To QRE or not to QRE? An Evaluation of Alternative Models in Behavioural Game Theory

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A dissertation submitted to Auckland University of Technology in partial fulfilment of the requirements for the degree of Master of Business (Economics)

2025

School of Business

Abstract

I conduct a targeted conceptual synthesis and evaluation of competing stochastic models of behaviour for normal form games. I present a review of Quantal Response Equilibrium (QRE) followed by a detailed discussion of two new stochastic models of behaviour, Noisy Belief Equilibrium (NBE) which introduces noise to beliefs instead of actions, and M equilibrium which introduces noise to both beliefs and actions. I contribute to existing literature by expanding on the relationship between NBE, Nash equilibrium, and QRE while also providing further analysis of the relationship between M equilibrium and Nash equilibrium. Additionally, I contribute to existing knowledge by proving key relationships between QRE, NBE, and M equilibrium in specific instances of generic 2×2 games. I briefly compare the parametric forms of each equilibrium model in two theoretical games before discussing areas of future research.

Contents

Attestation of Authorship

I hereby declare that this submission is my own work and that, to the best of my knowledge and belief, it contains no material previously published or written by another person (except where explicitly defined in the acknowledgements), nor used artificial intelligence tools (unless it is clearly stated, and referenced, along with the purpose of use), nor material which to a substantial extent has been submitted for the award of any other degree or diploma of a university or other institution of higher learning.

Acknowledgements

I would like to thank Professor Matthew Ryan for his guidance and for teaching me much of what I know about behavioural game theory. Without him this work would not have been possible.

1 Introduction

One of the foundational ideas in game theory, that of Nash equilibrium, is employed in a wide range of economic analysis. However, a substantial amount of economic research conducted in laboratory experiments has shown that often individuals do not follow the specified Nash equilibriums across different types of normal form and extensive form games (Goeree and Holt, 2001). To better model and understand how individuals make decisions within games, models have attempted to build on the foundational concept of Nash equilibrium providing possible solutions to the shortcomings of Nash equilibrium in practice. One of the prominent approaches is Quantal Response Equilibrium (QRE). Introduced by McKelvey and Palfrey (1995), under QRE individual players still choose amongst possible strategies based on their correctly formed beliefs, however, the best response functions of the individual now become probabilistic. In simpler terms, QRE relaxes the Nash equilibrium assumption that players choose their best response by adding "noise" to the players' action decisions. Responses that are associated with a higher level of utility are observed to be played with a higher probability, although not played with certainty (McKelvey and Palfrey, 1995). This approach was soon expanded beyond normal form games to extended form ones (see McKelvey and Palfrey, 1998). The research into QRE both theoretically and empirically is vast, with some key developments being regular QRE, heterogeneous QRE, and endogenous QRE (see Friedman, 2020; Goeree et al., 2005; Rogers et al., 2009). QRE has been frequently employed to more successfully model empirical results from experiments. For example, QRE was notably applied to private-value auctions to better model the overbidding that often occurred within experimental results, of which it was found to "closely [track] the exact distribution of bids" (Goeree et al., 2022, abstract). Although the advent of QRE has been highly successful, there are instances in which it has failed to fit experimental data (Haile et al., 2008). QRE is also known to over-predict individuals' level of sensitivity to changes in payoff magnitudes (Friedman, 2022). Furthermore, although QRE focuses on action noise, it does not address possible noise within players' beliefs. As a result, newer models of equilibrium have been developed, focusing instead on the Nash assumption that players have correctly formed beliefs. Some of these approaches include Cursed Equilibrium, M Equilibrium, Noisy Belief Equilibrium

(NBE), NLK,[1] and Inclusive Cognitive Hierarchy (see Eyster and Rabin, 2005; Friedman, 2022; Goeree and Louis, 2021; Koriyama and Ozkes, 2021; Levin and Zhang, 2022). In this paper I choose to focus on two alternative models to QRE. The first, NBE, can be viewed as a converse to QRE in that it introduces "noise" to players beliefs while maintaining the assumption that players best respond to these noisy beliefs. The second, M equilibrium, introduces "noise" to both beliefs and actions and can be viewed more as a meta theory (a point that will be discussed in detail later). These two models have been chosen based on their alternative approaches while being distinctly different from QRE as opposed to further investigating other forms of QRE. Each model has been established in fairly recent theoretical literature and are yet to see substantial comparison to QRE outside of the respective seminal papers for each equilibrium concept. Additionally, there currently exists no clear comparison between NBE and M equilibrium in current literature. This paper aims to fill these gaps through theoretically testing these models to understand their relationships and subsequently their relative strengths and weaknesses in specific normal form games. The questions which have guided the direction of this research are as follows:

When should new models of stochastic behaviour be used in place of Quantal Response Equilibrium (QRE) for normal form games?

- What relationships are present between QRE, NBE, and M Equilibrium?
- What do theoretical results tell us about the importance of stochastic actions versus stochastic beliefs?
- How does this influence which model to use theoretically and empirically?

This paper is comprised of six key sections. The first, introduces QRE detailing its development from its initial structural form through to its more commonly used form while also providing examples of two key parametric versions of QRE. The second section introduces NBE, detailing its construction, while also providing an example of its parametric model, logit transform NBE. I then contribute to existing literature by exploring NBE in the context of pure-mixed strategy equilibria and by expanding on the theoretical links between NBE and QRE. The third section introduces M equilibrium providing an example of its parametric model, μ-equilibrium.

[1] Levin and Zhang (2022) bridge Nash equilibrium and the level-k model and simply denote their new approach as NLK.

I then contribute to existing literature by further investigating M equilibrium's relationship to Nash equilibria and elaborating further on its connection to QRE. The fourth section focuses on the comparison of all three stochastic models, contributing to existing literature by finding novel results regarding the relationship of attainable equilibria across all models; this section also expands the notion of M equilibrium as a meta theory. The fifth section provides a brief general comparison of the parametric versions of each model for asymmetric matching pennies and asymmetric chicken. In the final section I summarise key findings and discuss areas for future research.

2 Quantal Response Equilibrium

Quantal response equilibrium (QRE) was first introduced by McKelvey and Palfrey (1995) to address some of the observed shortcomings of Nash equilibrium when attempting to model behaviour from experimental economic studies. Under QRE individual players still choose amongst possible strategies based on their correctly determined expected utility however, the best response functions of the individuals now become probabilistic. In simpler terms, QRE relaxes the Nash equilibrium assumption that players choose their best response by adding "noise" to the players' action decisions. Responses that are associated with a higher level of utility are observed to be played with a higher probability, although not played with certainty (McKelvey and Palfrey, 1995). QRE weakens the concept of 'best response' to that of a 'better response' while maintaining the equilibrium structure (Friedman, 2020). Since the seminal paper of McKelvey and Palfrey in 1995, many new forms of QRE have been developed and applied to various games. Some key developments to the initial model of QRE are regular QRE, reduced-form QRE, heterogeneous QRE, and endogenous QRE (see Friedman, 2020; Goeree et al., 2005; Rogers et al., 2009). This section provides a general overview of the initial development of structural QRE as stipulated by McKelvey and Palfrey, then discusses the developments beyond this into the more commonly used form of QRE. It should be noted that QRE has also been developed for extensive form games, however, given the scope of this research I confine attention to only the normal form versions of QRE.

2.1 Structural QRE

What is often called structural QRE, is the initial starting point for QRE in general. Introduced by McKelvey and Palfrey (1995), structural QRE incorporates the notion of additive errors in the estimation of one's payoffs given a specific strategy. This is often interpreted as a player making calculation errors when determining their expected payoffs, although it can also be viewed as a player receiving unclear information regarding their exact payoffs for a given game. The formal development of this model in context of normal form games is now discussed.

2.1.1 Model foundations

Within a finite n-player normal form game Γ, the set of players is denoted as $N = \{1, \ldots, n\}$ where each player $i \in N$ has a set of J_i pure strategies $S_i = \{s_{i1}, \ldots, s_{iJ_i}\}$. For each player $i \in N$, a payoff function $\pi_i : S \to \mathbb{R}$ is defined, where $S = \prod_{i \in N} S_i$. The set of probability measures over player i's set of pure actions S_i, is denoted as Δ_i. The probability that player i plays a particular pure strategy $s_{ij} \in S_i$, is given by $\sigma_i : S_i \to \mathbb{R}$, where $\sigma_{ij} = \sigma_i(s_{ij}) \in \Delta_i$. A non-negativity constraint is imposed such that $\sigma_{ij} \geqslant 0$ while $\sum_{s_{ij} \in S_i} \sigma_i(s_{ij}) = 1$. The Cartesian product of the sets of probability measures for all players is denoted as $\Delta = \prod_{i \in N} \Delta_i$. For ease, the notation $\sigma = (\sigma_i, \sigma_{-i})$ is used. The payoff function is therefore extended to Δ_i as follows: $\pi_i(\sigma) = \sum_{s \in S} p(s)\pi_i(s)$ where $p(s) = \Pi_{i \in N}\sigma_i(s_i)$. The space of player i's possible payoffs for their given strategies is denoted $X_i = \mathbb{R}^{J_i}$ and $X = \prod_{i \in N} X_i$. The function that maps all possible strategies (mixed and pure) to payoffs, $\bar{\pi} : \Delta \to X$, is defined as $\bar{\pi}(\sigma) = (\bar{\pi}_1(\sigma), \ldots, \bar{\pi}_n(\sigma))$ where the expected payoff for a specific strategy j for player i is denoted $\bar{\pi}_{ij}(\sigma) = \pi_i(s_{ij}, \sigma_{-i})$. Therefore:

$$\bar{\pi}_{ij}(\sigma) = \sum_{s_{-i} \in S_{-i}} p(s_{-i})\pi_i(s_{ij}, s_{-i})$$

Now, for each player $i \in N$, there also exists a random error, ε_{ij}, which disturbs the expected payoff of each pure strategy $s_{ij} \in S_i$ resulting in the following equation:

$$\hat{\pi}_{ij}(\sigma) = \bar{\pi}_{ij}(\sigma) + \varepsilon_{ij}$$

It is assumed that for each player their set of payoff errors, $\varepsilon_i = (\varepsilon_{i1}, \ldots, \varepsilon_{iJ_i})$, has a joint distribution that can be represented by a density function, f_i. The vector of density functions $f = (f_1, \ldots, f_n)$ is considered admissible if f_i ensures errors are independent across players and errors are unbiased such that $E(\varepsilon_i) = 0 \ \forall i \in N$ (Goeree et al., 2005).

Based on the payoffs and errors for each player i for a given strategy j, an ij-response set is defined as the set of errors in which the disturbed payoff for player i of strategy j is greater than or equal to the disturbed payoffs from all other possible strategies.

$$R_{ij}(\bar{\pi}_i) = \{\varepsilon_i | \hat{\pi}_{ij}(\sigma) \geqslant \hat{\pi}_{ik}(\sigma) \forall k = 1, \ldots, J_i\}$$

The probability of which player i chooses strategy j is therefore equal to:

$$Q_{ij}(\bar{\pi}_i) = \int_{R_{ij}(\bar{\pi}_i)} f(\varepsilon_i) d\varepsilon_i$$

The function $Q_i : \pi_i \to \Delta_i$ is player i's structural quantal response function which maps expected utilities to some mixed strategy. From this, a quantal response equilibrium of a normal form game Γ is defined by McKelvey and Palfrey (1995) as any mixed strategy profile $\sigma' \in \Delta$ such that:

$$\sigma'_{ij} = Q_{ij}(\bar{\pi}_i(\sigma')).$$

2.1.2 Additional implicit restrictions in structural QRE

Disturbing expected payoffs by adding some random error gives rise to additional restrictions on choice probabilities which are highlighted by Goeree et al. (2005) as follows:

1. Translation Invariance

$$Q_{ij}(\bar{\pi}_i + \alpha e_{J_i}) = Q_{ij}(\bar{\pi}_i) \ \forall \alpha \in \mathbb{R} \text{ and } \forall \bar{\pi}_i \in \mathbb{R}^{J_i} \text{ where } e_{J_i} = (1, \ldots, 1)$$

   This implies that if a constant is added to all payoffs, in our case the errors added to the payoffs of each strategy, choice probabilities do not change.

2. Symmetry

$$\frac{\partial Q_{ij}}{\partial \bar{\pi}_{ik}} = \frac{\partial Q_{ik}}{\partial \bar{\pi}_{ij}} \forall i \in N \text{and } j, k = 1, \ldots, J_i \text{ and } \forall \bar{\pi}_i \in \mathbb{R}^{J_i}$$

   This means that the effect of an increase in strategy k's payoff on choosing some strategy j is the same as the effect of an increase in strategy j's on choosing strategy k.

3. Strong Substitutability

$$(-1)^{\ell} \frac{\partial Q_{ij}}{\partial \bar{\pi}_{ik_1} \ldots \partial \bar{\pi}_{ik_\ell}} \geqslant 0 \forall 1 \leqslant \ell \leqslant J_i - 1, k_1 \neq \cdots \neq k_\ell \neq j, \forall \bar{\pi}_i \in \mathbb{R}^{J_i}$$

   This means that if the payoff of strategy j increases, then the probability of choosing any other strategy falls.

Goeree et al. (2005, p. 360) argue that these three additional restrictions arising from the construction of the structural quantal response function "do not translate into sensible empirical restrictions on possible QRE outcomes, since they are not related to monotonicity". Here monotonicity means that players choose actions that have higher expected payoffs more frequently than actions that have lower expected payoffs. In other words, players assign greater probability to actions with higher expected payoffs. These restrictions on choice probabilities are not theoretically linked to economic logic and can prove unhelpful when modelling experimental data. Take for example, translation invariance. There are instances where payoff errors depend on the magnitude of payoffs, consider the impact to choice probabilities from an additional $1 when payoffs are all below $3 compared to when payoffs are all above $1000 (Goeree et al., 2005). At times the relative difference between payoffs matter and not just the absolute difference. Translation invariance implies that individuals will behave the same way regardless of the size of their payoffs so long as the absolute difference is kept equal. If some amount δ was added to or subtracted from all payoffs, the respective structural QRE would still map the same probabilities to each strategy. Here an individual would choose a strategy with an expected payoff of 10 instead of one with 5 at the same probability they choose a strategy with an expected payoff of 30 instead of one with 25. Regarding symmetry and strong substitutability Goeree et al. (2005) note as an example that the implied substitution patterns from these restrictions are viewed as weaknesses within empirical Industrial Organisation literature. A further point on strong substitutability is that there is not always a theoretical underpinning of why the probability of choosing some strategy j should decrease in the event of an increase in the expected payoff of strategy k (Goeree et al., 2005). Although structural QRE has been observed to better fit experimental data and explain non-Nash behaviour (see McKelvey and Palfrey, 1995), it is not always the most optimal in certain game structures.

2.1.3 Key limitations of structural QRE

As structural QRE focuses on disturbing players' expected payoffs with some random error ε_i, the distribution of this random variable dictates the intensity at which expected payoffs are adjusted. Given there are no other clear assumptions imposed on the structural quantal response function, the independent and identically distributed (i.i.d.) assumption that is imposed jointly on $(f_1, \ldots, f_n)$ becomes crucial to ensuring additive error terms do not perturb expected payoffs such that any possible strategy can be considered a structural QRE. Haile et al. (2008) prove precisely that when the i.i.d. assumption is relaxed, any choice behaviour from players can be modelled by a structural QRE. Haile et al. (2008) prove this in two ways, first by relaxing the identical distributions part of the i.i.d. assumption, then secondly by relaxing the independence component.

Their Theorem 1 (Haile et al., 2008, Theorem 1) states that when taking any finite normal form game Γ with each player $i \in N$ having $j = 1, \ldots, J_i$ pure strategies, any completely mixed strategy profile $\sigma \in \Delta_{int}$, where Δ_{int} is the interior of all possible strategies, can be obtained as a structural QRE by relaxing either the (i) identical distribution assumption or (ii) independence assumption of i.i.d. for some joint probability density function f. Specifically, denote the set of joint probability density functions for J independent and mean-zero random variables as $\mathcal{I}_J$ and denote the set of joint probability density functions for J mean-zero random variables with identical marginal distributions as $\mathcal{S}_J$. Therefore, (i) $\exists f \in \mathcal{I}_J$ such that $\forall i \in N, \sigma = Q(\bar{\pi}_i(\sigma))$ and (ii) $\exists f \in \mathcal{S}_J$ such that $\forall i \in N, \sigma = Q(\bar{\pi}_i(\sigma))$. The proof for this, taken from both Haile et al. (2008) and Goeree et al. (2005), is as follows for binary action games.[2] Assume that player i has two pure strategies so $s_{ij} \in \{s_{i1}, s_{i2}\}$, the probability of choosing pure strategy j given their opponents' strategies σ_{-i} in a QRE is:

$$\sigma_{ij} = Q_{ij}(\bar{\pi}_i(\sigma)) = \Pr[\bar{\pi}_{ij} + \varepsilon_{ij} > \bar{\pi}_{ik} + \varepsilon_{ik}] \forall k \neq j \in \{1, 2\}].$$

Given that the expected payoffs for player i only depend on their opponents' strategies we can rewrite the above as such:

$$\sigma_{ij} = \Pr[\varepsilon_{ij} > \varepsilon_{ik} + D_i(\sigma_{-i}) \forall k \neq j \in \{1, 2\}],$$

where $D_i(\sigma_{-i})$ is simply the difference between expected payoffs of player i's strategy j and k $(\bar{\pi}_{ij}(\sigma_{-i}) - \bar{\pi}_{ik}(\sigma_{-i}))$. This means we only need to define a probability density function $f_i \in \mathcal{I}_J$ and $f_i \in \mathcal{S}_J$ such that the above equation equals some arbitrarily chosen probability to exhibit that any $\sigma \in \Delta_{int}$ can be a QRE. Assign some arbitrary values to player i's expected payoffs such that $D_i(\sigma_{-i}) = \delta_i$. To show (i) take some

[2] The proof for a game with an arbitrary number of strategies can be found in the Appendix of Haile et al. (2008).

$f_i \in \mathcal{I}_J$ such that

$$\varepsilon_{ij} = \begin{cases} \alpha_{ij} & \text{probability } q_{ij} \\ -\left(\frac{q_{ij}}{1-q_{ij}}\right)\alpha_{ij} & \text{probability } 1 - q_{ij} \end{cases}$$

where $\alpha_{ij} > 0$ and $q_{ij} \in (0, 1)$. Each draw for action j is independent and by construction $E(\varepsilon_{ij}) = 0$. Assume that δ_i is positive.[3] Then all you need to do is choose some $\alpha_{ij} > \delta_i$ and set $\alpha_k = 0$ then:

$$\sigma_{ij} = \Pr[\varepsilon_{ij} > \varepsilon_{ik} + \delta_i] = q_{ij}.$$

To show (ii) take some $f_i \in \mathcal{S}_J$ such that ε_{ij} is perfectly correlated with ε_{ik} such that it is larger than ε_{ik} by $\delta_i + \eta$ for a large set of ε_{ik} values where $\eta > 0$ is some arbitrarily small number. Assume some random variable γ is uniformly distributed on $[0, (\delta_i + \eta)/(1 - q_i)]$, let $\varepsilon_{ik} = \gamma$ and $\varepsilon_{ij} = (\gamma + \delta_i + \eta) \mod (\delta_i + \eta)/(1 - q_i)$. In other words, we have $\varepsilon_{ij} = \varepsilon_{ik} + (\delta_i + \eta) > \varepsilon_{ik}$ for $\varepsilon_{ik} \in [0, (\delta_i + \eta)/(1 - q_i) - (\delta_i + \eta)]$ and $\varepsilon_{ij} = \varepsilon_{ik} + (\delta_i + \eta) - (\delta_i + \eta)/(1 - q_i) < \varepsilon_{ik}$ otherwise. Given both marginal distributions of ε_{ij} and ε_{ik} are uniform on $[0, (\delta_i + \eta)/(1 - q_i)]$, we get:

$$\sigma_{ij} = \frac{(\delta_i + \eta)/(1 - q_i) - (\delta_i + \eta)}{(\delta_i + \eta)/(1 - q_i)},$$

which simplifies to:

$$\sigma_{ij} = \frac{(\delta_i + \eta)}{(\delta_i + \eta)} - \frac{(\delta_i + \eta)(1 - q_i)}{(\delta_i + \eta)} = q_i.$$

To reiterate, Haile et al. (2008) prove that in the absence of the strict i.i.d. restriction on the payoff error distributions, a structural quantal response function can be constructed to model and predict any observed behaviour. This means that when it comes to modelling experimental data, structural QRE is not falsifiable when the i.i.d. assumption is relaxed, without additional constraints to the quantal response function. Under these circumstances, structural QRE could result in modelling incoherent behaviour and lead to incorrect predictions of equilibria. Goeree et al. (2005, p. 354) note however, that QRE just needs additional economically logical assumptions to draw more meaningful conclusions.

