Insurance Mathematics and Economics 115 (2024) 122–131

Contents lists available at ScienceDirect

Insurance: Mathematics and Economics

journal homepage: www.elsevier.com/locate/ime

Pricing guaranteed annuity options in a linear-rational Wishart mortality model

José Da Fonseca [a,b,*]

[a] Auckland University of Technology, Business School, Department of Finance, Private Bag 92006, 1142 Auckland, New Zealand
[b] Université Paris 1 Panthéon - Sorbonne, PRISM Sorbonne EA 4101, 17 rue de la Sorbonne, 75005 Paris, France

ARTICLE INFO

JEL classification:
G13
G22

Keywords:
Mortality risk
Interest rate risk
Guaranteed annuity option
Linear-rational Wishart model
Dependence

ABSTRACT

This paper proposes a new model, the linear-rational Wishart model, which allows the joint modelling of mortality and interest rate risks. Within this framework, we obtain closed-form solutions for the survival bond and the survival floating rate bond. We also derive a closed-form solution for the guaranteed annuity option, i.e., an option on a sum of survival (floating rate) bonds, which can be computed explicitly up to a one-dimensional numerical integration, independent of the model dimension. Using realistic parameter values, we provide a model implementation for these complex derivatives that illustrates the flexibility and efficiency of the linear-rational Wishart model.

1. Introduction

A guaranteed annuity option (GAO) is an important insurance derivative that gives the holder the right, but not the obligation, to purchase an annuity that will pay a specified amount at some pre-specified dates, conditional on the holder's survival at those dates.

In Dahl (2004), followed by Biffis (2005), the authors set up the mortality modelling framework that is based on the standard vector affine framework of Duffie and Kan (1996). Following these works, many papers have addressed the problem of pricing mortality derivatives. As these derivatives also have an interest rate dimension, they are inherently hybrid, *i.e.*, interest rate/mortality derivatives. Although most of the literature assumes that interest rates and mortality are independent, there is some recent evidence that interest rates and mortality should be correlated in some circumstances, see Li et al. (2023). From a theoretical point of view, there are a number of papers that make this assumption and focus on the pricing of guaranteed annuity options, see Jalen and Mamon (2009), Liu et al. (2014) and Deelstra et al. (2016).

Deelstra et al. (2016) work is important to us for two reasons. First, it is one of the very few papers in the insurance field to uses the Wishart process, a multidimensional matrix affine process that allows for a fairly general correlation structure between its components, a feature that is important when tackling the difficult problem of modelling dependencies, whether between interest rate and mortality intensity or between different mortality intensities. For another application of the Wishart process to actuarial science, see Chiarella et al. (2014). Secondly, the authors rightly point out that the guaranteed annuity option is similar to an option on a coupon bearing bond, and it is well known that this type of derivative is numerically very difficult to price. In fact, in an exponential affine model the zero coupon bond, or a survival bond when considering a mortality derivative, is an exponential affine function of the state variable. As a result, a coupon bearing bond, or a survival annuity when considering a mortality derivative, is a sum of variables that are exponential affine in the state variable and the density of this sum is unknown, making the pricing of a derivative on this sum, either an option on a coupon bearing bond or a guaranteed annuity option, impossible to compute.

There are some price approximations available in the literature. In finance, an important contribution is Collin-Dufresne and Goldstein (2002) which proposes to approximate the density of the sum of zero coupon bonds using the moments of that sum. Their result relies crucially on the exponential affine property of the zero coupon bond and the fact that the moments of a sum of zero-coupon bonds are also an exponential affine function in the state variable. In the actuarial literature, Zhao and Mamon (2018) also raise the problem that to price a guaranteed annuity option one needs the law of the sum of correlated

---

[*] Correspondence to: Auckland University of Technology, Business School, Department of Finance, Private Bag 92006, 1142 Auckland, New Zealand.
E-mail address: jose.dafonseca@aut.ac.nz.

https://doi.org/10.1016/j.insmatheco.2024.01.004
Received August 2023; Received in revised form November 2023; Accepted 8 January 2024

random variables that is unknown and the get around this difficulty they develop comonotonic upper and lower bounds that are used to approximate the annuity and GAO prices. Apart from these approximations, the pricing of a guaranteed annuity option is a difficult problem that can always be solved using a generic numerical method such as Monte Carlo, see *e.g.*, Liu et al. (2014), Deelstra et al. (2016).

Having to rely on a generic numerical method such as Monte Carlo, or even the finite element method, is not in itself a problem, since as the complexity of the derivatives increases, one will inevitably rely on this solution. However, having a framework that allows exact and fast pricing of (not so) complex derivatives such as the guaranteed annuity option is certainly appealing. To solve this problem, an alternative framework to the exponential affine framework commonly used in actuarial science is needed.

This paper contributes to the literature by developing a joint mortality and interest rate risk model in which the prices of both the survival bond and the survival floating bond are linear-rational functions of the process. The model is based on the Wishart process and thanks to its analytical properties it allows for a fairly general correlation between mortality and interest rate risks. We also show that the joint survival annuity is also a linear-rational function of the Wishart process. More importantly, we show how a guaranteed annuity option can be priced using a simple one-dimensional numerical integration. This is made possible by the linear-rational structure of the model. Using realistic parameter values, we analyse the sensitivity of this derivative to the model parameters and its dependence on the correlation structure.

The structure of the paper is as follows. Section 2 sets out the modelling framework. Section 3 contains the main properties of the linear-rational Wishart model, while Section 4 focuses on the pricing of derivatives. Section 5 implements the model by providing some numerical examples. Section 6 concludes the paper.

2. The modelling framework

Let $(\Omega, \mathcal{G}, (\mathcal{G}_t)_t, \mathbb{P})$ be a filtered probability space where $\mathbb{P}$ stands for the historical probability measure. Denote by τ_x the positive random variable corresponding to the future lifetime of an individual aged x at time 0, admitting a random intensity $(\mu_x(t))_{t\geq 0}$. Following Dahl (2004) and Biffis (2005), τ_x is the first jump time of a nonexplosive $\mathcal{G}$-counting process $(N_t)_{t\geq 0}$ recording at each time $t \geq 0$ whether the individual has died ($N_t \neq 0$) or not ($N_t = 0$). Following Deelstra et al. (2016), let $\mathcal{R}_t$ be the filtration generated by the interest rate process and $\mathcal{M}_t$ the one associated with the mortality intensity. Denote by $\mathcal{F}_t := \mathcal{R}_t \vee \mathcal{M}_t$ the minimal σ-algebra containing $\mathcal{R}_t \cup \mathcal{M}_t$. The filtration $(\mathcal{G}_t)_{t\geq 0}$ denotes the flow of information available over time and includes the knowledge of the evolution of the two state variables above up to time t as well as whether the policyholder has died by that time. As a result, $\mathcal{G}_t := \mathcal{F}_t \vee \mathcal{H}_t$, with $\mathcal{H}_t := \sigma(\mathbf{1}_{\{\tau_x \leq s;\, 0\leq s\leq t\}})$ is the smallest filtration with respect to which τ_x is a stopping time. Still following Deelstra et al. (2016), we assume that $(N_t)_{t\geq 0}$ is a doubly stochastic process or Cox process driven by the sub-filtration $\mathcal{F}$ of $\mathcal{G}$, it implies the following survival probability

$$\mathbb{Q}(\tau_x > T|\mathcal{G}_t) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T \mu_x(s)ds} \middle| \mathcal{G}_t \right],$$

where t is the current time, T is a future date (*i.e.*, $t < T$) often the maturity of a survival bond, $\mathbb{Q}$ stands for an equivalent to $\mathbb{P}$ risk-neutral pricing measure while $\mathbb{E}^{\mathbb{Q}}\left[.|\mathcal{G}_t\right]$ is the expectation under $\mathbb{Q}$ conditional to $\mathcal{G}_t$. Regarding the choice of $\mathbb{Q}$ we follow Deelstra et al. (2016) and references therein and assume that it can be chosen on the basis of available market data. For papers dealing with the non-diversifiable nature of mortality risk leading to an incomplete market, see *e.g.*, Dahl and Møller (2006), Bayraktar et al. (2009), Huang et al. (2014) and Ceci et al. (2015).

To price a contingent insurance claim that pays c_T, which is $\mathcal{F}_T$-measurable, if the insured (with age x at time 0) survives at time T, also known as the survival benefit, one must determine

$$SB_t(c_T, T) := \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(s)ds} \mathbf{1}_{\{\tau_x > T\}} c_T \middle| \mathcal{G}_t \right], \tag{1}$$

where $(r(t))_{t\geq 0}$ is the risk-free rate that is adapted to the filtration $\left(\mathcal{F}_t\right)_{t\geq 0}$ and is correlated with mortality intensity $(\mu_x(t))_{t\geq 0}$. Following the literature the expectation (1) is known to be given by

$$\begin{aligned} SB_t(c_T, T) &:= \mathbf{1}_{\{\tau_x > t\}} \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T (r(s)+\mu_x(s))ds} c_T \middle| \mathcal{G}_t \right] \\ &= \mathbf{1}_{\{\tau_x > t\}} \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T (r(s)+\mu_x(s))ds} c_T \middle| \mathcal{F}_t \right], \end{aligned} \tag{2}$$

where Biffis (2005, Appendix C) is used to show that the conditioning on $\mathcal{G}_t$ can be reduced to that on $\mathcal{F}_t$, see also Deelstra et al. (2016, Eq. (4)) which uses this argument. In the following, $\mathbb{E}^{\mathbb{Q}}\left[.|\mathcal{F}_t\right]$ will be denoted as $\mathbb{E}_t^{\mathbb{Q}}[.]$.

An important quantity is the special case where $c_T = 1$ in (2), it is denoted $P(t,T)$ and is given by

$$P(t,T) := \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T (r(s)+\mu_x(s))ds} \right]. \tag{3}$$

It is commonly called the survival zero-coupon bond or survival bond and corresponds to a life insurance derivative that pays one dollar at time T upon the survival of the insured who is aged x at time 0 (and is still alive at time t). Options on such a survival bond are often analysed in the literature.

