{\partial \mu} \right) + \gamma^{PS} \left(\frac{\partial \beta_{\pi^e}}{\partial \pi} + \mu_\pi \frac{\partial \beta_{\pi^e}}{\partial \mu} \right)}{\frac{1}{(\varrho k^{\varrho-1}+\pi^e)^2} + \omega^{PS} \frac{\partial \alpha_{\pi^e}}{\partial \pi^e} + \gamma^{PS} \frac{\partial \beta_{\pi^e}}{\partial \pi^e} + \beta_{\pi^e} \frac{\partial \gamma^{PS}}{\partial \pi^e} + \alpha_{\pi^e} \frac{\partial \omega^{PS}}{\partial \pi^e}}.
\end{aligned}$$

For simplicity, we rewrite the above equation as

$$\pi^{e\prime} = \Psi^{\pi^e}(k, \pi; \mu, \pi^e). \tag{5.37}$$

Equilibrium of the feedback loop Nash solution concept is described as follows

$$k' = \left(1 - \frac{\theta(1-\tau)}{\sigma(\varrho k^{\varrho-1} + \pi^e)}\right)k^{\varrho} - \bar{g} + (\Omega - n)k, \tag{5.38a}$$

$$\pi' = \psi(k, \pi; \pi^e)\left((\mu - \pi - n) - \phi(k, \pi; \pi^e)\alpha(k; \pi^e)\right), \tag{5.38b}$$

$$\mu' = \Psi^{\mu}(k, \pi; \mu, \pi^e), \tag{5.38c}$$

$$\pi^{e\prime} = \Psi^{\pi^e}(k, \pi; \mu, \pi^e). \tag{5.38d}$$

5.2.3 Stackelberg Equilibrium

If the moves of some players are prioritized to others the game develops a hierarchical-moves and is known as the Stackelberg game. This section derives the open loop Stackelberg equilibrium define in the previous chapter. In this duopoly game model, both players (leader and follower) commit to the time paths of their planned strategies. In other words, (5.2) and (5.3) remain valid in this section.

As described in Chapter 4, two additional constraints are imposed on the leader. This constraint are the dynamics of the co-state variables of the follower, i.e., (4.38a) and (4.38b). The Hamiltonian of the central bank is as follows

$$\begin{aligned}\mathcal{H}^{CB} = &\frac{1}{2}\lambda_1 (k - k^n)^2 + \frac{1}{2}\lambda_2 (\pi - \hat{\pi})^2 + \omega^{CB}\alpha(k; \pi^e) + \gamma^{CB}\beta(k, \pi; \mu, \pi^e)\\ &+ \xi^{CB}\Delta(k, \pi, \omega^{PS}; \mu, \pi^e) + \zeta^{CB}\Lambda(k, \pi, \gamma^{PS}; \mu, \pi^e),\end{aligned} \tag{5.39}$$

where ω^{CB}, γ^{CB}, ξ^{CB} and ζ^{CB} are the co-state variables of the system. The first order conditions of the central bank are given as

$$\omega^{CB\prime} - \rho\omega^{CB} = \lambda_1 (k - k^n) - \omega^{CB}\alpha_k - \gamma^{CB}\beta_k - \xi^{CB}\Delta_k - \zeta^{CB}\Lambda_k, \tag{5.40a}$$

$$\gamma^{CB\prime} - \rho\gamma^{CB} = -\lambda_2 (\pi - \hat{\pi}) - \gamma^{CB}\beta_\pi - \xi^{CB}\Delta_\pi - \zeta^{CB}\Lambda_\pi, \tag{5.40b}$$

$$\xi^{CB\prime} - \rho\xi^{CB} = -\xi^{CB}\left(\rho - \alpha_k - \beta_k\frac{\alpha_{\pi^e}}{\beta_{\pi^e}}\right), \tag{5.40c}$$

$$\zeta^{CB\prime} - \rho\zeta^{CB} = -\zeta^{CB}(\rho - \beta_\pi), \tag{5.40d}$$