[3] The complementary case where δ_i is negative follows the same logic.

2.2 Regular QRE

Following from the initial developments of structural QRE, the concept of regular QRE was formalised by Goeree et al. (2005), addressing some of the limitations of structural QRE as highlighted by Haile et al. (2008) but initially raised in their 2004 worker paper (Haile et al., 2004). Following the critique of Haile et al. (2008), there was a need for additional constraints to be placed on the quantal response function rooted in economic logic such that QRE would no longer be able to predict any observed behaviour from experimental data. Goeree et al. (2005) first introduce four key axioms that they apply to structural QRE to ensure a set of sensible economic restrictions are placed on the quantal response function. They later treat these foundational axioms as primitive when developing what they call reduced form regular QRE (discussed later) which is often simply referred to as regular QRE. For Goeree et al. (2005), a regular quantal response function $Q_i : \mathbb{R}^{J_i} \to \Delta^{J_i}$, satisfies the following:

[QRE1] Interiority

$$Q_{ij}(\bar{\pi}_i) > 0 \; \forall j = 1, \ldots, J_i \text{ and } \forall \bar{\pi}_i \in \mathbb{R}$$

This guarantees that the model has full domain, capable of using all data sets. Without this, the quantal response function would not be able to be used with all sets of experimental data, greatly limiting the applicability of regular QRE (Goeree et al., 2005).

[QRE2] Continuity

$$Q_{ij}(\bar{\pi}_i) \text{ is differentiable (and therefore also continuous) } \forall \pi_i \in \mathbb{R}$$

This ensures that each set of probabilities found using regular QRE is not empty and each strategy maps to one probability rather than multiple. Additionally, it means that very small changes in the expected payoffs do not translate to large jumps in the probability of choosing respective related strategies.

[QRE3] Responsiveness

$$\frac{\partial Q_{ij}(\bar{\pi}_i)}{\partial \bar{\pi}_{ij}} > 0 \; \forall j = 1, \ldots, J_i \text{ and } \forall \bar{\pi}_i \in \mathbb{R}$$

This means that if the payoff of some specific strategy increases, ceteris paribus, the corresponding probability of choosing that strategy will also increase.

[QRE4] Monotonicity

$$\bar{\pi}_{ij} > \bar{\pi}_{ik} \Rightarrow Q_{ij}(\bar{\pi}_i) > Q_{ik}(\bar{\pi}_i)\ \forall j, k = 1, \ldots, J_i$$

This places the restriction on the model such that if a particular strategy has a higher expected payoff compared to another available strategy, then the individual will choose the former with a higher probability than the latter.

These axioms provide additional restrictions on structural QRE to ensure it is falsifiable even in the instance of relaxing the i.i.d. assumption. The general intuition underlying QRE is the smoothing of best response functions, such that strict rational choice is replaced with a weaker version of rational choice (Goeree et al., 2005). A key constraint on the quantal response function following from [QRE2] and [QRE4] is that strategies with the same expected payoffs are chosen with the same probability: $\bar{\pi}_{ij} = \bar{\pi}_{ik} \Rightarrow Q_{ij}(\bar{\pi}_i) = Q_{ij}(\bar{\pi}_i) \forall j, k \in 1, \ldots, J_i$ (Goeree et al., 2005). Intuitively, players can be expected to place the same weight on strategies where they expect to receive the same payoff. A regular QRE of a normal-form game Γ is similarly defined as a mixed-strategy profile $\sigma' \in \Delta$ such that:

$$\sigma' = Q(\bar{\pi}(\sigma')).$$

Structural QRE, under certain circumstances, can be considered regular QRE. In fact, if the payoff errors are admissible as previously defined and satisfy a full support condition, the first three axioms of regular QRE are met, leaving [QRE4] monotonicity to account for (Goeree et al., 2005). Under the assumption of i.i.d. payoff errors across strategies for given player i, monotonicity holds and [QRE4] is satisfied. This means that the commonly used parametric form structural QRE, logit QRE (discussed below), satisfies [QRE1]-[QRE4] and is therefore both a structural and a regular QRE. In a lot of cases i.i.d. errors are assumed, however, this is not a necessary condition for monotonicity and can be relaxed to allow for other approaches to modelling error distributions.

The concept of label independence can be applied to a structural quantal response function to ensure it satisfies [QRE4]. Label independence here simply means that regardless of the strategy, the probability of choosing any strategy is dependent only its expected payoff (i.e., if the labels were to be switched the expected payoffs would maintain their assigned probability) (Goeree et al., 2005). For example, given the following expected payoffs $\bar{\pi}_{i1} = 4, \bar{\pi}_{i2} = 8$ the choice probabilities are $\sigma_{i1} = 1/3, \sigma_{i2} = 2/3$ then, if the labels were switched such that $\bar{\pi}_{i1} = 8, \bar{\pi}_{i2} = 4$ the choice probabilities would be $\sigma_{i1} = 2/3, \sigma_{i2} = 1/3$. Goeree et al. (2005, Proposition 5) prove in their Proposition 5 that if the probability distribution function f used to construct the quantal response function satisfies interchangeability[4], defined as:

$$\forall \theta \in \Theta_{J_i}, f(\varepsilon_1, \ldots, \varepsilon_{J_i}) = f(\varepsilon_{\theta(1)}, \ldots, \varepsilon_{\theta_{(J_i)}}),$$

where Θ_{J_i} denotes the set of all possible permutations of J_i objects, then label independence is implied and therefore, monotonicity holds, satisfying [QRE4]. Together with satisfying admissibility as previously defined for f, a structural quantal response function satisfies [QRE1]-[QRE4] and is said to be regular.

2.3 Reduced-form Regular QRE

Although regular QRE provides a path to ensure that structural QRE is falsifiable while addressing the critique from Haile et al. (2008), the additional restrictions placed on a regular quantal response function that has been derived from the structural model prove problematic in certain circumstances as previously discussed. As these restrictions of translation invariance, symmetry, and strong substitutability arise "from a modeling assumption (of additive payoff disturbances) but are not derived from economic principles... they may lead to implausible or empirically false restrictions in certain contexts" (Goeree et al., 2005, p. 360). This is not to say that translation invariance, symmetry, and strong substitutability are always unhelpful restrictions, for example the well-known parametric logit QRE is a structural and regular QRE[5] meaning these three additional restrictions to the quantal response function occur when using logit QRE. Goeree et al. (2005) simply note that it is not necessary for QRE to be derived from the structural foundations and that treating the axioms [QRE1]-[QRE4] as primitives allows for a wider range of parametric models to be used while still maintaining the intuitive idea behind QRE. This is what Goeree et al. (2005) define as the reduced form approach to regular QRE. In

[4]Goeree et al. (2005) note that interchangeability is not always necessary to satisfy [QRE4]

[5]The logit QRE model satisfies regularity when ensuring i.i.d. payoff errors.

current literature this is often referred to as simply, regular QRE or reduced form QRE. An immediate result from treating [QRE1]-[QRE4] as primitive is that there exist many quantal response functions that satisfy [QRE1]-[QRE4] but would not be admissible under structural QRE. One such example is the Luce model, where choice probabilities from a Luce quantal response function (discussed below) do not satisfy translation invariance, symmetry, or strong substitutability and therefore cannot be derived from the structural approach yet are admissible as reduced form regular QRE (Goeree et al., 2005). Due to the advantages presented by reduced-form regular QRE and its frequency of use within current literature, I constrain further discussion of QRE to its reduced form and refer to this as regular QRE or simply just QRE.

2.3.1 Link to Nash Equilibrium

It is well established that in parametric models of QRE (discussed below) as the exogenous variable that represents the level of rationality of players, often denoted as λ, approaches infinity, the corresponding QRE approaches the Nash equilibrium. While this is the case, it is not always entirely clear what happens when there exist multiple Nash equilibria. Goeree et al. (2016) prove that for almost all games, there exists a connection between QRE as λ is varied and one specific Nash equilibrium, while all other Nash equilibria are connected as pairs to each other. I explore this relationship between QRE and Nash equilibrium further through my analysis of M equilibrium in section 3.2.5.

2.4 Parametric Models of QRE

In order to use QRE to model experimental results and better understand human behaviour within certain games, parametric models that satisfy the axioms of regular QRE have been developed. Although several of these parametric models exist, I focus on two different models. Firstly I will discuss logit QRE, one of the most commonly used in empirical literature. Following this, I will discuss the Luce QRE model which draws from R. Duncan Luce's (1959) work, specifically his choice axioms. These two models have been chosen as they act as a type of complement to each other, specifically logit QRE is translation invariant while Luce QRE is scale invariant. Here translation invariance is as previously defined and means that the QRE obtained by a specific model do not change due to equal additive errors being applied to each payoff in a given normal-form game Γ. Similarly, scale invariance means that the QRE obtained by a specific model do not change due to equal scaling of payoffs within Γ. The formal of scale invariance is:

$$Q_{ij}(\pi_i) = Q_{ij}(\beta\pi_i)\forall\pi_i \in \mathbb{R}^{J(i)}_{++} \text{ and } \beta > 0.$$

2.4.1 Logit QRE

Introduced within the seminal work of McKelvey and Palfrey (1995), logit QRE is one of the most common parametric QRE models used. As previously mentioned, assuming i.i.d. errors, logit QRE can be a structural QRE that has a quantal response function that satisfies [QRE1]-[QRE4] and is therefore also regular. The quantal response function for logit QRE is defined as:

$$Q_{ij}(\bar{\pi}_i) = \frac{e^{\lambda\pi_{ij}}}{\sum_{k=1}^{J_i} e^{\lambda\bar{\pi}_{ik}}}, i \in N \text{ and } j = 1, \ldots, J_i,$$

where λ is an exogenous variable that is used to represent the level of rationality of players, such that as $\lambda \to 0$ players resort to uniformly randomising among their strategies and as $\lambda \to \infty$ players converge to their best responses and therefore any QRE tends to a Nash equilibrium. In order to exhibit logit QRE I introduce the normal form 2×2 game of asymmetric matching pennies with the payoff structure defined in Table 1. The locus of fixed points of the relevant logit quantal response

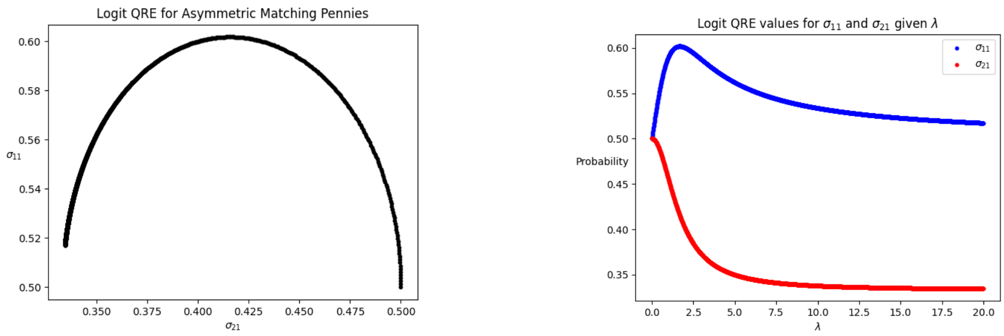
	H	T
H	(2,0)	(0,1)
T	(0,1)	(1,0)

Table 1: Asymmetric Matching Pennies Game

functions are plotted in Figure 1a for $\lambda \in [0, 20]$. The logit QRE choice probabilities for player 1 and player 2 as a function of λ are plotted in Figure 1b[6].

It is easy to observe that as $\lambda \to 0$ both players approach uniform randomisation between their given strategies and as $\lambda \to \infty$ both players approach the Nash equilibrium $\left(\sigma_1 = \frac{1}{2}, \sigma_2 = \frac{1}{3}\right)$.

[6]The logit QRE have been found using the the optimize package within the scipy Python library



(a) (b)

Figure 1: Logit QRE for Asymmetric Matching Pennies

2.4.2 Luce QRE

The Luce model arises from Luce's (1959) choice axioms that can be considered as a type of bounded rationality. Luce (1959) posited that individuals will make choices probabilistically relative to the attainable utility (or payoffs) of all options. The probability assigned by player $i \in N$ to action $j = 1, \ldots, J_i$ can be defined as

$$\sigma_{ij} = \frac{\pi_{ij}(\sigma_{-i})}{\sum_{k=1}^{J_i} \pi_{ik}(\sigma_{-i})}, i \in N \text{ and } j = 1, \ldots, J_i,$$

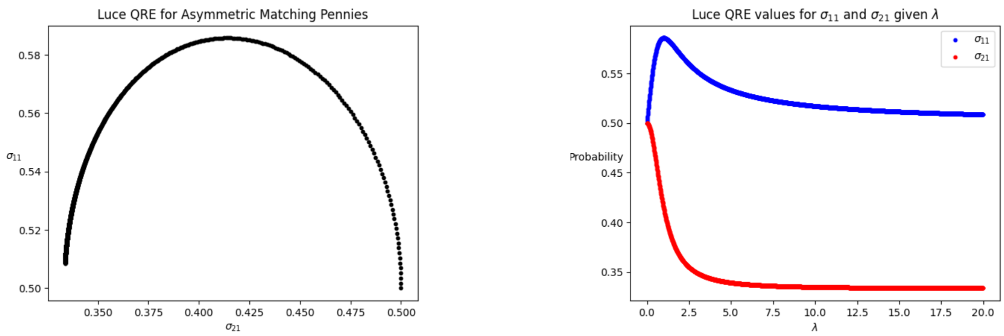
where σ_{-i} is the strategy profiles of all player i's opponents. This approach can then be generalised by transforming the expected utility associated with each action, in this case I follow that presented by Goeree et al. (2016) where the power function is used, raising each expected utility to the power of some variable λ. This forms the Luce power quantal response function for player i:

$$Q_{ij}(\pi_{ij}(\sigma_{-i})) = \frac{(\pi_{ij}(\sigma_{-i}))^{\lambda}}{\sum_{k=1}^{J_i} \pi_{ik}(\sigma_{-i})^{\lambda}}, i \in N \text{ and } j = 1, \ldots, J_i.$$

Similarly to the logit model, the fixed point of the Luce quantal response function is found for all λ to obtain all possible Luce power QRE[7] of a given game Γ.

Using the same approach as that presented in the logit QRE example, the Luce QRE are plotted in Figure 2a with individual players' choice probabilities for $\lambda \in [0, 20]$ plotted in Figure 2b.

[7] I refer to this simply as the Luce QRE



(a) (b)

Figure 2: Luce QRE for Asymmetric Matching Pennies

2.4.3 Comparison of Logit and Luce QRE

As previously mentioned, the logit QRE model can be a structural QRE and a regular QRE while the Luce QRE model is only regular in the reduced-from definition. The logit QRE model is translation invariant while not invariant to scale changes to payoffs. The reverse is true for the Luce model, which is scale invariant but not translation invariant. This difference can be observed when taking the asymmetric matching pennies game in Table 1 and adjusting the payoffs with (i) an additive constant α and (ii) a scale factor β. It should be noted that the scale invariance of the logit QRE is more subtle and as seen in Figure 3b scaling payoffs is equivalent to scaling λ by the same value. These adjusted games are presented below in Table 2 and Table 3 respectively.[8] The results for both logit and Luce QRE models are

	H	T
H	(7,5)	(5,6)
T	(5,6)	(6,5)

Table 2: Additive Constant M.P.

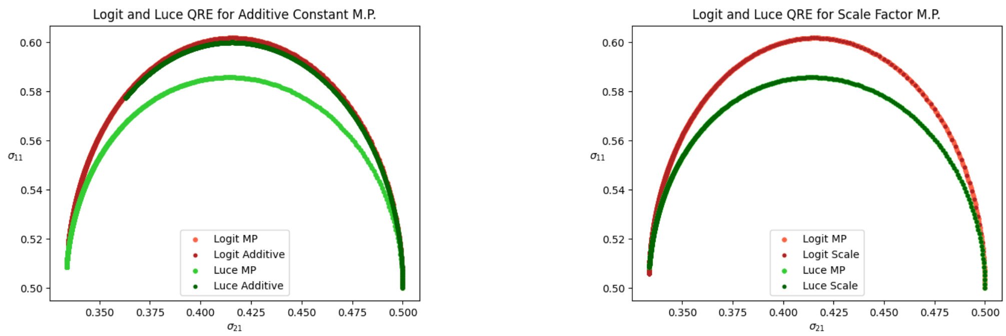
	H	T
H	(6,0)	(0,3)
T	(0,3)	(3,0)

Table 3: Scale Factor M.P.

presented below with Figure 3a showing the game presented in Table 2 and Figure 3b showing the game presented in Table 3. For both games, the logit and Luce QRE are compared to the results obtained for the initial asymmetric matching pennies game presented in Table 1, with these being the lighter colour lines.

In Figure 3a you can clearly see that the logit model remains unchanged between the asymmetric matching pennies game and the version with an additive constant, while the Luce QRE model is shifted further upwards while for $\lambda \in [0, 20]$ is no

[8]Note that the Nash equilibrium of each adjusted game in Table 2 and Table 3 are unchanged from that in the game presented in Table 1.



(a) (b)

Figure 3: Luce QRE for Asymmetric Matching Pennies

longer as filled out for choice probabilities $\sigma_{21} < 0.365$. In Figure 3b the Luce QRE model remains unchanged from the asymmetric matching pennies game when scaling payoffs by a factor of 3. The change in the logit QRE model is more difficult to discern in this specific game, however, the equilibria choice probabilities obtained by varying λ in the scaled version now occur more frequently where $\sigma_{21} < 0.425$.

3 Alternative Approaches to QRE

Since the introduction of QRE there have been alternative approaches to modeling the outcomes of games and subsequently modeling the behaviour of participants' beliefs and actions. Some concepts include that of Level-K and Cognitive Hierarchy which have both been further developed. There also exists more specific and more general versions of QRE, although given these are not as widely used as regular QRE, they are left for discussion later. More recently, other approaches to equilibrium modeling have been introduced, attempting to better account for deviations from the Nash equilibrium when compared to QRE. Two of the most recent theories that seek to explain deviations from Nash equilibria while addressing the drawbacks of QRE are M equilibrium and Noisy Belief equilibrium.

3.1 Noisy Belief Equilibrium

Noisy belief equilibrium (NBE), in contrast to QRE, focuses on relaxing the Nash equilibrium assumption of correct beliefs while maintaining the assumption of best responses. Within NBE, a certain "noise" is introduced to players' beliefs about their opponents' actions. Specifically, Friedman (2022) assumes that players draw their beliefs probabilistically, which they then best respond to, inducing some expected action. The "noise" or randomness introduced into players' beliefs can be interpreted as an individual error when players attempt to determine their best response or arise out of unclear signals from the other players in the game. From a broader perspective, randomness could be attributed to the heterogeneity of a wider population of which individual groups have belief sets that have formed deterministically.

3.1.1 Model Formulation

Friedman (2022) begins by introducing NBE in the context of binary action games, then extends this model to general normal-form games. Given the focus on 2×2 games within this paper I introduce NBE for the binary action case while briefly mentioning Friedman's generalised model leaving a more detailed review for the reader to explore (see Friedman, 2022, p. 104–107).

Binary Action Games

For games where there exist two pure actions that players can make, $J(i) = 2$, we can denote by $r_k \in [0,1]$ the probability that a player k will choose some specified action j. Player i's belief about player k's action is assumed to be drawn from a distribution that depends on r_k. For each possible value of r_k there will exist a random variable, $r_k^{i*}(r_k)$, which represents the belief that player i has over what player k's action will be. Collectively, this set of random variables forms what Friedman (2022) calls a *belief map* and is defined by a family of cumulative density functions (CDF). These CDFs provide the probability $F_k^i(\bar{r}|r_k) = \Pr(r_k^*(r_k) \leqslant \bar{r})$, of obtaining a belief less than or equal to any potential belief $\bar{r} \in [0,1]$. To ensure that the model is testable and incorporates noise in beliefs, while not being structurally biased, Friedman (2022) posits the following core axioms:

[NBE1′] Interior full support

For any $r \in (0,1)$, $F_k^i(\bar{r}|r)$ is strictly increasing and continuous in $\bar{r} \in [0,1]$; $r^*(0) = 0$ and $r^*(1) = 1$ with probability 1.

