Options on Leveraged ETF: Calibrations and Error Analysis

José Da Fonseca & Yahua Xu

AUCKLAND UNIVERSITY OF TECHNOLOGY

CEF, 2015

What is a Leveraged ETF?

A leveraged ETF is an asset that gives a multiple of the daily return of another asset. If s_t is the asset value and l_t a LETF with multiple m then we have

$$\frac{dl_t}{l_t} = m\frac{ds_t}{s_t} + (1-m)rdt$$

and if we specify the dynamic for s_t to be of Heston type

$$\begin{aligned}\frac{ds_t}{s_t} &= (r-q)dt + \sqrt{v_t}d\omega_t^1,\\ dv_t &= \kappa(\theta - v_t)dt + \sigma\sqrt{v_t}d\omega_t^2,\end{aligned}$$

with $(w_t^1, w_t^2)_{t\geq 0}$ a two-dimensional Brownian motion with correlation structure $d\langle w_\cdot^1, w_\cdot^2\rangle_t = \rho dt$ then l_t is known. We have

$$d\ln l_t = md\ln s_t + \underbrace{\frac{m-m^2}{2}v_t dt}_{\text{vol bias}} + (1-m)rdt,$$

For $m \in \{-3,-2,-1,1,2,3\}$ bias is negative.

What is a Leveraged ETF?

$$\ln l_t - \ln l_0 = m(\ln s_t - \ln s_0) + \frac{m - m^2}{2}\int_0^t v_u du + (1-m)rt$$

The bias depends on the integrated volatility and can be substantial. Also we rewrite as

$$l_t = l_0 \left(\frac{s_t}{s_0}\right)^m e^{\left(\frac{m-m^2}{2}\right)\int_0^t v_u du + (1-m)rt}$$

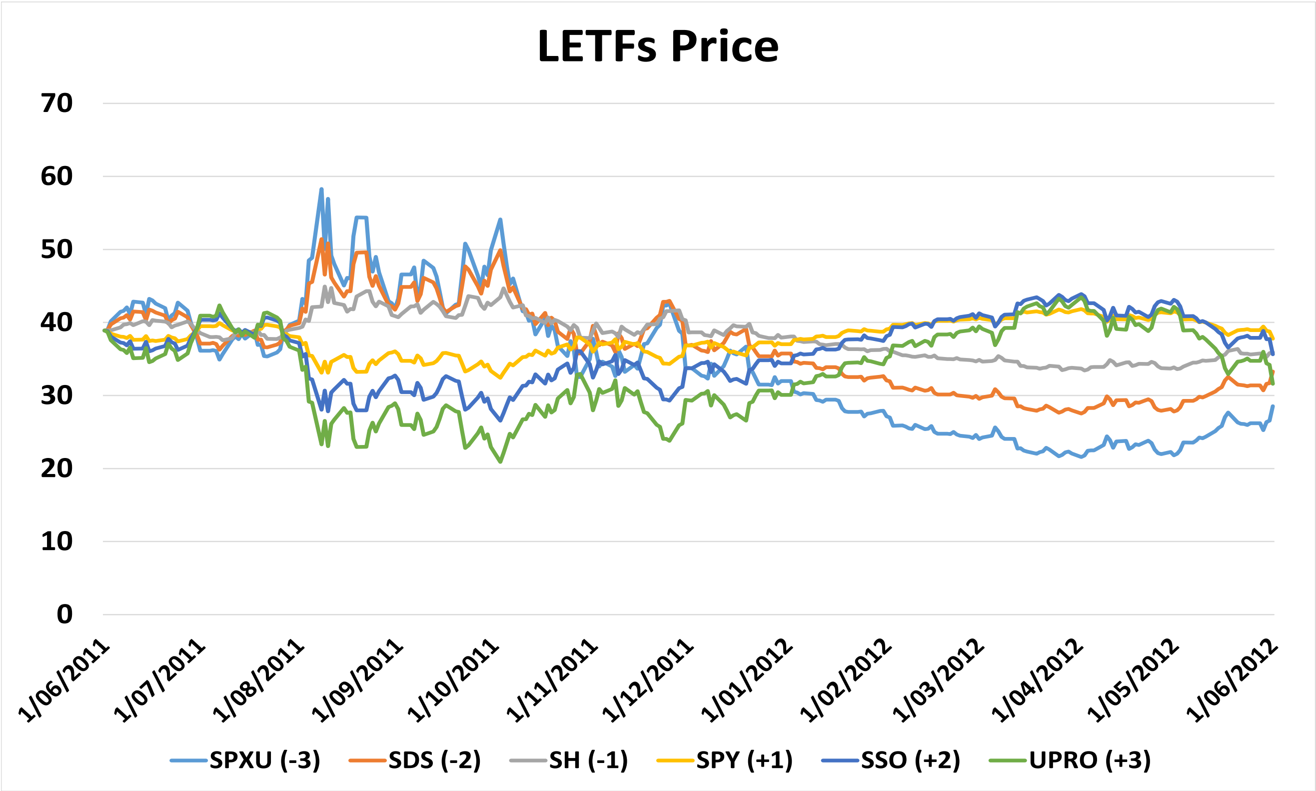
It is similar to the outcome of a CPPI of Black&Perold.

