

Counter-Examples in Teaching/Learning of Calculus: Students' Performance

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ABSTRACT: This paper presents a case study which involved the first year science and engineering students of the Auckland University of Technology, New Zealand. After very favourable feedback received from the students in the international study on their attitudes towards usage of counter-examples in teaching/learning of Calculus (Gruenwald & Klymchuk, 2003) it was decided to investigate how usage of counter-examples affects students' performance. The case study revealed that usage of counter-examples significantly improved students' performance on a test question that required conceptual understanding but did not affect their performance on the other test questions (applying familiar rules and algorithms, calculations, applications, etc.).

INTRODUCTION

Using counter-examples is a way of communication in mathematics. Actually the whole history of mathematics can be viewed as making conjectures and then either proving them or disproving them by counter-examples. Just a few examples:

1. For a long time mathematicians tried to find a formula for prime numbers. The numbers of the form $2^{2^n} + 1$, where n is natural once were considered as prime numbers until a counter-example was found. For $n = 5$ that number is composite: $2^{2^5} + 1 = 641 \times 6700417$.
2. Another conjecture about prime numbers stills waits to be proved or disproved. It is called the Goldbach's or Goldbach-Euler conjecture and it was posed by Goldbach in his letter to Euler in 1742. It looks very simple. It states that *every even number greater than 2 is the sum of 2 prime numbers*. For example, $12 = 5 + 7$, $20 = 3 + 17$, and so on. A powerful computer was used in 1999 to search counter-examples to that conjecture. No counter-examples have been found up to 4×10^{14} . In 2000 the book publishing company Faber offered a \$US 1 million prize to someone who could prove or disprove that conjecture within 2 years. Nobody has solved it until now.
3. In 19th century the great German mathematician Weierstrass constructed his famous counter-example – the first known fractal - to the statement 'a function continuous on (a,b) cannot be non-differentiable at any point on (a,b) '. Many mathematicians at that time thought that such 'monster-functions' that were continuous but not differentiable at any point were absolutely useless for

practical applications. About a hundred years later Norbert Wiener, the founder of cybernetics pointed out in his book “I am a mathematician” such curves exist in nature – for example, they are trajectories of particles in the Brownian motion. In recent decades such curves have been investigated in the theory of fractals – a fast growing area with many applications.

Using counter-examples in teaching/learning of Calculus can be beneficial for:

- deeper conceptual understanding
- reducing or eliminating some common misconceptions
- improving advanced mathematical thinking - neither algorithmic nor procedural
- enhancing generic critical thinking skills – analysing, justifying, verifying, checking, proving which can benefit students in other areas of life
- expanding the ‘example set’ - a number of examples of interesting functions for better communication of ideas in mathematics and in practical applications
- making learning more active and creative

The international study on students’ attitudes towards usage of counter-examples in teaching/learning of Calculus as a pedagogical strategy (Gruenwald & Klymchuk, 2003) that involved more than 600 students from 10 universities in different countries showed that the students’ attitudes were very positive. 92% of the participants reported that the pedagogical strategy was very effective. Many of them commented that it helped them to understand concepts better, prevent mistakes in future, develop logical and critical thinking, and made their participation in lectures more active. But the students’ attitudes and their exam performance are different matters. This study investigates how usage of counter-examples affects students’ performance.

THE STUDY

Two groups of the students of the Auckland University of Technology, New Zealand were selected for this case study. The students were majoring in science and engineering. In group A there were 14 students and in group B (the control group) there were 11 students. All the students in both groups had similar age and mathematics background and all were Chinese. There were 3 lectures and 1 tutorial per week in both groups. In both groups there was the same lecturer.

The only difference was - in group A one of the 3 lectures a week was given by another lecturer. That lecturer used counter-examples in his lectures for 5-6 minutes (out of a 50 minute lecture). Usually there were 1-2 examples per lecture. There were 8 weeks before the mid-semester test so there were 8 lectures in which counter-examples were used. During the 8 week period the total time for that activity was about 45 minutes.