This ensures that, when $0 < r < 1$, the belief distribution is atomless while having full support. Furthermore, the boundary cases where player k's action is a pure strategy $r \in \{0,1\}$, player i's beliefs over player k are completely accurate and collapse to a single atom, i.e., in the event of pure strategies players have perfect knowledge of their opponent's actions.

[NBE2′] Continuity

For any $\bar{r} \in (0,1)$, $F_k^i(\bar{r}|r)$ is continuous in $r \in [0,1]$. This requirement ensures the existence of equilibria.

[NBE3′] Belief responsiveness

For all $r < r' \in [0,1]$, $F_k^i(\bar{r}|r') < F_k^i(\bar{r}|r)$ for $\bar{r} \in (0,1)$.

This ensures that as player k becomes more likely to play a particular strategy, player i's belief over this will also shift in the same direction. In other words, if player k's probability of playing the designated action A increases, the probability that player i realises a belief less than or equal to some arbitrary possible belief $\bar{r}$ will decrease, as player i is now more likely to realise a belief that player k will be more likely to play action A. Intuitively, it is reasonable to assume if the payoffs and conditions of a game change such that an opponent is more likely to play a particular strategy, a similar but not necessarily equal change to one's own beliefs of an opponent's actions would follow.

[NBE4′] Unbiasedness

$F_k^i(r|r) = \frac{1}{2}$ for $r \in (0, 1)$.

This means that when player i's potential belief of player k's action is equal to the action taken by player k, the probability of realising a belief less than or equal to that possible belief is $\frac{1}{2}$. This also means that beliefs will be unbiased on median instead of mean (Friedman, 2022, p. 102).[9]

Axioms [NBE1′] and [NBE2′] are technical to the model and ensure the existence of equilibria, while [NBE3′] and [NBE4′] impose behavioural restrictions on the model and act to sensibly restrict the set of attainable equilibria (Friedman, 2022, p. 101). It is also important to note that the inclusion of both [NBE1′] and [NBE2′] imply there exist discontinuities in beliefs however, this only occurs in sets of realised beliefs near the boundaries as r approaches that same boundary (e.g. the belief $r' = 0$ as $r \to 0^+$) and are easy to characterise (Friedman, 2022, p. 101).

Friedman (2022, p. 102) provides an example of NBE in the context of binary action games in the form of generalised matching pennies denoted as Γ^m. He defines the following payoff structure where $a_L, a_R, b_U, b_D \in \mathbb{R}$ and $c_L, c_R, d_U, d_D > 0$.

	L	R
U	$a_L + c_L, b_U$	$a_R, b_U + d_U$
D	$a_L, b_D + d_D$	$a_R + c_R, b_D$

Table 4

Furthermore, r_U and r_L denote the probabilities of player 1 choosing the action U and player 2 choosing the action L, respectively. The complementary probabilities represent the probability of choosing the action D and R, respectively. This game represents the set of all 2×2 games with a unique Nash equilibrium, (r_U^{NE}, r_L^{NE}), that is in completely mixed strategies and which is determined solely by the payoff differences. Given that the players' points of indifference are conditional on payoff differences we know that this will occur at an interior point. From here, due to the axiom of interior full support [NBE1'], the probability of having a specific belief at the point of indifference is zero. Therefore, both players' expected best response functions can be single-valued functions of strategies r_U and r_L respectively, that determine the probability that actions L and U are best responses. For the specific generalised matching pennies game structure, the best response functions are de-

[9] Friedman (2022) notes that mean unbiasedness would still be consistent with [NBE1′] - [NBE4′]

fined:

$$\Psi_U(r_L) = 1 - F_2^1(r_L^{NE} \mid r_L)$$

$$\Psi_L(r_U) = F_1^2(r_U^{NE} \mid r_U).$$

An NBE is a fixed point of these functions. Specifically, an NBE is any $(r_U, r_L) \in [0,1]^2$ such that $\Psi_U(r_L) = r_U$ and $\Psi_L(r_U) = r_L$. Friedman (2022, p. 104) proves that when fixing some generalised matching pennies game, an NBE exists and is both unique and interior for any given belief maps $r^* = \left(r_k^{i*}(r_k)\right)_{i,k\neq i}$ that satisfy [NBE1′]-[NBE4′].

Normal-Form Games

This section aims to provide a brief introduction to Friedman's (2022) generalisation of NBE to normal-form games, referring the reader to the more detailed construction of the extended axioms in Friedman (2022, p. 105–107). It should be noted that the binary action model of NBE previously presented is nested within the generalised model. Similarly to the binary action case, player i will form beliefs over the action of player k. Given an action $\sigma_k \in \Delta_k$, from player k, player i's belief is given as a random vector $\sigma_k^{i*}(\sigma_k) = (\sigma_{k1}^{i*}(\sigma_k), \ldots, \sigma_{kJ(k)}^{i*}(\sigma_k))$ which is supported on Δ_k. This set of random vectors across all possible actions of player k form the belief map of player i. The vector of all players' belief maps is denoted as $\sigma^* = (\sigma_k^{i*})_{i,k\neq i}$ and the vector of belief maps of player i over all their opponents is $\sigma_{-i}^* = (\sigma_k^{i*})_{k\neq i}$. A player's belief map $\sigma_k^{i*}(\sigma_k)$, which states the probability of beliefs being realised given some action taken by player k, is formed using the probability measure $\mu_k^i(\cdot \mid \sigma_k)$ on the Borel σ-algebra on Δ_k, $\mathcal{B}(\Delta_k)$. Here, for any given $B \in \mathcal{B}(\Delta_k)$ the measure will assign some probability value i.e., $\mu_k^i(B \mid \sigma_k)$. Given there exists an assumption of independent belief formation: all player k's opponents form their beliefs about player k's actions independently and conditional only on σ_k and any player's beliefs over their opponents are formed independently from opponent to opponent solely conditional on the respective opponents' actions. The belief map of player i, $\sigma_{-i}^*(\sigma_{-i})$, over all their opponents is associated with the product of all measures for any Borel set given their opponents' actions. Specifically, $\mu_{-i}(B \mid \sigma_{-i}) = \prod_{k\neq i} \mu_k^i(B \mid \sigma_k)$ for all B, where $B = \times_{k\neq i} B_k \in \bigotimes_{k\neq i} \mathcal{B}(\Delta_k) = \mathcal{B}(\Delta_{-i})$.

From here the equilibrium concept can now be outlined. First, player i's response set for a given action j, $R_{ij} \subseteq \Delta_{-i}$, is defined as the set of rival action profiles for which player i's action j is a best response (i.e., the expected utility from playing action j is greater than or equal to the expected utility of playing any other action.

$$R_{ij} = \{\sigma^{'}_{-i} : \bar{u}_{ij}(\sigma^{'}_{-i}) \geqslant \bar{u}_{ij}(\sigma^{'}_{-i}) \ \forall k = 1, \ldots, J(i)\}$$

Building on this, for every player there exists a strategy function that maps a player's developed beliefs of their opponents' actions to a respective action $s_i = (s_{i1}, \ldots, s_{iJ(i)}) : \Delta_{-i} \to \Delta_i$. Here, for each possible action of player i the strategy function assigns a non-negative probability to that action based on some action taken by their opponents $s_{ij}(\sigma^{'}_{-i}) \geqslant 0 \forall \sigma^{'}_{-i} \in \Delta_{-i}$, where the total probability across all possible actions of player i is 1, $\sum_{j=1}^{J(i)} s_{ij}(\sigma^{'}_{-i}) = 1$. Furthermore, s_i is said to be rational if and only if it only places positive probability on player i's best responses as determined by the ij-response set defined above, specifically, $s_{ij}(\sigma^{'}_{-i}) = 0 \iff \sigma^{'}_{-i} \notin R_{ij}$.

To determine the expected action of player i, an integral over the probability distribution formed on player i's strategy function s_i is taken with respect to player i's realised beliefs of their opponents from the measure $\mu_{-i}(\cdot \mid \sigma_{-i})$. Given we are only interested in strategies that are considered best responses, these are limited to those that are defined as rational. From this, player i's expected best response correspondence is defined as

$$\Psi_i(\sigma_{-i}; \sigma^{*}_{-i}) = \left\{ \int_{\Delta_{-i}} s_i(\sigma^{'}_{-i}) d\mu_{-i}(\sigma^{'}_{-i} \mid \sigma_{-i}) : s_i \text{ is rational} \right\}$$

Similar to the case in the binary action game, an NBE is defined as the fixed point of $(\Psi; \sigma^{*}) = ((\Psi_1; \sigma^{*}_{-i}), \ldots, (\Psi_n; \sigma^{*}_{-n})) : \Delta \rightrightarrows \Delta$. Specifically, when some normal form game and belief maps are fixed, an NBE will be any $\sigma \in \Delta$ such that for all players $i \in N$, $\sigma_i \in \Psi_i(\sigma_{-i}; \sigma^{*}_{-i})$. Friedman (2022) introduces four key axioms for the belief map which like the binary action case, ensure full support, continuity, belief responsiveness, and unbiasedness. For the specific construction of these axioms see Friedman (2022, p. 104–107). The implications and reasoning behind these axioms follows that discussed in the binary action case, only here they are expanded on to ensure they function appropriately for the generalised normal-form model. Given the nesting of the binary action case, when $J(i) = 2$, the axioms are equivalent to [NBE1′]-[NBE4′].

3.1.2 Link to Rationalizability

In his discussion linking NBE to other established concepts in game theory, Friedman (2022, Lemma 2), proves that if some strategy profile is an NBE, then the actions which support that strategy profile are rationalizable. This is a significant result that has other implications for NBE that are not discussed further. Together with [NBE1′] in the event of a game with a unique Nash equilibrium and this equilibrium is in pure strategies, the interior of the set of attainable NBE must be empty.

Corollary 0.1. *For any game with a unique pure strategy Nash equilibrium,* $\{NBE\} = NE$*, where* $\{NBE\}$ *is the set of attainable NBE and* NE *is the set of of Nash equilibria.*[10]

3.1.3 Logit Transform NBE

In order to use NBE in the analysis of empirical data collected from experiments, Friedman (2022, p. 119) provides a parametric model which he calls Logit Transform NBE. The model is derived using the logit transform function $\mathcal{L}(r_k) = \ln(\frac{r_k}{1-r_k})$ to map player k's actions $r_k \in [0,1]$ to the extended real number line and where it is assumed that $\mathcal{L}(0) = -\infty$ and $\mathcal{L}(1) = \infty$. From here noise is added to the actions in the form of a random variable that is normally distributed $\varepsilon_i \sim_{iid} \mathcal{N}(0,1)$ and the magnitude of this noise is determined by $\tau \in (0,\infty)$. This τ can be interpreted as a rationality parameter: as τ approaches 0, player i's beliefs approach precisely the actions taken by their opponents, similar to the role of λ^{-1} within the QRE literature. Following the addition of noise to player i's logit-transformed beliefs, these are then mapped back to $[0,1]$ using the inverse of the logit transform to provide the noisy beliefs of player i for a given action of their opponents, $r_k^{i*}(r_k)$. In particular:

$$r_k^{i*}(r_k;\tau) = \mathcal{L}^{-1}(\mathcal{L}(r_k) + \tau\varepsilon_i) = \frac{\exp(\ln(\frac{r_k}{1-r_k}) + \tau\varepsilon_i)}{1 + \exp(\ln(\frac{r_k}{1-r_k}) + \tau\varepsilon_i)}$$

For the instances of binary action games, the belief map conveniently allows for a closed form cumulative density function (Friedman, 2022, p. 119):

$$F_k^i(\bar{r} \mid r_k;\tau) = \Phi\left(\frac{1}{\tau}\left[\ln\left(\frac{\bar{r}}{1-\bar{r}}\right) - \ln\left(\frac{r_k}{1-r_k}\right)\right]\right). \tag{1}$$

[10]In this case NE is a single equilibrium.

Here Φ is defined as the cumulative density function of the standard normal distribution.[11] Note that as $\tau \to 0^+, \frac{1}{\tau} \to \infty$ which in turn means the CDF will return either 1 or 0 depending on the negativity of the log odds between $\bar{r}$ and r_k. Thus, the players will tend towards their best responses under correct beliefs as $\tau \to 0^+$. When $\tau \to \infty, \frac{1}{\tau} \to 0$ meaning that the value returned by the CDF will approach 0.5 and players randomise uniformly.

To extend this parametric model to general normal form games, Friedman (2022, Appendix G) uses a normalising constant when developing the belief map for player i, such that the belief map takes the form

$$\sigma_{kj}^{i*}(\sigma_k;\tau) = \frac{\exp\left(\ln\left(\frac{\sigma_k}{1-\sigma_k}\right) + \tau\varepsilon_{kj}^i\right)}{1+\exp\left(\ln\left(\frac{\sigma_k}{1-\sigma_k}\right) + \tau\varepsilon_{kj}^i\right)} \cdot \left(\sum_{l=1}^{J(k)} \frac{\exp\left(\ln\left(\frac{\sigma_{kl}}{1-\sigma_{kl}}\right) + \tau\varepsilon_{kl}^i\right)}{1+\exp\left(\ln\left(\frac{\sigma_{kl}}{1-\sigma_{kl}}\right) + \tau\varepsilon_{kl}^i\right)}\right)^{-1}.$$

Friedman (2022) notes that while this model functions well for binary action games, the parametric model introduces bias within the distributions of player i's beliefs when extended to the general normal form games. This bias arises due to the need to normalise player i's beliefs across all possible actions of their opponent(s) to ensure that player i's beliefs about each opponent sum to 1.

3.1.4 Logit Transform NBE Example

Keeping with the asymmetric matching pennies examples provided within the QRE review section, I use the same game structure for the logit transform NBE example. This game of asymmetric pennies also follows that of the generalised matching pennies game introduced by Friedman (2022, p. 102) as previously discussed. Here,

	H	T
H	(2,0)	(0,1)
T	(0,1)	(1,0)

Table 5: Asymmetric Matching Pennies Game

r_1 and r_2 are used to denote the actions of each player: the probability of playing heads (H) for player 1 and 2 respectively. Consequently, the complements of r_1 and r_2 denote the probability of playing tails (T) for player 1 and 2 respectively. Under the NBE framework, both players will form beliefs about each other's actions then best respond to these beliefs. Given the structure of the game, player 1's and 2's

[11] In order to make the CDF well defined Friedman (2022, p. 119) resolves indeterminacies: $-\infty - (-\infty) = \infty$ and $\infty - \infty = \infty$. Additionally, $\Phi(-\infty) = 0$ and $\Phi(\infty) = 1$.

best responses are dependent on their realised beliefs relative to the Nash equilibrium. The Nash equilibrium of this game is $\left(r_1^{NE}, r_2^{NE}\right) = \left(\frac{1}{2}, \frac{1}{3}\right)$. Therefore, player 1 will, play heads when their belief is $r_1^*(r_2) > \frac{1}{3}$, play tails when their belief is $r_1^*(r_2) < \frac{1}{3}$, and be indifferent between playing both heads or tails when their belief is $r_1^*(r_2) = \frac{1}{3}$. For player 2, they will play heads when their belief is $r_2^*(r_1) < \frac{1}{2}$, play tails when their belief is $r_2^*(r_1) > \frac{1}{2}$, and be indifferent between playing both heads or tails when their belief is $r_2^*(r_2) = \frac{1}{2}$. In the event of indifference both players may randomise between their two possible strategies. As previously discussed, given [NBE1′] the probability of either player having their belief equal that of the indifference condition is zero and thus the players' best response functions are single valued and denoted as

$$\Psi_1(r_2) = 1 - F_2^1(r_2^{NE} \mid r_2)$$

$$\Psi_2(r_1) = F_1^2(r_1^{NE} \mid r_1).$$

Substituting from equation 1 we obtain the following equations:

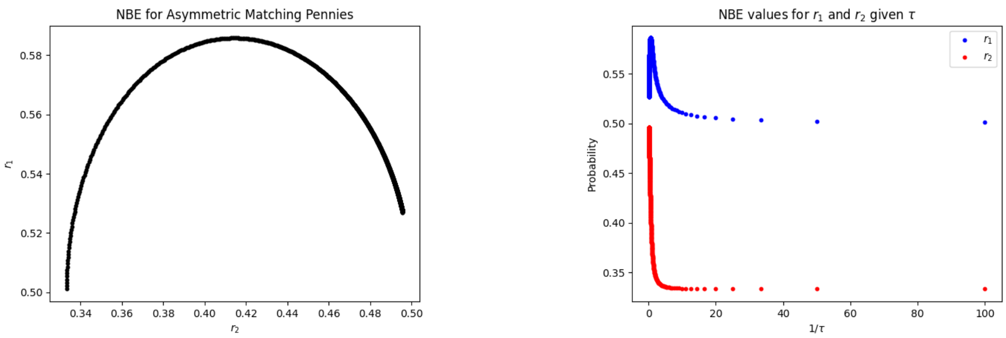
$$\Psi_1(r_2; \tau) = 1 - \Phi\left(\frac{1}{\tau}\left[\ln\left(\frac{r_2^{NE}}{1 - r_2^{NE}}\right) - \ln\left(\frac{r_2}{1 - r_2}\right)\right]\right),$$

$$\Psi_2(r_1; \tau) = \Phi\left(\frac{1}{\tau}\left[\ln\left(\frac{r_1^{NE}}{1 - r_1^{NE}}\right) - \ln\left(\frac{r_1}{1 - r_1}\right)\right]\right).$$

Using the above equations we find the fixed point $(r_1, r_2) \in [0, 1]^2$ such that $\Psi_1(r_2; \tau) = r_1$ and $\Psi_2(r_1; \tau) = r_2$, thus finding the NBE for any value of τ. The result of this exercise, obtained using code produced in Python with $\tau \in (0, 10]$, are exhibited below.[12]

The NBE for various values of τ can be observed in Figure 4a, where the range of NBEs follow a similar curve to that identified within the Logit and Luce QRE examples previously. Additionally, the values for both r_1 and r_2 are plotted against $1/\tau$ in Figure 4b, showing how each player best responds as their beliefs approach that of perfect information ($\tau \to 0$).

[12] Note that given the presence of $1/\tau$ within the best response functions, 0 has had to be excluded from the range provided, although this does not materially impact the results as we still observe players approaching best responses under perfect beliefs as $\tau \to 0$.



(a) (b)

Figure 4: NBEs for Asymmetric Matching Pennies

3.1.5 Games with Pure-Mixed Strategy Equilibria

It is evident that any pure strategy Nash equilibria will also be a NBE. What is somewhat less clear initially is how NBE behaves when dealing with games that have pure-mixed equilibria. Here, pure-mixed equilibria are defined as equilibria in which player i plays a pure strategy while player k plays a mixed strategy. In other words, player i assigns probability 1 to some action and player k assigns probability $\sigma \in (0, 1)$ to some action. Take the game presented in Table 6. Let r_1 and r_2

	L	R
U	(1,0)	(2,1)
D	(1,1)	(3,0)

Table 6: Game with Mixed-Pure Nash Equilibrium

be the probabilities that player 1 chooses U and player 2 chooses L, respectively. This game has one pure strategy Nash equilibrium and an infinite amount of pure-mixed strategy Nash equilibria. The pure strategy Nash equilibrium occurs when player 1 plays pure strategy D and player 2 plays pure strategy L. The infinite set of pure-mixed strategy Nash equilibria occur when player 2 plays pure strategy L and player 1 randomises between pure strategy U and D. Specifically, the set of mixed-pure Nash equilibria occur when $(r_1, r_2) = (x, 1)$, for $x \in (0, 0.5]$. We already know that a pure strategy Nash equilibrium of a binary action game will also be an NBE. In this specific instance, an NBE of the game in Table 6 will be $(r_1, r_2) = (0, 1)$. Now we turn our attention to the other Nash equilibria, the pure-mixed strategy case. We know from [NBE1′] that player 1's beliefs about player 2's pure strategy $r_2 = 1$ will be correct with probability 1 and therefore, player 1 will be indifferent between strategy U and D, and we get $r_1 \in [0, 1]$. For player 2, playing the pure strategy L will only be a best response to the instances where $r_1 \in [0, 0.5]$. From [NBE1′] we know that player 2's strategy must be supported in $r_1 \in (0,1)$ and since their response will be an average best response we need to have pure strategy L be a best response for all $r_1 \in (0,1)$. This is not the case as for $r_1 \in (0.5, 1]$, playing pure strategy R is the best response for player 2. As a result the equilibrium does not hold and for the game presented in Table 6 there exist no pure-mixed strategy NBE. This is not to say pure-mixed strategy NBE do not exist however, as there are instances where they can occur. I now generalise this finding for any generic 2×2 game.