$$k' = \left(1 - \frac{\theta(1-\tau)}{\sigma(\varrho k^{\varrho-1} + \pi^e)}\right)k^\varrho - \bar{g} + (\Omega - n)k, \tag{5.40e}$$

$$\pi' = \psi\left((\mu - \pi - n) - \phi\alpha\right), \tag{5.40f}$$

$$\omega^{PS\prime} = \left(\rho - \alpha_k - \beta_k\frac{\alpha_{\pi^e}}{\beta_{\pi^e}}\right)\omega^{PS} + \frac{\beta_k}{\beta_{\pi^e}}\mathcal{U}^{PS}_{\pi^e} - \mathcal{U}^{PS}_k, \tag{5.40g}$$

$$\gamma^{PS\prime} = (\rho - \beta_\pi)\gamma^{PS}, \tag{5.40h}$$

$$0 = \lambda_2\epsilon(\pi - \hat{\pi}) + \gamma^{CB}\beta_\mu + \xi^{CB}\Delta_\mu + \zeta^{CB}\Lambda_\mu, \tag{5.40i}$$

$$0 = \lim_{t\to\infty} k(t)\omega^{CB}e^{-\rho t}, \tag{5.40j}$$

$$0 = \lim_{t\to\infty} \pi(t)\gamma^{CB}e^{-\rho t}, \tag{5.40k}$$

$$0 = \lim_{t\to\infty} \omega^{PS}(t)\xi^{CB}e^{-\rho t}, \tag{5.40l}$$

$$0 = \lim_{t\to\infty} \gamma^{PS}(t)\zeta^{CB}e^{-\rho t}, \tag{5.40m}$$

$$k(0) = k_0,\ \ \pi(0) = \pi_0,\ \ \omega^{PS}(0); free,\ \ \gamma^{PS}(0); free. \tag{5.40n}$$

Rearranging (5.40c) and (5.40d), we obtain the pair of differential equation

$$\xi^{CB\prime} - \left(\alpha_k + \beta_k\frac{\alpha_{\pi^e}}{\beta_{\pi^e}}\right)\xi^{CB} = 0, \tag{5.41}$$

$$\zeta^{CB\prime} - \beta_\pi\zeta^{CB} = 0, \tag{5.42}$$

Solution to these two equations can be expressed as

$$\xi^{CB} = \xi^{CB}(k, \pi; \mu, \pi^e), \tag{5.43}$$

$$\zeta^{CB} = \zeta^{CB}(k, \pi; \mu, \pi^e). \tag{5.44}$$

To find γ^{CB}, we substitute (5.43) and (5.44) into (5.40i). Hence, we obtain

$$\gamma^{CB} = -\frac{\lambda_2 \epsilon(\pi - \hat{\pi})}{\beta_\mu} - \frac{\Delta_\mu}{\beta_\mu}\xi^{CB}(k, \pi; \mu, \pi^e) - \frac{\Lambda_\mu}{\beta_\mu}\zeta^{CB}(k, \pi; \mu, \pi^e). \tag{5.45}$$

Substitute (5.43), (5.44) and (5.45) into (5.40a) and rearranging, we get

$$\begin{aligned} \omega^{CB\prime} - (\rho - \alpha_k)\,\omega^{CB} &= \lambda_1 (k - k^n) + \lambda_2 \epsilon \frac{\beta_k}{\beta_\mu} (\pi - \hat{\pi}) \\ &\quad + \left(\Delta_\mu \frac{\beta_k}{\beta_\mu} - \Delta_k\right) \xi^{CB}(k, \pi; \mu, \pi^e) \\ &\quad + \left(\Lambda_\mu \frac{\beta_k}{\beta_\mu} - \Lambda_k\right) \zeta^{CB}(k, \pi; \mu, \pi^e). \end{aligned} \tag{5.46}$$