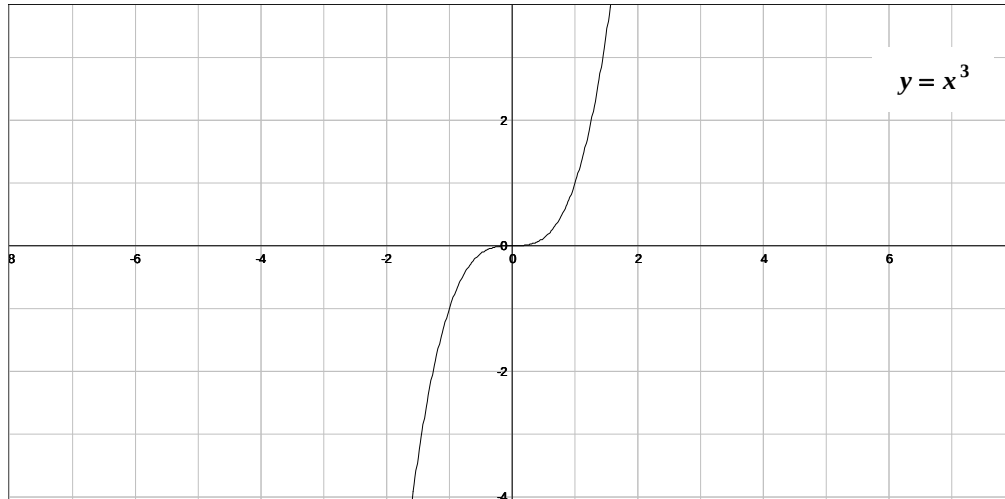
The research question was to check how usage of counter-examples affected students’ performance on a test question that required conceptual understanding.

Below are some statements and counter-examples to them that were discussed in lectures in group A.

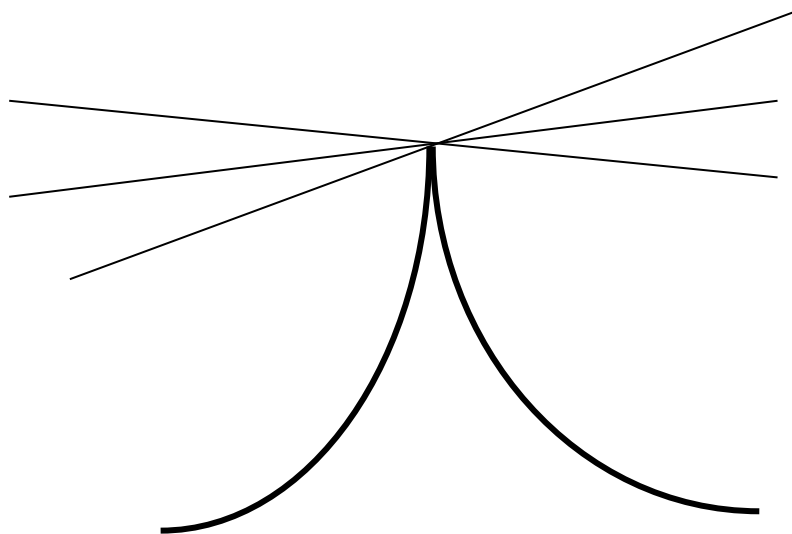
Statement 1. The tangent to a curve at a point is the line which touches the curve at that point but does not cross it there.

Counter-example.

a) The x-axis is the tangent line to the curve $y = x^3$ but it crosses the curve at the origin.