Definition 1 (Generic game)**.** A generic game is defined here as any game in which the set of mixed strategy profiles that make player i indifferent between two actions has measure zero in Δ_{-i}.

Proposition 1. *For any generic 2×2 game Γ, pure-mixed strategy NBE only exist when either both players have weakly dominant strategies or one player has a weakly dominant strategy while the other player has a strictly dominant strategy. Where there exists pure-mixed strategy Nash equilibria in Γ but no pure-mixed strategy NBE, all attainable NBE will be in pure actions.*

Proof. I will prove the first part of this proposition by contradiction. First assume that a pure-mixed NBE exists for a generic game where only one player has a weakly or strictly dominant strategy and will be a fixed point of $\Psi = (\Psi_{ij}, \Psi_{kl}) : [0,1]^2 \to [0,1]^2$ such that $r_i \in (0,1)$ and $r_k \in \{0,1\}$ for some $i, k \in \{1,2\}$ with $i \neq k$. When $r_i^{NBE} \in (0,1)$ and $r_k^{NBE} = 0$ we know that player i must be indifferent between their two strategies, which implies player i has a weakly dominant strategy. We know this because if both of player i's strategies had equal payoffs across any action of player k, then the game would be non-generic. As $r_i^{NBE} \in (0,1)$, for player k to play $r_k^{NBE} = 0$, $r_k = 0$ has to be a best response to each $r_i \in (0,1)$ by [NBE1′]. This can only be true when player k has a weakly dominant or a strictly dominant strategy, otherwise there would exist some $r_i' \in (0,1)$ in which $r_k = 0$ would not be a best response and therefore the average best response $r_k^{NBE} \neq 0$. This is a contradiction as player i must have a weakly dominant strategy and player k must have either a weakly or strictly dominant strategy.

Now all that is left to do is prove that when a pure-mixed Nash equilibrium exists in Γ and a pure-mixed NBE does not exist, there will only exist pure strategy

NBE. We know from the first part of this proof that when only player i has a weakly dominant strategy, there exists no action for player k that is a best response to any mixed strategy $r_i \in (0,1)$ and therefore by [NBE1′] will not be a NBE. Only when $r_i \in \{0,1\}$ will there exist a strategy for player k that is always a best response with full support. Subsequently, as player k has correct beliefs with probability 1 when $r_i \in \{0,1\}$, their best response will be a pure strategy. As a result, we have a pure strategy NBE. Futhermore, there exist no completely mixed strategy NBE, when there exists pure-mixed strategy Nash equilibria. This is because the existence of a pure-mixed strategy Nash equilibrium implies that at least player i has a weakly dominant strategy, which is a unique best response to any non-pure strategy of player k. As a result, there cannot exist any completely mixed strategy NBE. □

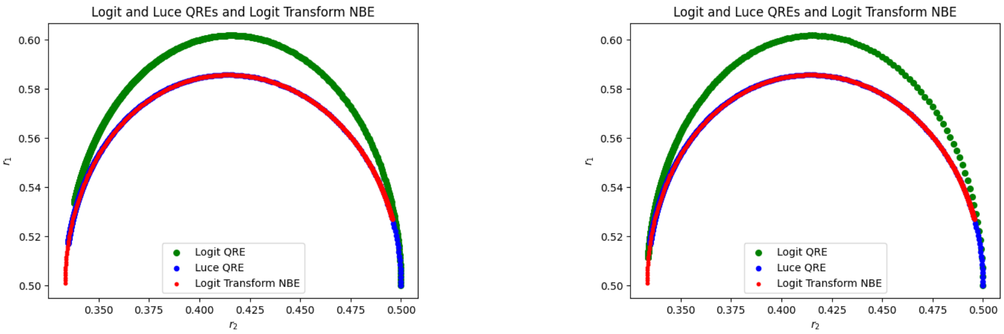
3.1.6 Relationship to QRE

An interesting finding of Friedman's (2022) relates to the scale and translation invariance issues that arise with QRE. As previously discussed, QRE cannot be both scale invariant and translation invariant, there exists a trade off for each type of quantal response function. Friedman (2022, Theorem 4; Theorem 5) proves this in his Theorem 4 before then proving that NBE is both scale invariant and translation invariant in his Theorem 5. This is a significant comparison between QRE and NBE and in some instances highlights a strength of NBE but also a potential weakness when considering if we expect individuals to make different decisions within affine transformations of some base game's payoffs.

For the generalised matching pennies game, Friedman (2022, Theorem 3) proves that "the set of attainable NBE is equal to the set of attainable QRE".[13] An interesting observation is that in certain instances of asymmetric matching pennies, the parametric Luce QRE is observationally equivalent to the parametric logit transform NBE (see Figure 5a)[14]. It should be noted that this does not generalise to all 2×2 games. Further, due to the scale invariance of both NBE and Luce QRE, we see these models remain unchanged when scaling payoffs by a factor of 3 (see Figure 5b).

[13]Friedman (2022, p. 133) proves this by construction after incorporating a differentiability condition to [NBE2']: $\forall \bar{r} \in (0,1), F^i_k(\bar{r} \mid r)$ is differentiable in $r \in (0,1)$. It is noted that this additional condition does not impact the set of attainable NBE.

[14]These instances occur for games following the structure shown in Table 5, in that if you alter player 1's payoff that is assigned the value 2 in Table 5 observationally the Luce QRE and logit transform NBE follow very similar paths.



(a) (b)

Figure 5: Example Asymmetric Matching Pennies and Scaled Version

It is possible to extend Friedman's Theorem 3 to all 2×2 games that have a completely mixed Nash equilibrium, with one minor caveat that will be addressed below. This expansion of Theorem 3 largely follows the same logic as Friedman (2022).

First, I introduce a generalised chicken game which is denoted Γ^c and has the following payoff structure: where $a_L, a_R, b_U, b_D \in \mathbb{R}$ and $c_L, c_R, d_U, d_D > 0$.

	L	R
U	(a_L, b_U)	$(a_R + c_R, b_U + d_U)$
D	$(a_L + c_L, b_D + d_D)$	(a_R, b_D)

Table 7: Generalised Chicken

The notation for each player's actions follows that of the generalised matching pennies game Γ^m previously discussed. Together the generalised matching pennies game and the generalised chicken game allow for the extension of Friedman's Theorem 3 as they cover all generic 2×2 games with a completely mixed Nash equilibrium.

Proposition 2. *$\Gamma^m \cup \Gamma^c$ covers all generic 2×2 games that have a completely mixed Nash equilibrium.*

Proof. All games can be explained in a similar form to that used within Γ^m and Γ^c, such that player i's payoffs are noted as some values relative to each other. Furthermore, as the completely mixed Nash equilibrium within generic 2×2 games occurs at the point of each players' indifference between two available actions, we know that the player i's strategy in the completely mixed Nash equilibrium will depend solely on the payoff structure of player k. From this we can construct a generalised game structure similar to that of Γ^m and Γ^c where there exist payoffs $a_L, a_R, b_U, b_D \in \mathbb{R}$ and payoff differences $c_L, c_R, d_U, d_D \in \mathbb{R}\backslash\{0\}$. As we know that indifference is determined based purely on differences in payoffs, we can ignore a_L, a_R, b_U, b_D initially. Additionally, as points of in generic games are solely reliant on the opponent's strategy we can split the game structure between player 1's and player 2's payoff differences as follows:

	L	R
U	c_L	c_R
D		

(a)

	L	R
U	c_L	
D		c_R

(b)

	L	R
U		
D	c_L	c_R

(c)

	L	R
U		c_R
D	c_L	

(d)

Table 8: Player 1 Structures (a), (b), (c), and (d)

	L	R
U	d_U	
D	d_D	

(a)

	L	R
U	d_U	
D		d_D

(b)

	L	R
U		d_U
D		d_D

(c)

	L	R
U		d_U
D	d_D	

(d)

Table 9: Player 2 Structures (a), (b), (c), and (d)

Due to the symmetry present within these possible payoff structures for both players, there only exist two unique points of indifference for each player across all possible structures. For both players, (a) and (c) are equivalent and (b) and (d) are equivalent. For player 1, indifference in (a) and (c) is given by $r_2^{MNE} = c_R/(c_R - c_L)$ and indifference in (b) and (d) is given by $r_2^{MNE} = c_R/(c_R + c_L)$. For player 2, indifference in (a) and (c) is given by $r_1^{MNE} = d_D/(d_D - d_U)$ and indifference in (b) and (d) is given by $r_1^{MNE} = d_D/(d_D + d_U)$. For payoff structures (a) and (c) indifference is possible when $c_L \times c_R < 0$ and when $d_U \times d_D < 0$, for players 1 and 2 respectively. For payoff structures (b) and (d) indifference is possible when $c_L \times c_R > 0$ and when $d_U \times d_D > 0$, for players 1 and 2 respectively. In the event that indifference for structures (a) and (c) are met given the values assigned to c_R, c_L, d_U, d_D, it is possible to maintain the absolute payoff structure by simply shifting to structure (b) or (d) changing one of the payoff differences such that both c_L, c_R or d_U, d_D have the same negativity. Consequently this means that all completely mixed Nash equilibria occur in a generic game such that $r_1^{MNE} = d_D/(d_D + d_U)$ and $r_2^{MNE} = c_R/(c_R + c_L)$. These occur when the payoff structures are combined to form the following set of games:

Given the payoff structures and not labels of actions matter here, these possible

	L	R
U		c_R, d_U
D	c_L, d_D	

(a)

	L	R
U	c_L	d_U
D	d_D	c_R

(b)

	L	R
U	d_U	c_R
D	c_L	d_D

(c)

	L	R
U	c_L, d_U	
D		c_R, d_D

(d)

Table 10: 4 Generalised Game Types with Completely Mixed Nash Equilibria

games reduce to simply just (a) and (b) which are the generalised chicken game and the generalised matching pennies game respectively. In the case where only one payoff difference exists for either player 1 or player 2, indifference for each player only occurs when the other plays a pure strategy, meaning there would exist a mixed-pure Nash equilibrium but not a completely mixed Nash equilibrium for this payoff structure. Lastly, in the instance where a player's respective payoff differences are both set to zero, this will cause the game to become non-generic and therefore irrelevant. This completes the proof, showing that $\Gamma^m \cup \Gamma^c$ covers all generic 2×2 games with a completely mixed Nash equilibrium. □

Γ^c represents all 2×2 generic games that have 2 pure strategy Nash equilibria and one completely mixed strategy Nash equilibrium. Like that of Γ^m, the completely mixed Nash equilibrium (r_1^{MNE}, r_2^{MNE}) of Γ^c relies solely on payoff differences. Specifically, $(r_1^{MNE}, r_2^{MNE}) = (\frac{d_D}{d_D+d_U}, \frac{c_R}{c_R+c_L})$, meaning that players' indifference conditions will occur on the interior. For completeness, the pure strategy Nash equilibria are $(r_1^{PNE}, r_2^{PNE}) = (0,1)$ and $(1,0)$. It follows from [NBE1'] that the best response functions $\Psi_i(\bar{r} \mid r_k)$ are single valued for $r_k \in (0,1)$. Due to the presence of equilibria on the boundary in Γ^c, additional care is required to ensure that the best response function is still single valued. From [NBE1'] $r^*(0) = 0$ and $r^*(1) = 1$, meaning that when opponent's play their pure strategies player i maintains perfect beliefs, we can infer that $F_k^i(\bar{r} \mid 0) = 1$ and $F_k^i(\bar{r} \mid 1) = 0$ providing single values from the CDF on the boundaries. As a result we have the following single valued best response functions for the generalised game of chicken for $(r_1, r_2) \in [0,1]^2$:

$$\Psi_1(r_2) = F_2^1(r_2^{MNE} \mid r_2)$$

$$\Psi_2(r_1) = F_1^2(r_1^{MNE} \mid r_1).$$

I now discuss the caveat to the generalisation of Friedman's Theorem 3; although rather trivial it should nevertheless be stated prior to the formal proof. From

[QRE1], we know that all QRE are interior each possible action is assigned some positive probability while no single action is assigned probability 1. Consequently, QRE cannot generate pure strategy equilibria. By contrast, NBE allows for pure strategy equilibria due to the existence of atoms at the boundary of the action space. In other words, as opponents' behaviour approaches playing pure strategies, the beliefs of players about these actions converge to reality. This allows for the existence of pure strategy equilibria. Thus, Theorem 3 can only be generalised for completely mixed equilibria.

Proposition 3. *For all generic 2×2 games with completely mixed Nash equilibria, defined by $\Gamma^m \cup \Gamma^c$, fix some $\Gamma \in \Gamma^m \cup \Gamma^c$. The set of attainable completely mixed NBE is equal to the set of attainable regular QRE.*

Proof. Following Friedman (2022, p. 133) the generalisation of Theorem 3 is shown through proof by construction. Firstly, every NBE is shown to be a QRE then subsequently the converse is shown, completing the proof. In order to avoid unnecessary repetition, I refer simply to Friedman's (2022) proof where no adjustments are required.

To show that every NBE is a QRE, an appropriate QRE can be constructed for every NBE. To begin, fix some $\{\Gamma^c, \sigma^*\}$ where player i's belief map $r^*(r)$ induces a NBE response function $\Psi_{ij} : [0,1] \to [0,1]$ which maps some belief to a strategy. As we fix some completely mixed NBE, all attainable NBE will be interior, meaning that any unique NBE will be a fixed point of $\Psi = (\Psi_{1j}, \Psi_{2l}) : [\varepsilon, 1-\varepsilon]^2 \to [\varepsilon, 1-\varepsilon]^2$ for some sufficiently small $\varepsilon > 0$ (Friedman, 2022, p. 133). A prequantal response function is constructed, that maps some set of utility vectors to actions, $\tilde{Q}_{ij} : U_i(\varepsilon) \to [0,1]$, in which the set of utility vectors is defined as $U_i(\varepsilon) = \bar{u}_i([\varepsilon, 1-\varepsilon]) = (\bar{u}_{i1}(r), \bar{u}_{i2}(r))_{r \in [\varepsilon, 1-\varepsilon]} \subset \mathbb{R}^2$. For the generalised chicken game we assume that $\partial \bar{u}_{i1}(r)/\partial r < 0$ and $\partial \bar{u}_{i2}(r)/\partial r > 0$, where r is the pair (r_1, r_2), the probability that player 1 chooses action U and player 2 chooses action L, respectively. The prequantal response function, $\tilde{Q}_{ij}$, is constructed so that for $r \in [\varepsilon, 1-\varepsilon]^2, \tilde{Q}_{ij} \circ \bar{u}_i(r) = \Psi_{ij}(r)$ and that $\tilde{Q}_{ij}$ satisfies the analogues of regular QRE's axioms [QRE1]-[QRE4] where [AQRE3] has been adjusted to suit Γ^c

[AQRE1]: $\tilde{Q}_{ij} \circ \bar{u}_i(r) \in (0,1) \forall r \in [\varepsilon, 1-\varepsilon]$.

[AQRE2]: $\tilde{Q}_{ij} \circ \bar{u}_i(r) \forall r \in [\varepsilon, 1-\varepsilon]$ is differentiable (and therefore continuous).

[AQRE3]: $\left(\partial\tilde{Q}_{i1} \circ \bar{u}_i(r)\right)/\partial r < 0, \left(\partial\tilde{Q}_{i2} \circ \bar{u}_i(r)\right)/\partial r > 0 \forall r \in [\varepsilon, 1-\varepsilon]$.

[AQRE4]: $\bar{u}_{ij}(r) > \bar{u}_{il}(r) \Rightarrow \tilde{Q}_{ij} \circ \bar{u}_i(r) > \tilde{Q}_{il} \circ \bar{u}_i(r) \forall r \in [\varepsilon, 1-\varepsilon]$.

From here Friedman's (2022) proof for showing that every NBE is a QRE suffices for Γ^c (with the pure strategy equilibria exclusion) with one change. Friedman (2022) relies on the fact that the composition of the prequantal response function and the utility function equals the NBE best response function as specified from the generalised matching pennies game, $\tilde{Q}_{i1} \circ \bar{u}_i(r) = \Psi_{i1}(r) = 1 - F_k^i(\bar{r} \mid r)$ and $\tilde{Q}_{i2} \circ \bar{u}_i(r) = \Psi_{i2}(r) = F_k^i(\bar{r} \mid r)$ for $\bar{r} \in [\varepsilon, 1-\varepsilon]$ where $\bar{r}$ is the indifference condition (e.g., $u_{i1}(\bar{r}) = u_{i2}(\bar{r})$). Using the payoff structure of Γ^c, the composition of the prequantal response functions are as follows: $\tilde{Q}_{i1} \circ \bar{u}_i(r) = \Psi_{i1} = F_k^i(r^{MNE} \mid r)$ and $\tilde{Q}_{i2} \circ \bar{u}_i(r) = \Psi_{i2}(r) = F_k^i(r^{MNE} \mid r)$ for $r^{MNE} \in [\varepsilon, 1-\varepsilon]$ where r^{MNE} is the completely mixed Nash equilibrium. Relying on largely the same logic as Friedman (2022) I show that these adjusted best response functions satisfy the analogues of the QRE axioms for $r \in [\varepsilon, 1-\varepsilon]$. $\tilde{Q}_{ij}$ satisfies [AQRE1] given that $\tilde{Q}_{i1} \circ \bar{u}_i(r) = F_k^i(\bar{r} \mid r) \in (0,1) \forall r \in [\varepsilon, 1-\varepsilon]$ from [NBE1']. Given that $F_k^i(\bar{r} \mid r)$ is continuous and differentiable from [NBE2'] and the additional differentiability condition, it follows that $\tilde{Q}_{ij}$ satisfies [AQRE2] for all $r \in [\varepsilon, 1-\varepsilon]$. $\tilde{Q}_{ij}$ satisfies [AQRE3] as $\tilde{Q}_{i1} \circ \bar{u}_i(r) = F_k^i(\bar{r} \mid r), \tilde{Q}_{i2} \circ \bar{u}_i(r) = 1 - F_k^i(\bar{r} \mid r)$ and given $\partial\bar{u}_{i1}(r)/\partial r < 0, \partial\bar{u}_{i2}/\partial r > 0$ and $\partial F_k^i(\bar{r} \mid r)/\partial r < 0$ from [NBE3] therefore, $(\partial\tilde{Q}_{i1} \circ \bar{u}_i(r))/\partial r < 0$ and $\partial(\tilde{Q}_{i2} \circ \bar{u}_i(r))/\partial r > 0$. Lastly, we know that $\bar{u}_{i1}(r) = \bar{u}_{i2}(r) \iff r = \bar{r}$ and subsequently from [NBE4'] $\tilde{Q}_{i1} \circ \bar{u}_i(\bar{r}) = F_k^i(\bar{r} \mid \bar{r}) = \frac{1}{2}, \tilde{Q}_{i2} \circ \bar{u}_i(\bar{r}) = 1 - F_k^i(\bar{r} \mid \bar{r}) = \frac{1}{2}$. As a result, we know that $\tilde{Q}_{i1} \circ \bar{u}_i(r)$ will only equal $\tilde{Q}_{i2} \circ \bar{u}_i(r)$ when $r = \bar{r}$, so given [AQRE], it follows that [AQRE4] is satisfied. The remaining step to show that every NBE is a QRE for Γ^c is to extend the prequantal response function to a quantal response function $Q_{ij} : \mathbb{R}^2 \to [0,1]$ that satisfies the regular QRE axioms [QRE1]-[QRE4]. For this, Friedman's (2022, p.135) proof suffices for Γ^c. Similarly, the proof to show that every QRE is an NBE for all mixed strategy equilibria in Γ^c remains unchanged from Friedman's proof showing this relationship for Γ^m. □

Furthermore, given that QRE are always completely mixed by [QRE1], and given that there exists no pure-mixed NBE for this specific class of games by Proposition 1 the following holds:

Corollary 3.1. *For any* $\Gamma \in \Gamma^m \cup \Gamma^c$, *QRE* $\subseteq$ *NBE*

3.2 M Equilibrium

In developing M equilibrium, Goeree and Louis (2021) maintain the critique of Nash equilibrium, specifically that within many laboratory experiments it fails to accurately model and predict outcomes. Similarly to QRE, M equilibrium relaxes the Nash assumption of best responses, instead focusing on the notion of 'better responses' over that of best responses. M equilibrium similar to NBE relaxes the assumption of rational expectations of correct beliefs, as there exists considerable criticism of this assumption reinforced by empirical results (Goeree and Louis, 2021, p. 4005). M equilibrium was developed to be consistent with monotonicity and consequential unbiasedness, while also being set-valued. Monotonicity, as previously discussed, ensures that the frequency with which individuals choose particular actions are positively associated with their respective expected payoffs based on the individual's beliefs. Consequential unbiasedness is what relaxes the rational expectations assumption, allowing for beliefs to be biased while ensuring that the ordering of expected payoffs from beliefs remains the same as that informed by observed choices (Goeree and Louis, 2021, p. 4004). By being set-valued, it means that M equilibrium is not a fixed point like QRE or NBE, but a set defined by a group of finite equalities and inequalities. As a result, M equilibrium only applies weak conditions to the data, allowing a certain baseline rooted in key empirical observations to be formed, in which further restrictions can be developed (Goeree and Louis, 2021, p. 4004). In the event that the M equilibrium is not predictive of empirical outcomes, it contributes significantly by highlighting possible deviations from economic norms in the observed behaviour (Goeree and Louis, 2021). Given M equilibrium's weak restrictions linked specifically to monotonicity and consequential unbiasedness, failure of M equilibrium points to possible violations of these established norms allowing for further investigation into other possible behavioural explanations of the observed behaviour.

