Measuring Cascade Effects in Interdependent Networks by Using Effective Graph Resistance

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***Abstract*—Understanding the correlation between the underlie network structure and overlay cascade effects in the interdependent networks is one of major challenges in complex network studies. There are some existing metrics that can be used to measure the cascades. However, different metrics such as average node degree interpret different characteristic of network topological structure, especially less metrics have been identified to effectively measure the cascading performance in interdependent networks. In this paper, we propose to use a combined Laplacian matrix to model the interdependent networks and their interconnectivity, and then use its effective resistance metric as an indicator to its cascading behavior. Moreover, we have conducted extensive comparative studies among different metrics such as average node degree, and the proposed effective resistance. We have found that the effective resistance metric can describe more accurate and finer characteristics on topological structure of the interdependent networks than average node degree which is widely adapted by the existing research studies for measuring the cascading performance in interdependent networks.**

***Keywords*—Interconnected networks, Network robustness, Topogical matrics, Effective resistance, Average node degree, cascade effects**

I. Introduction

In recent year, modern critical infrastructures are highly networked. Interdependency among those infrastructures is becoming a common strategy for the future of networking. For example, Smart Grid is the next generation of electrical power grid where the traditional power grids and the communication networks are interconnected and interdependent. Smart Grid is a modernisation of the traditional power network which provides advanced two-way communications with highly capability in term of control, reliability, efficiency and safety [1]. Interdependency in this context is the bidirectional relationship between two or more critical infrastructures. In addition, those infrastructures have great influence on each other [2].

In the past decade, there has been much interest in the study of complex networks. Cascading failures in complex networks is an active field of research. Failure trigger in one network might causes a malfunction to the dependent network. As a result, an avalanche of cascading failures widely spread within the interdependent netowrks. The events of power blackout caused by cacading failures have happened in the past and will continue to happen today. As examples, two blackout events happened in 1996 in west America led to 11 states without power [3]. In August 2003, the largest blackout in the history of the U.S. triggered in the power grid of the U.S. and Canada [4]. Understanding the cascading phenomena and measuring the network structures to find root cause of fragilities is a major challenge in complex network research.

Modelling and measuring cascading failures in complex networks with given topologies have been subjected to many research in recent year. The average node degree is a metric which has been widely adopted in many previous studies as a measurement indicator in quantifying the relationship between network connectivity and the dynamical characteristic of interdependent network. In particular, measuring the largest connected component of an interdependent network, where an attack occured on a node of a proposed "one-to-one correspondence" network model [5]. In this model, each node in one network is depending on only one node in another network and vice-versa. More realistic models have been introduced with a consideration of random and targeted attacks on large degree vertices such as: multiple support dependent relation [6]; two fully dependent networks [7]; correspondently coupled network [8]; interdependent networks with different degree distribution [9]; regular inter-edge allocation method for the purpose of improving the robustness of the system [10]; real power grid in Italy [11]; and network of networks [12]. These studies adopted average node degree and node degree distribution combined with the percolation framework in assessing the size of network components after failures of vertices.

Despite the average node degree has been adopted in many robustness measures studies, it does not provide sufficient information of the underlying structure of interdependent networks. In this study, we propose to use a combined Laplacian matrix to model the interdependent networks and their interconnectivity, and then use its effective resistance metric as a measure to its overlay cascading behavior. Moreover, we have conducted extensive comparative studies between average node degree, and the proposed effective resistance. In addition, we adopted a well-known Bak-Tang-Wiesenfeld (BTW) sandpile model as dynamic load distribution. The extensive simulation studies have confirmed that the effective resistance metric can generate more accurate and finer grained indicator on topological structure of interdependent networks than the average node degree which is a popularly used metric reported in literature for measuring the cascading behavior in interdependent networks.