Solution to (5.46) can be represented as

$$\omega^{CB} = \omega^{CB}(k, \pi; \mu, \pi^e). \tag{5.47}$$

The dynamics of the monetary policy instrument are obtained by (5.40i). For simplicity, we rewrite this equation as follows

$$\Xi^1(k, \pi, \gamma^{PS}, \omega^{PS}; \mu, \pi^e) = 0 \tag{5.48}$$

Differentiating (5.48) with respect to time, we obtain

$$\Xi^1_k k' + \Xi^1_\pi \pi' + \Xi^1_{\gamma^{PS}} \gamma^{PS\prime} + \Xi^1_{\omega^{PS}} \omega^{PS\prime} + \Xi^1_\mu \mu' + \Xi^1_{\pi^e} \pi^{e\prime} = 0, \tag{5.49}$$

Substituting the known terms $\gamma^{PS\prime} = \Lambda$, $\omega^{PS\prime} = \Delta$, $k' = \alpha$ and $\pi' = \beta$ into (5.49) and rearranging, we obtain the dynamics of the public expectations as

$$\mu' = \Xi^{11} + \Xi^{12}\pi^{e\prime}, \tag{5.50}$$

where

$$\begin{aligned}
\Xi^{11} &= -\frac{\Xi^1_k\alpha + \Xi^1_\pi\beta + \Xi^1_{\gamma^{PS}}\Lambda + \Xi^1_{\omega^{PS}}\Delta}{\Xi^1_\mu}, \\
\Xi^{12} &= -\frac{\Xi^1_{\pi^e}}{\Xi^1_\mu}.
\end{aligned}$$

To find the dynamics of the public expectations, we differentiate (5.12e) with respect to time

$$\begin{aligned}
&\mathcal{U}^{PS}_{\pi^e k}k' + \mathcal{U}^{PS}_{\pi^e\pi^e}\pi^{e\prime} + \left(\alpha_{\pi^e k}k' + \alpha_{\pi^e\pi^e}\pi^{e\prime}\right)\omega^{PS} \\
&\quad + \left(\omega^{PS}_\pi\pi' + \omega^{PS}_k k' + \omega^{PS}_\mu\mu' + \omega^{PS}_{\pi^e}\pi^{e\prime}\right)\alpha_{\pi^e} \\
&\quad + \left(\beta_{\pi^e\pi}\pi' + \beta_{\pi^e k}k' + \beta_{\pi^e\mu}\mu' + \beta_{\pi^e\pi^e}\pi^{e\prime}\right)\gamma^{PS} \\
&\quad + \left(\gamma^{PS}_\pi\pi' + \gamma^{PS}_k k' + \gamma^{PS}_\mu\mu' + \gamma^{PS}_{\pi^e}\pi^{e\prime}\right)\beta_{\pi^e} = 0,
\end{aligned} \tag{5.51}$$

Inserting the known terms $k' = \alpha$ and $\pi' = \beta$ into (5.51) and rearranging, we obtain

$$\pi^{e\prime} = \Xi^{21} + \Xi^{22}\mu', \tag{5.52}$$

where

$$\begin{aligned}
\Xi^{21} &= -\frac{\alpha\left(\mathcal{U}^{PS}_{\pi^e k} + \alpha_{\pi^e k}\omega^{PS} + \omega^{PS}_k\alpha_{\pi^e} + \beta_{\pi^e k}\gamma^{PS} + \beta_{\pi^e}\gamma^{PS}_k\right)}{\mathcal{U}^{PS}_{\pi^e\pi^e} + \alpha_{\pi^e\pi^e}\omega^{PS} + \omega^{PS}_{\pi^e}\alpha_{\pi^e} + \beta_{\pi^e\pi^e}\gamma^{PS} + \gamma^{PS}_{\pi^e}\beta_{\pi^e}} \\
&\quad - \frac{\beta\left(\omega^{PS}_\pi\alpha_{\pi^e} + \beta_{\pi^e\pi}\gamma^{PS} + \gamma^{PS}_\pi\beta_{\pi^e}\right)}{\mathcal{U}^{PS}_{\pi^e\pi^e} + \alpha_{\pi^e\pi^e}\omega^{PS} + \omega^{PS}_{\pi^e}\alpha_{\pi^e} + \beta_{\pi^e\pi^e}\gamma^{PS} + \gamma^{PS}_{\pi^e}\beta_{\pi^e}}, \\
\Xi^{12} &= -\frac{\omega^{PS}_\mu\alpha_{\pi^e} + \beta_{\pi^e\mu}\gamma^{PS} + \gamma^{PS}_\pi\beta_{\pi^e}}{\mathcal{U}^{PS}_{\pi^e\pi^e} + \alpha_{\pi^e\pi^e}\omega^{PS} + \omega^{PS}_{\pi^e}\alpha_{\pi^e} + \beta_{\pi^e\pi^e}\gamma^{PS} + \gamma^{PS}_{\pi^e}\beta_{\pi^e}}.
\end{aligned}$$