3.2.1 Model Formulation

Any finite normal-form game Γ is denoted by $\Gamma = (N, \{S_i, \Pi_i\}_{i \in N})$ where $N = \{1, \ldots, n\}$ is the set of players. The set of player i's pure strategies is denoted as $S_i = \{s_{i1}, \ldots, s_{iK_i}\}$ and $S = \prod_{i=1}^{n} S_i$ is the Cartesian product of all players' strategy sets giving the set of all possible strategy profiles. The payoff function $\Pi_i : S \to \mathbb{R}$ maps possible strategy profiles to some real-valued payoff for player i. The set of probability distributions over S_i is denoted as Σ_i and subsequently the set of all possible mixed strategy profiles is $\Sigma = \prod_{i=1}^{n} \Sigma_i$. In order to extend this model to include individual beliefs, $\Omega_i = \prod_{j \neq i} \Sigma_j$ is introduced which represents player i's set of beliefs about all other players' probability distributions over their pure strategies. Additionally, a player is assumed to form beliefs about their opponents independently. Define the set of all players' beliefs as $\Omega = \prod_{i \in N}^{n} \Omega_i$. The payoff function Π_i is therefore, extended to include player i's probability distributions Σ_i and beliefs Ω_i. For $(\sigma_i, \omega_i) \in \Sigma_i \times \Omega_i$, player i's expected payoff is now $\sum_{k=1}^{K_i} \sigma_{ik} \pi_{ik}(\omega_i)$ in which σ_{ik} is the probability player i chooses strategy k and $\pi_{ik}(\omega_i) = \sum_{s_{-i} \in S_{-i}} p_i(s_{-i}) \Pi_i(s_k, s_{-i})$ is the expected payoff of s_{ik} with $p_i(s_{-i}) = \prod_{j \neq i} \omega_{ij}(s_j)$. In the circumstance where player i's beliefs are correct ($\omega_i = \sigma_{-i}$), player i's vector of expected payoffs is denoted as $\pi_i(\sigma_{-i})$. Lastly, Σ_{int} is defined as the set of completely mixed strategies and $\overline{A'}$ and $\overline{B'}$ are the closures of $A' \subset \Sigma$ and $B' \subset \Omega$ respectively, relative to both Σ and Ω.

An M equilibrium of a game Γ is defined as $M = \overline{M^c} \times \overline{M^b} \subseteq \Sigma \times \Omega$, such that $\overline{M^c}$ and $\overline{M^b}$ are the closures of maximal sets $M^c \subseteq \Sigma_{int}$ and $M^b \subseteq \Omega$ that satisfy the following two conditions:

[M1] $\pi_{ij}(\omega_i) < \pi_{ik}(\omega_i) \Rightarrow \sigma_{ij} < \sigma_{ik}$

[M2] $\pi_{ij}(\omega_i) < \pi_{ik}(\omega_i) \Leftrightarrow \pi_{ij}(\sigma_{-i}) < \pi_{ik}(\sigma_{-i})$

for all $i \in N, 1 \leqslant j, k \leqslant K_i, \sigma \in M^c, \omega \in M^b$. The set of all M equilibria for a given game Γ is subsequently denoted as $\mathcal{M}(\Gamma)$.

The concept of colourability is introduced by Goeree and Louis (2021), such that an M equilibrium is considered colourable if the relationship between expected utility based on player i's beliefs ($\pi_{ij}(\omega_i)$) and player i's strategy (σ_{ij}) is strengthened such that for any action k, $\pi_{ij}(\omega_i) < \pi_{ik}(\omega_i)$ if and only if $\sigma_{ij} < \sigma_{ik}$. This means that there exist no equilibria in the colourable M equilibria set such that $\sigma_{ij} \neq \sigma_{ik}$ when $\pi_{ij}(\omega_i) = \pi_{ik}(\omega_i)$. Here, denote the set of all colourable M equilibria for a game Γ as $\mathcal{M}_c(\Gamma)$.

3.2.2 M Equilibrium Example

In order to further exhibit the base model of M equilibrium, the 2×2 normal form game of asymmetric matching pennies is used as previously discussed in both QRE and NBE sections. The payoff structure for the game is as follows:

	H	T
H	(2,0)	(0,1)
T	(0,1)	(1,0)

Table 11: Asymmetric Matching Pennies Game

Denote σ_1 and σ_2 as the probability player 1 and player 2 choose heads, respectively. Subsequently, player 1 and player 2's beliefs of their opponent's probability of choosing heads are denoted as ω_1 and ω_2 respectively. Each player's ordered choice probabilities can be observed in Figure 6a. The ordered expected payoffs informed by each players' beliefs are presented in Figure 6b, where $\pi_{ij}(\omega_i)$ is the expected payoff of player i for action j when holding the belief ω_i. Notice that at $\omega_1 = \frac{1}{3}$ player 1 is indifferent between each action and similarly at $\omega_2 = \frac{1}{2}$, player 2 is indifferent between expected payoffs of each action. These indifference curves divide the unit square into four distinct areas where there exist strict orderings of expected payoffs. Now to ensure the ordering of choice probabilities matches that of expected payoffs, Figure 6b can be superimposed onto Figure 6a to obtain the M choice set seen in Figure 6c. This set of possible choices is supported by the set of beliefs highlighted in Figure 6d.

3.2.3 μ Equilibrium

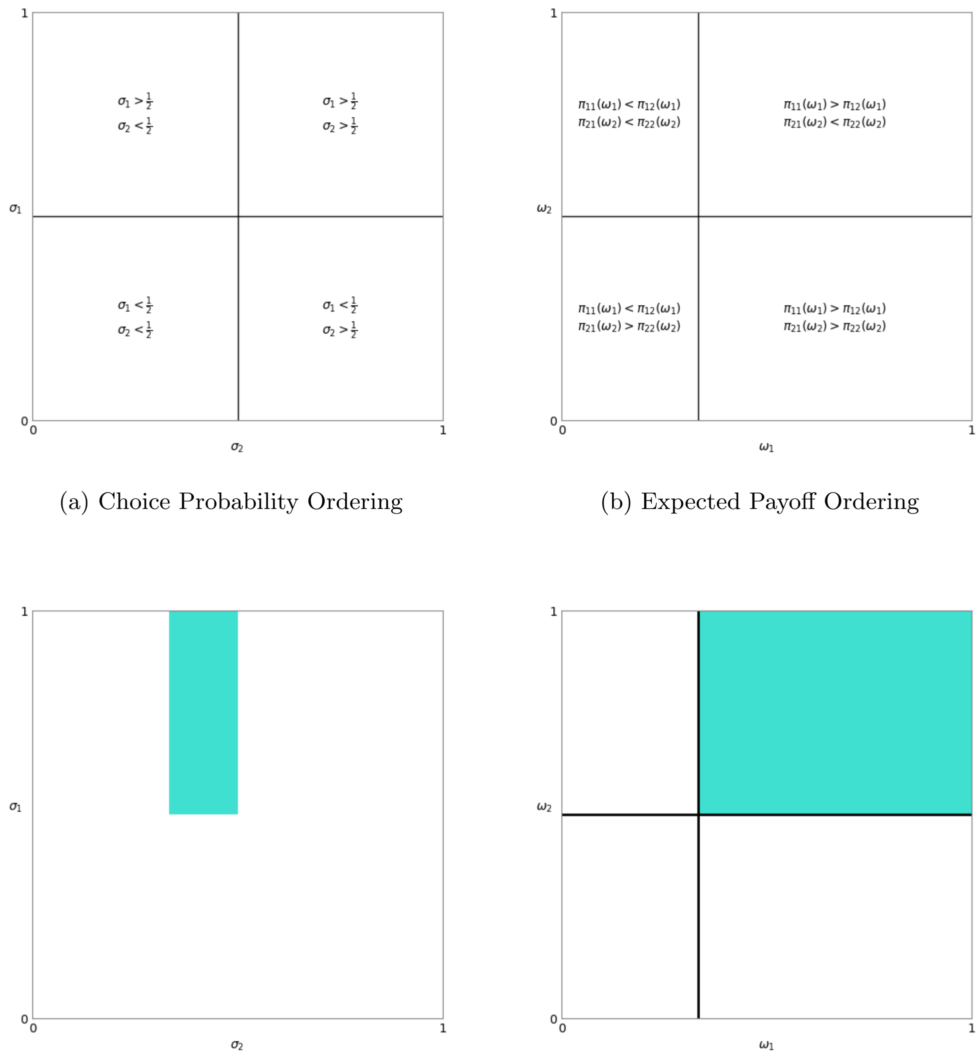
In order to provide a parametric fixed-point model to which data can be fitted to and subsequently used for out of sample predictions, Goeree and Louis (2021) establish μ Equilibrium. They base this fixed-point model on rank correspondences which I will briefly discuss prior to introducing the μ Equilibrium model.

Rank Correspondences

To begin, for each player $i \in N$, let $V_i(\mu_i)$ denote the set of all points in player i's set of probability distributions over their pure strategies Σ_i that result from permuting the entries in $\mu_i \in \Sigma_i$. Strategy μ_i is defined as regular if it is completely mixed and all entries are different. If μ_i is regular then the convex hull of the set $V_i(\mu_i)$, $P_i(\mu_i) = \text{co}(V_i(\mu_i))$, is the permutohedron generated by μ_i. Goeree and Louis (2021, p. 4013) define player i's rank correspondences as:

$$rank_i^{\mu_i}(\pi_i(\sigma_{-i})) = \text{co}\left(\{\sigma_i \in V_i(\mu_i) \mid \pi_{ij}(\sigma_{-i}) < \pi_{ik}(\sigma_{-i}) \Rightarrow \sigma_{ij} < \sigma_{ik} \text{ for } 1 \leqslant j,k \leqslant K_i\}\right).$$

Given that M equilibrium is dependent on comparing the ranking of expected payoffs



(a) Choice Probability Ordering

(b) Expected Payoff Ordering

(c) M Choice Set

(d) M Belief Set

Figure 6: M Equilibrium Asymmetric Matching Pennies

informed by some set of beliefs to the ranking of expected payoffs informed by a player's own actions, this can also be shown through the following relationship of rank correspondences:

$$rank_i^{\mu_i}(\pi_i(\sigma_{-i})) = rank_i^{\mu_i}(\pi_i(\omega_i)) \ \forall i \in N \text{ and regular } \mu_i.$$

Goeree and Louis (2021, p. 4013) note that "if this equality holds for some regular μ_i it holds for all regular μ_i" and therefore define $rank_i$ as player i's rank correspondence for the regular strategy where

$$\mu_i(\varepsilon) = \frac{(1-\varepsilon)}{(1-\varepsilon^{K_i})}(1, \varepsilon, \varepsilon^2, \dots, \varepsilon^{K_i - 1}).$$

for some fixed $\varepsilon \in (0, 1)$. Here we can see that ε acts as a rationality parameter, similar to λ and τ in QRE and NBE respectively. Where as $\varepsilon \to 0$, player i's rank correspondence will approach their standard best responses while as $\varepsilon \to 1$, player i's rank correspondence will place equal weighting on all possible strategies randomising between them. The Cartesian product of all players' rank correspondences is defined as $rank = \prod_{i \in N} rank_i$ and $\mathcal{P} = \prod_{i \in N} \mathcal{P}_i$ is the Cartesian product of the sets of faces of the permutohedrons ($P_i(\mu_i)$) as previously defined. Furthermore, Goeree and Louis (2021) prove that for $r \in \mathcal{P}$ if the closure of the set M_r is non-empty then it is an M equilibrium of Γ, where M_r is defined as:

$$M_r = \{(\sigma, \omega) \in \Sigma_{int} \times \Omega \mid rank(\sigma) \subseteq rank(\pi(\omega)) = rank(\pi(\sigma)) = r\}.$$

Parametric Model

This model differs significantly to those previously presented for QRE and NBE in that it is not defined in terms of functions that must satisfy specific axioms.[15] Now that rank correspondences have been reviewed, I introduce μ equilibrium, the fixed-point parametric model outlined by Goeree and Louis (2021). First, define the concatenations of μ_i and $rank_i^{\mu}$ as μ and $rank^{\mu}$ respectively. Now for some $\mu \in \Sigma$, the pair (σ, M^b) with $\sigma \in \Sigma$ and $M^b \subseteq \Omega$ will be defined as a μ equilibrium of Γ, if $\sigma \in rank^{\mu}(\pi(\omega)) = rank^{\mu}(\pi(\sigma))\ \forall \omega \in M^b$. Additionally, for some regular μ, $rank \circ rank^{\mu} = rank$ and when applying $rank$ to the definition of μ equilibrium, any $(\sigma, \omega) \in \Sigma_{int} \times \Omega$ therefore, belongs to an M equilibrium of some game Γ, if and only if $rank(\sigma) \subseteq rank(\pi(\omega)) = rank(\pi(\sigma))$ (Goeree and Louis, 2021). The set of all μ equilibria of Γ is denoted as $E_{\mu}(\Gamma)$.

Following with the same example of asymmetric matching pennies (Table 1), denote σ_1 and σ_2 as the probability of player 1 and player 2 choosing H, respectively. Assume that both players have the same rationality parameter such that $\mu_1 = \mu_2 = (1, \varepsilon)/(1 + \varepsilon)$. Given the payoff structure of the asymmetric matching pennies game, we get the following rank correspondences for any given $\varepsilon \in [0, 1]$ for player 1 and

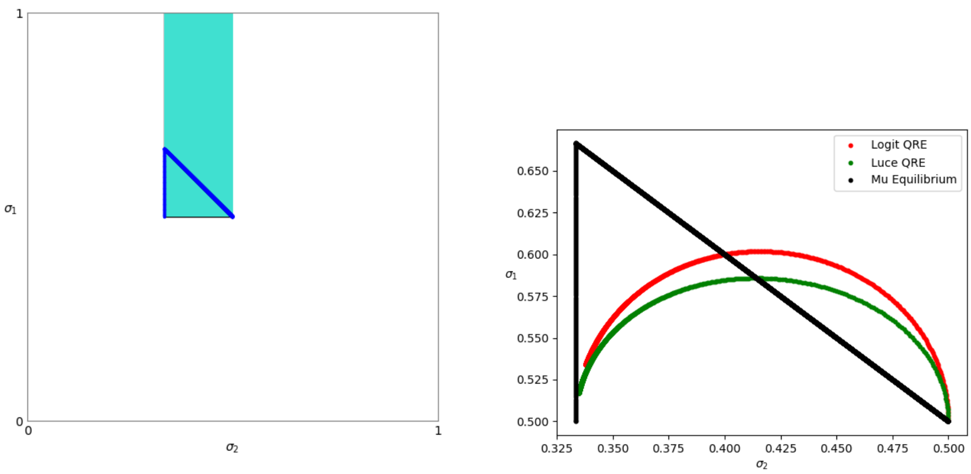
[15] As a result of this, μ-equilibrium is computationally faster to solve for.

2:[16]

$$rank_1^{\varepsilon}(\sigma_2) = \begin{cases} \frac{1}{1+\varepsilon} & \text{if } \sigma_2 > \frac{1}{3} \\ [\frac{1}{1+\varepsilon}, \frac{\varepsilon}{1+\varepsilon}] & \text{if } \sigma_2 = \frac{1}{3} \\ \frac{\varepsilon}{1+\varepsilon} & \text{if } \sigma_2 < \frac{1}{3} \end{cases}$$

$$rank_2^{\varepsilon}(\sigma_1) = \begin{cases} \frac{1}{1+\varepsilon} & \text{if } \sigma_1 < \frac{1}{2} \\ [\frac{1}{1+\varepsilon}, \frac{\varepsilon}{1+\varepsilon}] & \text{if } \sigma_1 = \frac{1}{2} \\ \frac{\varepsilon}{1+\varepsilon} & \text{if } \sigma_1 > \frac{1}{2} \end{cases}$$

Here it becomes more obvious that as $\varepsilon \to 0$, both players tend towards their best responses and as $\varepsilon \to 1$ they tend towards completely randomising between their possible actions. To further exhibit the concept, if we take $\sigma' = (\sigma_1', \sigma_2') = (\frac{3}{5}, \frac{1}{3})$ and set $\varepsilon = \frac{1}{2}$ we get the following rank correspondences: $rank_1^{\varepsilon}(\frac{1}{3}) = [\frac{1}{3}, \frac{2}{3}]$ and $rank_2^{\varepsilon}(\frac{3}{5}) = \frac{1}{3}$. Now to ensure the μ equilibrium condition, we must observe the initial value pair σ' within the respective rank functions. In this specific instance, $\sigma_1' \in rank_1^{\varepsilon}(\sigma_2') = [\frac{2}{3}, \frac{1}{3}]$ and $\sigma_2' \in rank_2^{\varepsilon}(\sigma_1') = \frac{1}{3}$, therefore $\sigma' \in E_{\mu}(\Gamma)$. The set of all μ equilibria for the asymmetric matching pennies game for $\varepsilon \in [0, 1]$ is plotted below in Figure 7a, while the logit and Luce QRE of the same game for $\lambda \in (0, 15]$ are plotted on top of this in Figure 7b.



(a) Mu Equilibrium in M Choice Set (b) Mu Equilibrium, Logit and Luce QRE

Figure 7: Mu Equilibrium Asymmetric Matching Pennies

This parametric model presents as a piecewise linear function, compared to that

[16]It should be noted that while the input to these rank correspondences are the expected payoffs for each player, given the expected payoffs are purely determined by the strategy of the other player, for simplicity I follow that of Goeree and Louis (2021) and only focus on the respective strategies of each player when dealing with rank correspondences in this example.

of both the logit and Luce QRE models as well as the logit transform NBE which generally exhibit curves.[17] You can see that the μ equilibrium falls within the M-choice sets as previously formed for the asymmetric matching pennies game (Figure 7a). Additionally, the logit and Luce QRE also fall within the M-choice set; a connection I will now discuss.

3.2.4 Relationship to QRE

As previously mentioned, the motivations for M equilibrium arise from critiques of QRE and critiques of a general focus on players' actions rather than their beliefs. Providing a set-based approach enables M equilibrium to focus on foundational behavioural economic assumptions and to act as a sort of meta theory, encapsulating other equilibrium models. Most notably, Goeree and Louis (2021, Proposition 3) highlight the relationship between regular QRE and M equilibrium. For ease of reference, I will henceforth refer to regular QRE simply as QRE, unless otherwise stated. Goeree and Louis (2021, Proposition 3) show and prove that for almost all $\sigma \in \Sigma$, σ belongs to a colourable M-choice set of Γ iff it is also a QRE of Γ. For generic games σ belongs to an M-choice set of Γ iff it also belongs to a colourable M-choice set iff it is a μ equilibrium for some regular $\mu \in \Sigma$, and iff it is a QRE for some quantal response function.