A slightly more complex derivative is the survival floating rate zero coupon bond or survival floating rate bond and following Biffis (2005, Eq. (17)) we define it as

$$P_{\mathrm{flo}}(t,T) := \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T (r(s)+\mu_x(s))ds} (1+\gamma r(T)) \right], \tag{4}$$

where $\gamma > 0$ is a percentage of the level of the short rate on the payment date T. We can see that $P(t,T)$ is just a special case of $P_{\mathrm{flo}}(t,T)$ if we take $\gamma = 0$.

Having introduced these survival bonds, we can now define the (life) annuity, which is a collection of survival bonds, these bonds being either survival bonds or survival floating rate bonds, *i.e.*, a derivative that pays an amount on certain pre-specified dates, possibly date-dependent, conditional on whether the policyholder is still alive on those dates. Let $T_1, \dots, T_N$ be a set of yearly spaced dates such that $T_N = x^* - x - 1$, where x^* is the maximum age the policyholder can reach. Consider the derivative that pays out at time T_i the amount $(1 + \gamma r(T_i))$ conditional on the survival of the policyholder, the value of the derivative at time $t = 0$ is given by

$$\sum_{i=1}^N P_{\mathrm{flo}}(0, T_i), \tag{5}$$

with $P_{\mathrm{flo}}(0,T)$ given by (4) and in the special case $\gamma = 0$ we get

$$\sum_{i=1}^N P(0, T_i). \tag{6}$$

The derivative of interest to us is the guaranteed annuity option, *i.e.*, an option on either (5) or (6). We follow Deelstra et al. (2016, Section 3.2) and references therein for the definition. Consider a guaranteed annuity option with maturity T and let be $T_1, \dots, T_N$ a set of yearly spaced dates such that $T_1 = T + 1\mathrm{Y}$ (where 1Y stands for 1 year) and $T_N = x^* - (x+T) - 1$ where x^* is the maximum age the policyholder can reach. The option gives the holder of the derivative at time T, the maturity of the option, the amount

$$\begin{aligned} C(T, T_N) &:= \max\left(g \sum_{i=1}^N P_{\mathrm{flo}}(T, T_i), 1 \right) \\ &= 1 + g\left(\sum_{i=1}^N P_{\mathrm{flo}}(T, T_i) - 1/g \right)_+ \end{aligned}$$

$$= 1 + g\bar{C}(T, T_N),$$

where g is the fixed rate called the guaranteed annuity rate and

$$\bar{C}(T, T_N) := \left(\sum_{i=1}^{N} P_{\mathrm{flo}}(T, T_i) - 1/g \right)_+ , \tag{7}$$

with $(x)_+ = \max(x, 0)$. Therefore, the guaranteed annuity option value at time $t = 0$ is given by

$$\begin{aligned} C(0, T, T_N) &= \mathbb{E}^{\mathbb{Q}} \left[e^{-\int_0^T (r(s)+\mu_x(s))ds} C(T, T_N) \right] \\ &= P(0, T) + g\bar{C}(0, T, T_N), \end{aligned}$$

with

$$\begin{aligned} \bar{C}(0, T, T_N) &:= \mathbb{E}^{\mathbb{Q}} \left[e^{-\int_0^T (r(s)+\mu_x(s))ds} \bar{C}(T, T_N) \right] \\ &= \mathbb{E}^{\mathbb{Q}} \left[e^{-\int_0^T (r(s)+\mu_x(s))ds} \left(\sum_{i=1}^{N} P_{\mathrm{flo}}(T, T_i) - 1/g \right)_+ \right], \end{aligned} \tag{8}$$

and it is sufficient to focus on the calculation of (8), which is a call option on a survival annuity with a strike of $1/g$, as the payoff (7) clearly shows.

Remark 2.1. The formula (8) shows the difficulty of pricing such a derivative in the exponential affine framework. To compute the expectation, the density of the sum of survival zero-coupon bonds is needed, and if each (survival) zero-coupon is an exponential affine function of the state variable, this density is unknown. This difficulty is well explained in Biffis and Millossovich (2006, p. 36), Cairns (2004, p. 117) and Collin-Dufresne and Goldstein (2002), with the latter work proposing an approximation. The problem, then, is to develop a framework for which the survival zero-coupon bond has a functional form such that the annuity has an explicit distribution, thereby leading to a guaranteed annuity option price that can be exactly computed.

3. The linear-rational Wishart model

We follow Deelstra et al. (2016) and use the Wishart process for modelling the dependence between interest rates and mortality intensity. We recall the main property of the process and refer to Deelstra et al. (2016) for a more detailed bibliography.

Given a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ we denote by $\mathbb{E}[\cdot]$ (resp. $\mathbb{E}_t[\cdot] := \mathbb{E}[\cdot|\mathcal{F}_t]$) the expectation (resp. conditional expectation) under the historical probability measure $\mathbb{P}$. The Wishart process satisfies the matrix stochastic differential equation

$$dv_t = (\omega + mv_t + v_t m^\top)dt + \sqrt{v_t} dw_t \sigma + \sigma^\top dw_t^\top \sqrt{v_t}, \tag{9}$$

where v_t is an $n \times n$ matrix belonging to the set of positive definite matrices denoted $\mathbb{S}_n^{++}$, m, σ belong to the set of $n \times n$ real matrices denoted $\mathsf{M}(n)$, $\{w_t; t \geq 0\}$ is a matrix Brownian motion of dimension $n \times n$ (*i.e.*, a matrix of n^2 independent scalar Brownian motions) under the probability measure $\mathbb{P}$ and $\cdot^\top$ stands for the matrix transpose.

The matrix $\omega \in \mathbb{S}_n^{++}$ satisfies certain conditions involving $\sigma^\top \sigma$ to ensure the positivity of the matrix process v_t. The variable $\sqrt{v_t}$ is well defined, since $v_t \in \mathbb{S}_n^{++}$. The matrix m is such that $\{\Re(\lambda_i^m) < 0; i = 1, \ldots, n\}$ where $\lambda_i^m \in \mathsf{Spec}(m)$ for $i = 1, \ldots, n$ and $\mathsf{Spec}(m)$ is the spectrum of the matrix m, while $\Re(\cdot)$ is the real part. The matrix σ belongs to $\mathsf{GL}_n(\mathbb{R})$ the general linear group over $\mathbb{R}$ (*i.e.*, the set of real invertible matrices). Thanks to the invariance of the law of the Brownian motion to rotations and the polar decomposition of σ, we can assume that $\sigma \in \mathbb{S}_n^{++}$. We denote by e_{ij} the basis of $\mathsf{M}(n)$, it is the $n \times n$ matrix with 1 in the (i, j) place and zero elsewhere.

Remark 3.1. Note that the dynamic of $(v_t)_{t\geq 0}$ is given under the historical probability measure $\mathbb{P}$.

The infinitesimal generator of the Wishart process is given by Bru (1991):

$$\mathcal{L} = \mathrm{tr}[(\omega + mv + vm^\top)D + 2vD\sigma^2 D], \tag{10}$$

where D is the $n \times n$ matrix operator $D_{ij} := \partial_{v_{ij}}$.

When $n = 2$, using (9), the following relations can be established for the quadratic covariation of the components of the Wishart process:

$$d\langle v_{11,\cdot}, v_{11,\cdot} \rangle_t = 4v_{11,t}(\sigma^2)_{11} dt, \tag{11}$$

$$d\langle v_{22,\cdot}, v_{22,\cdot} \rangle_t = 4v_{22,t}(\sigma^2)_{22} dt, \tag{12}$$

$$\begin{aligned} d\langle v_{12,\cdot}, v_{12,\cdot} \rangle_t &= v_{11,t}(\sigma^2)_{22} dt + 2v_{12,t}(\sigma^2)_{12} dt \\ &\quad + v_{22,t}(\sigma^2)_{11} dt, \\ d\langle v_{11,\cdot}, v_{12,\cdot} \rangle_t &= 2v_{11,t}(\sigma^2)_{12} dt + 2v_{12,t}(\sigma^2)_{11} dt, \\ d\langle v_{12,\cdot}, v_{22,\cdot} \rangle_t &= 2v_{12,t}(\sigma^2)_{22} dt + 2v_{22,t}(\sigma^2)_{12} dt, \\ d\langle v_{11,\cdot}, v_{22,\cdot} \rangle_t &= 4v_{12,t}(\sigma^2)_{12} dt, \end{aligned} \tag{13}$$

where $(\sigma^2)_{ij}$ is the (i, j) element of the σ^2 matrix.

The Wishart process is affine, *i.e.*, the moment generating function is exponentially affine in the state variable and is given by

$$\Phi(t, \theta_1, \theta_2, v) := \mathbb{E}\left[\exp\left(\mathrm{tr}[\theta_1 v_t] + \int_0^t \mathrm{tr}[\theta_2 v_u] du \right) \right], \tag{14}$$

where θ_1, θ_2 belong to $\mathbb{S}_n$ the set of real $n \times n$ symmetric matrices, $\mathrm{tr}[\cdot]$ stands for the trace of a matrix.

Following Grasselli and Tebaldi (2008), it is possible to prove that

$$\Phi(t, \theta_1, \theta_2, v_0) = \exp\left(\mathrm{tr}[a(t, \theta_1, \theta_2)v_0] + b(t, \theta_1, \theta_2) \right), \tag{15}$$

with the deterministic functions $(a(t, \theta_1, \theta_2), b(t, \theta_1, \theta_2))$, where $a(t, \theta_1, \theta_2)$ is an $n \times n$ matrix function and $b(t, \theta_1, \theta_2)$ a scalar function, satisfying the system

$$a' = am + m^\top a + 2a\sigma^2 a + \theta_2, \tag{16}$$

$$b' = \mathrm{tr}[\omega a], \tag{17}$$

with initial conditions $a(0, \theta_1, \theta_2) = \theta_1$ and $b(0, \theta_1, \theta_2) = 0$. As usual $\cdot'$ denotes the time derivative.