The structure of this paper is outlined as follow: section II proposes to use a combined Laplacian matrix to model the interdependent network and their interdependency-links. In section III, we introduce the concepts of average node degree, effective graph resistance, and Bak-Tang-Wiesenfeld sandpile model a representation of load distribution in interdependent network. We conduct the extensive comparative studies including adding and repositioning interdependency-links as well as results analysis in section IV. Final conclusion and future work are drawn in secition V.

II. Modelling Interdependent Networks

Lets defines G as a graph of interdependent network. According to graph theory, given a graph $G(N, L)$ where N is a set of nodes and L is a set of links. Suppose we have two graphs representation of two networks $G_1(N_1, L_1)$ and $G_2(N_2, L_2)$, each of which has a set of nodes N_1, N_2 and a set of links L_1, L_2. These two networks are undireted and unweighted [13].

In this case, G has a set of nodes $N = N_1UN_2$ and a set of links $L = L_1UL_2 UL_{Int}$, where L_{Int} is the interdependency-links connecting G_1 and G_2. Lets denote N_i as the number of nodes in $|N_i|$ and L_i as the number of links in $|L_i|$. Thus $N = N_1 + N_2$ and $L = L_1 + L_2$. Assign $A_1^{'}$ and $A_2^{'}$ be the adjancency matrices of G_1 and G_2 and $A^{'}$ be the adjacency matrix of graph G. The adjacency matrix:

$$A_{ij} = \begin{cases} 1, & \text{if } (i,j) \in L \\ 0, & \text{otherwise} \end{cases}$$

In an adjacency matrix, the element $a_{ij} = 1$ only if node i is connected to node j, otherwise $a_{ij} = 0$. When two networks are completely disconnected $L_{int} = 0$

$$A^{'} = \begin{bmatrix} A_1^{'} & 0 \\ 0 & A_2^{'} \end{bmatrix} \tag{1}$$

The adjacency matrix of the interdependency links L_I is denotes as B:

$$B = \begin{bmatrix} 0 & B_{Int} \\ B_{Int}^T & 0 \end{bmatrix} \tag{2}$$

The adjacency matrix of G

$$A = A^{'} + \alpha B = \begin{bmatrix} A_1^{'} & \alpha B_{Int} \\ \alpha B_{Int}^T & A_2^{'} \end{bmatrix} \tag{3}$$

where α is the coupling strength between the two networks. Based on the equations (2), (3), and (4), we define the Laplacian matrix L by taking the diagonal degree matrix D subtract the adjacency matrix A of G:

$$L = D - A \tag{4}$$

The diagonal degree matrix D can be written as:

$$D = \begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix}$$

Based on equation (3), the adjacency matrix of G is $A^{'} + \alpha B$ which consists of matrix $A^{'}$ and matrix B. Thus, the Laplacian matrices L_A for $A^{'}$ and L_B for B can be written as follow:

$$L_A = D - A = \begin{bmatrix} D_1 - A_1 & 0 \\ 0 & D_2 - A_2 \end{bmatrix} = \begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix} \tag{5}$$

$$L_B = D - \alpha B = \begin{bmatrix} D_1 & -\alpha B_{Int} \\ -\alpha B_{Int}^T & D_2 \end{bmatrix} \tag{6}$$

The total Laplacian matrix of interdependent network G can be written as:

$$L = L_A + \alpha L_B = \begin{bmatrix} L_1 + \alpha D_1 & -\alpha B_{Int} \\ -\alpha B_{Int}^T & L_2 + \alpha D_2 \end{bmatrix} \tag{7}$$

The eigenvalues of the Laplacian matrix is also known as the Laplacian Spectrum which is $0 = \lambda_1(G) \leqslant \lambda_2(G) \leqslant ... \leqslant \lambda_N(G)$.

III. Measuring Cascade Effects

A. Average node degree

Average node degree $\langle k \rangle$ is a degree metric which describes the characteristic of network connectivity. Given an undirected and unweighted graph $G(N, L)$, where N is the number of vertices and L is the number of edges, the average node degree can be written as:

$$\langle k \rangle = \frac{2 \times L}{N}$$

A network that is known to be better-connected, if it has higher average node degree [14]. However, this metric has a limitation such that two networks with the same average node degree but their topological structures are different.

