Computational Modeling of Stochastic Processes in Electron Amplifiers.

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Abstract A computational method for simulation of stochastic processes of an electron multiplication in microchannel electron amplifiers is developed. The method is based on 3D Monte Carlo (MC) simulations and theorems about serial and parallel amplification stages proposed here. Splitting a stochastic process into a number of different stages, enables a contribution of each stage to the entire process to be easily investigated. The method preserves all advantages of the MC simulations which are used only once for one simple stage. The use of the theorems allows to conduct any further investigations and optimizations without additional MC simulations. The method provides a high calculation accuracy with minimal cost of computations. The model is relevant to unsaturated operation of the amplifier with exponential output distribution. The mean gain and the variance of the amplitude distribution at the output of the amplifier are predicted. In this paper the method is used to show how the input ratio of the signal to the noise is transforming to the output one, and how different multiplication stages contribute to the noise factor of the system. Finally, the effect of variations in channel diameters on noise characteristics of microchannel electron amplifiers is investigated.

Keywords Stochastic process, electron multiplication, Monte Carlo simulations, microchannel electron amplifier, noise factor.

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1 Introduction

Amplification of the charged particles in electronic devices, particularly in microchannel electron amplifiers, is a complicated stochastic process, which is usually simulated by the use of Monte Carlo (MC) methods ([1], [2], [5], [6], [11], [13]). However, the direct simulation of the entire amplification process by the MC methods require considerable computer time (for example, see [8]), especially for optimization problems. Moreover, it is difficult to evaluate a contribution of different system's parameters to the entire amplification process, and their effect on output characteristics.

The essence of the approach proposed here consists of separating the amplification process into serial and parallel stages where the mean and variance of the amplitude distribution at the output of the entire system are obtained using the mean and variance of each stage. The advantage of the method is that MC simulations are used once for one simple stage, and any further investigations and optimizations do not require additional use of the MC methods. However, the method preserves all advantages of the MC simulations: represents real physical processes fully and adequately, and uses experimental characteristics completely in the model. Moreover, splitting a stochastic process into a number of different stages, allows a contribution of each stage to the entire process to be easily investigated. The method provides a high calculation accuracy with minimum expenditures of the computer time.

The limitations of the proposed model are that the model does not take into account saturation effects, inter channel interference and does not evaluate timing resolution. The amount of output charge is predicted by the model but the output energy and angular distribution require the MC simulations.

The proposed model is used for an investigation of the noise characteristics of microchannel electron amplifiers (Fig.1 [15]), which have found wide applications in different areas of science, engineering, medicine etc. However, the loss of information caused by the statistical fluctuations in the gain of the channels, and by loss of primary electrons when they strike the closed area of a microchannel plate (MCP), increases a noise factor which is a measure of the loss of available information [6], [12], [18].

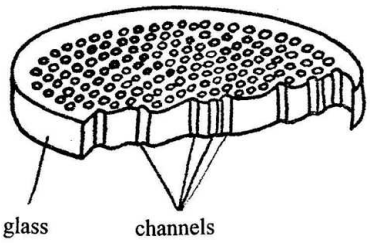

Fig. 1 Microchannel plate

Sect. 2 describes the method and includes a description of the 3D MC simulations (Sect. 2.2) and a proof of the theorems of serial and parallel amplification stages (Sect. 2.3 and 2.4). Sect. 2.1 describes a physical model and a computational algorithm of the electron multiplication inside the channel multiplier. It is shown (Sect. 2.5) that the amplitude distribution at the output is determined by the effective length of the channel where the amplitude distribution is stabilized. The MC simulations are carried for one electron emitted at the beginning of the channel, along the effective length. The gain and variance as functions of the distance from the channel entrance are calculated. The use of these functions and the theorems enable to find the mean and variance of the amplitude distribution at the output of the system and conduct further investigations without MC simulations. The output charge is predicted here for an MCP operating with exponential output pulse height distribution, not for an MCP operating in a saturated mode. Sect. 3 shows the efficiency and accuracy of the proposed method.

Sect.4 and 5 illustrate the application of the computational method. In Sect. 4, expressions for the noise factor of a single channel and an array of channels are obtained. It is shown, how the input ratio of the signal to the noise is transforming to the output one, and how different multiplication stages contribute to the noise factor of a system.

Sect.5 is devoted to evaluation of the effect of variations in channel diameters in the MCP on the noise factor. Even though, the nonuniformity of the channel diameters is not the most problematic for the present generations MCP, such application shows the effectiveness of the method when it is difficult or not possible to conduct an experiment. Except this application, the method enables to optimize characteristics of the channel amplifier in terms of the noise factor, such as an incidence angle and energy of the input electron beam, the spread in incidence coordinates of the primary electrons, nonuniformity of the emitter surface, the length of a contact conducting layer, or the depth and secondary emission yield of a high-efficiency emitter etc. All these investigations can be done without MC simulations for the same channel geometry. Even if a geometry or applied voltage is changed, the MC simulations should be conducted again only on the effective channel length for one electron emitted at the beginning. Moreover, the method can be applied for the optimization of the whole electronic devise, such as image intensifier, where MCP is considered as an intermediate stage in the entire amplification process. The model easily implements new experimental data without any changes in the algorithm.