The simultaneous solutions of (5.50) and (5.52) give the time paths of μ' and $\pi^{e\prime}$. Thus, we solve

$$\begin{pmatrix} \mu' \\ \pi^{e\prime} \end{pmatrix} = \begin{pmatrix} 1 & -\Xi^{12} \\ -\Xi^{22} & 1 \end{pmatrix}^{-1} \begin{pmatrix} \Xi^{11} \\ \Xi^{21} \end{pmatrix}. \tag{5.53}$$

Suppose that $1 - \Xi^{12}\Xi^{22} \neq 0$. We conclude that

$$\mu' = \frac{1}{1-\Xi^{12}\Xi^{22}}\left(\Xi^{11} - \Xi^{12}\Xi^{21}\right), \tag{5.54a}$$

$$\pi^{e\prime} = \frac{1}{1-\Xi^{12}\Xi^{22}}\left(\Xi^{21} - \Xi^{22}\Xi^{11}\right). \tag{5.54b}$$

In the open loop Stackelberg solution concept to the previous differential game is expressed as follows

$$k' = \left(1 - \frac{\theta(1-\tau)}{\sigma\left(\varrho k^{\varrho-1} + \pi^{e}\right)}\right) k^{\varrho} - \bar{g} + (\Omega - n)k, \tag{5.55a}$$

$$\pi' = \psi\left((\mu - \pi - n) - \phi\alpha\right), \tag{5.55b}$$

$$\omega^{PS\prime} = \left(\rho - \alpha_k - \beta_k \frac{\alpha_{\pi^e}}{\beta_{\pi^e}}\right)\omega^{PS} + \frac{\beta_k}{\beta_{\pi^e}}\mathcal{U}^{PS}_{\pi^e} - \mathcal{U}^{PS}_{k}, \tag{5.55c}$$

$$\gamma^{PS\prime} = \left(\rho - \beta_{\pi}\right)\gamma^{PS}, \tag{5.55d}$$

$$\mu' = \frac{1}{1-\Xi^{12}\Xi^{22}}\left(\Xi^{11} - \Xi^{12}\Xi^{21}\right), \tag{5.55e}$$

$$\pi^{e\prime} = \frac{1}{1-\Xi^{12}\Xi^{22}}\left(\Xi^{21} - \Xi^{22}\Xi^{11}\right). \tag{5.55f}$$

where ω^{PS} and γ^{PS} are calculated by (5.14) and (5.16), respectively.

5.3 The Chaotic Behavior

chaotic dynamics are commonly tested by an empirical method that finds the possibility of chaos in the time path generated by a dynamical system. In this section, the possible occurrences of chaos in the trajectories of both players are detected by the largest

Lyapunov exponent[1]. The exponential divergence rate of initially close trajectories is a measure of chaotic dynamics. Thus, we must first find the trajectories of both players under the Nash and Stackelberg solution concepts.

