Exploring Tertiary Students' Mathematical Modelling Experiences: Insights for Practice

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Abstract

The purpose of this study is to investigate tertiary student learning experiences for mathematical modelling. This study is based in New Zealand and involves three different tertiary modelling courses. For students participating in these modelling courses, this was their first time studying modelling at tertiary level. This study uses Interpretive Description as the methodology to look at the student experience with the aim of establishing a better understanding of student experiences with mathematical modelling, the role the lecturer plays in these experiences, and how any new insights can better inform teaching practices.

This study aims: to provide new insights and understandings of the student perspective to inform and advance tertiary teaching practices; to gain a deeper understanding of the lecturer's role in the student experience; to contribute to the general discussion on how to provide rich student learning experiences in mathematical modelling; and to inform the general discussion for students to be able to contribute to society through mathematical modelling.

One overarching theme and three subthemes were constructed from the data. The overarching theme *pushing our boundaries,* captures how all students had experiences that are new, unfamiliar, and outside their normal range of experiences with mathematics while mathematically modelling. Subtheme one, *moving forward on our own,* captures how students needed to learn how to move forward on their own in order to make progress. Moving forward involved taking responsibility, both individually and as a group, to actively find ways forward. Subtheme two, *being moved forward,* focusses on the use of resources and the lecturer's role in moving students forward, while also enabling student groups to work independently of the lecturer. Subtheme

three, *going forward together as a collective,* captures the interwoven and collective nature of learning to mathematical model.

This research shows how modelling is a change of culture for students and the lecturer has a role to play. This research presents nuances of the dynamics between independent student work and the lecturer's role.

The findings of this study suggest students can undertake modelling without the lecturer or teacher being present if they have prepared for the experience beforehand and have access to appropriate resources. Lecturer behaviours that help students to take responsibility for their own modelling process include: providing students with material to refer to; using facilitatory questions/prompts aligned with the modelling process; and creating environments for students to share their thoughts and workings. Lecturer support needs to allow for students to experience struggle, explore, be creative and become self-reliant. Experiencing these key student behaviours allows students to move from being uncertain to finding direction and thus take responsibility and ownership for their own modelling process.

Implications of this research for general mathematics education practices include the suggestion of lecturers and teachers incorporating connecting mathematics to context as a normal part of their teaching practice and develop a learning environment where being uncertain and the need for exploration is a normal part of learning mathematics.

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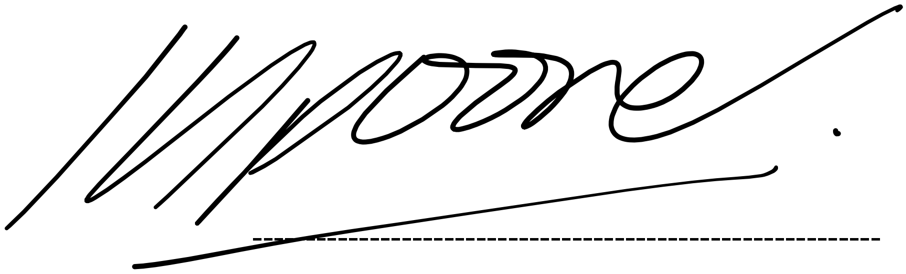
Attestation of authorship

I hereby declare that this submission is my own work and that, to the best of my knowledge and belief, it contains no material previously published or written by another person (except where explicitly defined in the acknowledgements), nor material which to a substantial extent has been submitted for the award of another degree of a university or other institution of higher learning.



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Ethics approval

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Chapter 1. Introduction

This thesis explores the student experience of learning to mathematically model. The term ‘mathematical modelling’ is used to describe the process by which we represent a situation in useful mathematical concepts and terms (Dym, 2004; Wake, 1997). This can be experienced in education through multiple ways including open-ended problems, case studies and fictional scenarios. The primary objective of this doctoral work is to develop an understanding of the student experience during mathematical modelling activities. These experiences are used to better inform lecturer practices for mathematical modelling.

Figure 1 illustrates an example of the modelling process, how the modelling process involves moving backwards and forwards between the real world and the mathematical world, and how this study focusses on the student experience of learning to mathematically model.

What is the Student Experience while learning to Mathematically Model?