LETF are written on many different assets (Index, Fx, commodities) but by far the most important ones are those on the SP500.

Leveraged ETF on the S&P500

Fund Name	Ticker Name	Leverage Ratio
Proshares UltraPro Short S&P500 ETF	SPXU	-3
Proshares UltraShort S&P500 ETF	SDS	-2
Proshares Short S&P500 ETF	SH	-1
SPDR S&P500 ETF	SPY	+1
Proshares Ultra S&P500 ETF	SSO	+2
Proshares UltraPro S&P500 ETF	UPRO	+3

Leveraged ETF on the S&P500



Note. Times series for LETF from 2011/06/01 to 2012/06/01.

Pricing Options on Leveraged ETF

Option on l_t are available and their price is:

$$\begin{aligned} c(t, l_0, v_0) &= e^{-rt}\mathbb{E}\left[(l_t - K)_+\right] \\ &= e^{-rt}\mathbb{E}\left[\left(l_0\left(\frac{s_t}{s_0}\right)^m e^{\left(\frac{m-m^2}{2}\right)\int_0^t v_u du + (1-m)rt} - K\right)_+\right] \end{aligned}$$

It is like a Power option with a volatility bias component. Also, it an hybrid product:

- Vanilla options $\rightarrow$ ”pure” stock derivative product.
- VIX options $\rightarrow$ ”pure” volatility derivative product.
- LETF options $\rightarrow$ stock/volatility derivative product.

Pricing Options on Leveraged ETF

$$
\begin{aligned}
c(t, l_0, v_0) &= e^{-rt}\mathbb{E}\left[(l_t - K)_+\right] \\
&= e^{-rt}\mathbb{E}\left[\left(l_0\left(\frac{s_t}{s_0}\right)^m e^{\left(\frac{m-m^2}{2}\right)\int_0^t v_u du + (1-m)rt} - K\right)_+\right] \\
&= e^{-rt}\mathbb{E}\left[\left(l_0 e^{x_t} - K\right)_+\right] \\
&= e^{-rt}\int_{-\infty}^{+\infty}\left(l_0 e^x - K\right)_+ f(x)dx,
\end{aligned}
$$

where $f(x)$ is the density of $x_t = m\ln\left(\frac{s_t}{s_0}\right) + \left(\frac{m-m^2}{2}\right)\int_0^t v_u du + (1-m)rt$. We denote by $\varphi(t, z) = \mathbb{E}\left[e^{izx_t}\right]$ the characteristic function of x_t, and we have

$$
\begin{aligned}
c(t, l_0, v_0) &= \frac{e^{-rt}}{2\pi}\int_{-\infty+i\gamma}^{+\infty+i\gamma}\varphi(t, z)\int_{-\infty}^{\infty}\left(l_0 e^x - K\right)_+ e^{-izx}dxdz \\
&= \frac{e^{-rt}}{2\pi}\int_{-\infty+i\gamma}^{+\infty+i\gamma}\varphi(t, z)\frac{K^{1-iz}l_0^{iz}}{iz(iz-1)}dz,
\end{aligned}
$$

where $\gamma = \Im(z) < -1$. Letting $k_0 = \ln\left(\frac{K}{l_0}\right)$, the above equation can be simplified to

$$
c(t, l_0, v_0) = \frac{Ke^{-rt}}{\pi}\int_{0+i\gamma}^{+\infty+i\gamma} e^{-izk_0}\frac{\varphi(t, z)}{iz(iz-1)}dz.
$$

