Options on Leveraged ETF: Calibrations and Error Analysis

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Abstract

Within the standard affine stochastic volatility framework we price options on leveraged and inverse leveraged ETFs using Fourier transform. We perform a calibration analysis for a given day on options written on leveraged and inverse leveraged ETFs tracking the S&P500 that is the most actively traded ETF derivatives. We analyze the calibrated parameters and assess the ability of the Heston model to price consistently all the options. Overall we find that the Heston model allows a good fit of the smiles and that the different option sets lead to consistent underlying spot distributions.

JEL Classification: G12; G13; C61

Keywords: Option on ETF, Option on Leveraged ETF, Affine model, Calibration, Fourier Transform.

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1 Introduction

The LETFs are designed to track the multiple daily returns of an underlying asset. Certainly, the most traded LETFs are those tracking the S&P500. For this index a large range of multiples is available. From a multiple of +1, whose ticker is SPY, that tracks the S&P500 and started to be traded in 1993 to multiple as small as -3 or as large as +3, LETFs related to this index constitute an active market. It was further enhanced by the introduction around 2010 of options on these LETFs. In the interesting work of Zhang (2010), the author develops a first analysis of option prices on these LETFs. Developing both option pricing formulas as well as an empirical analysis the author provides the first entry point on this important subject.

Our aim is to develop an empirical analysis of options written on LETFs related to the S&P500 index. Using a complete set of option prices on LETFs for multiples ranging from -3 to 3 we calibrate the Heston (1993) model. We analyze the pricing errors as well as the parameters provided by the different calibrations. It allows us to assess to which extent the Heston model can provide an unifying framework for all the derivatives. We find that the model performs reasonably well although, as expected, options on LETF with a multiple of -3 are more difficult to handle.

The paper is organized is as follows. We present in the first section the analytical framework for pricing options on LETF. A brief data description analysis is given in the second section. The third part of the paper is devoted to empirical results. All graphs and tables are relegated to the appendix.

2 The Model and Derivative Pricing

2.1 The Underlying Assets

The main asset in this work is denoted by $(s_t)_{t\geq 0}$ and is an exchange traded fund, that we may qualify as a stock, and its dynamic is given by the set of stochastic differential equations (SDE in the sequel) under the risk neutral probability

$$ds_t = (r-q)s_t dt + s_t\sqrt{v_t}dw_t^1, \tag{1}$$

$$dv_t = \kappa(\theta - v_t)dt + \sigma\sqrt{v_t}dw_t^2, \tag{2}$$

with $(w_t^1, w_t^2)_{t\geq 0}$ a two-dimensional Brownian motion with correlation structure $d\langle w_.^1, w_.^2\rangle_t = \rho dt$. We will denote by $\mathbb{E}[.]$ the expectation under the risk neutral probability Q. This model is the classical Heston model and belongs to the affine class. It means that the moment generating function of the log-stock $\ln s_t$, denoted by y_t, and the integrated volatility is known in closed form. Indeed, this function is given by the following proposition.

Proposition 2.1 *Let (y_t, v_t) be a vector with $y_t = \ln s_t$ where $(s_t, v_t)_{t\geq 0}$ is given by Eq.(1) and Eq.(2), then its moment generating function is given by:*

$$G(t, z_1, z_2, y_0, v_0) = \mathbb{E}[e^{z_1 y_t + z_2 \int_0^t v_u du}] = e^{z_1 y_0 + a(t) + b(t) v_0} \tag{3}$$

where

$$a(t) = \frac{2\kappa\theta}{\sigma^2}\left(t\lambda_- - \ln\left(\frac{\lambda_+ - \lambda_- e^{-\sqrt{\Delta}t}}{\lambda_+ - \lambda_-}\right)\right) + (r-q)z_1 t, \tag{4}$$

$$b(t) = (z_1^2 - z_1 + 2z_2)\frac{1 - e^{-\sqrt{\Delta}t}}{\lambda_+ - \lambda_- e^{-\sqrt{\Delta}t}}, \tag{5}$$

and

$$\lambda_\pm = \frac{(\kappa - z_1\rho\sigma) \pm \sqrt{\Delta}}{2}, \tag{6}$$

$$\Delta = (\kappa - z_1\rho\sigma)^2 - \sigma^2(z_1^2 - z_1 + 2z_2). \tag{7}$$

The exchange traded fund or stock being defined we denote by $(l_t)_{t\geq 0}$ a leveraged exchange traded fund that provides a multiple, denoted by m, of the daily return of $(s_t)_{t\geq 0}$. It was shown in Avellaneda and Zhang (2010) that these two assets are related through the relation

$$l_t = l_0 \left(\frac{s_t}{s_0}\right)^m e^{\left(\frac{m-m^2}{2}\right)\int_0^t v_u du + (1-m)rt}. \tag{8}$$