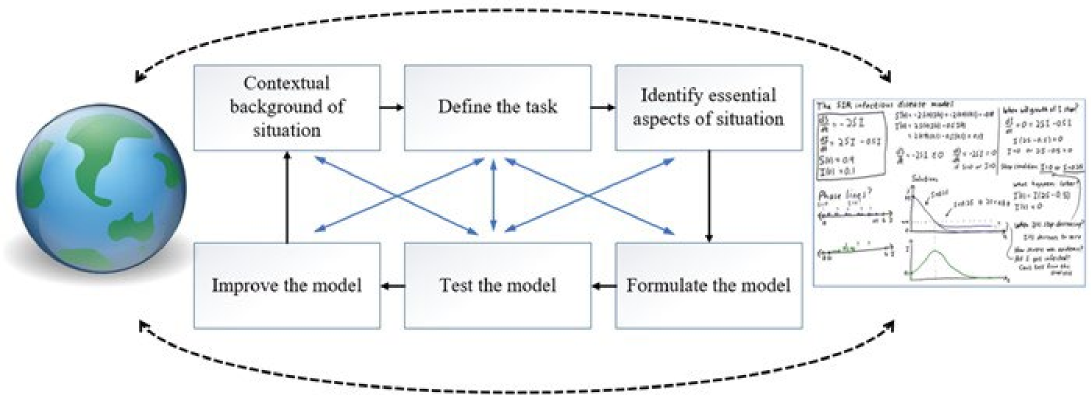

Figure 1. What is the student experience while learning to mathematically model?

1.1 Background of study

It is well established that students have difficulties with mathematical modelling (Kaiser, 2017; Klymchuk, Zverkova, Gruenwald & Sauerbier 2010; Soon, Tirtasanjaya Lioe, McInnes 2011). Though these difficulties are evident, little research has been undertaken into understanding student difficulties. One factor involved in removing, creating or exacerbating student difficulties is the teacher or lecturer (Crouch & Haines, 2004; Blum & Lieb, 2007; Stillman, Brown & Galbraith, 2013; Blum, 2011; Soon et al., 2011) who constitutes an important variable in student learning of mathematical modelling. However, Kaiser (2020) states clearly the role of the teacher in mathematical modelling has not been researched sufficiently enough. With most work focussed on studies at secondary level even less is known about the role of the lecturer in tertiary modelling experiences. This is not specific to modelling education however. In general, there is a lack of research into teaching at tertiary level (Jaworski, 2016). In particular, research on a lecturers' actions and goals and their relation to student learning (Jaworski, 2016).

This study will address the gap in understanding student experiences with mathematical modelling and the role that a lecturer can play in those experiences. This study seeks to identify ways to support students to learn mathematical modelling, while promoting mathematical modelling as a creative activity. The study also intends to provide a qualitative study to better understand the student perspective to balance and enrich the many quantitative studies in the field.

Lyon & Magana (2020) summarise the need for further research into understanding how to help students model by stating:

> Given that mathematical modelling is important to both education and professional practice, and the frequency by which we see these

> activities is on the rise, an understanding of how both instructors and students interact with these mathematical modeling activities would prove beneficial. (p. 1020)

Understanding the student experience has consequential implications for teaching practice. Educational benefits include improved student preparation for workforce (Julie, 2002; Robitaille & Dirks, 1982; Wedelin & Adwai, 2014) and development of mathematically equipped citizens (Tam, 2011). Wedelin and Adwai (2014) assert that it is through modelling and problem solving that students get to apply their mathematical knowledge in practice. Chaachoua and Saglam (2006) concluded that the current teaching of mathematics needs to address developing transverse competencies between mathematics and other disciplines involving real situations. The process of mathematical modelling is one way to do this (Chaachoua & Saglam, 2006). For example, the Domínguez, de la Garza and Zavala (2015) study involving first-year tertiary students confirmed that the use of mathematical modelling as the main teaching approach in courses is successful in developing links between mathematics and other disciplines, in this case physics. As Wedelin and Adwai (2014), Chaachoua and Saglam (2006) and Domínguez et al. (2015) illustrate, mathematical modelling is an important skill for preparing students for a variety of applications within education and the workforce. To identify teaching implications for lecturers' practice will have the benefit of an improved educational experience for both the lecturer and student and flow on consequences for the workforce.

