Spatial modelling of extreme precipitation in the Basque country, Spain

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Outline

1. Introduction
2. Univariate/multivariate extremes
3. Inference
4. Application to precipitation data : Basque country, Spain
5. Conclusion and issues

Introduction

Basic problem :
Simplest case, $X_1, X_2, \cdots, X_n \sim F$. Require accurate inference on tail F.

- Few observations in the tail of the distribution.
- Standard density estimation technique is often biased in estimating tail probabilities.
- Base tail models on asymptotically-motivated distributions : Statistics of extremes

Applications :

- Environmental : sea levels, pollution concentration, precipitation levels (rainfall, snow), river flow
- Reliability modelling : finance, insurance, telecommunication, ...

Univariate extremes

Framework for block maxima distribution Let $Y_1, \cdots, Y_n \sim F$ and define

$$M_n = \max\{Y_1, \cdots, Y_n\}\ .$$

The distribution of M_n is

$$Pr\{M_n \leq x\} = Pr\{Y_1 \leq y, \cdots, Y_n \leq y\} = F(y)^n$$

Approximate F^n by limit distribution as $n \to \infty$.

Extremal types theorem

For an i.i.d. sample $Y_1, \ldots, Y_n \in \mathbb{R}$.
If there exist sequences of constants $\{a_n > 0\}$ and $\{b_n\}$ such that

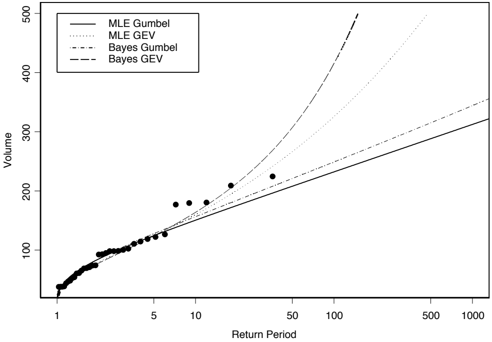
$$\lim_{n\to\infty} \Pr\left(\frac{\max\{Y_i\} - b_n}{a_n} \leq z\right) \to G(z)$$

is a non degenerate distribution function, then G is a generalised extreme value distribution (GEV)

$$G(z) = \exp\left\{-\left[1 + \xi\left(\frac{z-\mu}{\sigma}\right)\right]_+^{-1/\xi}\right\}.$$

- In analogy with central limit theorem for mean of samples

- In practice: fit GEV to sample maxima & predict levels of future extremes
- E.g. prob of maximum daily (n=365.25) rainfall exceeding a given level in the next 100 years



- Extension to spatial data via hierachical models
    - Spatially/temporally varying parameters
    - Spatial dependence through parameters only
    - No dependence at data level
- Annual maxima of daily spatial rainfall might be approximated by a stationary max-stable process
    - Still spatially/temporally varying parameters
    - Dependence at data level retained

Multivariate extremes

Max stable process (de Haan 1984)

A max-stable process $Z(x)$ is the limit process of maxima of i.i.d. random fields $Y_i(x)$, $x \in \mathbb{R}^d$

$$Z(x) = \lim_{n\to\infty} \frac{\max\{Y_i(x)\} - b_n(x)}{a_n(x)}, \qquad x \in \mathbb{R}^d \tag{1}$$

for two sequences of functions $a_n(\cdot) > 0$, $b_n(\cdot)$ (where they exist).

- Max-stable processes generalize multivariate extreme value distributions to the infinite dimensional case
- Incorporates dependence between locations x
- Can obtain GEV margins for each $x \in \mathbb{R}^d$ (assume unit Frećhet margins for the moment ...)

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for two sequences of functions $a_n(\cdot) > 0$, $b_n(\cdot)$ (where they exist).

- Without loss of generality, if $a_n(x) = n, b_n(x) = 0$, $\{Z(x)\}_{x\in\mathbb{R}^d}$ has a unit Fréchet margins with distribution function $F(z) = \exp(-1/z)$, $z > 0$.
- If $\{Z(x)\}_{x\in\mathbb{R}^d}$ is a stationary process, it can be expressed through spectral representation (de Haan and Pickands 1986).