3.2.5 Relationship to Nash equilibrium

The relationship between Nash equilibrium and M equilibrium is explored by Goeree and Louis (2021) through their discussion of proper equilibria and subsequently, proper Nash equilibria. Proper equilibrium is a refinement of Nash equilibrium by Myerson (1978), which builds on the work by Selten (1975) of ε-perfect equilibrium. I will briefly discuss proper equilibrium adapting notation to suit that of M equilibrium, for a formal review of this and consequently ε-proper equilibrium see Myerson (1978). An ε-proper equilibrium is any $\sigma \in \Sigma_{int}$ such that for some significantly small $\varepsilon > 0$, $\pi_{ij}(\sigma_{-i}) < \pi_{ik}(\sigma_{-i}) \Rightarrow \sigma_{ij} \leqslant \varepsilon\sigma_{ik}$ for any $j, k = 1, \ldots, K_i$ and $j \neq k$. Myerson (1978) then defines a proper equilibrium as any limit of ε-proper equilibrium, specifically $\sigma \in \Sigma_{int}$ is a proper equilibrium if and only if there exist sequences $\{\varepsilon_m\}_{m=0}^{\infty}$ and $\{(\sigma_1^m, \ldots \sigma_n^m)\}_{m=0}^{\infty}$ such that each $\varepsilon_m > 0$ and $\lim_{m\to\infty} \varepsilon_m = 0$, each $(\sigma_1^m, \ldots, \sigma_n^m)$

[17] When modelling certain games, notably those with only completely pure strategy Nash equilibria, the logit and Luce QRE models appear linear, although this is merely an observation and no formal proof is provided in this paper. The logit transform NBE will simply provide the pure strategy Nash equilibria.

is an ε_m-proper equilibrium and $\lim_{m\to\infty} \sigma_{ik}^m = \sigma_{ik}$ $\forall i$ and $\forall k = 1, \ldots, K_i$. Goeree and Louis (2021, Proposition 3) prove that almost all equilibria in the M choice set are also ε-proper equilibria. Given that a proper equilibrium is the limit of ε-proper equilibria, as the M choice set is closed, it therefore includes proper equilibria and consequently proper Nash equilibria (Goeree and Louis, 2021). Goeree and Louis (2021) briefly mention that some non-proper Nash equilibria can be included within the M choice set, however, they do not directly investigate the relationship between M equilibrium and non-proper Nash equilibria. In this section I aim to provide further clarity to the relationship between M equilibrium and Nash equilibrium constraining my analysis to that of generic 2×2 games. Here, I expand on the concept of M equilibrium as a meta theory and relate the attainable Nash equilibria to $\overline{M^c}$. From simple visual inspection it is sometimes obvious that Nash equilibria are often contained within $\overline{M^c}$. Take for example, $\overline{M^c}$ seen in Figure 6c, the Nash equilibrium of this game is $(\sigma_1, \sigma_2) = (1/2, 1/3)$ and located in the bottom left corner of the M choice set. A similar exercise can be conducted with other generic 2×2 games. There are, however, some instances in which some of the Nash equilibria of a given generic 2×2 game are not contained within the M choice set. All these equilibria happen to also be non-proper Nash equilibria, however, this cannot be used as a general exception rule as there exist cases where non-proper Nash equilibria are in $\overline{M^c}$. To specifically highlight instances where not all possible Nash equilibria are included within $\overline{M^c}$, I define a set of Nash problem games which is made up of generalised game structures in which not all Nash equilibria are in $\overline{M^c}$.

Definition 2 (Nash Problem Game)**.** A game is defined as a Nash problem game if the game structure follows one of the following generalised structures:

- player i and player k both have weakly dominant strategies
- player i has a strictly dominant strategy and player k has a weakly dominant strategy such that player k's higher payoff does not occur when player i plays their dominant strategy
- player i has a weakly dominant strategy and player k has a matching pennies type structure

I denote the set of problem games by Γ^{NP} and generalised examples of these can be found in the Tables below:

	L	R
U	a_L, b_U	a_R, b_U
D	a_L, b_D	$a_R + c_R, b_D + d_D$

(a)

	L	R
U	a_L, b_U	$a_R, b_U + d_U$
D	a_L, b_D	$a_R + c_R, b_D$

(b)

	L	R
U	a_L, b_U	$a_R, b_U + d_U$
D	$a_L + c_L, b_D$	$a_R + c_R, b_D$

(c)

	L	R
U	a_L, b_U	$a_R, b_U + d_U$
D	$a_L, b_D + d_D$	$a_R + c_R, b_D$

(d)

	L	R
U	$a_L, b_U + d_U$	a_R, b_U
D	a_L, b_D	$a_R + c_R, b_D + d_D$

(e)

Table 12: Set of Nash Problem Games

Proposition 4. *Denote the set of generic* 2×2 *games* $\Gamma^{2\times 2}$*. For any* $\Gamma \in \Gamma^{2\times 2} \backslash \Gamma^{NP}$*, the following relationship is true:* $NE(\Gamma) \subset \overline{M^c}$*, where* $NE(\Gamma)$ *is the set of all Nash equilibria of* Γ *and* $\overline{M^c}$ *is the M choice set of some M equilibrium of* Γ*.*

Proof. This proof is lengthy and involves analysis of several generalised cases of games. It takes a similar approach to the proof for Proposition 2 where I draw on generalised versions of different game structures. In this instance, to find all possible Nash equilibria for any generic 2×2 game that are contained within $\overline{M^c}$. In total I will examine 10 different generalised structures of games. To assist with the proof, visual representations of each game have been produced. Within the belief set figure, the thick black lines represent a player's indifference between the expected payoff of two strategies. For the second figure, the blue rectangles represent the relevant $\overline{M^c}$ of the given generalised game structure, while the red points and lines represent the corresponding Nash equilibria of the given game. The first two cases, generalised matching pennies and generalised chicken, have previously been introduced.

Case (i)

For generalised matching pennies the unique Nash equilibrium will always be interior, therefore, simply checking that if this point meets the M equilibrium criteria will suffice to establish that it is always included in $\overline{M^c}$. Out of ease for the reader the definition of an M equilibrium is restated:

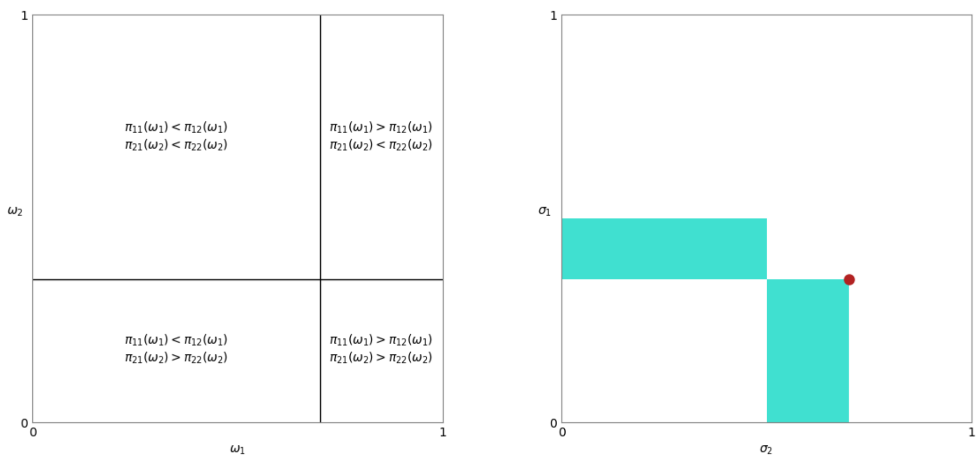
An M equilibrium of a game Γ is defined as $M = \overline{M^c} \times \overline{M^b} \subseteq \Sigma \times \Omega$, such that $\overline{M^c}$ and $\overline{M^b}$ are the closures of maximal sets $M^c \subseteq \Sigma_{int}$ and $M^b \subseteq \Omega$ that satisfy the following two conditions:

[M1] $\pi_{ij}(\omega_i) < \pi_{ik}(\omega_i) \Rightarrow \sigma_{ij} < \sigma_{ik}$

[M2] $\pi_{ij}(\omega_i) < \pi_{ik}(\omega_i) \Leftrightarrow \pi_{ij}(\sigma_{-i}) < \pi_{ik}(\sigma_{-i})$

for all $i \in N, 1 \leqslant j, k \leqslant K_i, \sigma \in M^c, \omega \in M^b$. The set of all M equilibria for a given game Γ is subsequently denoted as $\mathcal{M}(\Gamma)$.

We know that the unique Nash equilibrium point in the structure will be completely mixed, therefore we only need to check that the Nash equilibrium conditions satisfy [M1] and [M2]. Denote σ_1 and σ_2 as the probability assigned to action U and action L by player 1 and player 2, respectively. The unique Nash equilibrium point (σ_1, σ_2) is supported by the singleton belief set that gives $\pi_{11}(\omega_1) = \pi_{12}(\omega_1)$ and $\pi_{21}(\omega_2) = \pi_{22}(\omega_2)$ where given the Nash equilibrium is on the interior and players are indifferent we subsequently have $\pi_{11}(\sigma_2) = \pi_{12}(\sigma_2)$ and $\pi_{21}(\sigma_1) = \pi_{22}(\sigma_2)$ satisfying [M2]. Given that expected payoffs informed by beliefs are equal, players can mix between their actions and [M1] is satisfied, hence the unique completely mixed Nash equilibrium is contained within $\overline{M^c}$.



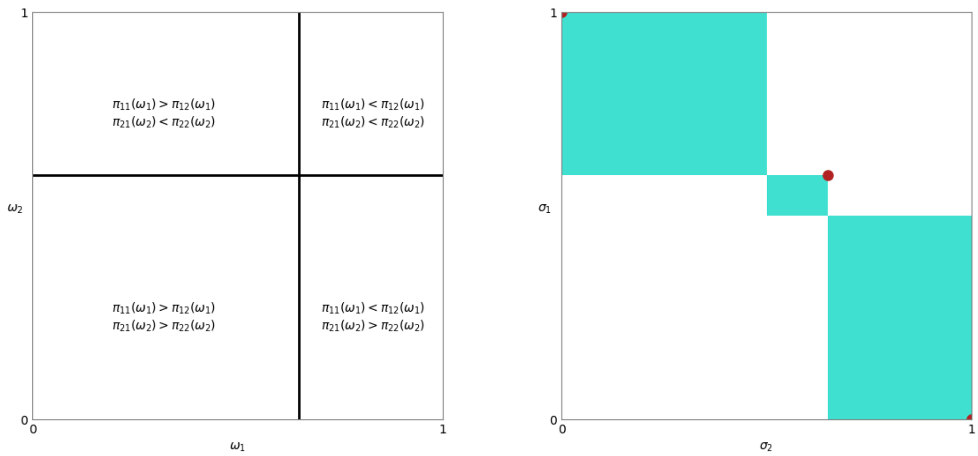
(a) M Belief Set

(b) M Choice Set and Nash Equilibria

Figure 8: Case i

Case (ii)

For generalised chicken, the proof for the mixed strategy Nash equilibrium being included in $\overline{M^c}$ follows that of case (i). I now turn attention to the instances of pure strategy Nash equilibria. This is trickier as $\overline{M^c}$ takes the closure of the maximal set of interior strategy profiles that satisfy [M1] and [M2], thus to prove the pure strategy Nash equilibria are included in $\overline{M^c}$ we must show that the interior strategy profiles that satisfy [M1] and [M2] approach the pure strategy Nash equilibria. If we take the pure strategy Nash equilibria in the generalised chicken game, $(\sigma_1, \sigma_2) = (0, 1)$ and $(1, 0)$, and perturb these by some significantly small $\varepsilon > 0$ then we obtain mixed strategy profiles at the limits of Σ_{int}. It should be clarified that ε is chosen to be small enough that each player's perturbed strategy preserves the optimality of the other player's Nash equilibrium strategy. If these mixed strategy profiles combined with some belief profile satisfy [M1] and [M2], then when the closure of $M^c \subseteq \Sigma_{int}$ and $M^b \subseteq \Omega$ is taken it will include the Nash equilibria. The first pure strategy Nash equilibrium occurs when $\pi_{12}(\sigma_2) > \pi_{11}(\sigma_2)$ and $\pi_{21}(\sigma_1) > \pi_{22}(\sigma_1)$. By perturbing these points we get $(\sigma_1, \sigma_2) = (\varepsilon, 1 - \varepsilon)$. As both players are indifferent only on the interior, the perturbations will not impact expected payoff orderings and [M1] will be satisfied. As players' beliefs are informed by their indifference conditions, we know that payoff ordering based on beliefs will match those based on their opponent's strategy. This is because the indifference conditions for each player occur in the interior. Subsequently, [M2] is satisfied and the pure strategy Nash equilibrium is in $\overline{M^c}$. The same argument follows for the second pure strategy Nash equilibrium.



(a) M Belief Set

(b) M Choice Set and Nash Equilibria

Figure 9: Case ii

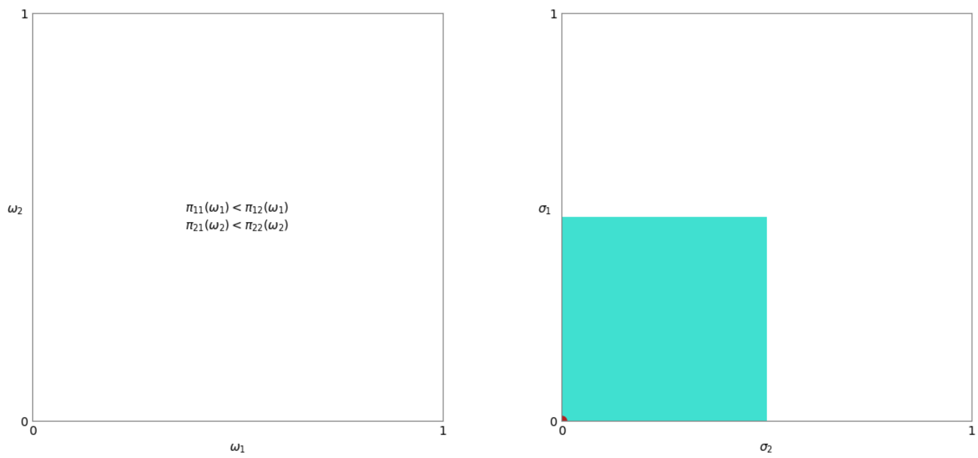
Case (iii)

Attention is now turned to games were both players have strictly dominant strategies. This generalised game is seen below in Table 13. Here, there exists a unique

	L	R
U	a_L, b_U	$a_R, b_U + d_U$
D	$a_L + c_L, b_D$	$a_R + c_R, b_D + d_D$

Table 13

pure strategy Nash equilibrium at $(\sigma_1, \sigma_2) = (0, 0)$. Within this game structure neither player is indifferent at any point so we know their expected payoffs as informed by their beliefs will always be $\pi_{11}(\omega_1) < \pi_{12}(\omega_1)$ and $\pi_{21}(\omega_2) < \pi_{22}(\omega_2)$. These beliefs will span the entire strategy space and therefore will support one quadrant of the strategy space. By perturbing the pure strategy Nash equilibrium by $\varepsilon > 0$, expected payoff orderings will remain unchanged. As the ordering of strategies is also unchanged by the perturbations, [M1] is satisfied. Given that the expected payoff ordering informed by strategies is unchanged and the expected payoff ordering informed by beliefs is constant for any section of the belief space, [M2] is satisfied. Subsequently, the unique pure strategy Nash equilibrium will be included in the closure of M^c.



(a) M Belief Set

(b) M Choice Set and Nash Equilibria

Figure 10: Case iii

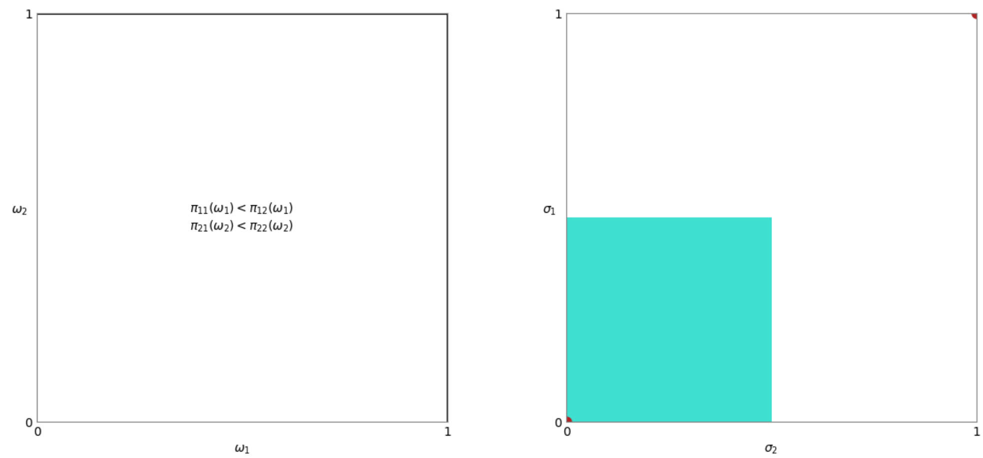
Case (iv)

I now examine the case where both players have a weakly dominant strategy generalised below in Table 14. Here each player is indifferent between their two strategies on the boundaries (i.e., when $\sigma_1 = 1$ and $\sigma_2 = 1$). Within this game

	L	R
U	a_L, b_U	a_R, b_U
D	a_L, b_D	$a_R + c_R, b_D + d_D$

Table 14

there exists two pure strategy Nash equilibria at $(\sigma_1, \sigma_2) = (0, 0)$ and $(1, 1)$. Due to indifference at the boundaries, we get $\pi_{11}(\omega_1) < \pi_{12}(\omega_1)$ and $\pi_{21}(\omega_2) < \pi_{22}(\omega_2)$ across the entire interior belief space. The strategies these beliefs support are seen in the bottom left quadrant of Figure 11b. It is easy to see that any perturbation of the second pure strategy Nash equilibrium will not satisfy [M1] and therefore, this Nash equilibrium is excluded from $\overline{M^c}$. Perturbing the other pure strategy Nash equilibrium to $(\sigma_1, \sigma_2) = (\varepsilon, \varepsilon)$, will still satisfy [M1] as payoff orderings informed by beliefs will match that of strategy orderings. Additionally, [M2] is satisfied as $\pi_{11}(\varepsilon) < \pi_{12}(\varepsilon)$ and $\pi_{21}(\varepsilon) < \pi_{22}(\varepsilon)$. As a result, only one of the pure strategy Nash equilibria is included in $\overline{M^c}$. Linking this to the concept of proper equilibrium, the excluded Nash equilibrium is non-proper. In fact, this game structure is a generalised version of the game analysed by Myerson (1978, Figure 1) in his Figure 1.



