

On Wijsman Convergence

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ICAA2010, Sultan Qaboos University, 24-26 January 2010

R. Wijsman and D. Burkholder, *Optimum properties and admissibility of sequential tests*, Ann. Math. Statist. **34** (1963), 1-17.

Question

A_n, A closed convex in \mathbb{R}^2 ; x is in none of the A_n nor in A .
Consider the supporting lines through x of A_n and of A .
If $A_n \xrightarrow{?} A$, is the same true for the supporting lines?

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Definition

Call $A_n \xrightarrow{\text{Wijsman}} A$, if $d(A_n, x) \rightarrow d(A, x)$ for every $x \in \mathbb{R}^2$, where

$$d(A, x) = \inf\{d(a, x) : a \in A\}.$$

This can be extended to the setting of a metric space.

Fenchel Transform

X normed linear, $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ **nontrivial**. Then $f^* : X^* \rightarrow \mathbb{R} \cup \{+\infty\}$ defined by

$$\forall p \in X^*, \quad f^*(p) = \sup_{x \in X} [\langle p, x \rangle - f(x)]$$

is called the **convex** conjugate of f , and $\mathcal{F} : f \mapsto f^*$ is called the Fenchel transform.

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Epigraph

The epigraph of $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ is defined by

$$\text{epi}(f) = \{(\mathbf{x}, \alpha) : \mathbf{x} \in X, \alpha \in \mathbb{R} \text{ and } \alpha \geq f(\mathbf{x})\}.$$

- 1 f is l.s.c. if and only if $\text{epi}(f)$ is closed in $X \times \mathbb{R}$.
- 2 f is convex if and only if $\text{epi}(f)$ is convex in $X \times \mathbb{R}$.

Theorem (Wijsman; 1966)

On \mathbb{R}^n , for nontrivial l.s.c. convex functions,
 $\text{epi}(f_n) \xrightarrow{\text{Wijsman}} \text{epi}(f)$ if, and only if $\text{epi}(f_n^*) \xrightarrow{\text{Wijsman}} \text{epi}(f^*)$.

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$CL(X)$ = the set of nonempty closed sets of (X, d) . Under

$$A \leftrightarrow d(A, \cdot) : X \rightarrow \mathbb{R},$$

$CL(X) \hookrightarrow \mathbb{R}^X$. The Wijsman topology $\tau_{w(d)}$ on $CL(X)$ is the just the subspace topology.

Example (Discrete Metrics)

If d is the 0-1 discrete metric on X , for any $\emptyset \neq A \subseteq X$, $d(A, x) = 0$ for $x \in A$ and $d(A, x) = 1$ for $x \notin A$. Thus, $(CL(X), \tau_{w(d)}) \simeq \{0, 1\}^X \setminus \{\mathbf{1}\}$.

In 1998, C. Costantini constructed a 3-valued discrete metric ϱ on \mathbb{R} such that $(CL(\mathbb{R}), \tau_{w(\varrho)})$ is not Čech complete.

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Basic Facts

- 1 The Fenchel transform $\mathcal{F} : (\Gamma(\mathbb{R}^n), \tau_{w(d)}) \rightarrow (\Gamma(\mathbb{R}^n), \tau_{w(d)})$ is a homeomorphism, where $\Gamma(\mathbb{R}^n)$ is the family of nontrivial l.s.c. convex functions on \mathbb{R}^n .
- 2 On $CL(X)$, $A_\lambda \xrightarrow{\text{Wijsman}} A$ if, and only if $A_\lambda \xrightarrow{\tau_{w(d)}} A$.
- 3 It is a Tychonoff topology.
- 4 It is metric dependent.
- 5 It is the weakest topology such that $d(\cdot, x)$ is continuous for all $x \in X$.

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Research Problems

- 1 Investigating properties of the Wijsman topology.
- 2 Extending Wijsman's theorem to infinite dimensions?
- 3 Determine relations of Wijsman convergence with others.
- 4 Using graphical approach to study function spaces.

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Deeper Properties of Wijsman Topology

Theorem (Lechicki and Levi; 1987)

A metric (X, d) is separable if and only if $(CL(X), \tau_w(d))$ is metrizable.

In 1991, Beer proved that if (X, d) is a Polish space, then so is $(CL(X), \tau_w(d))$.

Theorem (Costantini; 1995)

If X is a Polish space, then $(CL(X), \tau_w(d))$ is Polish for every compatible metric d on X .