Mathematical modelling is a valuable educational experience that contributes to students developing their ability to apply mathematics to solve real-life problems. Mathematical modelling enables the development of mathematically equipped citizens who can use mathematics to contribute to society as well as to aspects of their daily lives (Kaiser, Blomhøj, Sriraman, 2006; Maaß, 2005; Treilibs, Burkhardt & Low, 1980). Examples of modelling are found in the commercial, economic and political world. Examples include but are not limited to models for: successful aircraft designs and computer programmes; predicting global warming and the greenhouse effect; establishing inflation rates; predicting future trends in derivatives and stock markets; and understanding likely future economic outcomes. “The world of today cannot go on without mathematical modelling” (Lingefjärd, 2006, p. 96).

1.2 Problem statement summary

Not enough is known about the lecturer’s role in students’ learning to mathematically model. More qualitative research in the field is needed to understand students’ experiences with mathematical modelling. Students have difficulties with modelling that may impede their development as modellers and thus their potential future contributions to society as mathematically equipped citizens. An understanding into student experiences and the lecturer’s role in these experiences has an impact on the student educational experience, the lecturer’s teaching approach and the potential contribution the student can make to society.

To contribute to societal development, it is important that students have an educational experience where they begin to understand and develop models, while at the same time develop their own modelling concepts (Buckhardt, 2006; Lingefjärd, 2006). An understanding of the student experience is needed to find ways lecturers can support the development of students as mathematical modellers. By better understanding these student experiences this study identifies ways to constructively help navigate the difficulties students experience and provides additional insights into the student experience so lecturers can make decisions concerning how to construct learning experiences for mathematical modelling.

1.3 Purpose of the study

The purpose of this study is to investigate tertiary student learning experiences for mathematical modelling. This study is based in New Zealand and involves three different tertiary modelling courses (see Research aims and study design, Section 1.7, page 12 of this chapter and Background context for study, Chapter 3, page 119 for details). For students participating in these modelling courses, this was their first time studying modelling at tertiary level. This study uses Interpretive Description as the methodology to look at the student experience with the intention of establishing a better understanding of student experiences with mathematical modelling, the role the lecturer plays in these experiences and how any new insights can better inform teaching practices. The study attempts to provide new understandings of the student perspective to inform teaching practices.

The outcomes of this study will be of interest to the mathematical modelling research community, lecturer and teacher practitioners of mathematical modelling and, implicitly, students, as the experiences of fellow students are captured and used to inform teaching practices for mathematical modelling and mathematics education practitioners in general.

1.4 Introducing the methodology

Interpretive Description methodology provides the theoretical underpinning and theoretical framework for this study.

Interpretive Description addresses a real-world question from a position of understanding of what we do and do not know from considering all available evidence. The methodology is located within a "conceptual and contextual realm within which a target audience is positioned to receive the answer" (Thorne, 2008, p. 35) it generates. According to Thorne (2008), Interpretive Description:

> Constitutes a methodological direction that generates questions from applied disciplinary grounding, pushes one into the "field" in a logical, systematic, and defensible manner, and creates the context in which engagement with the data extends the interpretive mind beyond the self-evident – including both accumulated knowledge (such as clinical wisdom) and what has already been established (such as through empirical means) – to see what else might be there. As such, it offers the potential to deconstruct the angle of vision upon which prior understandings have been erected and to generate new insights that not only shape new inquiries but also translate them into practice. (p. 35).

The theoretical (philosophical) underpinning of an Interpretive Description study are the principles of naturalistic inquiry[1] (Hunt, 2009; Thorne et al., 2004; Thorne, 2008). Accordingly, an Interpretive Description study has a strong orientation towards constructivism as it is the researcher and participant that co-construct the knowledge (Hunt, 2009; Thorne et al., 2004; Thorne, 2008). Analysis is inductive in nature and generally follows an iterative process. The analysis produces findings that are generated from data, include multiple realities, and are contextualised (Thorne, 2008; Thorne et al., 2004).

An Interpretive Description study has scaffolding that positions the study. The theoretical scaffold (framework) is made up of positioning within the literature, the discipline, and the personal experience the researcher brings to the study (Thorne, 2008). The researcher understanding and experience of the phenomenon, along with the

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[1] In a naturalistic inquiry study "there is no manipulation [of the study conditions] on the part of the inquirer and the investigation is void of prior outcomes" (McInnes, Peters, Bonney & Halcomb, 2017).

discipline knowledge inform the scaffold. Please see Literature review, chapter 2 for the positioning of the study within the literature and discipline. I will discuss the positioning of the researcher in the following section (Positioning the researcher, Section 1.5, page 7).