(a) M Belief Set

(b) M Choice Set and Nash Equilibria

Figure 11: Case iv

Case (v)

Similar to *case (iv)*, the generalised game where both players have a weakly dominant strategy as defined in Table 15 is examined. For this game we again have indifference at the boundaries for both players, however, the game structure now al-

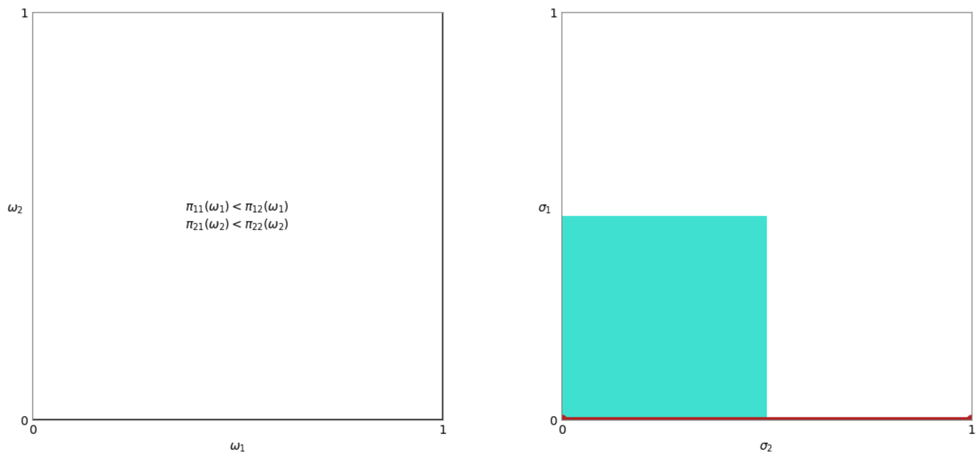
	L	R
U	a_L, b_U	$a_R, b_U + d_U$
D	a_L, b_D	$a_R + c_R, b_D$

Table 15

lows for pure-mixed strategy Nash equilibria. The game has two pure strategy Nash equilibria at $(\sigma_1, \sigma_2) = (0, 1)$ and $(0, 0)$. The pure-mixed strategy Nash equilibria occur at $(\sigma_1, \sigma_2) = (0, x),\ x \in (0, 1)$. The orderings of expected payoffs informed by beliefs are: $\pi_{11}(\omega_1) < \pi_{12}(\omega_1)$ and $\pi_{21}(\omega_2) < \pi_{22}(\omega_2)$ for the entire interior belief set. Perturbing the first pure strategy Nash equilibrium to $(\sigma_1, \sigma_2) = (\varepsilon, 1-\varepsilon)$ it is easy to see that the strategy ordering does not match the expected payoff orderings informed by beliefs and as a result this pure strategy Nash equilibrium is not in $\overline{M^c}$. For the other pure strategy Nash equilibrium, perturbing this to $(\sigma_1, \sigma_2) = (\varepsilon, \varepsilon)$ we can see that the strategy orderings match the expected payoff orderings informed by beliefs satisfying [M1]. Additionally, for $(\sigma_1, \sigma_2) = (\varepsilon, \varepsilon)$ we have the ordering of expected payoffs informed by beliefs match those informed by strategies, satisfying [M2] and therefore meaning this pure strategy Nash equilibrium is in $\overline{M^c}$. For the pure-mixed strategy Nash equilibria, when $\sigma_2 \geqslant 0.5$, the strategy ordering and expected payoff ordering informed by beliefs for player 2 do not satisfy [M1]. For $\sigma_2 < 0.5$, when perturbing player 1's strategy such that $(\sigma_1, \sigma_2) = (\varepsilon, x),\ x \in (0, 0.5)$ [M1] is satisfied and subsequently [M2] is also satisfied as orderings of expected payoffs informed by beliefs and informed by strategies match. As a result, the pure-mixed strategy Nash equilibria $(\sigma_1, \sigma_2) = (0, x),\ x \in (0, 0.5)$ are in $\overline{M^c}$. An interesting observation linked to non-proper Nash equilibria is that these are both excluded and included in $\overline{M^c}$, given this no clear distinction can be drawn between proper equilibria and those excluded from $\overline{M^c}$, other than that if a Nash equilibrium is excluded it will be non-proper. Similar instances of this occur in other cases analysed within this proof, however, given the inability to make a clear distinction this is left for potential future investigation.

Case (vi)

I now examine the game where one player has a strictly dominant strategy while the other has a weakly dominant strategy. The generalisation of this game can be seen in Table 16. Within this game there exists a unique pure strategy Nash equilibrium at $(\sigma_1, \sigma_2) = (0, 0)$. Player 1 is never indifferent and player 2 is indifferent only on the boundary when $\sigma_1 = 1$. Expected payoffs informed by beliefs are ordered as



(a) M Belief Set

(b) M Choice Set and Nash Equilibria

Figure 12: Case v

	L	R
U	a_L, b_U	a_R, b_U
D	$a_L + c_L, b_D$	$a_R + c_R, b_D + d_D$

Table 16

follows for the entire interior belief set: $\pi_{11}(\omega_1) < \pi_{12}(\omega_1)$ and $\pi_{21}(\omega_2) < \pi_{22}(\omega_2)$. By perturbing the Nash equilibrium to $(\sigma_1, \sigma_2) = (\varepsilon, \varepsilon)$, strategy orderings still match expected payoff orderings as informed by beliefs satisfying [M1]. Similarly, the expected payoff orderings informed by beliefs match those informed by the perturbed equilibrium and [M2] is satisfied. As a result, the unique pure strategy Nash equilibrium is in $\overline{M^c}$.

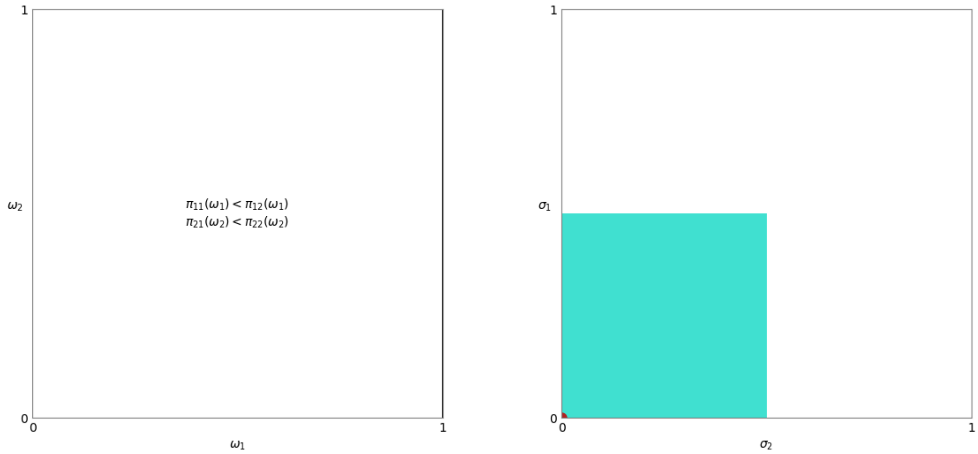
Case (vii)

The complementary case to *case (vi)* is now examined where the weakly dominant strategy does not coincide with the other player's strictly dominant strategy. The generalised form of this game is presented in Table 17. In this game there exist two

	L	R
U	a_L, b_U	$a_R, b_U + d_U$
D	$a_L + c_L, b_D$	$a_R + c_R, b_D$

Table 17

pure strategy Nash equilibria, $(\sigma_1, \sigma_2) = (0, 1)$ and $(0, 0)$, and an infinite number of pure-mixed strategy Nash equilibria $(\sigma_1, \sigma_2) = (0, x),\ x \in (0, 1)$. Player 1 is never indifferent and player 2 is indifferent on the boundary when $\sigma_1 = 0$. From visual



(a) M Belief Set

(b) M Choice Set and Nash Equilibria

Figure 13: Case vi

inspection and examples in the other cases presented thus far, it is obvious that a large set of Nash equilibria will be excluded from $\overline{M^c}$. Specifically, all equilibria where $\sigma_2 \in (0.5, 1]$ this is because by perturbing σ_1 such that $\sigma_1 = \varepsilon$, the expected payoff ordering informed by ω_2 will not match the ordering of player 2's strategies, therefore not satisfying [M1]. For $\sigma_2 \in [0, 0.5]$ the strategy ordering will match that of expected payoffs informed by ω_2. It is obvious that perturbing player 1's strategy will leave orderings unchanged, as a result [M1] is satisfied. For player 1 it follows that [M2] is satisfied for any $\sigma_2 \in [0, 1]$ and for player 2, when $\sigma_1 = \varepsilon$ expected payoff orderings hold between strategies and beliefs, $\omega_2 \in [0, 0.5]$ and [M2] is satisfied. As a result, the Nash equilibria $(\sigma_1, \sigma_2) = (0, x),\ x \in [0, 0.5]$ are included in $\overline{M^c}$.

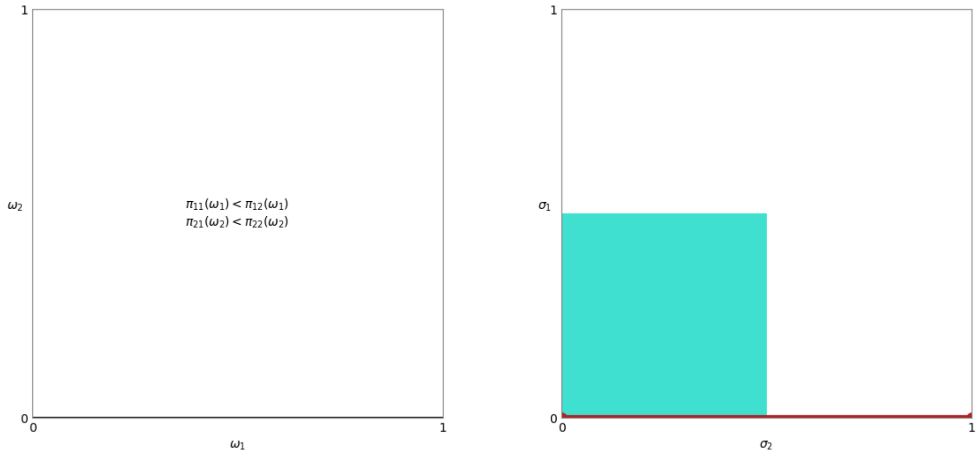
Case (viii)

The case where only one player has a strictly dominant strategy and the other player has no weakly dominant strategy is now examined. The generalised form of this game is seen in Table 18. Here, while player 1 is never indifferent, player 2 is

	L	R
U	$a_L, b_U + d_U$	a_R, b_U
D	$a_L + c_L, b_D$	$a_R + c_R, b_D + d_D$

Table 18

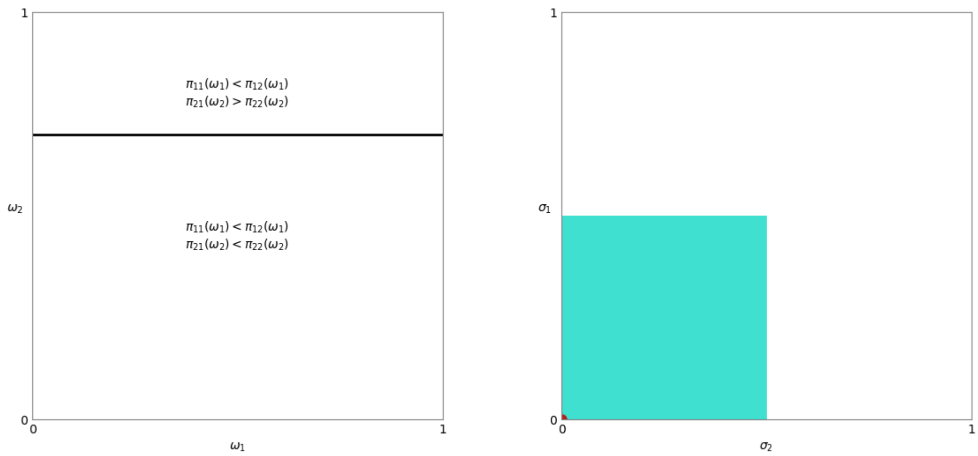
indifferent on the interior when $\sigma_1 = d_D/(d_D + d_U)$. This game has a unique pure strategy Nash equilibrium at $(\sigma_1, \sigma_2) = (0, 0)$. For the entire belief set we get the



(a) M Belief Set (b) M Choice Set and Nash Equilibria

Figure 14: Case vii

following ordering of expected payoffs informed by beliefs for player 1: $\pi_{11}(\omega_1) < \pi_{12}(\omega_1)$. For player 2 there exist both orderings, where for $\omega_2 > d_D/(d_D + d_U)$ we get $\pi_{21}(\omega_2) > \pi_{22}(\omega_2)$ and for $\omega_2 < d_D/(d_D + d_U)$ we get $\pi_{21}(\omega_2) < \pi_{22}(\omega_2)$. Perturbing the Nash equilibrium to $(\sigma_1, \sigma_2) = (\varepsilon, \varepsilon)$ we match player 1's expected payoff ordering informed by beliefs and for $\omega_2 < d_D/(d_D + d_U)$ we match player 2's ordering satisfying [M1]. For $(\sigma_1, \sigma_2) = (\varepsilon, \varepsilon)$ the expected payoff orderings based on strategies match that based on beliefs and [M2] is satisfied. As a result, the unique pure strategy Nash equilibrium is in $\overline{M^c}$.



(a) M Belief Set (b) M Choice Set and Nash Equilibria

Figure 15: Case viii

Case (ix)

In this case I examine the game where only one player has a weakly dominant strategy and the other player has no weakly dominant strategy. The generalised version of this game is seen in Table 19. This game has only one pure strategy

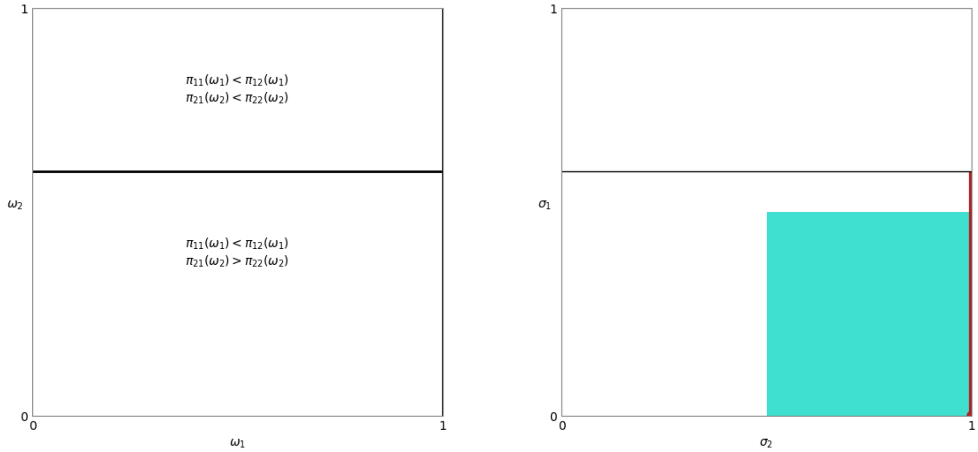
	L	R
U	a_L, b_U	$a_R, b_U + d_U$
D	$a_L, b_D + d_D$	$a_R + c_R, b_D$

Table 19

Nash equilibrium at $(\sigma_1, \sigma_2) = (0, 1)$ with an infinite number of pure-mixed strategy Nash equilibria at $(\sigma_1, \sigma_2) = (x, 1),\ x \in (0, d_D/(d_D + d_U)]$. From the prior cases it should be trivial now that the pure strategy Nash equilibrium is included within $\overline{M^c}$. What is less trivial are the cases of pure-mixed strategy Nash equilibria. From visual inspection there exist some pure-mixed Nash equilibria that will be included in $\overline{M^c}$ while others will not. As player 1 will have a mixed strategy we only need to perturb σ_2 for all possible Nash equilibria values of σ_1. The game produces the following expected payoff orderings informed by beliefs: $\omega_1 < 1,\ \pi_{11}(\omega_1) < \pi_{12}(\omega_1)$ and $\omega_2 < d_D/(d_D + d_U),\ \pi_{21}(\omega_2) > \pi_{22}(\omega_2)$ and $\omega_2 > d_D/(d_D + d_U),\ \pi_{21}(\omega_2) < \pi_{22}(\omega_2)$. By perturbing σ_2 such that $\sigma_2 = 1 - \varepsilon$, player 2's strategy ordering will match their expected payoff ordering up until $\sigma_1 = d_D/(d_D + d_U)$. However, beyond $\sigma_1 = 0.5$ we no longer have a match between player 1's strategy ordering and expected payoff ordering informed ω_1. This means that [M1] is satisfied only for pure-mixed strategy Nash equilibria $(\sigma_1, \sigma_2) = (x, 1),\ x \in (0, 0.5]$. Obviously if $d_D/(d_D + d_U) \leqslant 0.5$ then all pure-mixed strategy Nash equilibria satisfy [M1]. For [M2], it is easy to verify that the orderings of expected payoffs informed by beliefs match those informed by strategies. As a result, we see exclusion of pure-mixed strategy Nash equilibria from $\overline{M^c}$ only where $d_D/(d_D + d_U) > 0.5$.

Case (x)

For the final case, I examine the following game similar to that of *case ix* where one player has a weakly dominant strategy. The generalised structure of this game is seen in Table 20. In this game there exist two pure strategy Nash equilibria at $(\sigma_1, \sigma_2) = (1, 1)$ and $(0, 0)$. There also exist an infinite number of pure-mixed strategy Nash equilibria at $(\sigma_1, \sigma_2) = (x, 1),\ x \in [d_D/(d_D + d_U), 1)$. From previous cases it is now trivial that the pure strategy Nash equilibria $(\sigma_1, \sigma_2) = (0, 0)$ is



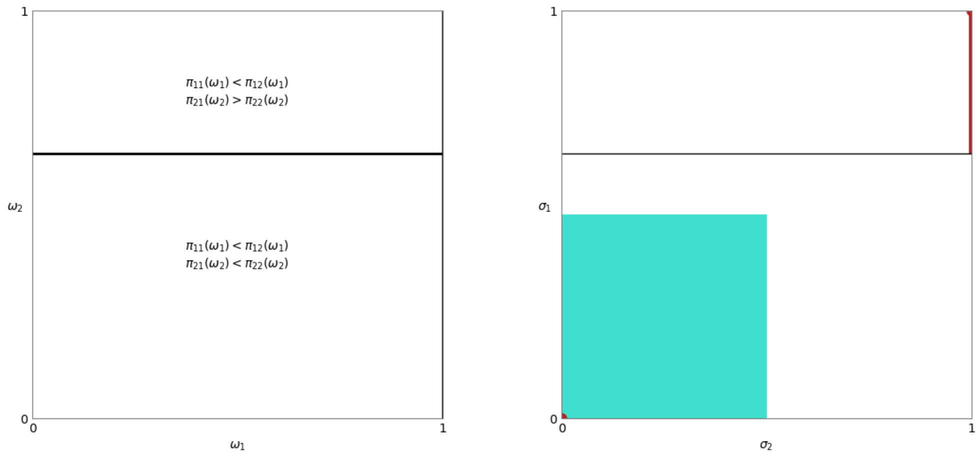
(a) M Belief Set (b) M Choice Set and Nash Equilibria

Figure 16: Case ix

	L	R
U	$a_L, b_U + d_U$	a_R, b_U
D	a_L, b_D	$a_R + c_R, b_D + d_D$

Table 20

contained within $\overline{M^c}$. For the other pure strategy Nash equilibria, perturbing this to $(\sigma_1, \sigma_2) = (1-\varepsilon, 1-\varepsilon)$ clearly results in [M1] not being satisfied. Similarly, for all pure-mixed strategy Nash equilibria, perturbing player 2's strategy such that $(\sigma_1, \sigma_2) = (x, 1-\varepsilon),\ x \in [d_D/(d_D+d_U), 1-\varepsilon]$ also results in [M1] not being satisfied. As a result, only one Nash equilibrium is included in $\overline{M^c}$.



(a) M Belief Set (b) M Choice Set and Nash Equilibria

Figure 17: Case x

□

From Goeree and Louis's (2021) result that all points on the relative interior of $\overline{M^c}$ is also a QRE for a given constructed quantal response function, it is evident that taking the closure of the set of attainable QRE will lead to a similar result as Proposition 4 for QRE.

Corollary 4.1. *Denote the set of generic* 2×2 *games* $\Gamma^{2\times2}$. *For any* $\Gamma \in \Gamma^{2\times2}\backslash\Gamma^{NP}$, *the following relationship is true:* $NE(\Gamma) \subset \overline{\{QRE\}}$, *where* $NE(\Gamma)$ *is the set of all Nash equilibria of* Γ *and* $\overline{\{QRE\}}$ *is the closure of the set of attainable QRE of* Γ.

It is clear from the proof of Proposition 4 and Corollary 4.1 that we have the following relationship:

Corollary 4.2. *For any* $\Gamma \in \Gamma^m \cup \Gamma^c$, *for all* $\sigma \in NE(\Gamma)$ *we have* $\sigma \in \overline{M^c}$ *and therefore,* $\sigma \in \overline{\{QRE\}}$.

3.3 Quantal Noisy Belief Equilibrium

This model takes an earlier working paper of Friedman's and expands this by introducing noise into actions simply incorporating QRE. This is a very natural progression of NBE and provides an intriguing approach that differs to the more simply equilibrium concept proposed in M equilibrium by Goeree and Louis (2021). Unfortunately, this unpublished paper of Friedman and Ward's (2022) only came to my attention later in my research and is therefore only briefly discussed in this paper. Friedman and Ward (2022) also spend limited time discussing the theoretical relationship between QNBE and other equilibrium concepts that introduce noise into both actions and beliefs, like that of M equilibrium. Granted, at the time Friedman and Ward's (2022) paper was initially written, M equilibrium was only a working paper. This leaves open the opportunity for future research comparing QNBE and M equilibrium. One notable difference is that Goeree and Louis (2021) specifically develop the parametric complement to M equilibrium in the form of μ-equilibrium (see Goeree and Louis, 2021, Section IB) whereas Friedman and Ward (2022) keep QNBE as purely a set-based equilibrium model. It is also immediately present from Friedman and Ward's (2022) analysis of what they term "X games", known in a generalised form here as asymmetric matching pennies, shows that QNBE can lead to a larger set of possible equilibria compared to that of M equilibrium. It is unclear how these models differ in more complex 2×2 games and is another specific area for future investigation. As Goeree and Louis (2021) make the argument that as M equilibrium is based on specific conditions directly rooted in fundamental economic theory, any behaviour that is observed outside of the equilibrium solution provides an intriguing result that highlights very unexpected or economically "unnatural" behaviour. Given that it is clear for asymmetric matching penny games that the QNBE set is larger than that of the M choice set, it raises an interesting question as to the spaces in QNBE that fall outside of the M choice set.