2 Computational Method

2.1 The model and the algorithm of the electron multiplication

The following physical picture was considered in the modeling. The electrons of a parallel monochromatic beam are incident on the input plane of a microchannel amplifier (Fig.2). Electrons entering the channel hit the walls at different incidence coordinates, producing secondary electrons with different emission energy and directions. The secondary electrons are multiplied until they leave the channel. When all the electrons have emerged from the channel, the yield of the individual pulse is known.

The gain of individual pulses is fluctuated considerably, and the pulse amplitude distribution at the output of the channel has a negative exponential form [6], [12]. The mean gain and the variance of the amplitude distribution at the output of the system define the noise factor of the amplifier (Sect.4) which is greater for the negative exponential distribution than for a peaked amplitude distribution [6], [12]. The effective length of a channel, where the amplitude distribution changes from the peaked one to the negative exponential distribution, is determined (Sect.2.5).

The entire multiplication process can be split into sequential stages and/or parallel multiplication paths, and how it is done depends on particular investigation. For a example, each collision of the electron in the channel can be considered as a separate stage. On the other

hand, if the MCP is used in an electronic device, it can also be treated as one stage in the whole system [14]. Variations in the collision coordinates of the electrons of the primary beam (Fig.2) can be considered as a choice of the parallel amplification path.

In application described here (Sect.5) individual channels in the MCP are taken as the parallel amplification paths, and the following sequential stages constitute the amplification process in the channel (Sect.4.1). The first stage is the collision of the primary electrons with the front surface of the microchannel plate. The collision of the primary electrons, which entered the channel, with the wall inside the channel is considered as the second stage. The third stage is the multiplication of a single electron, emitted at the beginning of the channel, along the effective length of the channel. Further multiplication in the channel is considered as the last stage.

The multiplication process of a single electron with an initial departure coordinate of $z = 0$ (z is the coordinate directed along the channel axis and measured from its beginning) is simulated by 3D MC methods in a homogeneous field along the effective channel length (Sect.2.5) which is equal half of the channel. (The uniform electric field is relevant to MCP with straight channels. For the MCP with tilted channels the field is not uniform and electron trajectories should be calculated by solving the system of differential equations of motion in nonuniform fields.)

Functions $m(z)$, the mean, and $d(z)$, the variance, are calculated for $0 \le z \le L/2$, where L is the coordinate of the end of the channel. For n electrons leaving the first half of the channel, the incidence coordinates ($z > L/2$) and the values of the secondary electron emission coefficient are determined.

The amplification in the second half of the channel is considered to consist of n parallel paths. Each path has two sequential stages: first collision and multiplication of a single electron until it leaves the channel. Using the theorems of series and parallel amplification stages, the functions $m(z)$ and $d(z)$ along the entire channel length ($0 \le z \le L$) are calculated.

Thus, the mean and variance of each sequential stage are defined, and the mean and variance at the output of the system are calculated using the theorems of serial and parallel amplification stages.

The functions $m(z)$ and $d(z)$ for $0 \le z \le L$ and the theorems of serial and parallel amplification stages allow us to conduct further investigations and optimizations without any additional MC simulations and provide highly accurate results.

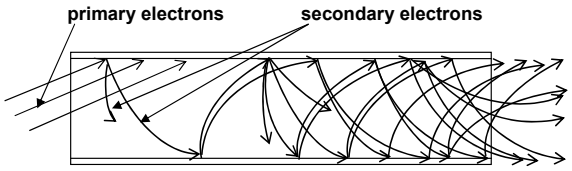

Fig. 2 Electron multiplication in the channel

2.2 Monte Carlo simulations

The computer model uses a random number generating procedure [6] to sample the various distributions which represent the statistical processes of the electron multiplication. They include the distribution of the actual yield of secondaries after each collision, the emission energy, and the direction of each secondary electron.

The actual number of secondaries generated by the particular collision is a random sample taken from the Poisson distribution:

$$P(\nu) = \frac{\sigma^{\nu} e^{-\sigma}}{\nu!},$$

where ν is the number of secondary electrons produced, σ is the Secondary Emission Yield (SEY).

The variation of the SEY is defined by a secondary emission function [6]:

$$\sigma = \sigma_m [\frac{V}{V_m}\sqrt{\cos\theta}]^{\beta} e^{\alpha(1-\cos\theta)+\beta(1-\frac{V}{V_m}\sqrt{\cos\theta})},$$

where V is the collision energy in eV, θ is the collision angle, σ_m is the maximum SEEC for the normal incidence ($\theta = 0$) which is achieved for $V = V_m$ (both V_m and σ_m are functions of θ); α and β are constants of the channel multiplier surface , and are chosen to fit experimental secondary emission curves at normal incidence. (The values $\sigma_m = 3.15$, $V_m = 300eV$ for $\theta = 0$, $\alpha = 0.62$ and $\beta = 0.6$ were used throughout the simulations).

The energy distribution is described by a Yakobson formula [16]:

$$p(\varepsilon) = 2.1\bar{\varepsilon}^{-3/2}\sqrt{\varepsilon} exp(-1.5\varepsilon/\bar{\varepsilon}),$$

where $\bar{\varepsilon}$ is the mean energy (the value $\bar{\varepsilon} = 5eV$ was used throughout the simulations).