The multiple m is also call the leverage factor. Obviously, if $m = 1$ then $l_t = s_t$. For a connection between the leveraged asset and the CPPI strategy proposed in Black and Perold (1992), see Bertrand and Prigent (2003). Following Avellaneda and Zhang (2010) other papers focused on this relation, see for example Haugh (2011) and Jarrow (2010). Taking the logarithm of Eq.(8) we can rewrite this equation as

$$\ln l_t - \ln l_0 = m(\ln s_t - \ln s_0) + \frac{m - m^2}{2}\int_0^t v_u du + (1-m)rt. \tag{9}$$

This equation illustrates better the fact that l_t provides a multiple of the stock return but it also underlines the presence of a bias due to the volatility. As $m \in \{-3, -2, -1, 2, 3\}$ then the volatility contribution will be negative, thus whatever the sign of m is it will reduce the return of the leveraged exchanged traded fund compared to the stock return.

Both options on $(s_t)_{t\geq 0}$ and $(l_t)_{t\geq 0}$ are available, and it is therefore of interest to focus on the pricing of options on these assets.

3 Option Pricing on LETF

The pricing of options on leveraged ETF was firstly presented in Zhang (2010) for the Heston model. A slightly different approach was proposed by Ahn et al. (2012) for the Heston model with jumps on the stock (a very judicious remark is made in this paper regarding option pricing that simplifies the initial exposition of Zhang (2010)). Other papers focusing on option pricing are Deng et al. (2014) for the Heston model; Leung and Sircar (2014) where the stock follows a non affine dynamic and option price approximations are given using expansions based on fast mean reverting decomposition of the volatility process. Let us briefly present the pricing of these products for the sake of completeness.

Consider a European call option on a leveraged exchange traded fund using standard argument, then we have

$$\begin{aligned} c(t, l_0, v_0) &= e^{-rt}\mathbb{E}\left[(l_t - K)_+\right] \\ &= e^{-rt}\mathbb{E}\left[\left(l_0\left(\frac{s_t}{s_0}\right)^m e^{\left(\frac{m-m^2}{2}\right)\int_0^t v_u du + (1-m)rt} - K\right)_+\right] \\ &= e^{-rt}\mathbb{E}\left[\left(l_0 e^{x_t} - K\right)_+\right] \\ &= e^{-rt}\int_{-\infty}^{+\infty}\left(l_0 e^x - K\right)_+ f(x)dx, \end{aligned} \tag{10}$$

where $f(x)$ is the density of $x_t = m\ln\left(\frac{s_t}{s_0}\right) + \left(\frac{m-m^2}{2}\right)\int_0^t v_u du + (1-m)rt$. We denote by $\varphi(t,z) = \mathbb{E}\left[e^{izx_t}\right]$ the characteristic function of x_t, and we have

$$\begin{aligned} c(t, l_0, v_0) &= \frac{e^{-rt}}{2\pi}\int_{-\infty+i\gamma}^{+\infty+i\gamma}\varphi(t,z)\int_{-\infty}^{\infty}\left(l_0 e^x - K\right)_+ e^{-izx}dxdz \\ &= \frac{e^{-rt}}{2\pi}\int_{-\infty+i\gamma}^{+\infty+i\gamma}\varphi(t,z)\frac{K^{1-iz}l_0^{iz}}{iz(iz-1)}dz, \end{aligned}$$

where $\gamma = \Im(z) < -1$. Letting $k_0 = \ln\left(\frac{K}{l_0}\right)$, the above equation can be simplified to

$$c(t, l_0, v_0) = \frac{Ke^{-rt}}{\pi}\int_{0+i\gamma}^{+\infty+i\gamma} e^{-izk_0}\frac{\varphi(t,z)}{iz(iz-1)}dz. \tag{11}$$

The characteristic function of $(x_t)_{t\geq 0}$ in Eq.(11) is linked to the moment generating function Eq.(3) as follows

$$\begin{aligned} \varphi(t,z) = \mathbb{E}\left[e^{izx_t}\right] &= \mathbb{E}\left[e^{iz(m\ln\frac{s_t}{s_0} + \frac{m-m^2}{2}\int_0^t v_u du + (1-m)rt)}\right] \\ &= e^{i(1-m)zrt}G\left(t, izm, iz\frac{m-m^2}{2}, 0, v_0\right). \end{aligned}$$