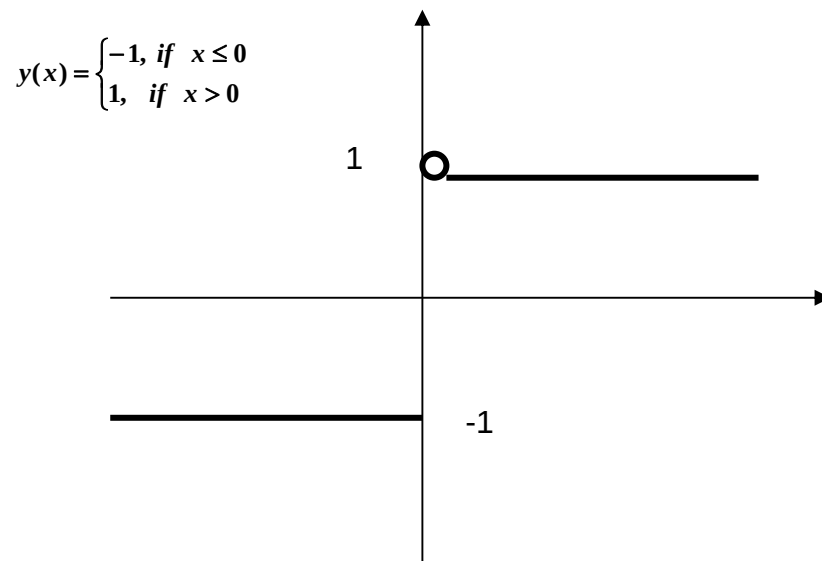
b) The three straight lines just touch and don't cross the curve below at the point but none of them is the tangent line to the curve at that point.



Statement 2. If the absolute value of the function $y = f(x)$ is continuous on (a,b) then the function is also continuous on (a,b) .

Counter-example. The absolute value of the function

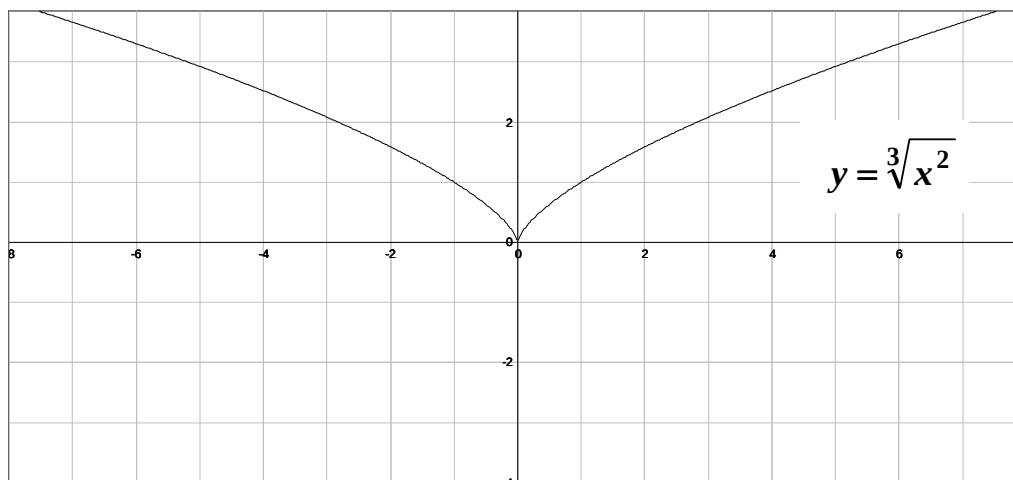
$y(x) = \begin{cases} -1, & \text{if } x \leq 0 \\ 1, & \text{if } x > 0 \end{cases}$ is $|y(x)| = 1$ for all real x and it is continuous but the function $y(x)$ is not continuous.



Statement 3. If a function is continuous on $\mathbb{R}$ and the tangent line exists at any point of its graph then the function is differentiable at any point on $\mathbb{R}$.

Counter-example.

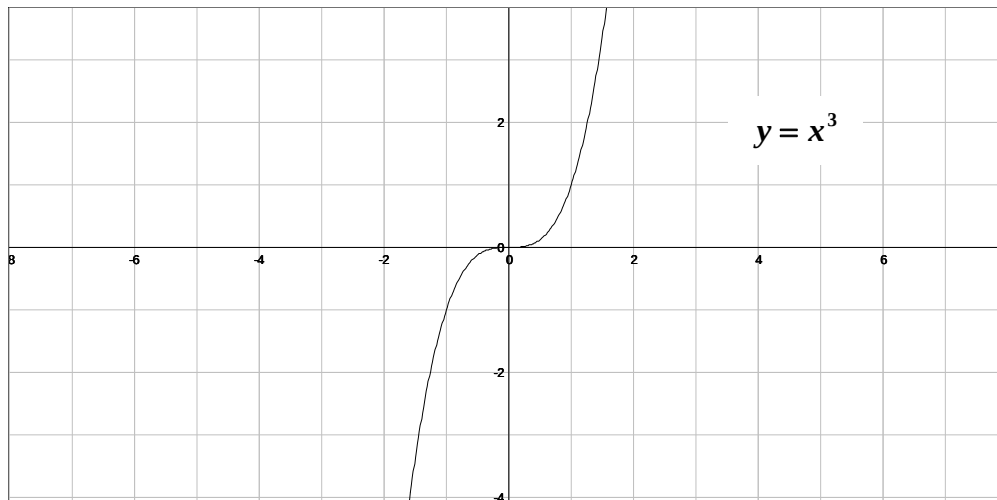
The function $y = \sqrt[3]{x^2}$ is continuous on $\mathbb{R}$ and the tangent line exists at any point of its graph but the function is not differentiable at the point $x = 0$.