The equation (16) is a matrix Riccati ordinary differential equation (ODE) whose solution is given by

$$a(t, \theta_1, \theta_2) = (\theta_1 A_{12}(t) + A_{22}(t))^{-1}(\theta_1 A_{11}(t) + A_{21}(t)), \tag{18}$$

where

$$\begin{pmatrix} A_{11}(t) & A_{12}(t) \\ A_{21}(t) & A_{22}(t) \end{pmatrix} := \exp\left\{ t \begin{pmatrix} m & -2\sigma^2 \\ \theta_2 & -m^\top \end{pmatrix} \right\},$$

while (17), together with the corresponding initial condition, leads to $b(t)$ after integration. Note that when using the vector affine process of Duffie and Kan (1996), instead of (16), one obtains a vector Riccati ODE that cannot be explicitly integrated, so a Runge-Kutta numerical scheme is used to solve the ODE. In this case, (17) is also integrated using a regular Runge-Kutta scheme. We have found that computing $b(t) = \int_0^t \mathrm{tr}[\omega a(s)]ds$ using a quadratic integration scheme is faster because the algorithm is adaptive and requires fewer calls or evaluations of the function $a(s)$ with $s \in [0, t]$. In fact, the quadratic adaptive integration scheme divides the integration interval into sub-intervals and evaluates the integrand on each sub-interval by further sub-division. The subdivision of a given interval stops as soon as the integral calculated over it does not vary much (as a result of an additional subdivision), with the consequence that intervals over which the integrand varies little lead to very few evaluations of the integrand.

Remark 3.2. Let us recall the following result, which appears in Benabid et al. (2009). Suppose $n = 2$, m is diagonal, $\theta_1 = u e_{11}$ with $u \in \mathbb{R}$ and $\theta_2 = 0_n$ with 0_n the null matrix in $\mathsf{M}(n)$. Then $\Phi(t, \theta_1, \theta_2, v_0)$ is the moment generating function of $v_{11,t}$ and the solution of (16) is a matrix, denoted a, whose only non-zero element is the $(1,1)$ element that solves the (scalar) ODEs:

$$a'_{11} = 2a_{11}m_{11} + 2a_{11}^2(\sigma^2)_{11},$$

$$b' = \omega_{11}a_{11}.$$

These ODEs are those of a CIR process with mean reverting parameter $-2m_{11}$, long term mean value $\omega_{11}/(-2m_{11})$ and volatility $(\sigma^2)_{11}$, see Lamberton and Lapeyre (2000, Proposition 6.2.5). A similar conclusion can be obtained for $v_{22,t}$. An important consequence is that $v_{11,t}$ and $v_{22,t}$ are two CIR processes that belong to an affine model and that can be correlated as $d\langle v_{11,.}, v_{22,.}\rangle_t$ given (13) is not zero if $(\sigma^2)_{12} \neq 0$ which is equivalent to $\sigma_{12} \neq 0$ since $(\sigma^2)_{12} = \sigma_{12}(\sigma_{11} + \sigma_{22})$ (*i.e.*, σ is not diagonal).

From (9) it is easy to get

$$d\mathbb{E}[v_t] = (\omega + m\mathbb{E}[v_t] + \mathbb{E}[v_t]m^\top)dt, \tag{19}$$

but when it comes to implementation, it is advisable to rewrite (9) using the vec operator, it leads to

$$\begin{aligned} d\mathrm{vec}\left(v_t\right) &= \left(\mathrm{vec}\left(\omega\right) + \left(I_n \otimes m + m \otimes I_n\right)\mathrm{vec}\left(v_t\right)\right)dt \\ &\quad + \left(\sigma \otimes \sqrt{v_t}\right)\mathrm{vec}\left(dw_t\right) + \left(\sqrt{v_t} \otimes \sigma\right)\mathrm{vec}(dw_t^\top) \\ &= \left(\mathsf{b} + \mathsf{A}\mathrm{vec}\left(v_t\right)\right)dt + \left(I_{n^2} + K_{nn}\right)\left(\sigma \otimes \sqrt{v_t}\right)\mathrm{vec}\left(dw_t\right), \end{aligned} \tag{20}$$

where I_n is the identity matrix of $\mathsf{M}(n)$, $\otimes$ denotes the Kronecker product while K_{nn} stands for the commutation matrix that operates on $\mathsf{M}(n)$, see for example Lütkepohl (2005, Appendix A). Using (20), it is easy to obtain

$$\frac{d}{dt}\mathbb{E}\left[\mathrm{vec}\left(v_t\right)\right] = \mathsf{b} + \mathsf{A}\mathbb{E}\left[\mathrm{vec}\left(v_t\right)\right],$$

which can be explicitly integrated

$$\mathbb{E}[\mathrm{vec}(v_t)] = e^{\mathsf{A}t}\mathrm{vec}\left(v_0\right) + \mathsf{A}^{-1}\left(e^{\mathsf{A}t} - I_{n^2}\right)\mathsf{b}. \tag{21}$$

The following lemma is useful as it gives the mean value of a linear function of the Wishart process.

Lemma 3.3. *Let* $u_0 \in \mathsf{M}(n)$ *then there exist* $a_0(t) \in \mathsf{M}(n)$ *and* $b_0(t) \in \mathbb{R}$ *such that*

$$\mathbb{E}\left[\mathrm{tr}[u_0 v_t]\right] = \mathrm{tr}[a_0(t)v_0] + b_0(t). \tag{22}$$

Proof. With (21) one can obtain

$$\begin{aligned} \mathbb{E}\left[\mathrm{tr}[u_0 v_t]\right] &= \mathrm{vec}(u_0^\top)^\top\mathbb{E}\left[\mathrm{vec}(v_t)\right] \\ &= \left(e^{\mathsf{A}^\top t}\mathrm{vec}(u_0^\top)\right)^\top\mathrm{vec}(v_0) \\ &\quad + \mathrm{vec}(u_0^\top)^\top\mathsf{A}^{-1}\left(e^{\mathsf{A}t} - I_{n^2}\right)\mathsf{b} \\ &= \mathrm{tr}[a_0(t)v_0] + b_0(t), \end{aligned}$$

with $a_0(t) \in \mathsf{M}(n)$ such that $\mathrm{vec}(a_0(t)^\top) = e^{\mathsf{A}^\top t}\mathrm{vec}(u_0^\top)$. ☐

An alternative proof of the previous lemma is also useful as it gives more precise results.

Lemma 3.4. *Let* $u_0 \in \mathsf{M}(n)$ *then there exist a matrix function* $a_0(t) \in \mathsf{M}(n)$ *and a scalar function* $b_0(t)$ *such that*

$$\mathbb{E}\left[\mathrm{tr}[u_0 v_t]\right] = \mathrm{tr}[a_0(t)v_0] + b_0(t), \tag{23}$$

with

$$a_0(t) = e^{tm^\top}u_0 e^{tm}, \tag{24}$$

$$b_0(t) = \mathrm{tr}\left[u_0\int_0^t e^{(t-s)m^\top}\omega e^{(t-s)m}ds\right]. \tag{25}$$

In particular, if $u_0 \in \mathbb{S}_n$ *then* $a_0(t) \in \mathbb{S}_n$ *and if* $u_0 \in \mathbb{S}_n^+$ *then* $a_0(t) \in \mathbb{S}_n^+$ *and* $b_0(t) > 0$.

Proof. Using (19) one gets

$$\mathbb{E}[v_t] = e^{tm}v_0 e^{tm^\top} + \int_0^t e^{(t-s)m}\omega e^{(t-s)m^\top}ds,$$

and as $\mathbb{E}[\mathrm{tr}[u_0 v_t]] = \mathrm{tr}\left[u_0\mathbb{E}[v_t]\right]$ the equality (23) follows with $a_0(t)$ and $b_0(t)$ given by (24) and (25), respectively. ☐

If the system is stationary, which requires the eigenvalues of m to be negative and therefore the eigenvalues of A are also negative thanks to the Kronecker product property, it leads to $\bar{v}_\infty := \lim_{t\to+\infty}\mathbb{E}\left[v_t\right]$, which solves the equation

$$\mathrm{vec}\left(\bar{v}_\infty\right) = -\mathsf{A}^{-1}\mathsf{b}, \tag{26}$$

and corresponds to the matrix equation

$$m\bar{v}_\infty + \bar{v}_\infty m^\top = -\omega. \tag{27}$$

The Wishart process was first defined and analysed in Bru (1991) under the assumption that $\omega = \beta\sigma^2$ with $\beta \in \mathbb{R}_+$ such that $\beta \geq n+1$ to ensure that $v_t \in \mathbb{S}_n^{++}$. This specification will be referred to as the Bru case. In this particular case (17) can be explicitly integrated, some computations lead to

$$\begin{aligned} \Phi(t, \theta_1, 0_n, v) &= \mathbb{E}\left[\exp\left(\mathrm{tr}[\theta_1 v_t]\right)\right] \\ &= \frac{\mathrm{etr}\left[\frac{\Lambda_t^\top}{2}(2\Xi_t\theta_1)(I_n - 2\Xi_t\theta_1)^{-1}\right]}{\det(I_n - 2\Xi_t\theta_1)^{\beta/2}}, \end{aligned} \tag{28}$$

with $\Xi_t := \int_0^t e^{(t-s)m}\sigma^2 e^{(t-s)m^\top}ds$, $\Lambda_t := \Xi_t^{-1}e^{mt}v_0 e^{m^\top t}$ and $\mathrm{etr}[.] = \exp(\mathrm{tr}[.])$, see Grasselli and Tebaldi (2008). We can recognise the non-central Wishart distribution, see Gupta and Nagar (2000, Chap. 3.5).

Following the introduction of the Wishart process in finance by Gouriéroux and Sufana (2010), it was later extended in Mayerhofer et al. (2011) (see also Cuchiero et al. (2011)) to the case $\omega \in \mathbb{S}_n^{++}$ and proved that if

$$\omega \succeq \beta\sigma^2, \tag{29}$$

with $\beta \geq n+1$ (where (29) means that $\omega - \beta\sigma^2 \in \mathbb{S}_n^{++}$) then $v_t \in \mathbb{S}_n^{++}$.