Each secondary electron is assigned two emission angles chosen from Lambert's law (Fig.3):

$$p_1(\theta) = \sin 2\theta \qquad p_2(\varphi) = 1/2\pi.$$

The trajectory of each electron is calculated in three dimensions from the ballistic equations , and so the position, energy, and angle of the subsequent collisions are determined. The result of each collision is calculated as before and the process is repeated for each secondary electron generated.

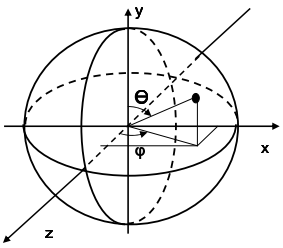

Fig. 3 Emission angles of the secondary electrons

The fringe field effects are taken into account in numerical simulations. The nonuniform field at the entrance of the channel is calculated by solving a system of linear algebraic equations which are the finite difference analog of the Laplace equation. The part of the channel with nonuniform electric field is considered as a separate stage.

The trajectories of the electrons in the nonuniform field [9] are calculated by solving the system of differential equations of motion in fields having axial symmetry by the Runge-Kutta method [3], [4], [10]. The trajectories of the electron motion inside the channel are calculated from the equations of motion in the uniform field.

2.3 Theorem about serial amplification stages

The mean and variance of the entire multiplication process can be calculated using the mean and variance of the separate sequential stages. Let $p_k(\nu)$ be the probability distribution of the number of particles at the output of the k-th stage, produced by one particle at its input. Let m_k be the mean and d_k be the variance of the $p_k(\nu)$. Then the generating function of the probability distribution $p_k(\nu)$ is:

$$q_k(u) = \sum_{\nu=0}^{\infty} u^{\nu} p_k(\nu), \qquad where \qquad |u| \le 1.$$

Using an approach similar to [17] the generating function for the probability distribution of the number of particles after the last (N-th) stage can be constructed as:

$$Q_N(u) = q_0(q_1(q_2(...(q_N(u))...))). \tag{1}$$

To find the mean M, and variance D of the amplitude distribution $P_N(\nu)$ after the N-th stage, we convert the expressions (1) to the logarithmic generating functions, introducing new variables:

$$v = \ln u, \qquad h_k(v) = \ln \sum_{\nu=0}^{\infty} e^{v\nu} p_k(\nu)$$

$$H_N(v) = \ln \sum_{\nu=0}^{\infty} e^{v\nu} P_N(\nu).$$

Then the expressions (1) can be written as:

$$H_N(v) = h_0(h_1(h_2(...(h_N(v))...))),$$

where $H_N(v)$ is the logarithmic generating function of the distribution $P_N(\nu)$; $h_k(v)$ is the logarithmic generating function of the distribution $p_k(\nu)$.

Differentiating $H_N(v)$ with respect to v once and using the properties of the logarithmic generating functions, with $v = 0$ we obtain the mean value of $P_N(\nu)$:

$$M = m_0 m_1 ... m_k ... m_N = \prod_{k=0}^{N} m_k. \tag{2}$$

Differentiating $H_N(v)$ with respect to v twice, with $v = 0$ we obtain the variance D after the N-th stage of this multistep sequential process.

$$\begin{aligned} D = {} & d_0(m_1 m_2 ... m_N)^2 + d_1 m_0 (m_2 m_3 ... m_N)^2 \\ & + d_2 m_0 m_1 (m_3 m_4 ... m_N)^2 + ... \\ & ... + d_k m_0 m_1 ... m_{k-1} (m_{k+1} m_{k+2} ... m_N)^2 + ... \\ & ... + d_N m_0 m_1 ... m_{N-1} \end{aligned}$$

or

$$D = \sum_{k=0}^{N} d_k \prod_{i=0}^{k-1} m_i \prod_{j=k+1}^{N} m_j^2. \tag{3}$$

The expressions (2) and (3) constitute the theorem of serial amplification stages. The theorem is a generalization of the Burgess theorem [18] for a multistep sequential process. The relations (2) and (3) are transformed to the Burgess theorem for N=1.

2.4 Theorem of parallel amplification paths

The mean and variance of the amplitude distribution at the output of the system with some parallel amplification paths, can be calculated using the mean and variance of each path. Let the primary particle be multiplied along one of n possible parallel paths, and ρ_k be the probability of choosing the k-th path. If each path gives an average of g_k particles at the output with a variance of v_k, then the mean G and the variance V of this multiplication process can be obtained.

Let $\varphi_k(\nu)$ be the probability distribution of the number of particles ν at the output of the k-th path produced by one particle at its input. Then the probability distribution $\Phi(\nu)$ of the number of particles at the output of the entire system of n parallel paths will be:

$$\Phi(\nu) = \sum_{k=1}^{n} \rho_k \varphi_k(\nu).$$

Then the mean G of such a multiplication process is equal to:

$$G = \sum_{\nu=0}^{\infty} \Phi(\nu)\nu = \sum_{k=1}^{n} \rho_k \sum_{\nu=0}^{\infty} \varphi_k(\nu)\nu = \sum_{k=1}^{n} \rho_k g_k. \qquad (4)$$

The variance V of the distribution at the output of the system can be written as:

$$V = \sum_{\nu=0}^{\infty} \Phi(\nu)\nu^2 - [\sum_{\nu=0}^{\infty} \Phi(\nu)\nu]^2. \qquad (5)$$

The first sum in (5) can be transformed to:

$$\sum_{\nu=0}^{\infty} \Phi(\nu)\nu^2 = \sum_{k=1}^{n} \rho_k v_k + \sum_{k=1}^{n} \rho_k g_k^2.$$

Taking into account that $[\sum_{\nu=0}^{\infty} \Phi(\nu)\nu]^2 = G^2$ finally the expression

$$V = \sum_{k=1}^{n} \rho_k v_k + \sum_{k=1}^{n} \rho_k g_k^2 - G^2 \qquad (6)$$

is obtained, where G is determined by (4).