Statement 4. If the derivative of a function is zero at a point then the function is neither increasing nor decreasing at this point.

Counter-example.

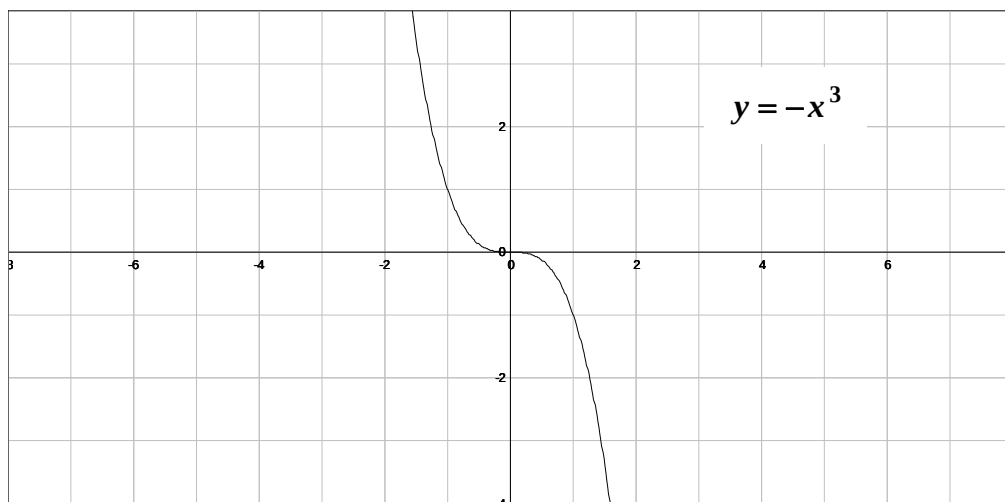
The derivative of the function $y = x^3$ is zero at the point $x = 0$ but the function is increasing at the point $x = 0$.



Statement 5. If a function is differentiable and decreasing on (a,b) then its gradient is negative on (a,b) .

Counter-example.

The function $y = -x^3$ is differentiable and decreasing on $\mathbb{R}$ but its gradient is zero at the point $x = 0$.