B. Effective graph resistance

This section presents the basic concept of the effective graph resistance and its computational method. The effective graph resistance is a function of the Laplacian eigenvalues of the interdependent networks graph. The theory behind effective graph resistance is related to an electrical circuit, where a link between nodes i and j corresponds to a resistor. According to [15], if two resistors r_1 and r_2 are connected in series, the total resistance $R = r_1 + r_2$. On the other hand, if r_1 and r_2 are connected in parallel, the total resistance $R = (r_1^{-1} + r_2^{-1})^{-1}$.

The total effective resistance is also known as the effective graph resistance. The effective graph resistance is the sum of the effective resistances over all pairs of nodes in a graph [16]. According to Klein and Randic [17], the effective graph resistance can be written as:

$$R_G = N \sum_{i=2}^{N} \frac{1}{\mu_i} \tag{8}$$

where N is the total number of nodes and μ_i is the eigenvalues of the Laplacian matrix. In this study, we focus on interdependent networks and the mathematical modelling of this type of network. The computational method of defining Laplacian matrix is defined in the next section.

C. Bak-Tang-Wiesenfeld (BTW) sandpile model

The original BTW sandpile model derived from a real sand pile behavior, which establish on 2-dimensional lattices. This model is initiated by dropping grains of sand uniformly at random on the network and every node has a threshold. After dropping grains of sand, if a node exceeds its capacity or threshold, this node will begins to topple sand to the neighboring nodes. At this stage, if the neighboring nodes become unstable or failed, grains of sand will continue to topple to other neighboring nodes. Eventually, there will be no other nodes exceed anymore capacity. However, upon dropping another grain of sand, the toppling process will initiate again [18].

Grains of sand represent as load and the degree of nodes represent as capacities which is also the threshold. We are most interested in the probability of avalanche size distribution, which is the chance that the load might topple a large number of nodes within the interconnected networks. In 2-dimensional finite lattices with open boundaries condition, some sands are lost when arriving at the boundaries naturally equivalent to delete grains of sand independently with probability f or dissipation rate [19]. Consider sandpile model of 2-dimentional finite $N \times N$ networks, every node in each network is denoted by Z_i where:

$$Z_i = Z(x, y) = 0, 1, 2, 3, ...$$
$$x, y = 1, 2, 3, ...N$$

At initial state, $Z_i << K$ where there is no grain of sand being dropped. K is the threshold of each node. Upon dropping grains of sand uniformly at random, eventually a grain of sand arrive at a node in which its capacity equal to threshold ($Z_i = K$), a toppling event will begin which might leads to a whole series of toppling events and it will continue stable state is reached where there is no other node topple [20]. A toppling event is represented by $Z_i \longrightarrow Z_i + 1$, when $Z_i = K$

IV. Case Studies

In this section, interdependent netowrk topology is presented. Python based Networkx tool [21] is being used to generate Erdös Rényi (ER) interdependent netowrk. We introduce two numerical simulation scenarios include adding interdependency links and repositioning interdependency links. The results from this numerical simulation can be used to calculate the effective graph resistance. Bak-Tang-Wiesenfeld sandpile dynamic model is used to generate load distribution.

The detail of the two scenarios are: (i) Measuring the cascade effects by using the effective graph resistance as an indicator in relation to the probability of avalanche size distribution. In this case, three interdependency-links are being constantly added to the network for eight set of experiments. (ii) In the second scenario, again we are most interested in correlation between the effective graph resistance and the probability of avalanche size distribution. However, we conduct different experiments by respositioning the interdependency-links ranging from connecting between small degrees nodes to large degrees nodes. These scenarios allow us to observe the cascading effects due to topological changes.