Equations (4) and (6) can be used for discrete and for continuous systems, where sums should be replaced by integrals. For example, for variations in the collision coordinates of the electrons of the primary beam the equations (4) and (6) can be written as:

$$G = \int_s \psi(s)g(s)ds, \qquad (7)$$

$$V = \int_s \psi(s)v(s)ds + \int_s \psi(s)g^2(s)ds - G^2, \qquad (8)$$

where s is the surface area stroked by particles; ψ is the probability density for the particle to strike the elementary surface ds; $g(s)$ is the average number of particles with variance $v(s)$ at the output of the path. Equations (4), (6), (7) and (8) constitute the theorem of parallel amplification paths.

2.5 Effective length of the channel

It is known [6], [12] that the pulse amplitude distribution at the output of the channel has a negative exponential form. Let us show that such shape of the pulse distribution is established on the initial section of the channel, the effective length l_{eff}. After l_{eff} the distribution is not changed significantly. Thus the effective length can be defined as a part of the channel where the amplitude distribution is stabilized, and the shape of the distribution is close to the negative exponential function.

Formulae (2) and (3) enable one to evaluate the number of stages n_s, after which the relative variance v_r has an error Δ compared with the relative variance of the amplitude distribution at the output of the entire channel with diameter d_c.

Let stages from $(l+1)$ to the end of the channel have the same mean m and variance d each. If M_l is the mean and D_l is the variance of the amplitude distribution after l first stages, then according to (2) and (3) the expressions for M and D at the output of the entire system can be obtained:

$$M = M_l m^k,$$

$$D = D_l m^{2k} + dM_l m^{k-1} \frac{m^k - 1}{m - 1},$$

where k is the number of the last amplification stages.

Suppose $m^k \gg 1$ then the formula for the relative variance v_r will be:

$$v_r = \frac{D}{M^2} = \frac{D_l}{M_l^2} + \frac{d}{(m-1)M_l m}.$$

The relative variance v_{rn} after n_s similar stages will be:

$$v_{rn} = \frac{D_l}{M_l^2} + \frac{d(m^{n_s} - 1)}{(m-1)M_l m^{n_s+1}}.$$

The absolute value of the error Δ of the relative variance after n_s stages compared with the relative variance at the output of the system is:

$$\Delta = \frac{|v_{rn} - v_r|}{v_r} = \frac{dM_l}{D_l(m-1)m + dM_l} \cdot \frac{1}{m^{n_s}}.$$

Since for the negative exponential distribution after l stages $D_l > M_l$ [12] we obtain:

$$\Delta < \frac{d}{(m-1)m + d} \cdot \frac{1}{m^{n_s}}.$$

Assuming that each separate stage of amplification in the channel is described by a negative exponential distribution (Furry's statistics) [12] with a mean gain of m $(m > 1)$ and variance $d = m(1+m)$ we obtain the number of stages n_s after which the relative variance has an error Δ compared with the relative variance at the output of the entire channel:

$$n_s < \ln(\frac{1+m}{2m\Delta}) / \ln m.$$

At the same time the effective length l_{eff} can be evaluated as $l_{eff} = \lambda n_s$ where λ is the average free path of electrons in the channel ($\lambda \approx d_c$).

For $\Delta = 0.01$, for typical values of the multiplier parameters, l_{eff} corresponds to half the channel length. The numerical experiment, using the MC methods, completely confirms this result. Fig.4 shows the relative variance v_r as a function of the length of the channel. It has been calculated for a single electron emitted at the