We build on the potential approach proposed by Rogers (1997), see also Björk (2009, Chap. 28.5), whose main idea is to first specify the pricing kernel with the right mathematical properties, it has to be a positive supermartingale, and then to derive the corresponding instantaneous short interest rate, which is positive by construction. The classical approach is the other way round, *i.e.*, one usually specifies the positive short interest rate and derives the corresponding pricing kernel, which gives the value of the zero coupon bond. Since the short interest rate must be positive, the square root process or the vector affine process of Duffie and Kan (1996) is often used. The potential approach thus reverses the modelling strategy, and this proved to be fruitful in the context of multi-curve interest rate derivatives analysed in Da Fonseca et al. (2022) on which this paper is based. As we shall see, it will prove to be a very powerful approach to solving the problem of pricing guaranteed annuity options.

First, we define a pricing kernel as:

$$\zeta_t := e^{-\alpha t}(1+\mathrm{tr}[u_0 v_t]), \tag{30}$$

with $\alpha \in \mathbb{R}_+$, $u_0 = u_1 + u_2$ with $u_1 \in \mathbb{S}_n^+$ (the set of positive semidefinite matrices) and $u_2 \in \mathbb{S}_n^+$ (so that $u_0 \in \mathbb{S}_n^+$) and $(v_t)_{t\geq 0}$ a Wishart process. The pricing kernel can be rewritten as follows. Define the positive function $f : \mathbb{S}_n^+ \to \mathbb{R}^+$ such that $f(v) := 1+\mathrm{tr}[u_0 v]$ then the pricing kernel is written as $\zeta_t = e^{-\alpha t} f(v_t)$ and is positive since f is. Define $g(v) := (\alpha - \mathcal{G}) f(v)$ and assume that it is a positive function for sufficiently large α. We see that the pricing kernel allows us to compute $P(t,T)$ given by (3), or any other mortality derivative, if we assume the following change of probability measure

$$\zeta_t = e^{-\int_0^t (r(s)+\mu_x(s))ds} \left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t}. \tag{31}$$

Indeed, suppose that an asset Π_t pays the amount c_T at time T (*i.e.*, we assume c_T is $\mathcal{F}_T$-measurable), conditional on the policyholder being alive at that time, then its value at time t is given by

$$\Pi_t = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T (r(s)+\mu_x(s))ds} c_T\right] \tag{32}$$

$$= \mathbb{E}_t\left[\frac{\zeta_T}{\zeta_t} c_T\right], \tag{33}$$

which, depending on the complexity of c_T, can be computed explicitly for an appropriate choice of the pricing kernel $(\zeta_t)_{t\geq 0}$. Note that Equations (31), (32) and (33) correspond for $c_T = 1$ to Rogers (1997, Eqs. (1.3),(1.2),(1.4)). The next section shows how to handle the derivatives presented in section 2.

Remark 3.5. Note that even if Π_t is expressed as an expectation under $\mathbb{Q}$, as (32) clearly shows, the change of probability measure (31) leads us to calculate the value of Π_t under the historical probability $\mathbb{P}$. It explains the Remark 3.1 which emphasises the fact that the dynamic of $(v_t)_{t\geq 0}$ is given under $\mathbb{P}$.

4. Pricing of derivatives

To illustrate the methodology, let us start with the survival bond, which can be explicitly calculated as shown in the following proposition.

Proposition 4.1. *In the linear-rational Wishart model the survival bond* $P(t,T)$ *given by* (3) *is known explicitly as we have*

$$P(t,T) = e^{-\alpha(T-t)} \frac{1+b_0(T-t)+\mathrm{tr}[a_0(T-t)v_t]}{1+\mathrm{tr}[u_0 v_t]}, \tag{34}$$

where $b_0(t) := b_1(t)+b_2(t)$ *with* $b_1(t)$ *and* $b_2(t)$ *two scalar functions and* $a_0(t) := a_1(t)+a_2(t)$ *with* $a_1(t)$ *and* $a_2(t)$ *two matrix functions that are explicitly known.*

Proof. Using (3) and the change of probability measure (31) and the expression for the pricing kernel (30) we get

$$\begin{aligned} P(t,T) &= \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T (r(s)+\mu_x(s))ds}\right] \\ &= \mathbb{E}_t\left[\frac{\zeta_T}{\zeta_t}\right] \\ &= e^{-\alpha(T-t)} \frac{1+\mathbb{E}_t[\mathrm{tr}[u_0 v_T]]}{1+\mathrm{tr}[u_0 v_t]}. \end{aligned}$$

But

$$\mathbb{E}_t[\mathrm{tr}[u_0 v_T]] = \mathrm{vec}(u_0^\top)^\top \mathbb{E}_t[\mathrm{vec}(v_T)],$$

and according to Lemma 3.3 or 3.4 we get that there exist $b_0(T-t) = b_1(T-t)+b_2(T-t)$ with $b_1(t)$ and $b_2(t)$ two scalar functions and $a_0(T-t) = a_1(T-t)+a_2(T-t)$ with $a_1(t)$ and $a_2(t)$ two matrix functions such that $\mathrm{vec}(u_0)^\top \mathbb{E}_t\left[\mathrm{vec}\left(v_T\right)\right] = b_0(T-t)+\mathrm{tr}[a_0(T-t)v_t]$. ☐

Remark 4.2. The survival bond (34) is a linear-rational function of the Wishart process whose dynamic is given under $\mathbb{P}$. Consequently, to estimate the model from a time series of survival bonds using a Kalman filter approach, the transition probability of the process under the historical probability $\mathbb{P}$ is required, and in the model proposed here the dynamic of $(v_t)_{t\geq 0}$ is already given under this probability measure. This contrasts with the standard approach used in the literature, where the dynamic of the state variable is given under the risk-neutral probability $\mathbb{Q}$. In this case, to estimate the model using a time series (*i.e.*, several days) of survival bonds, since the dynamic of the state variable under the historical probability measure $\mathbb{P}$ is required, it is necessary to specify the market price of risk, that is, the change in the probability measure from the risk-neutral probability measure to the historical probability measure. Following the work of Duffee (2002), it is common practice to specify the so-called completely affine parametrisation (in the terminology of Duffee (2002, p. 410-411)) for the risk premia.

According to Rogers (1997, Eq. (1.5)) or Björk (2009, Eq. (28.42)), we have

$$\begin{aligned} r(t)+\mu_x(t) &= \frac{(\alpha-\mathcal{L})f}{f} \\ &= \frac{\alpha+\alpha\mathrm{tr}[u_0 v_t]-\mathrm{tr}[u_0\omega]-2\mathrm{tr}[u_0 m v_t]}{1+\mathrm{tr}[u_0 v_t]}, \end{aligned}$$

and we define the short interest rate and the mortality intensity as

$$r(t) = \frac{\alpha/2+\alpha\mathrm{tr}[u_1 v_t]-\mathrm{tr}[u_1\omega]-2\mathrm{tr}[u_1 m v_t]}{1+\mathrm{tr}[u_0 v_t]}, \tag{35}$$

$$\mu_x(t) = \frac{\alpha/2+\alpha\mathrm{tr}[u_2 v_t]-\mathrm{tr}[u_2\omega]-2\mathrm{tr}[u_2 m v_t]}{1+\mathrm{tr}[u_0 v_t]}. \tag{36}$$

To understand how the positivity enters into the problem let us consider the case $n=2$. By assumption $u_1 \in \mathbb{S}_n^+$ and $u_2 \in \mathbb{S}_n^+$ so that $\mathrm{tr}[u_1\omega]$, $\mathrm{tr}[u_2\omega]$, $\mathrm{tr}[u_1 v_t]$ and $\mathrm{tr}[u_2 v_t]$ are positive as $\omega \in \mathbb{S}_n^{++}$ while v_t belongs to $\mathbb{S}_n^{++}$. If $\mathrm{tr}[u_1 m v_t] < 0$ and $\mathrm{tr}[u_2 m v_t] < 0$ then as long as $\alpha/2 > \max(\mathrm{tr}[u_1\omega], \mathrm{tr}[u_2\omega])$ then $r(t)$ and $\mu_x(t)$ are positive. To understand the positivity problem a bit better, consider the simple case $n=2$, m diagonal, $u_1 = e_{11}$ and $u_2 = e_{22}$, then in this particular case $r(t)$ and $\mu_x(t)$ can be rewritten as

$$r(t) = \frac{\alpha/2+\alpha v_{11,t}-\omega_{11}-2m_{11}v_{11,t}}{1+v_{11,t}+v_{22,t}}, \tag{37}$$

$$\mu_x(t) = \frac{\alpha/2+\alpha v_{22,t}-\omega_{22}-2m_{22}v_{22,t}}{1+v_{11,t}+v_{22,t}}, \tag{38}$$

where the condition $\alpha/2 > \max(\omega_{11}, \omega_{22})$ clearly shows that it is sufficient to ensure the positivity of both $r(t)$ and $\mu_x(t)$. Furthermore, the correlation between these two variables is driven by $d\langle v_{11,.}, v_{22,.}\rangle_t$ given by (13) which can have any sign. In the more general case, the instantaneous covariance between the short interest rate and the mortality intensity depends on $d\langle \mathrm{tr}[h_1 v_.], \mathrm{tr}[h_2 v_.]\rangle_t$ with $h_1 = \alpha u_1 - 2u_1 m$ and $h_2 = \alpha u_2 - 2u_2 m$. Note that for $h_1 \in \mathsf{M}(n)$, $\mathrm{tr}[h_1 v_t] = \mathrm{tr}[(h_1+h_1^\top)v_t]/2$ with $(h_1+h_1^\top)/2 \in \mathbb{S}_n$. Therefore using (20) together with $K_{nn}\mathrm{vec}(u) = \mathrm{vec}(u)$ if $u \in \mathbb{S}_n$ and the property of the Kronecker product we get

$$\begin{aligned} d\langle \mathrm{tr}[h_1 v_.], \mathrm{tr}[h_2 v_.]\rangle_t &= \frac{1}{4}\mathrm{vec}(h_1+h_1^\top)^\top \left(I_{n^2}+K_{nn}\right)\left(\sigma^2 \otimes v_t\right) \\ &\quad \left(I_{n^2}+K_{nn}\right)\mathrm{vec}(h_2+h_2^\top)dt \\ &= \mathrm{vec}(h_1+h_1^\top)^\top \left(\sigma^2 \otimes v_t\right)\mathrm{vec}(h_2+h_2^\top)dt \\ &= \mathrm{tr}[(h_1+h_1^\top)v_t(h_2+h_2^\top)\sigma^2]dt, \end{aligned}$$

which illustrates the effect of the parameters on the covariance. From now on we will assume that there exists α such that $r(t)$ given by (35) and $\mu_x(t)$ given by (36) are positive. Note also the difference from a modelling point of view with Deelstra et al. (2016, Eqs. (2),(3)).