Spectral representation

- Smith's storm model (Smith 1990)
- Schlather's model (Schlather 2002)

Smith's model

Let $\{\tau_i, k_i\}$ denote a non-homogeneous Poisson process on $\mathbb{R}^2 \times \mathbb{R}_+$ with intensity measure $k^{-2}dk \times \mu(d\tau)$ (μ is a +ive measure) then

$$Z(x) = \max_i k_i f(x - \tau_i) \qquad x \in \mathbb{R}^2$$

where f is a unimodal continuous pdf.
"Storm profile model" interpretation:
k_i=storm magnitude, τ_i=storm location, f=storm shape.

For $f = N_d(0, \Sigma)$, the bivariate CDF is
$Pr(Z(x_1) \leq z_1, Z(x_2) \leq z_2) =$
$$\exp\left[-\frac{1}{z_1}\Phi\left(\frac{a}{2} + \frac{1}{a}\log\frac{z_2}{z_1}\right) - \frac{-1}{z_2}\Phi\left(\frac{a}{2} + \frac{1}{a}\log\frac{z_1}{z_2}\right)\right]$$
where Φ is a standard normal CDF and $a^2 = \Delta x^T \Sigma^{-1} \Delta x$, $h = (x_2 - x_1)$.

Schlather's model

Let $\{Y_i\}$ denote a stationary process on $\mathbb{R}^2$ such that $\mathbb{E}[\max\{0, Y(x)\}] = 1$ and $\{k_i\}$ be the points of a Poisson process on $\mathbb{R}_+$ with intensity measure $k^{-2}dk$ then

$$Z(x) = \max_i k_i \max\{0, Y_i(x)\} \ .$$

Taking Y_i to be a stationary standard Gaussian process with correlation function $\rho(h)$, the bivariate CDF is

$$Pr(Z(0) \le z_1, Z(h) \le z_2) =$$

$$\exp\left[\frac{-1}{2}\left(\frac{1}{z_1}+\frac{1}{z_2}\right)\left(1+\sqrt{1-2(\rho(h)+1)\frac{z_1 z_2}{(z_1+z_2)^2}}\right)\right] \ .$$

Correlation functions : Three common parametric families for ρ are

Whittle-Matérn	$\rho(h) = c_1 \frac{2^{1-\nu}}{\Gamma(\nu)} \left(\frac{h}{c_2}\right)^{\nu} K_{\nu}\left(\frac{h}{c_2}\right)$
Cauchy	$\rho(h) = c_1 \left[1 + \left(\frac{h}{c_2}\right)^2\right]^{-\nu}$
Powered Exponential	$\rho(h) = c_1 \exp\left[-\left(\frac{h}{c_2}\right)^{\nu}\right]$

where c_1, c_2 and ν are the sill, range and smooth parameters.

Inference

- Estimator : probability-weighted moments, likelihood-based techniques
  For max-stable processes, as only the bivariate densities are known we will consider the pairwise likelihood

  $$logL_p(y;\theta) = \sum_{j} \sum_{i<j} \sum_{k=1}^{n} logf(y_k^i, y_k^j)$$

  (Lindsay 1998, Varin 2008)
- Model comparison : graphical model checking, composite likelihood information criterion (CLIC)
- Return levels and return periods
- Spatial dependent measures : the extremal coefficient function, geostatistics based approaches (F-madogram, λ-madogram)

The Extremal Coefficient

Let $Z(\cdot)$ be a stationary max-stable random field with unit Fréchet margins. The extremal dependence among N fixed locations in $\mathbb{R}^d$ can be summarised by the extremal coefficient which is defined as

$$Pr[Z(x_1) \leq z, \cdots, Z(x_N) \leq z] = \exp\left(-\frac{\theta_N}{z}\right)$$

where $1 \leq \theta_N \leq N$ with the lower and upper bounds corresponding to complete dependence and independence.

- θ_N can be regarded as the effective number of independent stations.

- We will especially focus on pairwise extremal coefficients

$$Pr(Z(x_1) \leq z, Z(x_2) \leq z) = \exp\left(-\frac{\theta(x_1 - x_2)}{z}\right)$$

  and $\theta(\cdot)$ is the extremal coefficient function.
- The extremal coefficient functions for the two models are

  $$\text{Smith}: \ \theta(x_1 - x_2) = 2\Phi\left(\frac{\sqrt{(x_1 - x_2)^T \Sigma^{-1} (x_1 - x_2)}}{2}\right)$$

  $$\text{Schlather}: \ \theta(x_1 - x_2) = 1 + \sqrt{\frac{1 - \rho(x_1 - x_2)}{2}}$$

  NOTE : Schlather's model has an upper bound of $1 + \sqrt{1/2}$ for a positive ρ.