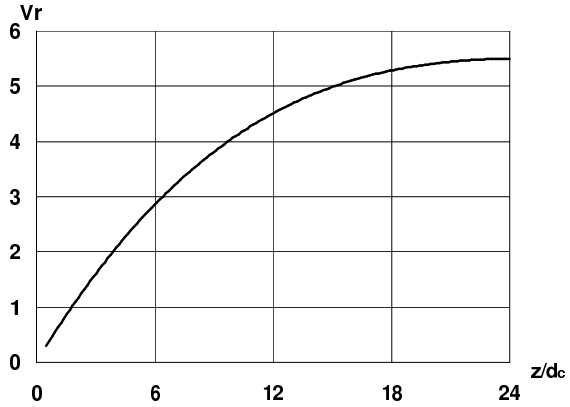

Fig. 4 The relative variance as a function of the channel length

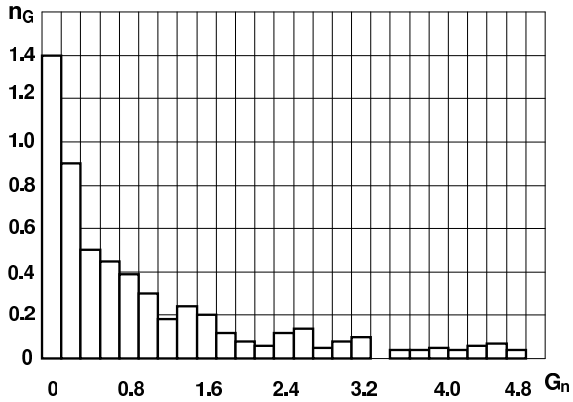

Fig. 6 The amplitude distribution for $z/d_c = 22$

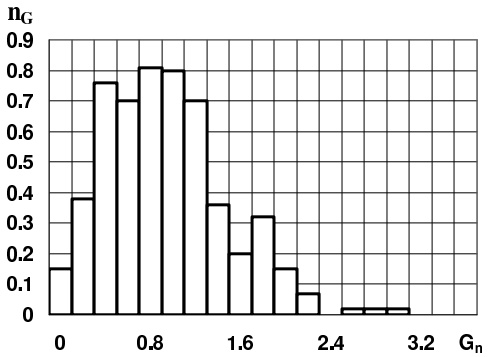

Fig. 5 The amplitude distribution for $z/d_c = 1$

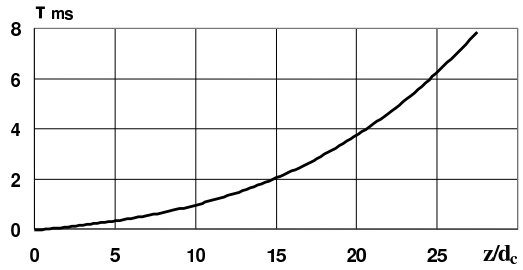

Fig. 7 Average time taken for simulation of one electron pulse

beginning of the channel (z is the length of the channel, and d_c is its diameter.)

Figures 5 and 6 show the amplitude distributions calculated by MC methods, for the different lengths of the channel (n_G is the number of pulses with gain G_n). For the small length ($z/d_c = 1$) the distribution is close to the Poisson distribution. As the length of the channel is increasing ($z/d_c = 22$, which is relevant to the half of the channel) the distribution changes to the negative exponential function.

All computational and experimental results here and further are obtained for the channel diameter $d_c = 10\mu m$, the channel length $l = 500\mu m$, the voltage on the channel $V_c = 800\ V$, the energy of the input electron beam V=2 kV and the angle of the input beam $\theta = 85^0$.

3 Evaluation of the Efficiency of the Method

The mean gain on the channel length z can be defined [18] as :

$$G_c(z) = e^{\eta z}, \tag{9}$$

where η is the electron gain on the length unit.

Let $x = z/d_c$, where d_c is the channel diameter. Then the time needed to calculate one electron pulse is proportional to the number of the emitted electrons. An average number of collisions on the length dx is proportional to $dG_c/dx = \eta e^{\eta x}$. Then the time needed to calculate the electron pulse can be declared as

$$\tau = \tau_0 \int_0^x \eta e^{\eta t} dt = \tau_0 (e^{\eta x} - 1), \tag{10}$$

where τ_0 is a proportional coefficient.

Values of τ_0 and η can be defined from computational experiments. Fig.7 shows the average time needed for MC simulations of one electron pulse as a function of the channel length (the data is relevant for the computer Pentium 4). From the graph, $\tau_0 = 0.44\ msec$ and $\eta = 0.12$.

If statistics is taken for 1000 MC iterations (pulses) for each data point [2] then simple calculations show time needed for particular simulations.

For example, MC simulations of one amplitude distribution at the output of the channel with a spread in the collision coordinates of the input beam of electrons (which is the practical case) will take more than 5 hours if the surface area stroked by particles is split into only 100 elementary surfaces. The use of the theorems proposed here and the algorithm described above reduces the time requirements to 5 - 10 seconds.

Calculations of the dependence of the average gain (G_c) and the noise factor (F) (see Sect.4) on the energy (E) of the input electron beam (Fig. 8 and 9) would take more than 2 days and nights of the constant work of the computer for one characteristic with only 10 data points. The use of the proposed theorems reduces the duration of calculations to 30 - 60 seconds.

It would require about 20 days and nights to find the optimal combination of the energy and the angle of the input electron beam which provides the minimal noise factor if only $[10 \times 10]$ data points are chosen, but about 1 - 2 minutes if the proposed theorems are used.

These evaluations are done for the uniform electrostatic field at the channel entrance. For the nonuniform electrostatic field the cost of calculations will be increased significantly for the direct MC simulations. Moreover, if some parameters of the input beam or of the channel were changed, the MC simulation should be conducted again.

Alternatively, the method proposed here does not require additional use of MC methods. The MC simulations in this case, should be conducted only once on the effective channel length for one electron emitted at the beginning.