Thanks to (34) the survival bond $P(t,T)$ (*i.e.*, (3)) is known, and so is the survival coupon bearing bond (6). We are left with the determination of (5), which can be computed explicitly in the linear-rational Wishart model, as the following proposition shows.

Proposition 4.3. *In the linear-rational Wishart model, the survival floating rate bond is given by* (4)*, that is*

$$P_{\text{flo}}(t,T) = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T (r(s)+\mu_x(s))ds}(1+\gamma r(T))\right], \tag{39}$$

and admits the following explicit expression

$$P_{\text{flo}}(t,T) = e^{-\alpha(T-t)}\frac{c_0 + b_3(T-t) + \text{tr}[a_3(T-t)v_t]}{1+\text{tr}[u_0 v_t]}, \tag{40}$$

where c_0 is as constant, $b_3(t)$ is a scalar function and $a_3(t)$ is a matrix function, which are explicitly known.

Proof. Starting with the expectation (39) and changing the probability using (31) we get

$$P_{\text{flo}}(t,T) = \mathbb{E}_t\left[\frac{\zeta_T}{\zeta_t}(1+\gamma r(T))\right],$$

and inserting the expressions (30) and (35) leads, after reorganizing the terms, to

$$P_{\text{flo}}(t,T) = \frac{e^{-\alpha(T-t)}}{1+\text{tr}[u_0 v_t]}\left(c_0 + \mathbb{E}_t\left[\text{tr}[u_3 v_T]\right]\right),$$

with $c_0 = 1 + \gamma\alpha/2 - \gamma\text{tr}[u_1\omega]$ and $u_3 = u_0 + \gamma\alpha u_1 - 2\gamma u_1 m$. Since $\mathbb{E}_t\left[\text{tr}[u_3 v_T]\right]$ is given by (22) we get that there exists $b_3(t)$ a scalar function and $a_3(t)$ a matrix function such that $\mathbb{E}_t\left[\text{tr}[u_3 v_T]\right] = b_3(T-t) + \text{tr}[a_3(T-t)v_t]$ and hence the announced result. ☐

Remark 4.4. The Remark 4.2 applies here without modification.

Note that in Deelstra et al. (2016), to compute the expectation (39), the authors have to derive the moment generating function of the Wishart process (28), as they consider the Bru case, which is feasible thanks to the remarkable analytical property of the process.[1] In the case of the linear-rational Wishart model the expression is much simpler as the proof clearly shows. Note also that the survival bond $P(t,T)$ given by (3) and the survival floating rate bond $P_{\text{flo}}(t,T)$ given by (4) are both linear-rational functions of the Wishart process, as shown by the Propositions 4.1 and 4.3. Note also that only the numerator of the survival bond $P(t,T)$ (or a survival floating rate bond $P_{\text{flo}}(t,T)$) depends on the time to maturity $T-t$, so that when summing several survival bonds (or survival floating rate bonds) with different maturities, as when calculating an annuity, the numerators will simply add up and give a joint survival annuity which is also a linear-rational function of the Wishart process. It is this functional form stability with respect to the summation of survival bonds (or survival floating rate bonds) that makes the linear-rational Wishart model far superior to the exponential affine models commonly used in the literature.

To fully grasp the difference with the linear-rational model and the standard exponential-affine model used in the literature, the expression of the joint survival annuity clearly illustrates the advantages. The proof is straightforward and will therefore be omitted.

Proposition 4.5. *Consider the joint survival annuity underlying the guaranteed annuity option* (8)*, then its value at time T is given by*

$$\sum_{i=1}^N P_{\text{flo}}(T,T_i) = \frac{\sum_{i=1}^N e^{-\alpha(T_i-T)}\left(c_0 + b_3(T_i-T)\right)}{1+\text{tr}[u_0 v_T]} + \frac{\sum_{i=1}^N \text{tr}\left[e^{-\alpha(T_i-T)}a_3(T_i-T)v_T\right]}{1+\text{tr}[u_0 v_T]}, \tag{41}$$

where c_0 is a constant, $b_3(t)$ is a scalar function and $a_3(t)$ is a matrix function given in Proposition 4.3.

Remark 4.6. The joint survival annuity is a linear-rational function of the Wishart process whose dynamic is given under the historical probability measure $\mathbb{P}$. Note that the density of the annuity can be computed explicitly thanks to Gurland (1948); this property contrasts with the numerical difficulties highlighted in Remark 2.1 and makes the linear-rational Wishart model superior when it comes to value derivatives on an annuity. There is an additional property that further simplifies the option pricing problem, it is the fact that the denominator in (41) is the pricing kernel, it strongly simplifies the problem of pricing a guaranteed annuity option.

The next proposition shows that in the linear-rational Wishart model, a guaranteed annuity option, whether the annuity pays a fixed or floating coupon (conditional on the survival of the policyholder), can be explicitly computed.

Proposition 4.7. *The value at time $t=0$ of a guaranteed annuity option* (8) *with maturity T is given by*

$$\bar{C}(0,T,T_N) := e^{-\alpha T}\frac{\mathbb{E}\left[(b_4(T,T_N) + \text{tr}[a_4(T,T_N)v_T])_+\right]}{1+\text{tr}[u_0 v_0]}, \tag{42}$$

where $b_4(T,T_N)$ is a constant and $a_4(t,T_N)$ is an explicitly known matrix.

Proof. Starting with (8) and using (31) together with (30) we get

$$\begin{aligned}\bar{C}(0,T,T_N) &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^T (r(s)+\mu_x(s))ds}\left(\sum_{i=1}^N P_{\text{flo}}(T,T_i) - 1/g\right)_+\right] \\ &= \mathbb{E}\left[\frac{\zeta_T}{\zeta_0}\left(\sum_{i=1}^N P_{\text{flo}}(T,T_i) - 1/g\right)_+\right] \\ &= \frac{e^{-\alpha T}}{1+\text{tr}[u_0 v_0]}\mathbb{E}\left[(1+\text{tr}[u_0 v_T])\bar{C}(T,T_N)\right].\end{aligned}$$

Using (7) and (41), the above equation can be rewritten after some simplifications as

$$\bar{C}(0,T,T_N) = \frac{e^{-\alpha T}}{1+\text{tr}[u_0 v_0]}\mathbb{E}\left[\left(b_4(T,T_N) + \text{tr}[a_4(T,T_N)v_T]\right)_+\right],$$

with

$$b_4(T,T_N) := \sum_{i=1}^N e^{-\alpha(T_i-T)}\left(c_0 + b_3(T_i-T)\right) - \frac{1}{g},$$

$$a_4(T,T_N) := \sum_{i=1}^N e^{-\alpha(T_i-T)}a_3(T_i-T) - \frac{1}{g}u_0.$$

This is the announced result. ☐

Remark 4.8. Note that the guaranteed annuity option is a function of the Wishart process whose dynamic is given under the historical probability, so the Remarks 4.2, 4.4 and 4.6 apply to this derivative. Note also that even though the joint survival annuity is a linear-rational function of the Wishart process, to price the option on that annuity only an expectation of an affine function of the process is needed. This is much simpler than in the exponential affine framework commonly used in the literature.

---

[1] At this level, the model of Deelstra et al. (2016) has an advantage over mortality models based on the standard vector affine framework of Duffie and Kan (1996), as the latter requires the simulation of Riccati equations using a Runge-Kutta scheme, which is time-consuming.

As the formula (42) shows, the pricing involves the expectation of an affine function of the Wishart process, which can be performed using a Fourier transform. The following proposition explains the details.

Proposition 4.9. *Let* $z_T = b_4(T,T_N) + \mathrm{tr}[a_4(T,T_N)v_T]$ *the random variable in* (42)*, then its characteristic function is given by*

$$\Phi_z(u) := \mathbb{E}\left[e^{\mathrm{i}uz_T}\right], \\ = e^{\mathrm{i}ub_4(T,T_N)}\Phi(T, \mathrm{i}ua_4(T,T_N), 0_n, v_0),$$

with $u \in \mathbb{C}$, $\mathrm{i} = \sqrt{-1}$ *and the function* $\Phi(t,\theta_1,\theta_2,v)$ *given by* (14)*. The expectation in* (42) *can be calculated as*

$$\mathbb{E}\left[(z_T)_+\right] = \frac{1}{\pi}\int_0^{+\infty} \Re\left(\frac{\Phi_z(u+\mathrm{i}u_i)}{(\mathrm{i}(u+\mathrm{i}u_i))^2}\right)du, \tag{43}$$

where $u_i < 0$ *and* $\Re(.)$ *is the real part.*

Remark 4.10. Note that the price of the guaranteed annuity option is obtained by one-dimensional integration and not by approximation. This contrasts with the Remark 2.1.

Remark 4.11. If, instead of a survival benefit floating rate bond, a death benefit floating rate bond is considered, see Biffis (2005, Eq. (14)), this leads to the calculation

$$\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T(r(s)+\mu_x(s))ds}(1+\gamma\mu_x(T))\right],$$

and it should be clear that this is feasible within the current model.

