When is a Volterra space Baire?

by

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Recall that a space X is said to be *a Baire space* if the intersection of any sequence of dense open sets of X is still dense.

A space X is called *a Volterra space* if for any pair of functions $f, g : X \to \mathbb{R}$ such that $C(f)$ and $C(g)$ are dense in X, $C(f) \cap C(g)$ are also dense in X (Gauld and Piotrowski, 1993).

Weakly Volterra spaces and spaces of the second Baire category can be defined similarly.

Easy fact: *All Baire spaces are Volterra spaces.*

Example: *There is a first countable, Tychonoff, paracompact and Volterra space that is not Baire.*

Question: *When is a Volterra space a Baire space?*

Here, I shall present two main theorems. Our first theorem is:

Theorem 1: *Let X be a stratifiable space. Then X is Volterra $\rightrightarrows$ X is Baire.*

A space X is *stratifiable* if X is regular and one can assign a sequence of open sets $\{G(n, H) : n \in \mathbb{N}\}$ to each closed set $H \subseteq X$ such that

(i) $H = \bigcap_{n\in\mathbb{N}} G(n, H) = \bigcap_{n\in\mathbb{N}} \overline{G(n, H)}$,

(ii) $H \subseteq K \rightrightarrows G(n, H) \subseteq G(n, K)$ for all $n \in \mathbb{N}$.

In 2000, Gruenhage and Lutzer proved that *metric and Volterra spaces are Baire*. Since metric spaces are stratifiable, they asked: *Must X be Baire if it is stratifiable and Volterra?*

Our Theorem 1 answers this question affirmatively.

Recall that a space X is said to be *resolvable* if it contains two disjoint dense subsets.

Sharm-Sharm Theorem (1988): *Let X be dense-in-itself Hausdorff. If $\lambda(X) = X$, then X is resolvable.*

$A \subseteq X$ is called *simultaneously separated* if each $x \in A$ has an open nbhd U_x so that $\{U_x : x \in A\}$ is pairwise disjoint. Then, $\lambda(X) = \bigcup\{A^d : A \text{ is s.s.}\}$.

Consequence of the Sharma-Sharma theorem:

Let X be a dense-in-itself and Hausdorff space.

(a) X *is sequential* $\rightrightarrows X$ *is resolvable* (Pytkeev, 83)

(b) X is a k-space $\rightrightarrows X$ is resolvable

(c) X *is countably compact* $\rightrightarrows X$ *is resolvable*

(d)$^{\circledast}$ X *is monotonically normal* $\rightrightarrows X$ *is resolvable*

(Dow, Tkachenko, Tkachuk and Wilson, 02)

Our second theorem concerns locally convex topological vector spaces (for short, tvs) over $\mathbb{R}$.

Recall that a tvs is *locally convex* if $\mathbf{0}$ has a neighbourhood base consisting of convex sets.

Theorem 2: *Let X be a locally convex tvs. Then X is Volterra $\rightrightarrows$ X is Baire.*

For a normed linear space E, let E^* denote the topological dual of E (i.e., the set of all continuous linear functionals on E), and let $\sigma(E, E^*)$ denote the weak topology on E generated by E^*.

Corollary 1: *Let E be a normed linear space. Then $(E, \sigma(E, E^*))$ is Volterra $\rightleftarrows$ $\dim E < +\infty$.*

Corollary 1 extends a result in Functional Analysis.

The proof of the second theorem depends on a consequence of the Hahn-Banach theorem.

If X is a locally convex tvs, then there exists a non-trivial $f \in X^*$. It follows that

$$X \approx \ker(f) \times X/\ker(f) \approx \ker(f) \times \mathbb{R}$$

Now, we need the following simple, but

Important fact: *If Z is of the firts Baire category, then $Z \times \mathbb{R}$ is not weakly Volterra.*

Suppose that X is Volterra. Then $\ker(f)$ is of the second Baire category. Thus, X must be of the second Baire category. Finally, apply the following

Simple fact: *A homogenous space X is of the second Baire category $\rightleftarrows$ X is a Baire space.*

To finish my talk, I shall state one more theorem.

For a vector space X, let X' denote the algebraic dual of X. A subset $A \subseteq X'$ is called *pointwise bounded* if $\sup\{|f(x)| : f \in A\} < +\infty$ for all $x \in X$.

Theorem 3: *Let X be a vector space. If $Y \subseteq X'$ contains an infinite linearly independent and pointwise bounded subset, then $(X, \sigma(X, Y))$ is not Baire.*

Corollary 2: *If a Tychonoff space Z contains an infinite relatively pseudocompact subset, then $C_p(Z)$ is not a Baire space.*

Corollary 3: *Let E be a normed linear space. If E is infinite-dimensional, then $(E, \sigma(E, E^*))$ is not Baire.*

——————— ⋇ *The End* ⋇ ———————