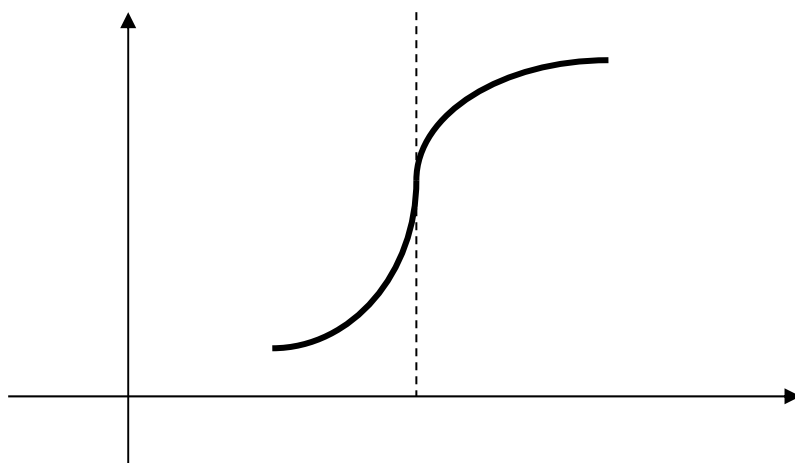


THE RESULTS

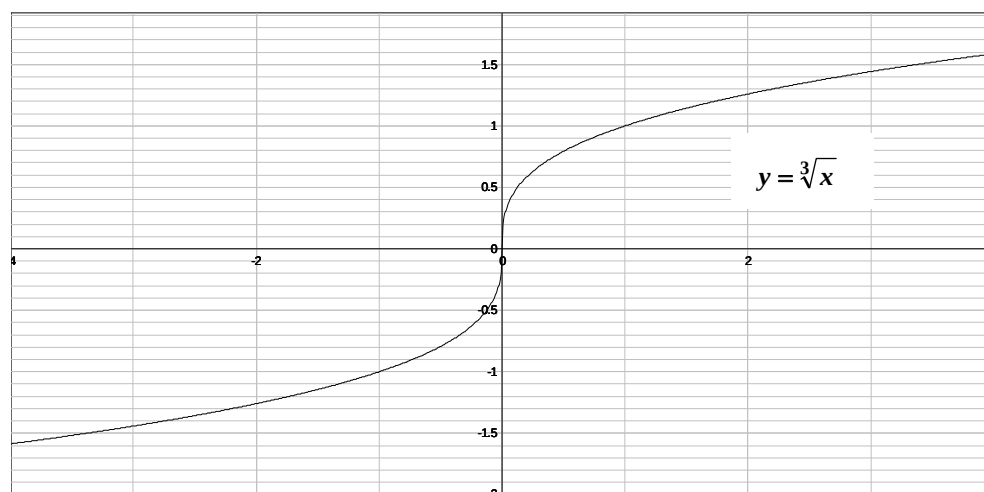
After 8 weeks of study both groups were taking the same mid-semester test that contained 11 questions: the first 10 questions were on techniques and question 11 was on conceptual understanding.

Question 11. Sketch a graph of a function whose graph is a continuous and smooth curve (no sharp corner) at a point but which is not differentiable at that point.

What we expected from the students was a simple sketch:



Or they could come up with a cube root function:



The results of the test and question 11 are below:

Group A: Passed the test 13/14=93%

Question 11: 11/14=79%

Group B: Passed the test 10/11=91%

Question 11: 5/11=45%

DISCUSSION AND CONCLUSION

The students' performance in both groups on each of the questions 1-10 was very similar. Their overall performance on the test also was very similar: 93% of the students in group A and 91% in group B passed the test (received more than 50% of the total marks). But there was a significant difference in giving the correct solution to question 11: 79% of the students in group A versus 45% in group B. This might indicate that usage counter-examples in group A improved students' conceptual understanding so they performed better in question 11.

As with any case study one of the major questions is – to which extent can we generalise its results? Regardless of the answer to this somewhat rhetorical question usage of counter-examples as a pedagogical strategy is certainly worth trying!

There is a well-known book on counter-examples in Calculus: “Counterexamples in Analysis” by B.R.Gelbaum and J.M.H.Olmsted (Holden-Day, Inc., San Francisco, 1964). It is an excellent resource for teaching/learning of Calculus at an advanced level but it is well beyond the scope of a basic first-year university Calculus course based for example on the popular textbook “Calculus: Concepts and Contexts” by J.Stewart (Brooks/Cole, Thomson Learning, 2nd ed., 2001). Another supplementary teaching resource can be a recently published book “Counter-Examples in Calculus” (Klymchuk, 2004). These two books on counter-examples are not overlapping – all statements and examples are different. The latter is aimed to fill the niche in the activity on using counter-examples as a pedagogical strategy in teaching/learning of Calculus in upper secondary school and first-year university Introductory Calculus.

REFERENCES

Gruenwald, N. & Klymchuk, S., Using counter-examples in teaching Calculus. *The New Zealand Mathematics Magazine* 40, 2, 33-41 (2003).

Klymchuk, S., *Counter-Examples in Calculus*. Maths Press. Auckland, New Zealand, ISBN 0-476-01215-5 (2004).