Case study : Basque country, Spain

- Daily precipitation records for 97 years, 1914-2010, over 234 catchments.
- Unbalanced records are infilled using a spacial-temporal model (Cowpertwait 206).

- Fréchet margin transformation : spatial GEV model (longitude, latitude, and altitude), AIC

$$g : Y(x) \rightarrow \left(1 + \xi(x)\frac{Y(x) - \mu(x)}{\sigma(x)}\right)^{1/\xi(x)}$$

  where $\mu(x), \sigma(x)$ and $\xi(x)$ are linear dependent to x. Hence, the bivariate distribution is rewritten as

$$Pr[Y(x_1) \leq y_1, Y(x_2) \leq y_2] = Pr[Z(x_1) \leq g(y_1), Z(x_2) \leq g(y_2)] .$$

- Spatial extreme model selection : composite likelihood information criteria (CLIC)

Figure: Spatial GEV model diagnostic plots

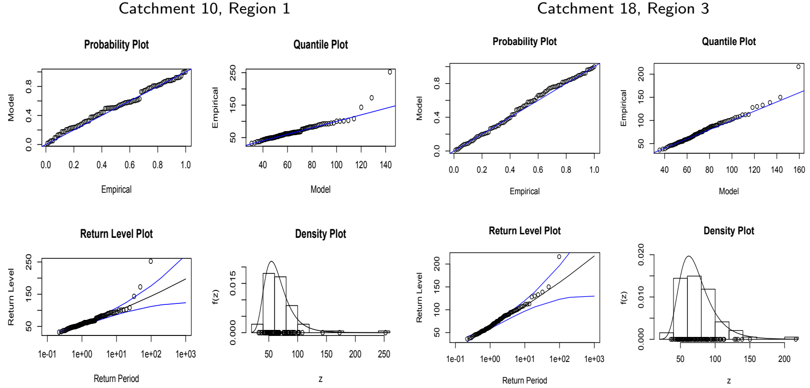
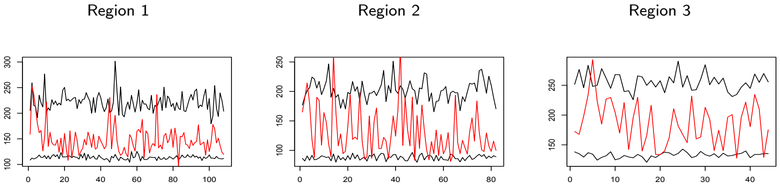


Figure: Spatial GEV model diagnostic plots; Return level of one in 97 years, 95% confidence intervals (red) and observations (black)



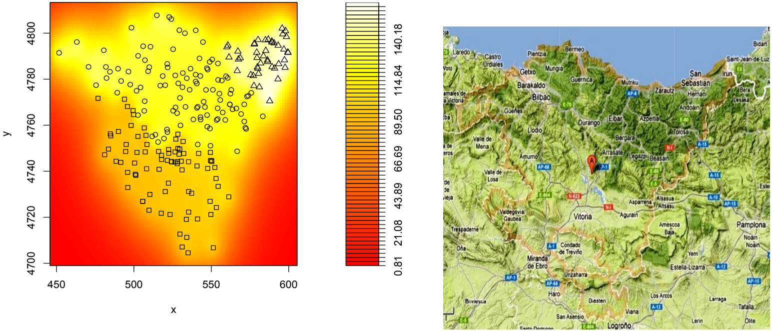
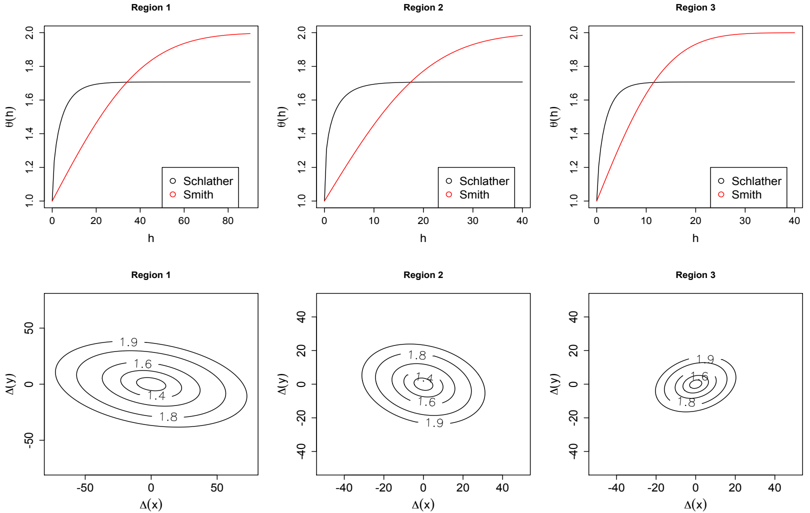