A. Interdependent Network Topologies

This section presents the detail and graph representation of the interdependent network topological structure which is shown in Fig.1. This interdependent network consist of network A and network B, nodes in each network are connected via internal-links and interconnected via interdependency links. The key characteristics of the interdependent netowrks used in this study are listed in TABLE I. First, network A and

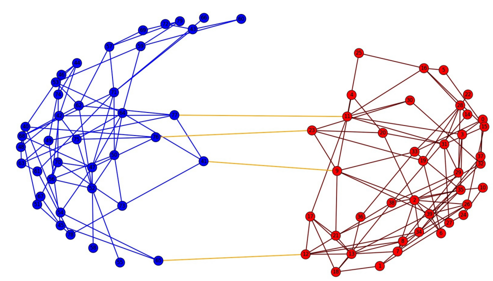

Fig. 1. Interdependent networks topology

network B have identical number of nodes $N_a = N_b$. Secondly, both network have the same average node degree $Z_a = Z_b$. Finally, in our numerical simulation, we are investigating the impact of topological changes in four different network sizes ($N_a = N_b = 40, 50, 60, 70$). These networks sizes will be used to quantify the cascade effects by using the effective graph resistance metric as an indicator.

TABLE I. Characteristics of the interdependent network topologies

Characteristics	ER1	ER2	ER3	ER4
$N = N_a + N_b$	80	100	120	140
$Z = Z_a + Z_b$	85	117	164	173

In both scenarios, we are focusing on the changes in interdependency-link. All four different network size will be investigated. First scenario, we are constantly adding three interdependency links 8 times included the first three initial interdependency-links. Following the process in section II, we can obtain the Laplacian matrix and then calculate the eigenvalues. Once we obtain the eigenvalues of Laplacian matrix, the effective graph resistance can be computed using equation (8). Second scenario, we reposition the interdependency links by considering the importance of node degree. The results of these two scenarios will be discussed in the following subsections.

B. Numerical Simulations

1) Adding Interdependency-Links: In this section, we investigate the cascading behaviours affecting by the increase in number of interdependency-links using the effective graph resistance. In addition, we conduct extensive comparative studies between the effective graph resistance and the average node degree. Table II shows the characteristics of interdependent networks with various cases of adding interdependency-links.

Two numerical simulations were conducted on two network sizes, $N_a = N_b = 40$ and $N_a = N_b = 70$. Three interdependency-links are constantly added to eight different cases. According to Fig.2 and Fig.3, while the number of interdependency-links increases, we found that the average node degree is also increased. In theory, when the average node degree incrases, the network is better connected. On the other hand, the effective graph resistance is decreasing when more interdependency-links being added to the network.

TABLE II. CHARACTERISTICS OF ADDING INTERDEPENDENCY-LINKS

N	$N_a = N_b = 40$							
$Z_a + Z_b$	85							
Inter-links	3	6	9	12	15	18	21	24
$\langle k \rangle$	2.20	2.28	2.35	2.43	2.50	2.58	2.65	2.73
R	3246	2730	2518	2394	2307	2248	2193	2122
N	$N_a = N_b = 70$							
$Z_a + Z_b$	173							
Inter-links	3	6	9	12	15	18	21	24
$\langle k \rangle$	2.51	2.56	2.60	2.64	2.69	2.73	2.77	2.81
R	9982	8471	7925	7633	7450	7281	7123	7012

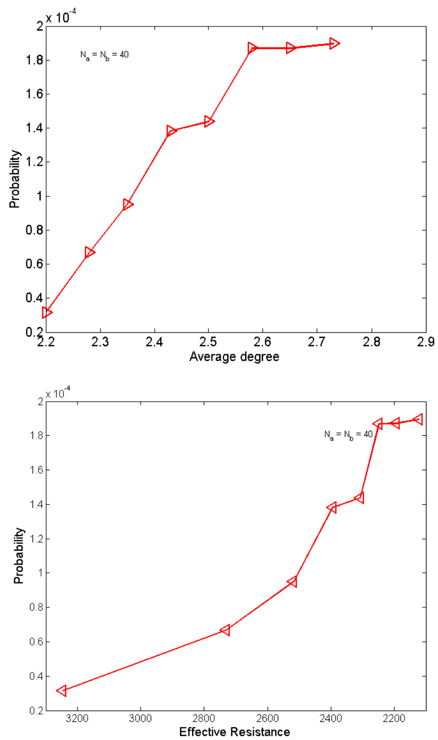

Fig. 2. Probability of avalanche size distribution vs. average node degree and effective graph resistance in the case of adding interdependency-links ($N_a = N_b = 40$)