Table 5.1: Baseline parameter and coefficient values of the specific model

Coefficient	Definition	Value	Acceptable Interval
θ	Marginal Propensity to Consumption (MPC)	0.70	$\theta \in (0,1)$
τ	Income Tax Rate	0.15	$\tau \in (0,1)$
δ	Incriment Rise of Inflation on Expectated Inflation	0.85	$\delta \in (0,1)$
Ω	Depreciation Rate	0.05	$\Omega \in (0,1)$
ϱ	Output Elasticity of Capital	0.65	$\varrho \in \mathbb{R}_+$
λ_1	Central Bank's Weight on Output	1	$\lambda_1 \in \mathbb{R}_+$
λ_2	Central Bank's Weight on Inflation	1.5	$\lambda_2 \in \mathbb{R}_+$
ρ	Subjective Discount Rate	0.03	$\rho \in (0,1)$
$\hat{\pi}$	Inflation Goal	0.02	$\hat{\pi} \in \mathbb{R}_+$
ϵ	Incriment Rise of Policy Rate on Inflation	0.80	$\epsilon \in [0,1]$
η	Coefficien of Output in Money Demand Function	0.30	$\eta \in \mathbb{R}_+$
ι	Coefficien of Interest Rate in Money Demand Function	0.10	$\iota \in \mathbb{R}_+$
n	Population Growth Rate	0.01	$n \in \mathbb{R}$
σ	Coefficien of Interest Rate in Consumption Function	0.20	$\sigma \in \mathbb{R}_+$
μ_π	Monetary Policy Reaction on Inflation	-0.2	$\chi \in \mathbb{R}_-$
ϖ	Coefficien of Inflation in Money Demand Function	0.20	$\varpi \in \mathbb{R}_+$
k^n	Natural Rate of Capital-Labor Ratio	85	$k^n \in \mathbb{R}_+$

Using the coefficients and parameters presented in Table 5.1, we simulate time paths of the strategies of both players in the model introduced in the previous section. Seventeen coefficients and parameters appear in the equations that characterize the behavior of the dynamical system. The value of the most of these coefficients and parameters are common to standard economic models (Walsh, 2003). Figures 5.1-5.3 plot the trajectories of the monetary policy rate and the public expectations in our three solution concepts. In all solution concepts, the public expectations are more volatile than the monetary policy rate.

For our purpose, the most interesting part of this study is to find the sign of the largest Lyapunov exponent from the series generated by the model. As mentioned earlier, chaotic dynamic systems are characterized by a positive Lyapunov exponent (λ).

[1]Among the many methods available for determining the largest Lyapunov exponent, but here we employ the BenSaïda's algorithm (BenSaïda, 2007, 2012; Bensaïda & Litimi, 2013; Bensaïda, 2014; BenSaïda, 2015).

We now detect possible chaotic behavior in our monetary dynamic games by applying the Lyapunov test. The null hypothesis of this test is $H_0 : \lambda \geq 0$, and its rejection provides a strong evidence of non-chaotic dynamics (Bensaïda, 2014; Moosavi Mohseni et al., 2015). In fact, the more chaotic the system, the higher positive the value of λ (Lynch, 2004).

Table 5.2 provides the Lyapunov exponent in the open loop Nash, feedback Nash and Stackelberg solution concepts.

Table 5.2: Results of the chaos test on the monetary policy games

Solution Concepts		Players	(L, m, q)	λ	p-value*	Confidence Interval**	Accepted H
Nash	Open Loop	Central Bank	(1, 4, 1)	0.12839	0.99900	[0.06007, ∞)	H_0
		Public Sector	(1, 6, 1)	0.01103	0.99990	[0.00617, ∞)	H_0
	Feedback	Central Bank	(1, 1, 1)	-0.57874	0.00000	[-0.78652, ∞)	H_1
		Public Sector	(1, 6, 3)	0.00198	0.72226	[-0.00355, ∞)	H_0
Stackelberg		Central Bank	(1, 1, 5)	-2.39456	0.00000	[-2.39456, ∞)	H_1
		Public Sector	(1, 6, 5)	0.01425	0.72546	[-0.02355, ∞)	H_0

* At the 5% significance level, H_0 is rejected for p-value less than 0.05.

** Confidence interval at the 5% significance level.

Open Loop Nash Staregy

The Lyapunov exponent of the central bank and the public sector are 0.12839 and 0.01103, respectively. Therefore, the trajectories of both players are chaotic at the 5% significance level.