If we wish to relax the assumption on $\Im(z)$ then we can firstly consider a put minus a cash position. The Fourier transform of this function is

$$\int_{-\infty}^{+\infty}\left(\left(K - l_0 e^x\right)_+ - K\right)e^{-izx}dx = \frac{Ke^{-iz\ln\frac{K}{l_0}}}{iz(iz-1)}.$$

It implies that the constraint $\Im(z) \in [-1, 0]$ and the call option price can be obtained by using the call-put parity relation. Numerically, the option pricing is performed using the Fast Fourier Transform following the exposition made by Carr and Madan (1999).

4 Data Description

We have a rich dataset which contains prices of options on a sextet of LETFs tracking the daily performance (or a multiple) of the S&P 500 index. The underlying LETFs are issued by ProShare company, which is a division of ProFunds Group and offers many different ETFs in terms of asset class besides the equity LETFs targeting the S&P 500. The detailed information of the LETFs is reported in the following table

[Insert Table I here]

For example, the SSO whose leverage ratio is +2 is designed to track twice the daily return of S&P 500 Index, before fees and expense charged. The target returns are achieved by daily rebalancing. SPY (+1) is actually an unleveraged ETF but it can be treated as a LETF with leverage ratio 1 here for the purpose of consistency.

We report in Figure 1 the evolution of the LETF for the period from 2011/06/01 to 2012/06/01 where the curves are consistent with the leverage ratios. Regarding the options we restrict the analysis to the day 2011/10/24 and to a certain range of option moneyness that increases as the absolute value of the leverage ratio increases to be consistent with the relation Eq.(9). Table II contains the number of options, the smallest and the largest maturity available as well as the moneyness range for each LETF.

[Insert Figure 1 here]

[Insert Table II here]

Not surprisingly options are more numerous for the leverage ratio of 1, that is to say, the exchange traded fund SPY that tracks the S&P 500. More options are available for positive leverage ratios although the results depend on the range of moneyness selected. Note that compared to Ahn et al. (2012) or Leung and Sircar (2014) our range of moneyness is quite large which translates into smiles that display more curvature (see the figures in Leung and Sircar (2014) for comparison). Also, let us mention the fact that we also consider the leverage ETF SH, with ratio -1, that is not considered in both Ahn et al. (2012) and Leung and Sircar (2014).

[Insert Figure 2 here]

[Insert Figure 3 here]

[Insert Figure 4 here]

[Insert Figure 5 here]

5 Numerical Results

5.1 Implementation Strategy

For each LETF (and leverage ratio m) we calibrate the model by solving the following optimisation problem:

$$\min_{v_0,\kappa,\theta,\sigma,\rho} \frac{1}{N}\sum_{i}^{N}\left(\sigma_{imp}^{market}(t_i,K_i,m)-\sigma_{imp}^{model}(t_i,K_i,m)\right)^2 \tag{12}$$

where $\sigma_{imp}^{market}(t_i,K_i,m)$ is the Black-Scholes implied volatility for the option with maturity t_i, strike K_i and leverage ratio m. N stands for the number of options available and will vary across the different leverage ratios. Similarly, σ_{imp}^{model} is the Black-Scholes model implied volatility. We restrict the sum in Eq.(12) to in the money options. Although the calibration is performed using the norm in volatility we will also report the option pricing error for a norm expressed in price and it is given by

$$\min_{v_0,\kappa,\theta,\sigma,\rho} \frac{1}{N}\sum_{i}^{N}\left(c^{market}(t_i,K_i,m)-c^{model}(t_i,K_i,m)\right)^2 \tag{13}$$

where $c^{market}(t_i,K_i,m)$ is the market price *normalized* by the underlying spot value of a call/put with maturity t_i, strike K_i and leverage ration m.