Fig. 8 demonstrates the accuracy of the method, where theoretical results are compared with experimental data for the same channel parameters. [1]

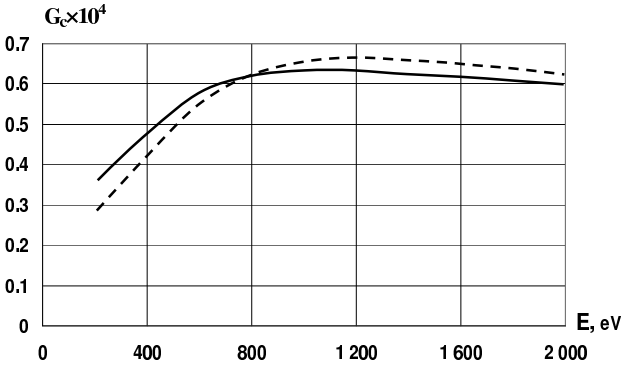

Fig. 8 Dependence of the average gain on the energy of the input electron beam (solid curve represents calculations and dashed curve represents experimental results).

4 The Noise Factor of the Channel Multiplier

4.1 The noise factor of a single channel

The noise factor F, which is a measure of the loss of available information [6], can be written as

$$F = \frac{(S/N)^2_{in}}{(S/N)^2_{out}}, \tag{11}$$

where $(S/N)_{in}$ and $(S/N)_{out}$ are ratios of the input signal (S) to the noise (N) and the output signal to the noise respectively. In the channel multiplier the noise is predominantly determined by the statistical fluctuations of the multiplication process (shot noise [18]), and defined by the spread of a number of the electrons about the mean gain.

Therefore, assuming that the number of electrons entering the multiplier fluctuates according to Poisson's law [12] with the mean and variance n_e, we obtain

$$(S/N)^2_{in} = (n_e/\sqrt{n_e})^2 = n_e. \tag{12}$$

Using the definition of the noise factor (11) and the theorems about serial amplification stages and parallel amplification paths expressions for calculating the noise factor can be obtained. The expressions depend on how the entire process is split into a sequence of amplification steps.

The entire amplification process in the channel multiplier can be represented in the form of a sequence of the next several stages.

1. The first observation of electrons, incident at the input of the multiplier (described by the Bernoulli distribution [12]), can be defined as a first stage. If γ is the fraction of the front surface of the multiplier exposed to electrons, then the average number of particles entering the channel and the variance can be given by

$$m_0 = \gamma, \qquad d_0 = \gamma(1-\gamma). \tag{13}$$

2. The collision of the primary electrons with the wall of the channel is defined as the second stage of the amplification. The distribution of the number of electrons knocked out by one primary electron (with the mean m_1 and the variance d_1) depends on the properties of the emitter. For a uniform emitter the number of electrons fluctuates according to Poisson's law with

$$m_1 = d_1 = \sigma_1,$$

where σ_1 is the secondary emission yield of the emitting surface (see Sect.2.2).

[1] The experimental data was provided by A.M.Tyutikov, State Optic Institute, St Petersburg.

3. Further amplification of the electrons in the channel is regarded as the third stage with the mean gain $m_2 = m(L)$ and the variance $d_2 = d(L)$.

Taking into account the contribution of each stage to the overall process of amplification and with the help of (2) and (3) we obtain:

$$(\frac{S}{N})^2_{out} = \frac{M^2}{D}, \qquad where \tag{14}$$

$$M = n_e\gamma m_1 m(L), \qquad and$$
$$D = n_e[\gamma m_1 m(L)]^2 + \gamma(1-\gamma)n_e[m_1 m(L)]^2 + d_1 n_e \gamma m^2(L) + d(L)n_e\gamma m_1.$$

Based on the definition of the noise factor (11) and using formulae (12)-(14) the noise factor for the three stages of amplification can be written as

$$F = \gamma^{-1}(1 + v_{r1} + v_{r2}/m_1),$$

where $v_{r1} = d_1/m_1^2$ is the relative variance of the distribution at the output of the second stage, and $v_{r2} = d(L)/m^2(L)$ is the relative variance of the stage of amplification of a single electron.

4.2 The noise factor of an array of the channels

Let us define the noise factor at the output of the system of n parallel channels (Fig.1) where diameters of the channels are not necessarily the same. The single channel can be defined as one of n parallel amplification paths with the mean g_k and the variance v_k of the amplitude distribution at the output of the k-th channel. Let ρ_k be the probability of entering the channel with radius R_k. The part of the front surface of the channel plate closed to electrons is considered as a separate path with the mean g_0, variance v_0 and probability $\rho_0 = 1 - \gamma$.

Using (4) and (6) the mean G and variance V of the distribution at the output of the system of n channels can be obtained as:

$$G = \sum_{k=1}^{n} \gamma\rho_k g_k + (1-\gamma)g_0,$$

$$V = \sum_{k=1}^{n} \gamma\rho_k v_k + (1-\gamma)v_0 + \sum_{k=1}^{n} \gamma\rho_k g_k^2 + (1-\gamma){g_0}^2 - G^2.$$

For $g_0 = 0$ and $v_0 = 0$, using the theorem about serial amplification stages and the definition (11), the noise factor F can be written as:

$$F = 1 + \frac{V}{G^2} = \frac{1}{\gamma}(1 + \frac{V_1}{G_1^2}), \qquad where \tag{15}$$

$$G_1 = \sum_{k=1}^{n} \rho_k g_k,$$

$$V_1 = \sum_{k=1}^{n} \rho_k v_k + \sum_{k=1}^{n} \rho_k g_k^2 - G_1^2.$$

The distribution of the radii in the system of parallel channels is a continuous function, and the probability ρ_k should be changed to the probability density function $\psi(R)$ for the electron to enter the channel with the radius R. Then, the expressions for G_1 and V_1 can be written as:

$$G_1 = \int_{R_{min}}^{R_{max}} \psi(R)g(R)dR, \tag{16}$$

$$V_1 = \int_{R_{min}}^{R_{max}} \psi(R)v(R)dR + \int_{R_{min}}^{R_{max}} \psi(R)g^2(R)dR - G_1^2, \tag{17}$$

where $g(R)$ and $v(R)$ are the mean and the variance of the amplitude distribution at the output of the channel with the radius R.