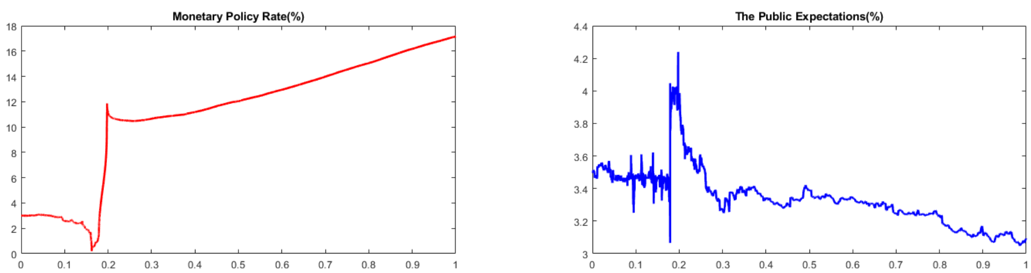

Figure 5.1: Trajectories of the monetary policy rate and the public expectations in the Nash open loop strategy

Feedback Nash Strategy

In this solution concept, the trajectory of the central bank shows the absence of any chaotic tendencies at 5% confidence level. However, the trajectory of the public sector still remains chaotic.

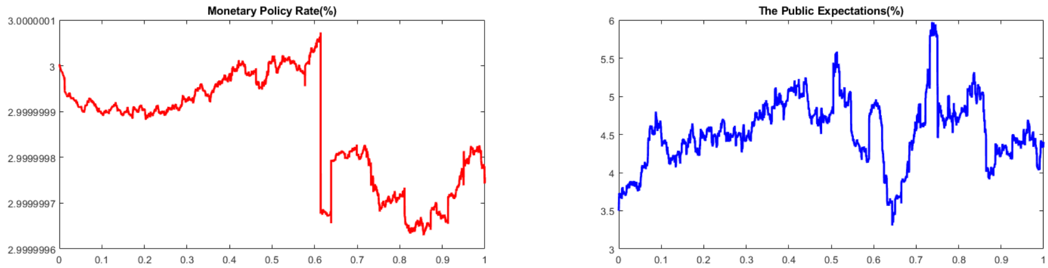

Figure 5.2: Trajectories of the monetary policy rate and the public expectations in the Nash feedback strategy

Stackelberg Strategy

In the Stackelberg strategy, no evidence of chaotic dynamics appears in the trajectory of the central bank, and the null hypothesis is rejected at the 5% significance level. On the other hand, the results confirm the test by accepting the null hypothesis of the chaotic dynamics in the trajectory of the public sector.

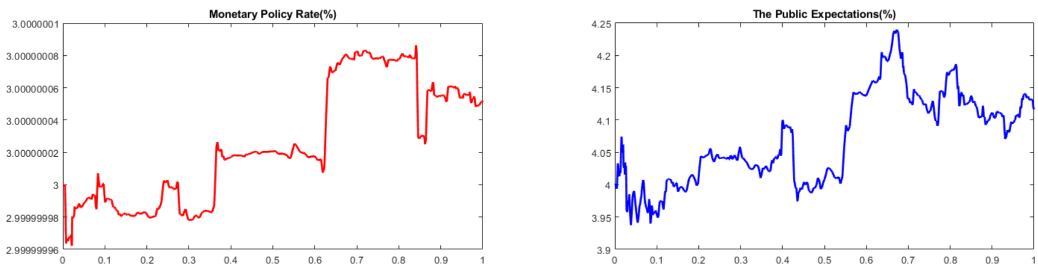

Figure 5.3: Trajectories of the monetary policy rate and the public expectations in the Stackelberg strategy

The above chaos test is robust to relative changes in the parameter values. Although changing the parameter values can alter the time paths and the Lyapunov exponents but the accepted hypotheses still remain unchanged. However, this analysis rises one interesting possibility. The source of any chaotic behavior in the dynamics of monetary policy games depend on the behavior of the public expectations. As established by the following proposition, the public expectation plays an important role in the emergence of chaotic dynamics in monetary policy games.