5.2 Empirical Results

Firstly, we analyze the calibration performance for each LETF and in Table III both the estimated parameters as well as the calibration errors in volatility and price are reported. All the LETFs lead to a negative sign for the spot-volatility correlation, which is consistent with the leverage effect, but SH (-1) clearly displays a higher value (or lower value if we consider the absolute value of the correlation coefficient) nearly half of the one obtained for the SPY (+1). Regarding the pair κ and θ, UPRO (+3) leads to rather small values (for both parameters) and contrast with the other LETFs. It might be more relevant to consider the ratio $\frac{\kappa}{\sigma^2}$ as it is this quantity that appears in the asymptotic distribution of the volatility[1]. Using the values of the

[1]The volatility process used in the Heston model has an asymptotic distribution that depends on $\frac{\kappa}{\sigma^2}$ and $\frac{\kappa\theta}{\sigma^2}$.

table we get 14.3, 2.9, 3.1, 0.8, 2.1 and 2.3 for the different LETFs (the values are reported following the order of the table). Theses values suggest that SPXU (-3) has the most different distribution. The calibration errors expressed using the implied volatility for the norm (i.e. Eq.(12)) are in line with those obtained in Da Fonseca and Grasselli (2011) for SPY (+1) and SH (-1) but deteriorate as the leverage ratio increases in absolute value terms. This might due to the larger moneyness range involved in the calibration procedure turning the fitting of the smile more difficult.

[Insert Table III here]

In order to assess the ability of information content extracted from options on a LETF to explain option prices written on another LETF we perform a repricing exercise. More precisely, for a given LEFT option set we report the ratio of the volatility error value obtained when this set is priced using parameters calibrated on another LETF option set and the volatility error value obtained when this the model is calibrated on this set. We also compute these ratios for the price error norm. The results are reported in Table IV for the first norm and in Table V for the second norm. The smaller these ratios are, the more the stock distributions implied by the option prices are similar. The smiles can therefore be qualified as consistent.

[Insert Table IV here]

[Insert Table V here]

It seems that SDS (-2) is the option set that leads to the largest repricing errors as the values are large. However, when the SDS (+2) parameters are used to reprice the other LETF options, whatever the LETF selected, the repricing is quite accurate. The parameters of SPXU (-3) lead to large repricing errors while the options on this LETF can be fairly correctly priced. As a result, we can conclude that options on LETF with negative ratios are priced with larger errors. Interestingly, the SPY (+1) do not lead to the lowest repricing errors although we could have expected such result.

6 Conclusion

In this work we propose a calibration analysis of options on leveraged exchange traded fund that tracks the S&P500. More precisely, we consider the LETFs, namely SPXU, SDS, SH, SPY, SSO and UPRO whose daily returns are multiple of the daily S&P500 returns with the multiple given by -3, -2, -1, 1, 2 and 3, respectively. Using the Heston (1993) model and a Fourier transform algorithm we price these options efficiently so that a calibration can be easily performed. For a given day, 2011/11/24, we carry out different calibrations. We found that calibration errors are larger for options written on LETFs with larger leverage ratios (in absolute value term). The

SDS(-2) model parameters lead to a rather good repricing error while the SPY(+1), which is the LETF with the largest number of options, does not lead to the smallest repricing error.

Or work suggests several extensions. First, we should consider rolling the calibration to analyze the stability of the calibrated parameters. Second, we should develop option price asymptotic expansions as it provides useful tools to simplify the calibration procedure, among this line see the first results in Leung and Sircar (2014). Thirdly, similar analysis should be performed on other options on LETFs. We leave these problems for future research.

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A Appendix

Table I: LETF Ticker and Ratio

Fund Name	Ticker Name	Leverage Ratio
Proshares UltraPro Short S&P 500 ETF	SPXU	-3
Proshares UltraShort S&P 500 ETF	SDS	-2
Proshares Short S&P 500 ETF	SH	-1
SPDR S&P 500 ETF	SPY	+1
Proshares Ultra S&P 500 ETF	SSO	+2
Proshares UltraPro S&P 500 ETF	UPRO	+3

Note. LETFs tracking the S&P 500, complete name as well as the ticker name along with the associated leverage ratio.

Table II: Options Properties

	Number opt.	Smallest mat.	Largest mat.	Moneyness range
SPXU	135	0.071	1.241	$[0.5, 2.5]$
SDS	144	0.071	1.241	$[0.5, 2.0]$
SH	126	0.071	1.241	$[0.8, 1.5]$
SPY	722	0.071	1.241	$[0.8, 1.3]$
SSO	211	0.071	1.241	$[0.5, 1.4]$
UPRO	181	0.071	0.819	$[0.3, 1.6]$

Note. Options of LETF properties for the day 2011/10/24.