Pricing Options on Leveraged ETF

The characteristic function of $(x_t)_{t\geq 0}$ is linked to the moment generating function of the stock s_t as follows

$$\begin{aligned}\varphi(t,z) &= \mathbb{E}\left[e^{izx_t}\right] = \mathbb{E}\left[e^{iz(m\ln\frac{s_t}{s_0}+\frac{m-m^2}{2}\int_0^t v_u du+(1-m)rt)}\right] \\ &= e^{i(1-m)zrt}G\left(t, izm, iz\frac{m-m^2}{2}, 0, v_0\right)\end{aligned}$$

with

$$G\left(t, z_1, z_2, \ln s_0, v_0\right) = \mathbb{E}[e^{z_1\ln s_t + z_2\int_0^t v_u du}] = e^{z_1\ln s_0 + a(t) + b(t)v_0}$$

where

$$\begin{aligned}a(t) &= \frac{2\kappa\theta}{\sigma^2}\left(t\lambda_- - \ln\left(\frac{\lambda_+ - \lambda_- e^{-\sqrt{\Delta}t}}{\lambda_+ - \lambda_-}\right)\right) + (r-q)z_1 t, \\ b(t) &= (z_1^2 - z_1 + 2z_2)\frac{1-e^{-\sqrt{\Delta}t}}{\lambda_+ - \lambda_- e^{-\sqrt{\Delta}t}}, \\ \lambda_\pm &= \frac{(\kappa - z_1\rho\sigma)\pm\sqrt{\Delta}}{2}, \\ \Delta &= (\kappa - z_1\rho\sigma)^2 - \sigma^2(z_1^2 - z_1 + 2z_2).\end{aligned}$$