In term of cascading behaviours, when the average node degree increases the chance of having a large avalanche size which is the probability of avalanche size distribution, is also increased. On the other hand, when the effective resistance is becoming smaller, the chance of having large avalanche size is becoming higher. Thus, from the results we can see that, the effective graph resistance is a better indicator than the average node degree.

2) Repositioning Interdependency-Links: We further investigate the impact of interdependency-links on cascading behaviours by adopting the link reposition approach and again using both average node degree and the effective graph resistance as the indicators. In this case, two network sizes, $N_a = N_b = 50$ and $N_a = N_b = 60$ were selected. TABLE III

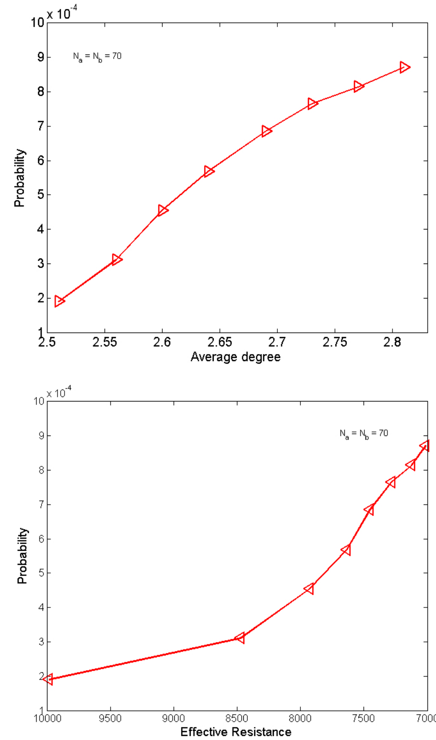

Fig. 3. Probability of avalanche size distribution vs. average node degree and effective graph resistance in the case of adding interdependency-links ($N_a = N_b = 70$)

shows the characteristics of interdependent networks in various cases of repositioning interdependency-links.

TABLE III. CHARACTERISTICS OF ADDING INTERDEPENDENCY-LINKS

N	$N_a = N_b = 50$		
$Z_a + Z_b$	85		
Inter-links	6		
Connectivity	Small degrees	Medium degrees	Large degrees
$\langle k \rangle$	1.82	1.82	1.82
R	41.77	41.55	41.17
Probability	0.00038	0.00051	0.00063
N	$N_a = N_b = 60$		
$Z_a + Z_b$	117		
Inter-links	6		
Connectivity	Small degrees	Medium degrees	Large degrees
$\langle k \rangle$	2.05	2.05	2.05
R	40.98	40.45	39.73
Probability	0.0024	0.0026	0.0027

As shwon in TABLE III, the average node degree have exactly the values for all three different repositioning cases. On the other hand, the effective graph resistance shows different values. In this scenario, we are most interested in the impact of connecting between nodes ranging from small degrees to large degrees on cascading behaviours. Fig.4 and Fig.5 show a clear evidences that the average node degree cannot tell differences between three cases, although the probability of avalanche size distribution shows differences. In cotrast, the effective graph resistance shows different values for different cases where network with higher effective graph resistance has lower chance of having a large avalanche size. From the results we found that, connecting between small degrees nodes has higher resistance than large degrees.