Notice also that for death benefit derivatives, it is often necessary to calculate the quantity

$$\int_t^T \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^u(r(s)+\mu_x(s))ds}\mu_x(u)\right]du,$$

see for example Biffis (2005, Eq. (13)). Within the current model it is simply a matter of computing the integral $\int_0^t e^{-\alpha t}\mathbb{E}[\mathrm{vec}(v_u)]du$ with $\mathbb{E}[\mathrm{vec}(v_u)]$ given by (21), which is easy to do.

Remark 4.12. Although the linear-rational Wishart model has remarkable analytical properties, one derivative that does not lead to a simple valuation formula is

$$\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T(r(s)+\mu_x(s))ds}\mu_x(T)r(T)\right].$$

For such a derivative a Monte Carlo method as in Deelstra et al. (2016) can be used.

Remark 4.13. In the linear-rational Wishart model, which is based on the potential approach applied to interest rate modelling proposed by Rogers (1997), the dynamic of the state variable is specified under the historical probability measure $\mathbb{P}$. This contrasts with the standard approach commonly used in the literature where the state variable dynamic is specified under the risk-neutral probability $\mathbb{Q}$. In the potential approach, and hence in the linear-rational Wishart model proposed in this paper, it is possible to obtain the dynamic under the risk-neutral probability measure $\mathbb{Q}$, as shown in Rogers (1997, p. 164) (it is the first unnumbered equation on that page). Under the risk-neutral probability measure $\mathbb{Q}$, the infinitesimal generator of the process $(v_t)_{t\geq 0}$ is given by

$$\mathcal{L}^* f(v) = \frac{1}{1+\mathrm{tr}[u_0 v]}\left(\mathcal{L}((1+\mathrm{tr}[u_0 v])f(v)) - f(v)\mathcal{L}((1+\mathrm{tr}[u_0 v]))\right), \tag{44}$$

where $f(.)$ is a sufficiently regular function defined on $\mathsf{M}(n)$ and $\mathcal{L}$ is given by Eq. (10). Note that there is a typo in this equation in Rogers (1997), the correct expression is the one above and can be found in the working paper version that is available on L. C. G. Rogers' website.

Table I
Model parameters.

Param.	Value
α	0.050
v_{11}	0.020
v_{12}	7.071×10^{-3}
v_{22}	0.010
ω_{11}	0.016
ω_{12}	4.326×10^{-3}
ω_{22}	0.013
m_{11}	−0.29
m_{22}	−0.5
σ_{11}	0.030
σ_{12}	1.549×10^{-2}
σ_{22}	0.050

Note: Model parameters for the linear-rational Wishart process of (9) while $u_1 = e_{11}$ and $u_2 = e_{22}$.

Table II
GAO parameters.

Param.	Value
g	0.23
T	1
N	5

Note: Parameters for the guaranteed annuity option of Proposition 4.7.

5. Implementation

To illustrate the methodology we price a guaranteed annuity option in a linear-rational Wishart model of size $n = 2$ with model parameters given in Table I. Using (37) and (38) together with the parameter values we obtain the initial values $r(0) = 2.097\times10^{-2}$ and $\mu_x(0) = 2.184\times10^{-2}$. Replacing v_0 in these two equations with the long-term mean of the Wishart process $\bar{v}_\infty$ solution of (27) gives an approximation to the long-term means of interest rate and mortality intensity, which we denote $r(\infty)$ and $\mu_x(\infty)$. We find $r(\infty) = 2.535\times10^{-2}$ and $\mu_x(\infty) = 2.464\times10^{-2}$. These values are consistent with those used in the literature, see for example Li et al. (2023).

We consider a guaranteed annuity option on an annuity with fixed payments, *i.e.*, we take $\gamma = 0$ in (8), so that the underlying annuity of the option is given by (6) instead of (5), with the survival bond given by (3) instead of (4). The option characteristics are reported in Table II. This choice simplifies the analysis and we leave to the end of this section the guaranteed annuity option on a floating annuity.

The formula (43) requires the specification of u_i, which is equal to -0.025. With these parameters, the option price is: 7.944×10^{-3}. The computation time is 7.10 seconds on an Intel i7-8665U CPU @ 2GHz using Anaconda3. One change in the model that increases the computation time is the introduction of time-dependent parameters, since it requires a special and more time-consuming numerical scheme to compute the moment generating function (14) (*i.e.*, time-ordered exponential).

To measure the impact of the parameters on the guaranteed annuity option, we perform a sensitivity analysis by varying the values of the parameters. We first consider the diagonal terms as they are easier to analyse. For each parameter, we apply a perturbation between −10% and 10%, reprice the option and report the results in Table III. From (37) we see that increasing ω_{11} decreases $r(t)$ and therefore increases the survival bond $P(t,T)$ (and the annuity), since the survival bond is inversely related to the interest rate, so the option price increases.

Table III
Guaranteed annuity option sensitivity I.

%	ω_{11}	ω_{22}	m_{11}	m_{22}	σ_{11}	σ_{22}
−10	6.161	7.127	10.202	9.146	7.492	7.416
−9	6.328	7.207	9.941	9.010	7.536	7.468
−8	6.498	7.288	9.688	8.878	7.581	7.521
−7	6.671	7.369	9.444	8.749	7.625	7.573
−6	6.846	7.450	9.208	8.625	7.670	7.626
−5	7.023	7.532	8.980	8.503	7.715	7.679
−4	7.202	7.614	8.759	8.385	7.761	7.732
−3	7.384	7.696	8.545	8.271	7.807	7.785
−2	7.569	7.779	8.339	8.159	7.852	7.838
−1	7.756	7.862	8.138	8.050	7.898	7.891
0	7.945	7.945	7.945	7.945	7.945	7.945
1	8.136	8.028	7.757	7.842	7.991	7.998
2	8.330	8.112	7.576	7.742	8.038	8.052
3	8.526	8.196	7.400	7.644	8.085	8.105
4	8.725	8.281	7.230	7.549	8.132	8.159
5	8.926	8.366	7.065	7.456	8.180	8.213
6	9.129	8.451	6.905	7.366	8.227	8.267
7	9.335	8.536	6.750	7.278	8.275	8.321
8	9.543	8.622	6.600	7.192	8.323	8.375
9	9.753	8.708	6.455	7.109	8.371	8.429
10	9.965	8.794	6.314	7.027	8.419	8.483

Note: Sensitivity of the guaranteed annuity option to the diagonal parameters. We apply a percentage perturbation, given in column %, to each parameter whose value is given in Table I and reprice the option. The price is equal to $\times 10^{-3}$ the value given in the table. The formula for pricing the guaranteed annuity option is given in Proposition 4.7. The annuity is with fixed payments.

From (38) we see that increasing ω_{22} decreases $\mu_x(t)$ and therefore increases $P(t,T)$ (and the annuity), since the survival bond is inversely related to the mortality intensity, so the option price increases. With respect to the m parameters, this matrix must have negative eigenvalues, and since it is diagonal, this implies that the diagonal terms are negative. The equation (38) shows that increasing $|m_{11}|$ increases $r(t)$ and therefore decreases $P(t,T)$ (and the annuity), so the option price decreases. A similar argument applies to $|m_{22}|$ but with an effect on $\mu_x(t)$. Regarding the volatility parameters, since $r(t)$ and $\mu_x(t)$ have the same numerator, it is convenient to introduce $\tilde{r}(t) = (1+\mathrm{tr}[u_0 v_t])r(t)$ and $\tilde{\mu}_x(t) = (1+\mathrm{tr}[u_0 v_t])\mu_x(t)$. Using these variables together with (11) and (12) we derive

$$d\langle \tilde{r}(.), \tilde{r}(.)\rangle_t = 4(\alpha - 2m_{11})^2 v_{11,t}(\sigma_{11}^2 + \sigma_{12}^2)dt, \tag{45}$$

$$d\langle \tilde{\mu}_x(.), \tilde{\mu}_x(.)\rangle_t = 4(\alpha - 2m_{22})^2 v_{22,t}(\sigma_{12}^2 + \sigma_{22}^2)dt. \tag{46}$$

An increase of σ_{11} increases the interest rate volatility and the survival bond $P(t,T)$ volatility (and the annuity volatility), the option is more likely to end in the money, so the option price should increase. A similar argument applies to σ_{22} but with an effect on $\tilde{\mu}_x(t)$.

The equations (45) and (46) allow us to clarify the covariation, and therefore the dependence, between the interest rate and the mortality intensity. Combining these two equations with

$$d\langle \tilde{r}(.), \tilde{\mu}_x(.)\rangle_t = (\alpha - 2m_{11})(\alpha - 2m_{22})v_{12,t}\sigma_{12}(\sigma_{11} + \sigma_{22})dt, \tag{47}$$

which follows from (13), we conclude that the conditional (instantaneous) correlation between $\tilde{r}(t)$ and $\tilde{\mu}_x(t)$ is given by

$$\mathrm{Corr}(\tilde{r}(.), \tilde{\mu}_x(.))_t := \frac{d\langle \tilde{r}(.), \tilde{\mu}_x(.)\rangle_t}{\sqrt{d\langle \tilde{r}(.), \tilde{r}(.)\rangle_t}\sqrt{d\langle \tilde{\mu}_x(.), \tilde{\mu}_x(.)\rangle_t}} \tag{48}$$

$$= \frac{v_{12,t}}{\sqrt{v_{11,t}v_{22,t}}}\frac{\sigma_{12}(\sigma_{11}+\sigma_{22})}{\sqrt{(\sigma_{11}^2+\sigma_{12}^2)(\sigma_{12}^2+\sigma_{22}^2)}} \tag{49}$$

$$= \rho_t \frac{\sigma_{12}(\sigma_{11}+\sigma_{22})}{\sqrt{(\sigma_{11}^2+\sigma_{12}^2)(\sigma_{12}^2+\sigma_{22}^2)}}, \tag{50}$$

where $\rho_t = v_{12,t}/\sqrt{v_{11,t}v_{22,t}}$ is the stochastic correlation associated with the Wishart process whose dynamic is given in Da Fonseca et al. (2014). The equation (49) (or (50)) is useful because it allows us to approximate the correlation between the interest rate and the mortality intensity by replacing ρ_t with $\bar{\rho}_\infty = \frac{\bar{v}_{12,\infty}}{\sqrt{\bar{v}_{11,\infty}\bar{v}_{22,\infty}}}$ where the values of $(\bar{v}_{11,\infty}, \bar{v}_{12,\infty}, \bar{v}_{22,\infty})$ are those of $\bar{v}_\infty$ which is the solution of (26) (or (27)). For the values of Table I we get 0.2028, which is positive and consistent with the value found in Li et al. (2023).