If all channels in assembly are identical and have radius R then G_1 and V_1 will be equal to the mean and variance of the amplitude distribution at the output of one channel. The noise factor of one channel with radius R will be equal to the noise factor of the assembly of channels with the same radii.

Fig. 9 compares theoretical and experimental results of the noise factor as a function of the energy of the input electron beam. The computational results are obtained for the array of identical channels, and experimental data corresponds to the real array of channels (MCP) where a spread in channel diameters (due to the old technology) increases the noise. On the other hand, the computational results are obtained for the ideal uniform emitter and without taking into account the effect of the contact conducting layer at the exit of the channels.

5 Effect of Variations in Channel Diameters in the Channel Plate on the Noise Factor

Variations of the channel diameters as a result of technological distortions of a channel's geometry lead to the variations of the amplitude distributions at the outputs of different channels, and therefore they increase the noise factor.

The effect of the variations of the diameters in the channel plate on the noise factor can be evaluated using

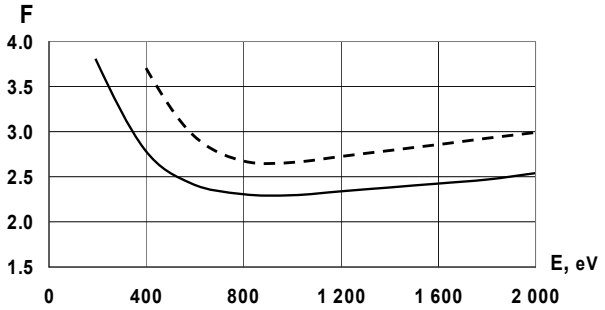

Fig. 9 Dependence of the noise factor on the energy of the input electron beam (solid curve represents calculations and dashed curve represents experimental results).

the equations (15) - (17). To find the probability density function $\psi(R)$, let n be the total number of channels, n_k is the number of channels with the radius R_k, and $\rho_k = n_k/n$ is the probability of choosing the channel with radius R_k (probability distribution of the radii in the array of channels). If S_k is the area of the channels with radius R_k, and S is the total area of all channels, then the probability of entering the channel with radius R_k is defined by:

$$P_k = \frac{S_k}{S} = \frac{\rho_k R_k^2}{\rho_1 R_1^2 + \rho_2 R_2^2 + ...}.$$

Therefore, the probability density distribution $\psi(R)$ can be written as

$$\psi(R) = \frac{R^2\varphi(R)}{\int_0^\infty R^2\varphi(R)dR}, \tag{18}$$

where $\varphi(R)$ is the probability density distribution of the channels' radii in the array.

Assume that $\varphi(R)$ is defined by the probability density function for the normal distribution:

$$\varphi(R) = \frac{1}{\sigma_x\sqrt{2\pi}} exp[-\frac{(R-\bar{R})^2}{2\sigma_x^2}], \tag{19}$$

where ${\sigma_x}^2$ is the variance, and $\bar{R}$ is the mean.

Substituting (19) into the equation (18) we obtain

$$\psi(R) = AR^2 exp[-\frac{(R-\bar{R})^2}{2\sigma_x^2}], \qquad where$$

$$A = [\sigma_x\sqrt{2\pi}\int_0^\infty R^2\varphi(R)dR]^{-1}.$$

According to the normalization condition:

$$A\int_0^\infty R^2 exp[-\frac{(R-\bar{R})^2}{2\sigma_x^2}]dR = 1. \tag{20}$$

Introducing a new variable $t = \frac{R-\bar{R}}{\sigma_x}$ the expression (20) can be written as:

$$A(\sigma_x^3\int_{-\bar{R}/\sigma_x}^\infty t^2 e^{-t^2/2}dt + 2\bar{R}\sigma_x^2\int_{-\bar{R}/\sigma_x}^\infty te^{-t^2/2}dt +$$

$$+\bar{R}^2\sigma_x\int_{-\bar{R}/\sigma_x}^\infty e^{-t^2/2}dt) = 1. \tag{21}$$

The integrals in (21) can be expressed via the integral (see the standard normal cumulative distribution function [7]):

$$I(x) = 2/\sqrt{2\pi}\int_0^x e^{-t^2/2}dt.$$

Therefore, we obtain

$$\int_{-\bar{R}/\sigma_x}^\infty e^{-t^2/2}dt = \frac{\sqrt{2\pi}}{2}[1 + I(\frac{\bar{R}}{\sigma_x})],$$

$$\int_{-\bar{R}/\sigma_x}^\infty te^{-t^2/2}dt = e^{-\bar{R}^2/2\sigma_x^2},$$

$$\int_{-\bar{R}/\sigma_x}^\infty t^2 e^{-t^2/2}dt = \frac{\sqrt{2\pi}}{2}[1 + I(\frac{\bar{R}}{\sigma_x})] - \frac{\bar{R}}{\sigma_x}e^{-\bar{R}^2/2\sigma_x^2}.$$