4 Theoretical Links

In this section, I discuss the connection between regular QRE, NBE, and M equilibrium, focusing on the relationship of attainable equilibria in each concept. A discussion and analysis of their respective parametric models is presented in the following section. This section serves to investigate the extent to which Proposition 3 in Goeree and Louis (2021) can be extended to relate to NBE. I find new links between these equilibrium concepts, building on existing literature and providing additional context for M equilibrium as a meta theory.

Revisiting my expansion of Friedman's (2022) Theorem 3 in Proposition 3, a further link can be made between NBE and M equilibrium.

Proposition 5. *For any $\Gamma \in \Gamma^m \cup \Gamma^c$ the following relationships are true: $\{QRE\} \subseteq \{NBE\} \subseteq \overline{M^c}$ and $\{QRE\} = \{NBE\} \cap \Delta^\circ = \overline{M^c} \cap \Delta^\circ$, where $\{QRE\}$ is the set of attainable QRE for Γ, $\{NBE\}$ is the set of attainable NBE for Γ, $\overline{M^c}$ is the M choice set of $\mathcal{M}(\Gamma)$, and Δ° is the set of completely mixed strategy profiles.*

Proof. To begin, from Proposition 3 we know that for any $\Gamma \in \Gamma^m \cup \Gamma^c$, $\{QRE\} = \{NBE\} \cap \Delta^\circ$, hence $\{QRE\} \subseteq \{NBE\}$. From Goeree and Louis's (2021) Proposition 3 we know that for any $\Gamma \in \Gamma^m \cup \Gamma^c$, $\{QRE\} = \overline{M^c} \cap \Delta^\circ$, therefore $\{QRE\} = \{NBE\} \cap \Delta^\circ = \overline{M^c} \cap \Delta^\circ$. For the final part of the proof I show that $NBE \backslash \Delta^\circ \subseteq \overline{M^c} \backslash \Delta^\circ$, thus proving the last link $\{NBE\} \subseteq \overline{M^c}$ for any $\Gamma \in \Gamma^m \cup \Gamma^c$. For $\{NBE\}$, from Proposition 1 there exist no pure-mixed strategy NBE for $\Gamma \in \Gamma^m \cup \Gamma^c$, while there does exist pure strategy NBE. As $\overline{M^c}$ may contain equilibria on the boundaries (e.g., both pure strategy and mixed-pure strategy equilibria) for any $\Gamma \in \Gamma^m \cup \Gamma^c$, we know that the attainable NBE set cannot be equal to this. Furthermore, as $\overline{M^c}$ contains all Nash equilibria for $\Gamma \in \Gamma^m \cup \Gamma^c$ from Proposition 4 then, it follows that $\{NBE\} \subseteq \overline{M^c}$. □

Continuing from here, we can explore the relationship between the attainable sets of each equilibrium concept further by expanding Proposition 5 to more generic 2×2 games. As I have already addressed all games that contain completely mixed strategy Nash equilibria, all remaining 2×2 games to address only involve equilibria on the boundaries (i.e., pure strategy and pure-mixed strategy equilibria). Some issues arise on the boundaries when dealing with specific game structures. As a result I constrain the set of all generic 2×2 games similar to that in Proposition 4. I now introduce what I call a 'problem game', which is a refinement of the previously defined Nash problem game. A problem game is by definition a Nash problem game, however, not all Nash problem games are problem games. The reason for this is to address specific instances where Nash equilibria that are not in $\overline{M^c}$ are also not in $\{NBE\}$.

Definition 3 (Problem Game)**.** A game is defined as a problem game if the game structure follows one of the following generalised structures:

- player i and player k both have weakly dominant strategies
- player i has a strictly dominant strategy and player k has a weakly dominant strategy such that player k's higher payoff does not occur when player i plays their dominant strategy
- player i has a weakly dominant strategy and player k has a matching pennies type structure such that player i's increased payoff coincides with player k's increased payoff

I denote the set of problem games by Γ^P. This set of games excludes the generalised game seen in Table 12d in Definition 2. Simply put, this definition excludes *case (ix)* from the proof of Proposition 4. The reason for this will be made apparent in the proof of Proposition 6.

It is no coincidence that these types of games include the games in which pure-mixed strategy NBE can exist as proved in Proposition 1. As previously stated these games are also the subject of interesting results from Proposition 4 in that they are examples of games where $\overline{M^c}$ excludes some Nash equilibria. It should now be apparent that these games make up all the instances within generic 2×2 games where there exist NBE that are not in $\overline{M^c}$.

Proposition 6. *Denote the set of all* 2×2 *games as* $\Gamma^{2\times2}$*. For any* $\Gamma \in \Gamma^{2\times2} \backslash \Gamma^P$*, the following relationship is true:* $\overline{\{NBE\}} \subseteq \overline{\{QRE\}} = \overline{M^c}$*, where* $\overline{\{NBE\}}$ *and* $\overline{\{QRE\}}$ *are the closures of* $\{NBE\}$ *and* $\{QRE\}$ *in* Δ *for* Γ*, respectively.*

Proof. From Proposition 1 we know that there exist some instances where $\{NBE\}$ does not include all possible Nash equilibria and therefore this extends to $\overline{\{NBE\}}$. From Proposition 4, we know that there are also some instances where $\overline{M^c}$ does not include all possible Nash equilibria and therefore this is also the case for $\{QRE\}$ and subsequently $\overline{\{QRE\}}$. From the proof for Proposition 4, there exist five generalised game structures in which $\overline{M^c}$ and $\overline{\{QRE\}}$ do not include all attainable Nash equilibria. It is also immediately evident from Proposition 1 that several of these same games contain Nash equilibria that are not included in $\overline{\{NBE\}}$. Reviewing the proof of Proposition 4 we can directly address the specific cases to show that Γ^P encompass all 2×2 games in which $\overline{\{NBE\}} \nsubseteq \overline{M^c}$. From Proposition 5, $\overline{\{NBE\}} \cap \Delta^\circ \subseteq \overline{M^c}$ so we are only concerned with equilibria on the boundaries. We know that all pure strategy Nash equilibria are included in $\{NBE\}$ (and subsequently $\overline{\{NBE\}}$) while this is not true for $\overline{M^c}$. This also means that there exist non-proper Nash equilibria explained by NBE but not M equilibrium. Similarly, there exist pure-mixed strategy Nash equilibria that are in $\overline{\{NBE\}}$ but not in $\overline{M^c}$. As a result, we can identify the specific cases where $\overline{\{NBE\}}$ account for some equilibria and $\overline{M^c}$ does not. Specifically, these games as discussed in the proof of Proposition 4 are *cases (iv), (v), (vii),* and *(x)*. For *case (iv)* this occurs at $(\sigma_1, \sigma_2) = (1, 1)$. For *cases (v)* and *(vii)*, this occurs at the pure strategy Nash equilibrium $(\sigma_1, \sigma_2) = (0, 1)$ and also for pure-mixed strategy Nash equilibria $(\sigma_1, \sigma_2) = (1, x)$, $x \in [0.5, 1)$ as by Proposition 1 pure-mixed NBE exist for these two game structures. For *case (x)*, this only occurs at $(\sigma_1, \sigma_2) = (1, 1)$ as the pure-mixed strategy Nash equilibria present within this game structure cannot be explained by NBE given Proposition 1. For *case (ix)*, although there exist Nash equilibria that are not in $\overline{M^c}$ these cannot be explained by NBE given Proposition 1. Furthermore, *case (ix)* differs from the structure presented in *case (x)* in that there exists one pure strategy Nash equilibrium instead of two. This pure strategy Nash equilibrium is in $\overline{M^c}$ and an NBE so for this game we do not have any issues.[18] In *cases (i), (ii), (iii), (vi)* and *(viii)* we see all Nash equilibria included in $\overline{M^c}$. To address these instances with respect to $\overline{\{NBE\}}$ each case is discussed. For *cases (i)* and *(ii)* from previous results we know the Nash equilibria are in $\overline{\{NBE\}}$. For *case (iii)* as there is a unique pure strategy Nash equilibrium, we know from the construction of NBE that this will be included in $\overline{\{NBE\}}$. For *case (vi)* there exists a unique pure strategy Nash equilibria and subsequently a unique pure strategy NBE which is included in $\overline{M^c}$. For *case (viii)* neither player has a weakly dominant strategy, therefore by Proposition 1 there can exist no pure-mixed strategy NBE and $\overline{\{NBE\}}$ collapses to the pure strategy Nash equilibrium which is also in $\overline{M^c}$. Excluding those games in which

[18]In fact, depending on the payoff differences for player k (player with the matching pennies structure) all Nash equilibria are contained in $\overline{M^c}$.

attainable Nash equilibria are included within $\{NBE\}$ but not $\overline{M^c}$, we are left with 2×2 games where all the attainable NBE on the boundary are within or equal to the set of attainable equilibria on the boundary of $\overline{M^c}$. Together with the fact that $\{NBE\} \cap \Delta^\circ \subseteq \overline{M^c} \cap \Delta^\circ$, we get $\overline{\{NBE\}} \subseteq \overline{M^c}$. The final step of including $\overline{\{QRE\}}$ follows from previous proofs. □

Another result that follows from Proposition 5 and Proposition 6 is that when constraining games to that of generalised matching pennies and generalised chicken, taking the closures of $\{QRE\}$ and $\{NBE\}$ gives the following:

Corollary 6.1. *For any $\Gamma \in \Gamma^m \cup \Gamma^c$ the following relationship is true: $\overline{QRE} = \overline{NBE} = \overline{M^c}$.*

5 Parametric Comparisons

Within this section I present two games, asymmetric matching pennies and asymmetric chicken, which are modelled by each parametric equivalent of the equilibrium concepts analysed.[19] This section aims to provide a visual analysis of the parametric models to compliment the discussion in previous sections. In many instances the models all follow relatively similar paths. It is evident from Section 4 that there will exist some notable differences between parametric forms of the equilibrium models analysed for games that contain pure-mixed strategy Nash equilibria and games that have unique pure strategy Nash equilibria.

5.1 Asymmetric Matching Pennies

As this game has been consistently discussed thus far, it is natural to show how all parametric versions of the equilibrium models presented compare for this game. Presented below again in Table 21 is the payoff structure for asymmetric matching pennies. Seen in Figure 18, are the logit and Luce QRE, the logit transform NBE,

	H	T
H	(2,0)	(0,1)
T	(0,1)	(1,0)

Table 21: Asymmetric Matching Pennies

and the μ-equilibrium. As previously noted, for this specific game the Luce QRE is observationally equivalent to the logit Transform NBE. The logit QRE model also follows a similar pattern, while the μ-equilibrium is linear. Interestingly, the μ-equilibrium reaches a further removed set of equilibria seen in the top left side of Figure 18.

5.2 Asymmetric Chicken

The game of chicken has been previously discussed within section 3.1.6 in a more generalised form. Here the game is given a defined payoff structure presented in Table 22. Seen in Figure 19, is the comparison of each parametric model for the game of asymmetric chicken presented in Table 22. Again, we see the fixed-point models of logit QRE, Luce QRE, and logit transform NBE follow similar paths while

[19]The parametric models have been coded in Python with rationality each model's rationality parameters varied as follows: for QRE, $\lambda \in (0, 20)$ in increments of 0.01; for NBE, $\tau \in (0, 15)$ in increments of 0.01; for μ-equilibrium, $\varepsilon \in [0, 1]$ in increments of 0.0001.

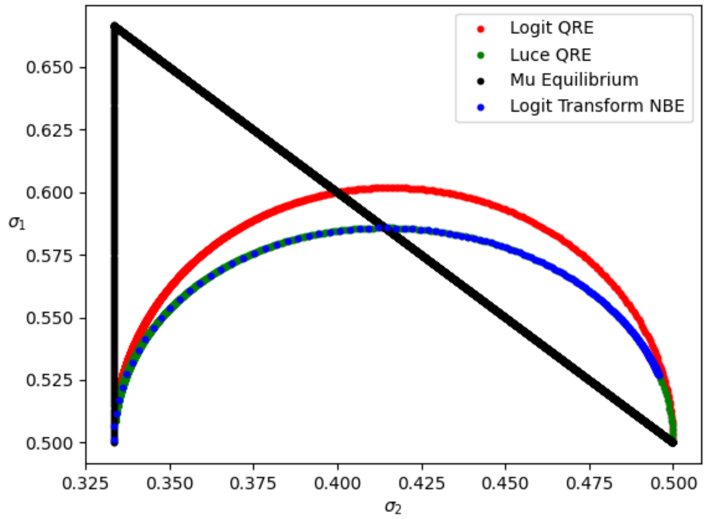


Figure 18: Parametric Comparison for Asymmetric Matching Pennies

	R	L
R	(0,0)	(8,2)
L	(3,12)	(4,4)

Table 22: Asymmetric Chicken

μ-equilibrium deviates more. The deviation however, is less than that observed in the asymmetric matching pennies game, with the μ-equilibrium remaining much closer to the paths of the other models.

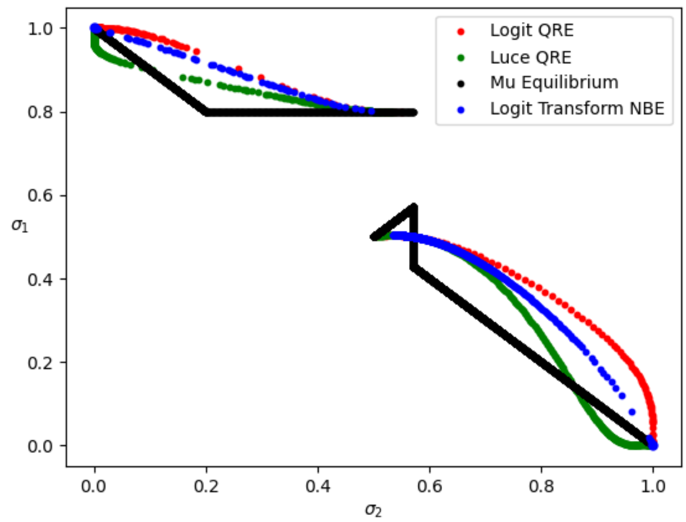


Figure 19: Parametric Comparison for Asymmetric Chicken

6 Conclusion and Direction for Future Research

In this concluding section I address the questions raised at the beginning of this paper and highlight my key findings. I then discuss key areas for future research that have been raised from my analysis of QRE, NBE, and M equilibrium.

The main question regarding when new models of stochastic behaviour, such as that of NBE and M equilibrium, should be used in place of QRE is undoubtedly a tricky question to answer and one where there does not yet exist a simple answer. To provide some framing for this it is important to discuss the additional questions which naturally flow from this main driving question. Within this paper significant focus has been placed on understanding the relationships between QRE, NBE, and M equilibrium. It is useful to understand how the broader picture of these models 'fit' together prior to exploring whether or not a specific parametric form of a new model is the 'best' new approach to use in analysing experimental data. A novel result is seen within my Proposition 5 where for a specific game, $\Gamma \in \Gamma^m \cup \Gamma^c$ the interior of the set of attainable of NBE is equal to the interior of the M choice set and is equal to the set of attainable QRE. Within Proposition 5 I also show the relationship between the M choice set and the attainable sets of QRE and NBE, highlighting the relationship between these concepts at the boundaries of possible equilibria. Within Proposition 6 I extend my finding of the relationship between these equilibrium models, to a wider set of generic 2×2 games. In order to do so, I find the specific instances where the relationship between models breaks down. Not only does this provide useful insight into the set of equilibria that can be explained by each model and their significant overlap within certain games, but it also provides insights into specific types of generalised game structures where certain equilibria can be explained by NBE but not by M equilibrium and QRE. This substantial analysis of generic 2×2 games contributes to existing literature by connecting the two new equilibrium models of M equilibrium and NBE. Additionally, I have expanded on the concept of using M equilibrium as a meta theory, proving that in many cases for generic 2×2 games, attainable NBE are contained in the M choice set. For the instances where this does not occur, I have clearly identified the generalised generic 2×2 game structures. As a result, this also provides further clarity to the "almost all" condition that is present within the meta theory Proposition in Goeree and Louis (2021, Proposition 3).

Given the different approaches to incorporating 'noise' within equilibrium models, a natural question was to investigate the extent to which the relationship between these models could provide further context to the importance of noise in beliefs, noise in actions, and noise in both. It is clear from my results that the majority of differences between these approaches when dealing with attainable sets of equilibria, occur on the boundaries. Specifically, as players approach both pure strategies and pure-mixed strategies. In general, the foundations of M equilibrium and the reasoning for its construction are compelling. Combined with the results presented in both Proposition 5 and Proposition 6 it is clear that M equilibrium could act as a meta theory including NBE for generic 2×2 games, with some notable exceptions.

From a practical and experimental perspective there are trade-offs involved in transitioning to new models if the further benefit they provide above that which researchers are already familiar with is minimal. The well-established use of QRE within modelling experimental data means it is difficult to easily replace with substantially new approaches. Furthermore, as QRE has various adaptations to suit specific types of games beyond the generic 2×2 games analysed within this paper, it is currently unclear how well the parametric forms of either NBE or M equilibrium would function as an improvement in a broad sense. The parametric form of M equilibrium, μ-equilibrium presents significant benefits over that of the parametric forms of QRE and NBE, as it does not rely on transcendental functions thereby making it less computationally complex. The parametric form of NBE, logit transform NBE, provides possible benefits in that it is invariant to affine transformations of payoffs. In certain cases however, this could be a flaw. Take for example, games in which researchers wish to study how individual's responses may vary dependent on payoff differences or different steaks (i.e., very high payoffs versus very low payoffs while keeping relative payoff differences constant). Another restriction of the logit tranform NBE is that it is bounded by rationalizability. In certain games equilibria might be observed in experimental data that are not rationalizable and logit transform NBE will not be able to model these.[20] Additionally, at this stage there is not currently a parametric form of NBE for normal form games with more than two actions that does not introduce bias. For a detailed discussion of how the parametric forms of NBE and M equilibrium compare to QRE when modelling experimental data in a specific selection of games see Friedman (2022, Section VI)

[20]Take for example the guess the average game.

and Goeree and Louis (2021, Section II). As M equilibrium applies a minimal set of well-established economic conditions, equilibria arising from experimental data that do not fall within the M choice set violate well-established economic norms and as a result lead to new questions about why the observed behaviour has occurred. Given this and the argument of using M equilibrium as a meta theory, a reasonable approach would be to first check the experimental data against M equilibrium prior to then specifying a specific type of parametric model to be used which best suits the requirements of the research.[21]

Following the key results of this paper in investigating the relationship between QRE, NBE, and M equilibrium there exist many areas for future research. I discuss these here, presenting direction for future research to better understand the relationships between these stochastic models of behaviour. From a theoretical perspective, continuing to explore the theoretical relationship between QRE, NBE, and M equilibrium by extending the analysis of generic 2×2 games to non-generic games would provide a more complete picture of how these equilibrium models relate. Equally, extending the analysis to games with n-dimensional players and j-dimensional actions would provide useful contributions to understanding how the relationship between these models presented in Proposition 6 changes as additional players and additional actions are incorporated. As there currently exists no literature comparing QNBE and M equilibrium, two comparable models which both incorporate noise to actions and beliefs, further research exploring the relationship between them would provide interesting contributions to understanding how the way in which noise is incorporated into both actions and beliefs changes the set of attainable equilibria. From an empirical perspective, one area that is immediately present is to test these models with empirical data to better observe how they perform in out of sample predictions. Specifically, to investigate the differences between μ-equilibrium and logit transform NBE. Due to the bias introduced in the logit transform NBE when applying this to games where players have more than two actions, developing alternative parametric forms of NBE that do not introduce bias in these situations could help provide an alternative model for fitting experimental data that could subsequently be compared to parametric versions of QRE in more complex games. Lastly, as both QRE and NBE have been extended for extensive form games, developing an equivalent for M equilibrium which could then be compared to the extensive form versions of QRE

[21]This relates the the previous discussion on how different parametric models of QRE and NBE provide unique benefits or drawbacks depending on the context in which they will be applied.

and NBE would allow for further exploration of M equilibrium as a meta theory.

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