Figure: Left : Pointwise return levels of one in 50 years events. The sites located in the regions 1,2 and 3 are indicated by the circle, square, and triangle marks respectively. The units for x and y are in kilometers.

- Extremal coefficient estimates, θ_{Smith} and $\theta_{Schlather}$.



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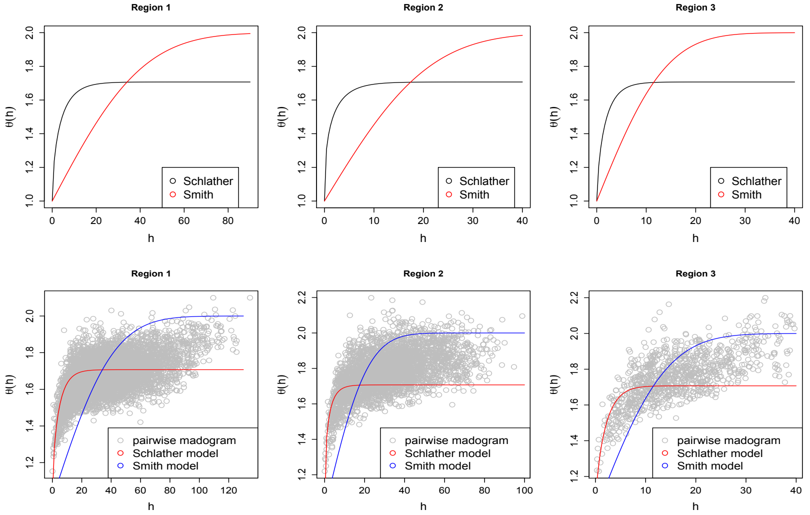
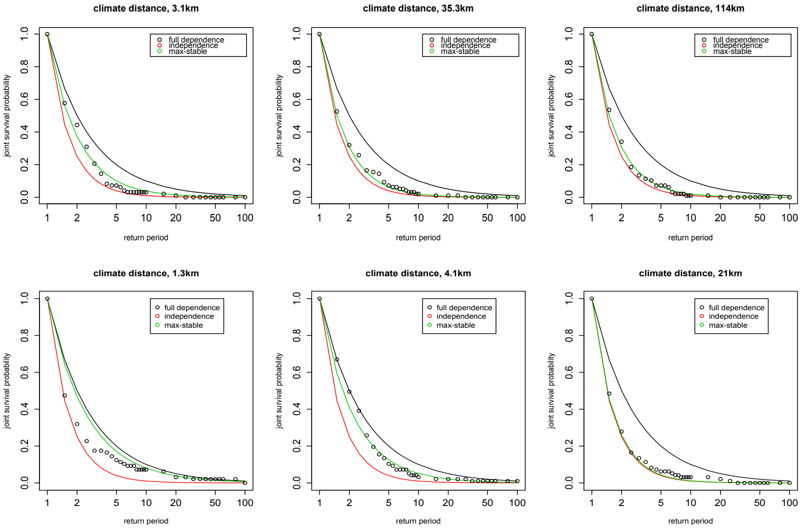


Figure: Risk analysis of pairwise annual maxima: joint survival probability versus return period.



Conclusion and issues

Conclusion :

- Max-stable models are powerful ad mathematically justified models of spatial extremes.
- More realistic predictions over hierachical models.
- Application to the precipitation in Basque country, Spain.
- Design storm : Average storm over the region (sort of).

Issues :

- Fréchet margin transformation techniques
- Infilled data technique (due to irregular observations) : splines (Neville, S. E. et al, 2011)

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Thank you for your attention!

This work has been done by using the R package *SpatialExtremes*

`http://spatialextremes.r-forge.r-project.org/`