Changing the off-diagonal parameters of ω and σ requires caution, since by definition these matrices belong to $\mathbb{S}_n^{++}$ and therefore we must have $|\omega_{12}| < \sqrt{\omega_{11}\omega_{22}}$ and $|\sigma_{12}| < \sqrt{\sigma_{11}\sigma_{22}}$. First we compute the corresponding correlations associated with the off-diagonal terms, they are defined as $\rho_\omega = \frac{\omega_{12}}{\sqrt{\omega_{11}\omega_{22}}}$ for ω and $\rho_\sigma = \frac{\sigma_{12}}{\sqrt{\sigma_{11}\sigma_{22}}}$ for σ. For the numerical values in Table I we get $\rho_\omega = 0.3$ and $\rho_\sigma = 0.4$. Then we apply a perturbation between -10% and 10% to ρ_ω and ρ_σ and denote the perturbed parameters by $\hat{\rho}_\omega$ and $\hat{\rho}_\sigma$. We then construct the perturbed matrices as follows

$$\hat{\omega} = \begin{pmatrix} \omega_{11} & \hat{\rho}_\omega\sqrt{\omega_{11}\omega_{22}} \\ \hat{\rho}_\omega\sqrt{\omega_{11}\omega_{22}} & \omega_{22} \end{pmatrix}, \tag{51}$$

$$\hat{\sigma} = \begin{pmatrix} \sigma_{11} & \hat{\rho}_\sigma\sqrt{\sigma_{11}\sigma_{22}} \\ \hat{\rho}_\sigma\sqrt{\sigma_{11}\sigma_{22}} & \sigma_{22} \end{pmatrix}, \tag{52}$$

and reprice the option with $(\hat{\omega}, m, \sigma)$ and $(\omega, m, \hat{\sigma})$. Table IV reports the results. For the set of parameters $(\hat{\omega}, m, \sigma)$ using (19) we get

$$d\mathbb{E}[v_{12,t}] = (\hat{\omega}_{12} + (m_{11} + m_{22})\mathbb{E}[v_{12,t}])dt, \tag{53}$$

so that increasing $\hat{\omega}_{12}$ (and assuming $\hat{\omega}_{12} > 0$) increases the long term mean value of $v_{12,t}$ and thus the correlation between $r(t)$ and $\mu_x(t)$ according to (49) if we assume $\sigma_{12} > 0$ in (49) or (47).[2] This implies that if $r(t)$ decreases then $\mu_x(t)$ is likely to decrease and therefore the survival bond $P(t,T)$ is likely to increase. Conversely, if $r(t)$ increases then $\mu_x(t)$ is likely to increase and therefore the survival bond $P(t,T)$ is likely to decrease. A positive correlation between $r(t)$ and $\mu_x(t)$ implies a widening of the support of the distribution of the annuity at the maturity of the option, which implies a higher option price. This intuition is confirmed by the values in Table IV. For the set of parameters $(\omega, m, \hat{\sigma})$, the equation (47) is rewritten as follows

$$d\langle \tilde{r}(.), \tilde{\mu}_x(.)\rangle_t = (\alpha - 2m_{11})(\alpha - 2m_{22})v_{12,t}\hat{\sigma}_{12}(\sigma_{11} + \sigma_{22})dt, \tag{54}$$

and shows that an increase of the correlation ρ_σ, which implies an increase of $\hat{\sigma}_{12}$, increases the correlation (if $\hat{\sigma}_{12} > 0$) between the interest rate and the mortality intensity if we assume that $v_{12,t}$ is positive, which is likely to happen if $\omega_{12} > 0$. Using the same arguments as above, we conclude that the option price should increase. Note, however, that increasing the off-diagonal term of σ also affects the volatilities of $\tilde{r}(t)$ and $\tilde{\mu}_x(t)$ as (45) and (46) clearly show.

Overall, the impact of the parameters on the option price is consistent with the intuition.

An important property of the linear-rational Wishart model is that it allows for a correlation between the interest rate and the mortality intensity. In Li et al. (2023) this correlation was found to be positive. A natural question is what happens if we neglect it? Note that changing σ_{12} to zero changes both the correlation between the interest rate and the mortality intensity given by (47) and the interest rate (45) and the mortality intensity volatilities (46). To disentangle the effects, we proceed as follows. We consider the matrix σ of Table I and build the matrix

$$\check{\sigma} = \begin{pmatrix} \sqrt{\sigma_{11}^2 + \sigma_{12}^2} & 0 \\ 0 & \sqrt{\sigma_{12}^2 + \sigma_{22}^2} \end{pmatrix}, \tag{55}$$

---

[2] The equations (53) and (47) show that in order to understand the model, one has to analyse the sign of ω_{12} and the sign of $v_{12,t}\sigma_{12}$.

Table IV
Guaranteed annuity option sensitivity II.

%	ρ_ω	ρ_σ
−10	7.889	7.692
−9	7.895	7.716
−8	7.900	7.741
−7	7.906	7.767
−6	7.911	7.792
−5	7.917	7.817
−4	7.922	7.842
−3	7.928	7.868
−2	7.934	7.893
−1	7.939	7.919
0	7.945	7.945
1	7.950	7.971
2	7.956	7.997
3	7.961	8.023
4	7.967	8.049
5	7.973	8.075
6	7.978	8.101
7	7.984	8.128
8	7.989	8.154
9	7.995	8.181
10	8.000	8.207

Note: Sensitivity of the guaranteed annuity option to the off-diagonal parameters. We apply a percentage perturbation, given in column %, to $\rho_\omega = \frac{\omega_{12}}{\sqrt{\omega_{11}\omega_{22}}}$ for ω and $\rho_\sigma = \frac{\sigma_{12}}{\sqrt{\sigma_{11}\sigma_{22}}}$ for σ. We denote by $\hat{\rho}_\omega$ and $\hat{\rho}_\sigma$ these perturbed parameters, then we build the perturbed matrices $\hat{\omega}$ given by (51) and $\hat{\sigma}$ given by (52). We price the option using the parameters $(v_0, \hat{\omega}, m, \sigma)$ and report the values in the column ρ_ω while the option prices for the parameters $(v_0, \omega, m, \bar{\sigma})$ appear in the column ρ_σ. The price is equal to $\times 10^{-3}$ the value given in the table. The formula for pricing the guaranteed annuity option is given in Proposition 4.7. The annuity is with fixed payments.

and use the Wishart process with parameters $(\omega, m, \check{\sigma})$, which we denote $\check{v}_t$. By construction we have $(\check{\sigma}^2)_{11} = (\sigma^2)_{11}$ and $(\check{\sigma}^2)_{22} = (\sigma^2)_{22}$ and thanks to the relations (11)-(12) we conclude that

$$\frac{d\langle v_{11,.}, v_{11,.}\rangle_t}{4v_{11,t}} = (\sigma^2)_{11} = (\check{\sigma}^2)_{11} = \frac{d\langle \check{v}_{11,.}, \check{v}_{11,.}\rangle_t}{4\check{v}_{11,t}}, \tag{56}$$

$$\frac{d\langle v_{22,.}, v_{22,.}\rangle_t}{4v_{22,t}} = (\sigma^2)_{22} = (\check{\sigma}^2)_{22} = \frac{d\langle \check{v}_{22,.}, \check{v}_{22,.}\rangle_t}{4\check{v}_{22,t}}, \tag{57}$$

and state that $v_{11,t}$ and $\check{v}_{11,t}$ have the same volatility and that $v_{22,t}$ and $\check{v}_{22,t}$ have the same volatility.[3] We also have that $d\langle v_{11,.}, v_{22,.}\rangle_t$ is given by (13) while $d\langle \check{v}_{11,.}, \check{v}_{22,.}\rangle_t = 0$ since $\check{\sigma}_{12} = 0$ and according to (50) the short rate and mortality intensity computed using $(\check{v}_t)_{t\geq 0}$ have an instantaneous covariation equal to zero. We price the option using $(\check{v}_t)_{t\geq 0}$ and find 6.707×10^{-3}, which we compare to 7.944×10^{-3}, the option price obtained using the parameters from Table I. The difference is about -15.5%. If we consider that the difference between the two models (*i.e.*, the two sets of parameters) is the lack of correlation, then -15.5% can be seen as the pricing error due to neglecting it.

It is also possible to consider a guaranteed annuity option written on an annuity that pays a variable coupon. In the linear-rational Wishart model this derivative is no more complicated to price and is given by Proposition 4.7. A similar analysis can be carried out, but for the sake of brevity we only report the difference between the price of the guaranteed annuity option using the parameters in Table I and the equivalent model "without" correlation, *i.e.*, assuming a transformation similar to (55). The option parameters are those in Table II, $\gamma = 0.05$ in (4), so that the survival annuity in (8) or (7) is given by (5) and depends on $P_{\mathrm{flo}}(t,T)$ of (4) with its price given by Proposition 4.3. The price of the guaranteed annuity option is 10.040×10^{-3}, which is of course higher than the option price with fixed payment (*i.e.*, with $\gamma = 0$), and if there is no correlation the option price is 8.632×10^{-3}, which is -14.02% lower.