Proposition 5.3.1. *Any complexity during the conduct of monetary policy is sourced from the chaotic behavior of the public expectations.*

We have already indicated the crucial role of the public expectations in the effectiveness of monetary policy. Also, in the monetary policy games, the expected inflation behaves as a punishment strategy of the public sector. Furthermore, as revealed in this section the public expectations is the sources of the transient to chaos and complexity in such games.

5.4 Conclusion

This chapter confirmed that chaotic dynamics can arise in monetary policy games. Consequently, the trajectories fail to replicate themselves (Baumol & Quandt, 1985) thereby limiting the predictability of the future behavior from the past history. Such complexity imposes a serious caveat on monetary policymakers. Policy designing is a difficult task, rendered more complex when the public sector exhibits chaotic behaviors. Chaotic behavior in the strategy of the public sector inhibits the ability of the policymakers to control the business cycles through predictive analysis.

Chapter 6

Conclusion and Future Works

Employing the dynamic two-player noncooperative monetary policy game introduced in Chapter 3 and the mathematics and economic concepts discussed in Chapter 2, this thesis provides numerous insights into the time inconsistency of monetary policy and chaotic games with the variation parameters. The numerical solutions revealed more volatile behavior in the public sector than in the central bank regardless of solution concepts. This aspect of chaotic games in monetary economic systems is veiled in previous theoretical and applied works. Rather than focusing on chaotic behavior in the monetary policy games, this study revealed the time inconsistency in the monetary policy games with varying parameters.

The main finding was the possibility of chaotic dynamics in the monetary policy games. The likely source of this complexity is the chaotic behavior of the public sector.

6.1 Concluding Remarks and Discussion

When discussing the dynamics of the monetary policy games, in Chapter 3 we emphasized aspects of the structural time inconsistency that were previously unmentioned in the literature, but which are important for studying the monetary policy dynamics. By relaxing the assumption of fixed coefficients in conventional growth models, we found that the co-state variables were uncontrollable in all solution concepts.

Remark 6.1.1. *According to the analysis, the monetary policy instrument is state and control contingent and responds to any changes in the state and control variables. In other words, the economic condition forces the central bank to deviate from its planned action. Moreover, this deviation is necessary to maintain optimality of the monetary policy game. We referred to this type of deviation as the structural time inconsistency.*

The importance of this analysis is twofold. First, it represents that even if the policymaker commits to the goals, the structure of the economy can influence the actual decisions of the central bank. Second, we understand how the central bank should conduct a monetary policy that maintains a low and stable rate of inflation.

Remark 6.1.2. *Propositions 4.2.1 and 4.2.4 imply that in an open loop solution concepts the trajectory of each player directly depends on the time path of the strategy of the other player. Consequently, the dynamic behavior of one player depends on the time path of the other player.*

It is worth noting that the public expectations largely influence the transmission mechanism of monetary policy and appearance of the expected inflation in this model is the result of relaxing the usual perfect foresight assumption. In our model, the expected inflation is a punishment strategy of the public sector.

Remark 6.1.3. *Results of the chaos test indicated that the trajectory of the public expectations introduces complexity and chaos in the monetary policy games. This complexity creates difficulties for policymakers.*

To elucidate the impact of the monetary policy on the economy, we must determine the response of the public expectations. This becomes possible when the monetary policy behaves in a systematic manner (Walsh, 2003). The emergence of chaotic dynamics which means the *inability to forecast* weakens the power of the monetary policymaker to stabilize the economic cycles.

6.2 Future Works

Even though Moosavi Mohseni et al. (2015) reported that expectations hypotheses increase the complexity in the behavior of the dynamical monetary systems, it seems further thought needs to be devoted into the different expectations hypotheses in the presence of chaotic dynamics in monetary policy games.

We have explored a two-player dynamical game. Adding a third player (the fiscal authority) to this game would refine the analysis. We know that the interaction between the fiscal and the monetary policies are through the government's budget constraint. One could analyze the chaotic dynamics when monetary and fiscal policies are coordinate, and not coordinate, and compare the results.

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