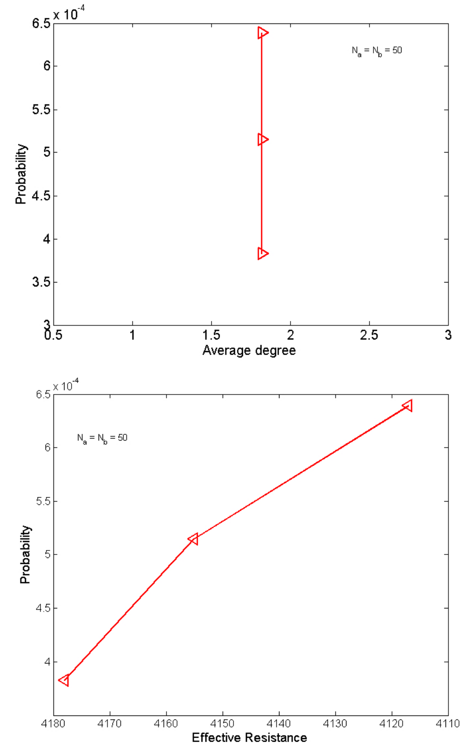

Fig. 4. Probability of avalanche size distribution vs. average node degree and effective graph resistance in the case of repositioning interdependency-links ($N_a = N_b = 50$)

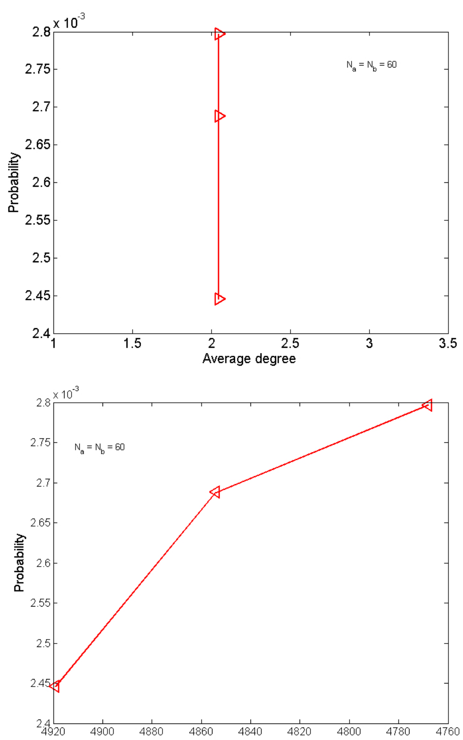

Fig. 5. Probability of avalanche size distribution vs. average node degree and effective graph resistance in the case of repositioning interdependency-links ($N_a = N_b = 60$)

V. Conclusion

In this paper, we proposed to use the effective graph resistance metric as a cascading effects indicator. In addition, we have selected the average node degree which have been widely used in many complex networks research and conducted extensive comparative studies with the effective graph resistance. The results from this paper lead to our arguement that the effective graph resistance is a better and effective metric than the average node degree that sufficiently describe the finest topological changes in interdependent networks.

In this study, we proposed to use Erös Rényi random graph to model the interdependent networks. Furthermore, we used a combine Laplacian matrix to mathematically model the networks. Numerical simulations were conducted using adding and repositioning interdependency-links approaches and computed the average node degree and the effective graph resistance. Bak-Tang-Wiesentfeld sandpile dynamic model is being used to generate load distribution as a dynamical behaviours in interdependent networks.

Results clearly shows that networks with same average node degree have different topological structures. In addition, average node degree can only tell how network are connected based on an average degrees of every node in the netowrk. In this case, repositioning the interdependency-links does not affect the average degree values because no links has been added or removed. On the other hand, the effective graph resistance is more sensitive to topological changes in the case of repositioning and adding interdependency links. In addition, when a network has high effective resistance, the probability chance of having a large avalanche size is low.

For future research, it would be interesting to consider a large scale interconnected networks and how both internal-links and inter-links affect the cascading performance of interconnected network, taking into account the cascading failures in relation to the dynamical behaviors of the system. In addition, there is scope for more accurate topological metric than our current proposed effective graph resistance waiting to be identified to better describe cascade effects in coupled networks.

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