6. Conclusion

This paper proposes a new model, the linear-rational Wishart model, for jointly modelling mortality risk and interest rate risk. Within this framework, we show how a survival bond and a survival floating rate bond can be priced. Thanks to the model's remarkable analytical properties, the pricing of these derivatives is explicit. We also show how a guaranteed annuity option, *i.e.*, an option on a sum of survival (floating) bonds, can be priced. Unlike existing models in the literature, the pricing of this derivative is explicit and can be done up to a one-dimensional integration and is independent of the dimension of the model. There is no approximation. Pricing is extremely fast and comparable to the pricing of an option on a single (longevity) bond. Using realistic parameter values, we provide a numerical implementation of the model and show the dependence of the derivative on the parameters. We also show that neglecting the correlation between the interest rate and the mortality intensity leads to a significant pricing error.

The framework can be extended in several directions. First, recent results in Li et al. (2023) show the importance of joint jumps in mortality and interest rate risks, so adding jumps to the current framework is a relevant objective. Although the guaranteed annuity option can be computed explicitly, other mortality derivatives may not lead to a simple pricing formula, an example of which is given in Remark 4.12. The development of some numerical approximations for these derivatives is of interest. In Da Fonseca et al. (2022) the moments of the Wishart process are obtained under certain parameter assumptions, these moments allow the use of the option price approximations of Jarrow and Rudd (1982) and Collin-Dufresne and Goldstein (2002). Obtaining these moments in the general case is interesting but difficult, Graczyk and Vostrikova (2007) may provide a way to solve this problem. Another important research topic is the estimation of the model. Note that the model parameters are matrices that satisfy certain constraints, such as negative eigenvalues or symmetric positivity; any numerical algorithm should take such properties into account. Finally, the present paper focuses on the dependence between interest rates and mortality, but it should be clear that the linear-rational Wishart model can be used to deal with other dependencies. For example, Xu et al. (2020) study the multi-cohort modelling problem using the vector affine process and the exponential affine framework. The analysis of this problem, and more generally multi-population and/or multi-annuitant problems, in the linear-rational Wishart framework is definitely an interesting topic. In this line of research, the GAO pricing problem is mainly (or only) relevant in a multi-annuitant context. The main modelling difficulty is to specify the dependence between mortality events of two (or more) populations/cohorts/annuitants. These are interesting problems worthy of investigation.

CRediT authorship contribution statement

José Da Fonseca: Conceptualization, Formal analysis, Investigation, Methodology, Project administration, Resources, Software, Validation, Visualization, Writing – original draft, Writing – review & editing.

Declaration of competing interest

The author declares that there is no competing interest.

Data availability

No data was used for the research described in the article.

---

[3] Obviously, strictly speaking this is not true, since the instantaneous variance of $v_{11,t}$ is $4v_{11,t}(\sigma^2)_{11}dt$ and not $4(\sigma^2)_{11}dt$.

Acknowledgements

We would like to thank the reviewers for their constructive comments, which have improved the quality of the paper. The usual caveat applies.

References

Bayraktar, E., Milevsky, M.A., Promislow, S. David, Young, V.R., 2009. Valuation of mortality risk via the instantaneous Sharpe ratio: applications to life annuities. Journal of Economic Dynamics and Control 33 (3), 676–691. https://doi.org/10.1016/j.jedc.2008.09.004.

Benabid, A., Bensusan, H., El Karoui, N., 2009. Wishart stochastic volatility: asymptotic smile and numerical framework. http://hal.archives-ouvertes.fr/hal-00458014/fr.

Biffis, E., 2005. Affine processes for dynamic mortality and actuarial valuations. Insurance: Mathematics and Economics 37, 443–468. https://doi.org/10.1016/j.insmatheco.2005.05.003.

Biffis, E., Millossovich, P., 2006. The fair value of guaranteed annuity options. Scandinavian Actuarial Journal 2006 (1), 23–41. https://doi.org/10.1080/03461230500462204.

Björk, T., 2009. Arbitrage Theory in Continuous Time, 3rd edition. Oxford University Press, Oxford.

Bru, M.F., 1991. Wishart processes. Journal of Theoretical Probability 4 (4), 725–751. https://doi.org/10.1007/BF01259552.

Cairns, A.J.G., 2004. Interest Rate Models: An Introduction. Princeton University Press.

Ceci, C., Colaneri, K., Cretarola, A., 2015. Hedging of unit-linked life insurance contracts with unobservable mortality hazard rate via local risk-minimization. Insurance: Mathematics and Economics 60, 47–60. https://doi.org/10.1016/j.insmatheco.2014.10.013.

Chiarella, C., Da Fonseca, J., Grasselli, M., 2014. Pricing range notes within Wishart affine models. Insurance: Mathematics and Economics 58, 193–203. https://doi.org/10.1016/j.insmatheco.2014.07.008.

Collin-Dufresne, P., Goldstein, R., 2002. Pricing swaptions within an affine framework. The Journal of Derivatives 3 (1–2), 167–179. https://doi.org/10.3905/jod.2002.319187.

Cuchiero, C., Filipović, D., Mayerhofer, E., Teichmann, J., 2011. Affine processes on positive semidefinite matrices. The Annals of Applied Probability 21 (2), 397–463. https://doi.org/10.1214/10-AAP710.

Da Fonseca, J., Grasselli, M., Ielpo, F., 2014. Estimating the Wishart Affine Stochastic Correlation model using the empirical characteristic function. Studies in Nonlinear Dynamics and Econometrics 18 (3), 253–289. https://doi.org/10.1515/snde-2012-0009.

Da Fonseca, J., Dawui, K.E., Malevergne, Y., 2022. A linear-rational multi-curve term structure model with stochastic spread. SSRN-eLibrary. https://doi.org/10.2139/ssrn.4176102.

Dahl, M., 2004. Stochastic mortality in life insurance: market reserves and mortality-linked insurance contracts. Insurance: Mathematics and Economics 35 (1), 113–136. https://doi.org/10.1016/j.insmatheco.2004.05.003.

Dahl, M., Møller, T., 2006. Valuation and hedging of life insurance liabilities with systematic mortality risk. Insurance: Mathematics and Economics 39 (2), 193–217. https://doi.org/10.1016/j.insmatheco.2006.02.007.

Deelstra, G., Grasselli, M., Van Weverberg, C., 2016. The role of the dependence between mortality and interest rates when pricing guaranteed annuity options. Insurance: Mathematics and Economics 71, 205–219. https://doi.org/10.1016/j.insmatheco.2016.09.010.

Duffee, G.R., 2002. Term premia and interest rate forecasts in affine models. Journal of Finance 57, 405–443. https://doi.org/10.1111/1540-6261.00426.

Duffie, D., Kan, R., 1996. A yield-factor model of interest rates. Mathematical Finance 6 (4), 379–406. https://doi.org/10.1111/j.1467-9965.1996.tb00123.x.

Gouriéroux, C., Sufana, R., 2010. Derivative pricing with multivariate stochastic volatility. Journal of Business and Economic Statistics 28 (3), 438–451. https://doi.org/10.1198/jbes.2009.08105.

Graczyk, P., Vostrikova, L., 2007. The moments of Wishart processes via Itô calculus. Theory of Probability and Its Applications 51 (4), 609–625. https://doi.org/10.1137/S0040585X97982682.

Grasselli, M., Tebaldi, C., 2008. Solvable affine term structure models. Mathematical Finance 18 (1), 135–153. https://doi.org/10.1111/j.1467-9965.2007.00325.x.

Gupta, A., Nagar, D., 2000. Matrix Variate Distributions. Chapman & Hall/CRC.

Gurland, J., 1948. Inversion formulae for the distribution of ratios. The Annals of Mathematical Statistics 19 (2), 228–237.

Huang, Y.-L., Tsai, J.T., Yang, S.S., Cheng, H.-W., 2014. Price bounds of mortality-linked security in incomplete insurance market. Insurance: Mathematics and Economics 55, 30–39. https://doi.org/10.1016/j.insmatheco.2013.11.008.

Jalen, L., Mamon, R., 2009. Valuation of contingent claims with mortality and interest rate risks. Mathematical and Computer Modelling 49 (9), 1893–1904. https://doi.org/10.1016/j.mcm.2008.10.014.

Jarrow, R., Rudd, A., 1982. Approximate option valuation for arbitrary stochastic processes. Journal of Financial Economics 10 (3), 347–369. https://doi.org/10.1016/0304-405X(82)90007-1.

Lamberton, D., Lapeyre, B., 2000. Stochastic Calculus Applied to Finance. Chapman & Hall/CRC.

Li, H., Liu, H., Tang, Q., Yuan, Z., 2023. Pricing extreme mortality risk in the wake of the Covid-19 pandemic. Insurance: Mathematics and Economics 108, 84–106. https://doi.org/10.1016/j.insmatheco.2022.11.002.

Liu, X., Mamon, R., Gao, H., 2014. A generalized pricing framework addressing correlated mortality and interest risks: a change of probability measure approach. Stochastics 86 (4), 594–608. https://doi.org/10.1080/17442508.2013.859388.

Lütkepohl, H., 2005. New Introduction to Multiple Time Series Analysis, 1 edition. Springer-Verlag.

Mayerhofer, E., Pfaffel, O., Stelzer, R., 2011. On strong solutions of positive definite jump-diffusions. Stochastic Processes and Their Applications 121 (9), 2072–2086. https://doi.org/10.1016/j.spa.2011.05.006.

Rogers, L.C.G., 1997. The potential approach to the term structure of interest rates and foreign exchange rates. Mathematical Finance 7 (2), 157–176. https://doi.org/10.1111/1467-9965.00029.

Xu, Y., Sherris, M., Ziveyi, J., 2020. Continuous-time multi-cohort mortality modelling with affine processes. Scandinavian Actuarial Journal 2020 (6), 526–552. https://doi.org/10.1080/03461238.2019.1696223.

Zhao, Y., Mamon, R., 2018. An efficient algorithm for the valuation of a guaranteed annuity option with correlated financial and mortality risks. Insurance: Mathematics and Economics 78, 1–12. https://doi.org/10.1016/j.insmatheco.2017.09.001.