In 1997, L. Zsilinszky gave a short proof of Costantini's theorem by using the Chouquet game.

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Wijsman's Theorem in Infinite Dimensions

Note that Wijsman's theorem fails in any infinite-dimensional Banach space.

Every infinite-dimensional Banach space admits an equivalent renorm such that the induced dual norm fails to have the sequential weak* Kadec property [Borwein & Fabian; 1993].

Definition

A dual norm $\|\cdot\|$ of a Banach space X is said to have the *sequential weak* Kadec property* provided $y = \text{weak}^*\text{-}\lim y_n$ and $\|y\| = \lim \|y_n\|$ jointly imply that $\lim \|y_n - y\| = 0$.

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Recall that $A_\lambda \xrightarrow{K} A$ if $A = \liminf A_\lambda = \limsup A_\lambda$. The topology associated with Kuratowski convergence is the Fell topology τ_F .

In finite dimensions, Wijsman convergence agrees with Kuratowski convergence.

For an infinite dimensional $(X, \|\cdot\|)$, consider the Fell topology τ_F^S w.r.t. the norm topology on $CL(X)$, and the the Fell topology τ_F^W w.r.t. the weak topology on the family of weakly closed sets of X . The supremum of τ_F^S and τ_F^W is Mosco topology.

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Theorem (Mosco; 1971)

On a reflexive Banach space X , for nontrivial l.s.c. functions, $\text{epi}(f_n) \xrightarrow{M} \text{epi}(f)$ if, and only if $\text{epi}(f_n^*) \xrightarrow{M} \text{epi}(f^*)$.

We call $\{A_n : n \in \mathbb{N}\}$ **Attouch-Wets converges** to A on $CL(X)$, denoted by $A_n \xrightarrow{A-W} A$, if $d(A_n, \cdot) \rightarrow d(A, \cdot)$ uniformly on every bounded set of (X, d) .

Theorem (Beer; 1990)

On any normed linear space X , for nontrivial l.s.c. convex functions defined, $\text{epi}(f_n) \xrightarrow{A-W} \text{epi}(f)$ if, and only if $\text{epi}(f_n^*) \xrightarrow{A-W} \text{epi}(f^*)$.

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Question

Given a metric space (X, d) , when is $(CL(X), \tau_w(d))$ **Baire**?

Recall that X is Baire if G_n 's are dense open in X , then $\bigcap_{n \in \mathbb{N}} G_n$ is dense in X . In case that every nonempty closed subspace is Baire, X is called hereditarily Baire.

Theorem

$(CL(X), \tau_w(d))$ a Baire space, if

- 1 (X, d) is *separable* and Baire (Zsilinszky; 1996);
- 2 (X, d) is *complete* (Zsilinszky; 1998);
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In 2002, J. Chaber and R. Pol showed that if the set of points in (X, d) without any compact nbhd has weight 2^{\aleph_0} , then $\mathbb{Q} \hookrightarrow (CL(X), \tau_w(d))$ as a closed subspace.

Example

$X := C(\beta\mathbb{N})$ with the sup-norm

$$\|f\|_\infty = \sup_{x \in \beta\mathbb{N}} |f(x)|.$$

Then $(CL(X), \tau_w(d))$ is not a hereditarily Baire space.

Theorem (Cao & Junnila, 20??)

Any *Dieudonné complete* space can be embedded into $(CL(X), \tau_w(d))$ as a closed subspace for some complete metric space (X, d) .

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Countably subcompactness

Example (Pol-Zsilinszky; 2008)

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Definition

- 1 \mathfrak{B} is a *π -base* if members of \mathfrak{B} are nonempty open, and every nonempty open set of X contains some $B \in \mathfrak{B}$.
- 2 X is called *almost countably subcompact* w.r.t. a π -base \mathfrak{B} if $\bigcap_{n \in \mathbb{N}} B_n \neq \emptyset$ for every sequence $\{B_n : n \in \mathbb{N}\} \subseteq \mathfrak{B}$ satisfying $\overline{B_{n+1}} \subseteq B_n$ for all $n \in \mathbb{N}$.

Note that the Pol-Zsilinszky metric space satisfying:

$$d(x, z) \leq \max\{d(x, y), d(y, z)\},$$

and also no ball can be covered by finitely many balls with smaller radii. The previous theorem motivates us to consider the Wijsman topology of spaces having properties similar to these.

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Thank You Very Much !

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