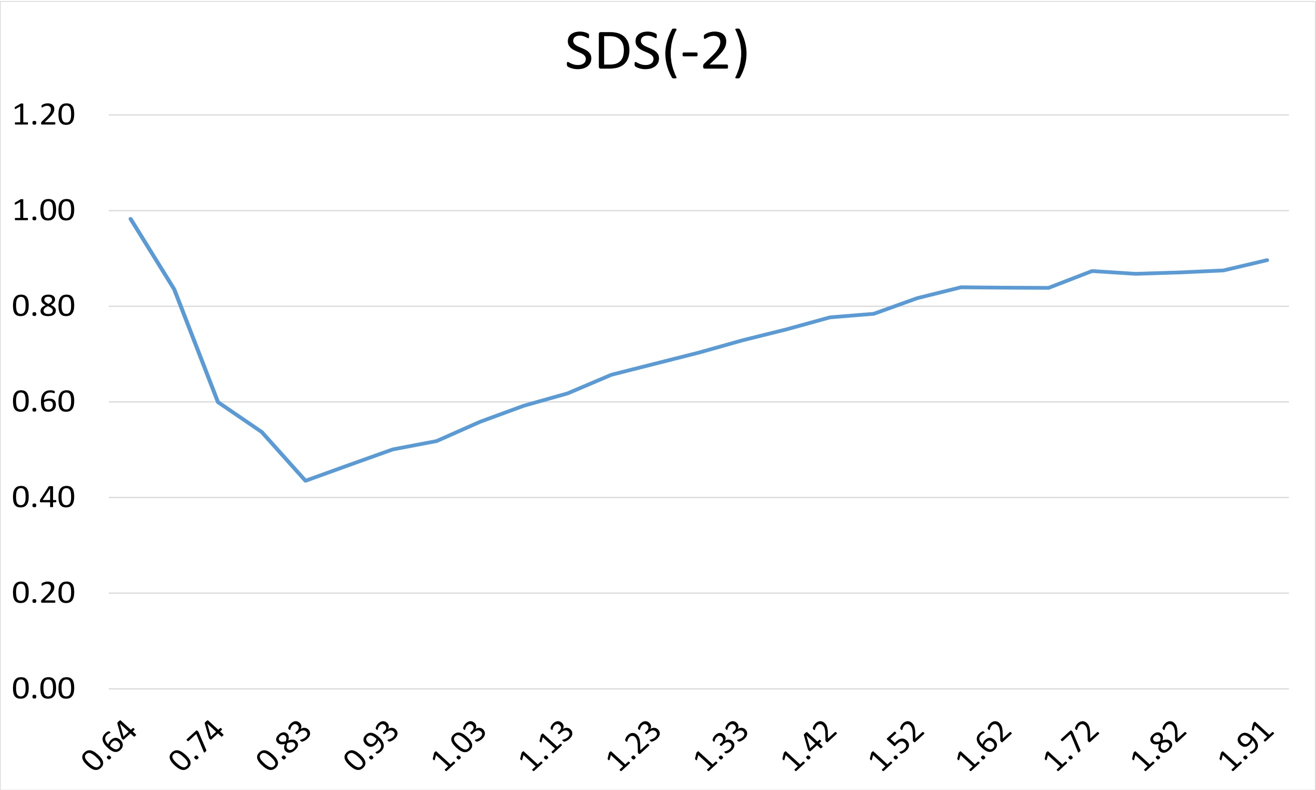
Pricing Options on Leveraged ETF

As a result

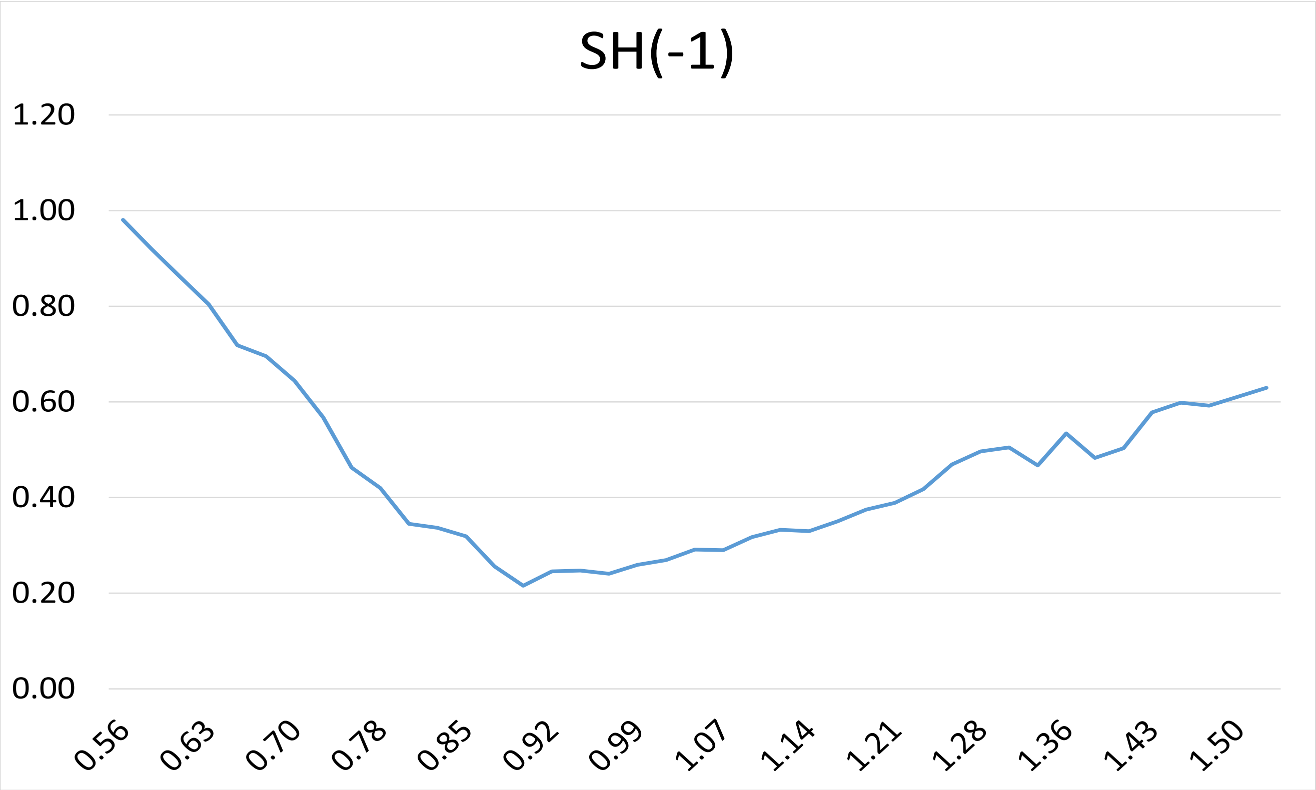
$$c(t, l_0, v_0) = \frac{Ke^{-rt}}{\pi} \int_{0+i\gamma}^{+\infty+i\gamma} e^{-izk_0} \frac{\varphi(t,z)}{iz(iz-1)} dz.$$

can be computed using FFT (as a vanilla option).