Table III: Calibration Results

	v_0	κ	θ	σ	ρ	ErrorVol	Error Price
SPXU	0.0253	2.6308	0.1438	0.4282	-0.664	3.949×10^{-2}	6.093×10^{-4}
SDS	0.0823	8.3064	0.0781	1.6757	-0.535	2.108×10^{-3}	3.891×10^{-5}
SH	0.0708	2.7108	0.0948	0.9343	-0.320	5.835×10^{-4}	5.152×10^{-6}
SPY	0.0854	2.4816	0.1345	1.6613	-0.739	4.654×10^{-4}	2.13×10^{-6}
SSO	0.0698	3.7320	0.0983	1.3296	-0.638	2.505×10^{-3}	3.697×10^{-5}
UPRO	0.0612	0.5308	0.1549	0.4747	-0.649	3.765×10^{-3}	3.946×10^{-5}

Note. Calibrated parameters for each LETF for the day 2011/10/24. "ErrorVol" reports the error as given by formula Eq.(12) while "Error Price" gives the value obtained using Eq.(13). Notice that option prices are normalized by the corresponding underlying spot value so the pricing errors can be compared.

Table IV: Repricing Errors - Volatility Norm

	SPXU	SDS	SH	SPY	SSO	UPRO
SPXU	1.00	2.55	1.24	5.31	2.86	1.65
SDS	20.64	1.00	5.97	29.85	16.08	10.92
SH	5.46	1.86	1.00	9.57	2.86	1.76
SPY	9.20	4.67	6.78	1.00	3.71	5.13
SSO	7.91	0.97	2.60	1.64	1.00	3.13
UPRO	4.01	1.94	1.82	3.25	1.74	1.00

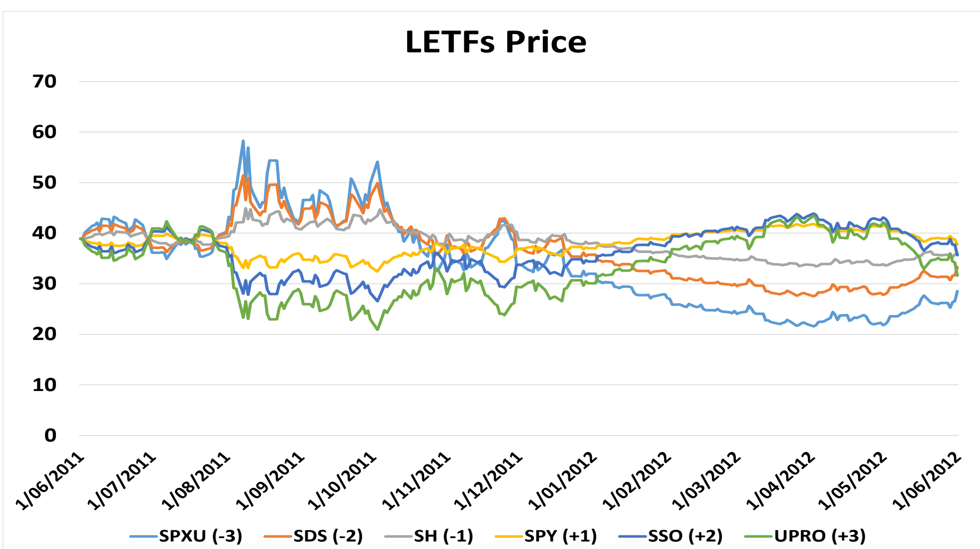
Note. For a given set of calibrated parameters obtained for a given LETF (in the top row), options written on other LETF (given in the left column) are priced and the ratio of error in volatility Eq.(12) is reported.

Table V: Repricing Errors - Price Norm

	SPXU	SDS	SH	SPY	SSO	UPRO
SPXU	1.00	23.05	10.88	37.69	3885.89	13.15
SDS	9.45	1.00	1.60	314.09	159.44	2.66
SH	10.56	1.31	1.00	30.86	2.87	1.79
SPY	0.00	0.00	0.00	1.00	0.00	0.00
SSO	8.55	0.71	1.83	2.74	1.00	1.26
UPRO	11.62	1.37	2.79	2.21	0.84	1.00

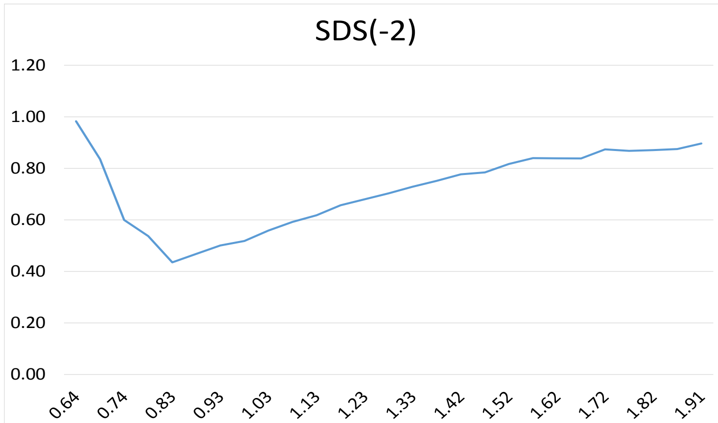
Note. For a given set of calibrated parameters obtained for a given LETF, options written on other LETF are priced and the ratio of error in price Eq.(13) is reported.