Finally, the expression for the probability density distribution $\psi(R)$ can be written as:

$$\psi(R) = \frac{2R^2 exp[\frac{-(R-\bar{R})^2}{2\sigma_x^2}]}{2\bar{R}\sigma_x^2 exp(\frac{-R^2}{2\sigma_x^2}) + \sigma_x\sqrt{2\pi}(\bar{R}^2+\sigma_x^2)[1 + I(\frac{\bar{R}}{\sigma_x})]}.$$

For the normal distribution, the maximum and minimum radii of the channels in the array can be given as

$$R_{max} = \bar{R} + 3\sigma_x \qquad R_{min} = \bar{R} - 3\sigma_x. \tag{22}$$

If δ is the variation of the channels' radii (as a percentage of the mean value $\bar{R}$) then

$$R_{max} = \bar{R} + \delta\bar{R}/100 \qquad R_{min} = \bar{R} - \delta\bar{R}/100. \tag{23}$$

Using the expressions (22) and (23) we find $\sigma_x = \delta\bar{R}/300$. For the values $\bar{R}/\sigma_x$ corresponding to $\delta \sim 1 - 50\%$, $I(\bar{R}/\sigma_x) \approx 1$ (see the standard normal table [7]).

Finally the probability density function $\psi(R)$ is defined as

$$\psi(R) = \frac{R^2 e^{-(\frac{300(R-\bar{R})}{\sqrt{2}\delta\bar{R}})^2}}{\bar{R}(\frac{\delta\bar{R}}{300})^2 e^{-(\frac{300R}{\sqrt{2}\delta\bar{R}})^2} + \frac{\sqrt{2\pi}\delta\bar{R}}{300}[\bar{R}^2 + (\frac{\delta\bar{R}}{300})^2]}. \tag{24}$$

Using formulae (15) - (17) and (24), the dependence of the noise factor F on the variations of the channels' diameters δ is calculated (Fig.10), where $\bar{R} = 4\mu m$. It is seen that even for $\delta = 5\%$ of the mean value $\bar{R}$, the noise factor F increases by 75% compared with the noise factor for $\delta = 0$. [2]

[2] It is clear that the variations of up to 25% about the mean channel radius is incompatible with the dimensions of real microchannel plates. However, the function $F(\delta)$ in Fig.10 is theoretical, and can be interesting from a mathematical point of view.

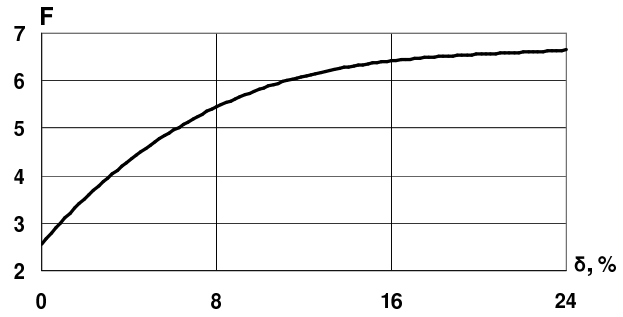

Fig. 10 The noise factor as a function of the variations of the channel diameters

The results obtained here can be used to calculate the noise factor F for the given values of δ and $\bar{R}$, to calculate δ which provides the required value F, and also to optimize parameters of the channel plate in terms of the minimum F.

6 Conclusions

The method for calculation of the stochastic processes has been developed where the entire process is represented in the form of the sequence of several stages.

The theorems for the multistep sequential processes and for the parallel amplification paths have been proved. Expressions for the mean and variance of the amplitude distribution at the output of the system have been obtained.

The transformation of the input ratio *signal/noise* to the output one has been shown, and the expressions for calculating the noise factor have been obtained. The noise factor, as a function of the variations of the channel diameters, has been calculated.

It has been shown that the amplitude distribution at the output of the channel is determined by the effective length of the channel. It enables MC simulations to be carried out only along this length for one electron, emitted at the beginning of the channel. The output of the entire amplification process is calculated by the use of the theorems and obtained characteristics. Any further investigations and optimizations can be conducted without MC simulations. The method provides highly accurate results and significantly reduces the cost of calculations. The contribution of different amplification stages to the entire stochastic process can be easily investigated.

The method enables to optimize characteristics of the channel amplifier in terms of the noise factor, such as an incidence angle and energy of the input electron beam, the spread in incidence coordinates of the primary electrons, nonuniformity of the emitter surface, the length of a contact conducting layer or the depth and secondary emission yield of a high-efficiency emitter etc. The model easily implements new experimental data without any changes in the algorithm.

The method can be applied for the optimization of the whole electronic devise, such as image intensifier, where MCP is considered as an intermediate stage in the entire amplification process. Moreover, the developed method is not limited to systems with the channel multipliers and can be used for many stochastic processes which require computer simulations.

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