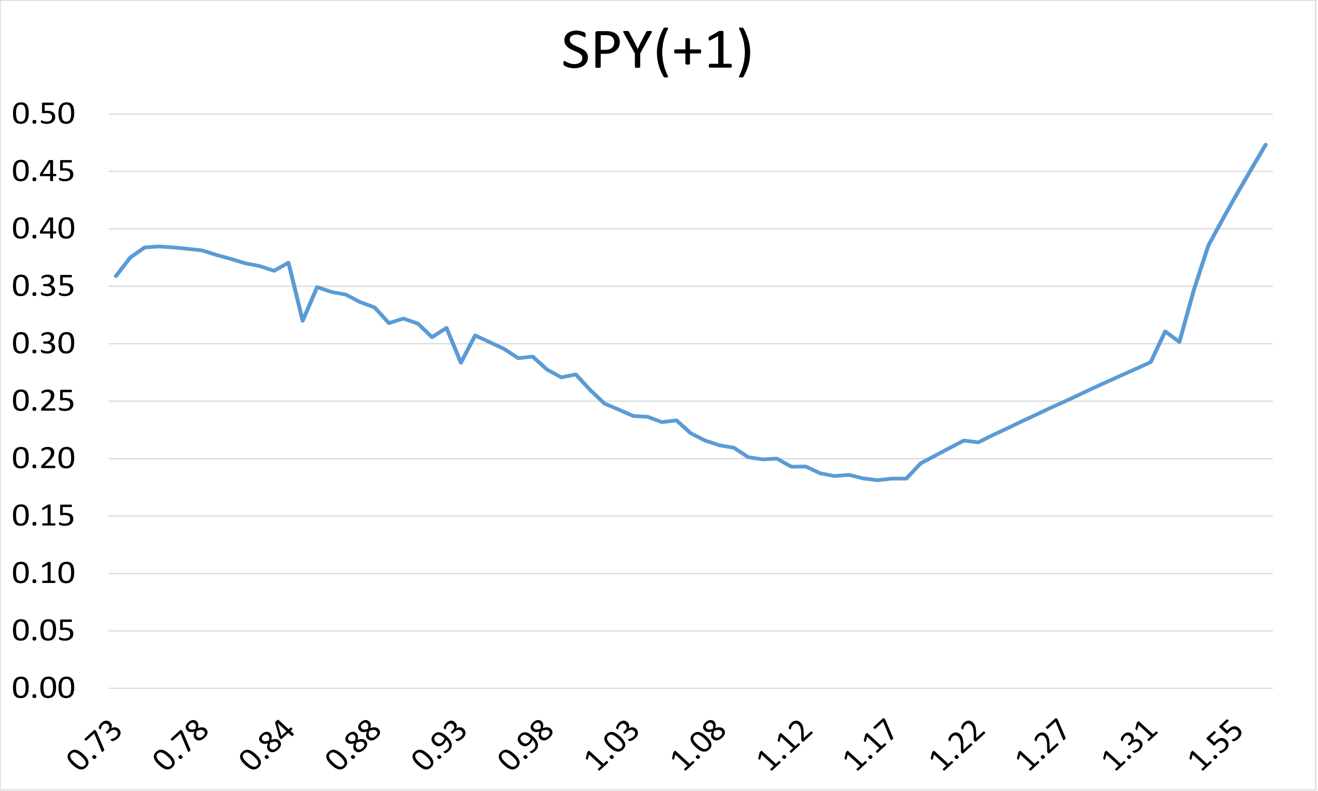
Options on Leveraged ETF



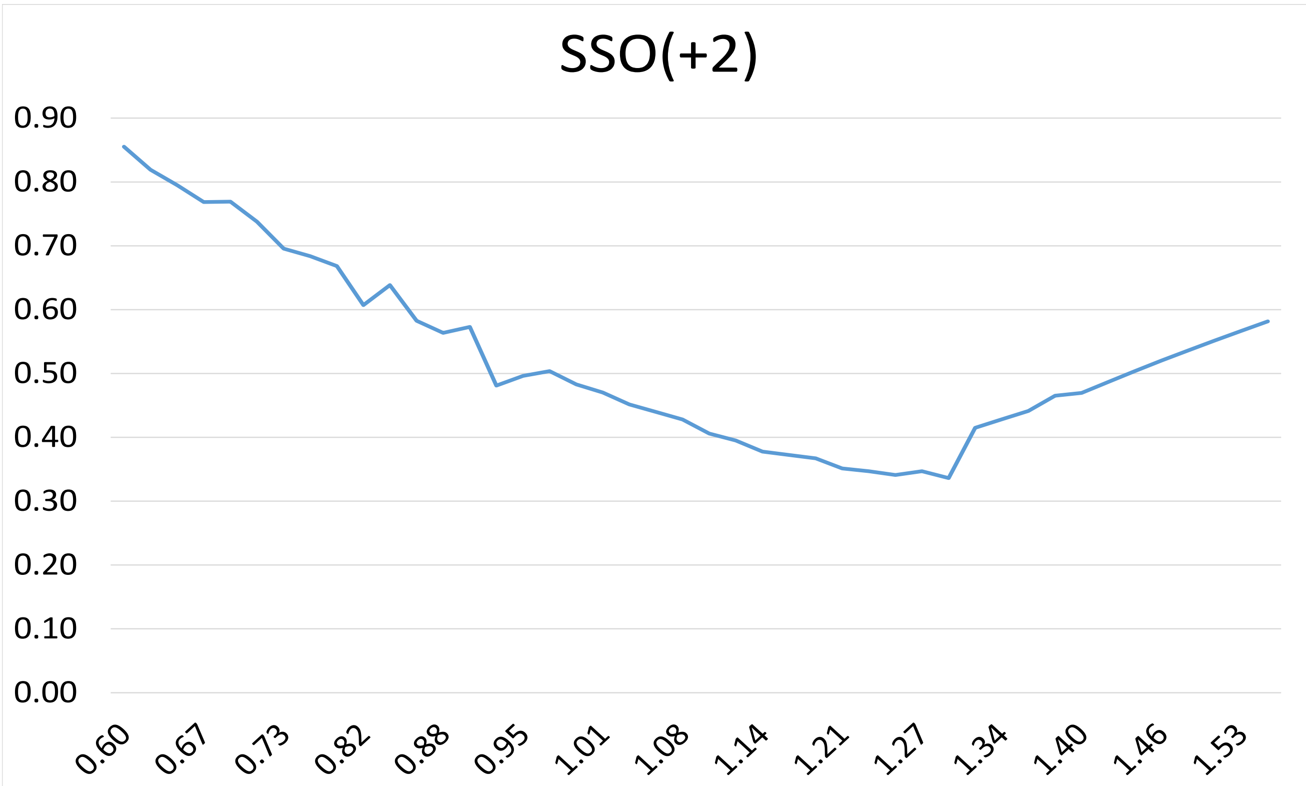
Options on Leveraged ETF



Options on Leveraged ETF



Options on Leveraged ETF



Calibration of options on Leveraged ETF

For each leverage ratio we calibrate the options by solving the following optimization problem for the day 2011/10/24:

$$\min_{v_0,\kappa,\theta,\sigma,\rho} \frac{1}{N} \sum_{i}^{N} \left(\sigma_{imp}^{market}(t_i, K_i, m) - \sigma_{imp}^{model}(t_i, K_i, m)\right)^2$$

	v_0	κ	θ	σ	ρ	ErrorVol	Error Price
SPXU	0.0253	2.6308	0.1438	0.4282	-0.664	3.949×10^{-2}	6.093×10^{-4}
SDS	0.0823	8.3064	0.0781	1.6757	-0.535	2.108×10^{-3}	3.891×10^{-5}
SH	0.0708	2.7108	0.0948	0.9343	-0.320	5.835×10^{-4}	5.152×10^{-6}
SPY	0.0854	2.4816	0.1345	1.6613	-0.739	4.654×10^{-4}	2.13×10^{-6}
SSO	0.0698	3.7320	0.0983	1.3296	-0.638	2.505×10^{-3}	3.697×10^{-5}
UPRO	0.0612	0.5308	0.1549	0.4747	-0.649	3.765×10^{-3}	3.946×10^{-5}

- Calibration errors smaller for -1 and +1.
- Large leverage ratio (absolute value) implies large error.
- The ratios lead to same kind of parameters.
- SH leads to different ρ $\rightarrow$ explains why not used in Ahn et al.?
- Feller condition not satisfied by most of parameter sets.

Options on Leveraged ETF: Open problems

We know that there is a moment explosion problem: for $z > 1$ we can have t^* s.t.

$$\mathbb{E}[s_{t^*}^z] = +\infty \tag{1}$$

The fact that $\rho < 0$ helps but

1. Large m requires higher moments, are they well defined?
2. Here we need to consider negative moments! not studied so far...

Conclusions

- We perform a calibration of options on LETF related to the SP500
- We find that higher leverage ratio leads to larger calibration errors
- We obtain similar parameter sets
- We calibrate all the parameters
- We raise some open problems