Figure 1: LETF Prices



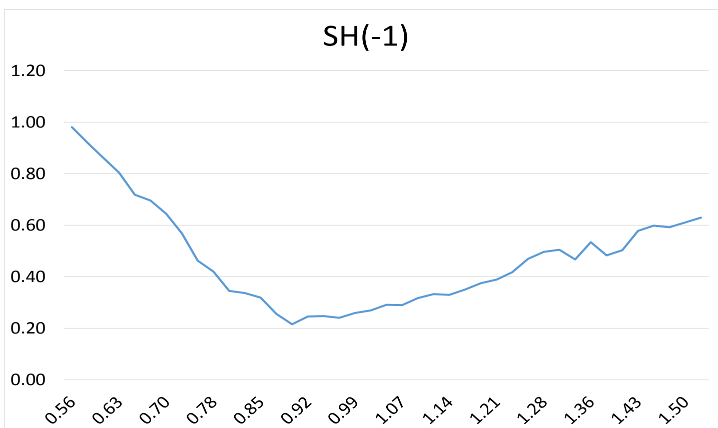
Note. Times series for LETF from 2011/06/01 to 2012/06/01.

Figure 2: Option Smile for SDS(-2)



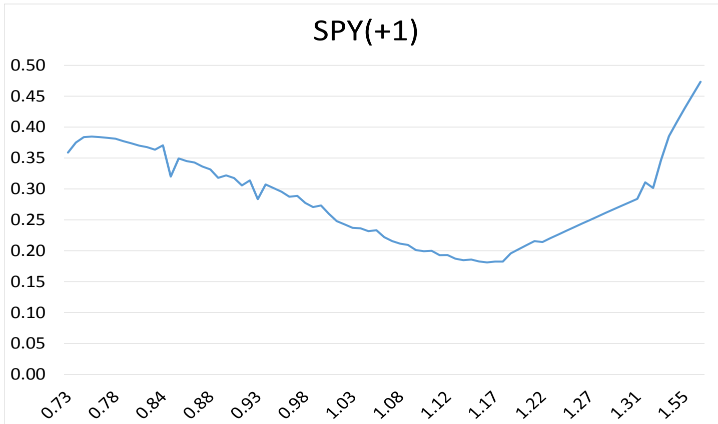
Note. Smile for the maturity 0.14 for the LETF SDS(-2) for the day 2011/10/24.

Figure 3: Option Smile for SH(-1)



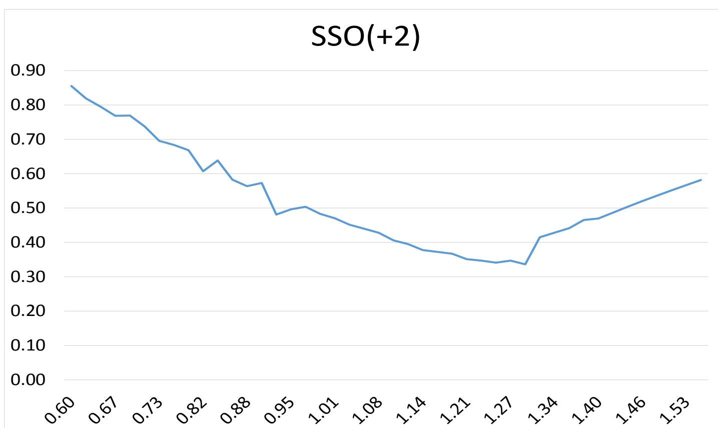
Note. Smile for the maturity 0.14 for the LETF SH(-1) for the day 2011/10/24.

Figure 4: Option Smile for SPY(+1)



Note. Smile for the maturity 0.14 for the LETF SPY(+1) for the day 2011/10/24.

Figure 5: Option Smile for SSO(+2)



Note. Smile for the maturity 0.14 for the LETF SSO(+2) for the day 2011/10/24.