1.5 Positioning the researcher

A crucial characteristic of Interpretive Description methodology is an acknowledgement of what the researcher brings to the study. The researcher's understanding and experience are acknowledged and inform the scaffold of the study (Hunt, 2009; Thorne, 2008). I will now present my experiences with mathematical modelling as a member of mathematical modelling teams, a student of mathematical modelling, and a secondary school teacher of mathematical modelling.

My interest in this area first developed when I attended my first Mathematics in Industry Study Group (MISG) in 2002. Here, I was part of a working mathematical team providing a practical working solution to a real-life problem occurring in Industry. At this time, I was working as a secondary school mathematics teacher. The intended outcome of this experience was to have a first-hand experience working with a real-world application of mathematics I could draw on in the classroom. I have since participated in several MISGs and later in Mathematics in Industry New Zealand (MINZ) workshops. As the researcher, and as an educator, my experience in MISG and MINZ, gave me first-hand knowledge working with others modelling, making me aware of the nature of modelling and the complexities of applying the modelling cycle to a real-world problem. My initial experience with MISG sparked my interest in mathematical modelling so much that I was left pondering if it was possible to create a truly authentic experience of mathematical modelling for secondary school students.

In 2011, while on a study award and Royal Society fellowship, I had the opportunity to further develop my modelling skills. I undertook further study in mathematical modelling and became a tertiary mathematical modelling student. I studied two modelling courses: MATH260 Differential Equations at Auckland University; and 160.319 Mathematical Modelling at Massey University. With differential equations being central to modelling systems of change MATH260 increased my understanding of differential equations and their applications for mathematical modelling. This course also touched on the modelling cycle and applications of modelling, though it was primarily focussed on differential equations as a mathematical modelling technique. In comparison, 160.319 was more focussed on the mathematical modelling cycle in conjunction with modelling techniques and involved creating mathematical models for a variety of situations. A range of modelling techniques were introduced, and multiple modelling scenarios were presented with the modelling cycle underpinning the whole course.

My experience as a tertiary mathematical modelling student has made me sensitive to some of the issues students face. 160.319 in particular gave me an awareness of students needing to be open to exploring new contexts, being prepared to learn new mathematics that will be suitable for the situation and being willing to ask for help and work with others. To me, this awareness and understanding I developed also reflected what I came to understand as modelling behaviours, which suggests that the course did prepare me for mathematical modelling.

I continually think about my experience as a tertiary student and reflect on what my lecturers did and how this impacted me. This helped me keep an open mind while reflecting on the actions of the lecturer participants and how these could impact the experience for their students. I had to step back and look at student responses to what the lecturer was doing and separate this from my own experience. I needed to remember that even though I will have a feel for what the student experience could potentially be, this may not necessarily be the case. I needed to be open to the experiences of the students that were alternative to my own.

In the latter half of 2011, I also had the opportunity to work as part of a professional mathematical modelling team. This involved working on a modelling project for a client within a modelling team. The project spanned three months, giving me the opportunity to participate in and observe the behaviours of a professional mathematical modelling team. See Spooner (2013) for more detail of this experience. By participating in the professional modelling team, MISGs and MINZs enabled me as the researcher to bring first-hand experience of modelling and knowledge of behaviours of modelling. This means my ideas of what constitutes modelling and modelling behaviours are based on my personal experience. It also means I needed to be conscious of being open to new modelling behaviours presented by my participants. This helped me to consider new student modelling behaviours of students that I may not have been aware of before.

Encouraged by Maaß's (2006) research project showing students as young as 13 years old modelling, in 2012 I took my modelling experiences and used these to develop and teach a holistic mathematical modelling unit for my secondary class. It was important for me that students had an authentic experience of mathematical modelling and were truly engaged with the process of mathematical modelling, as opposed to participating in superficial modelling tasks. From my experience at Massey University, I knew that the lecturers had observed that students were successful in creating models when they were given the factors and the base model to develop the model with. However, students found identifying the important variables of the situation to use to create the model around the most difficult aspect of modelling. With this in mind, I created a holistic mathematical modelling unit that allowed students to experience the whole modelling process, including identifying key variables for the situations. Discussion cycles were used to prompt and facilitate each stage of the modelling process. See Spooner (2013) and Spooner (2017) for further details of the modelling unit and activities created.

My personal experience of developing a teaching unit on modelling before undertaking this research means I already had several ideas of how modelling could be taught. First, I considered it important for students to have an authentic experience of modelling that reflected the nature of professional modelling. Second, I wanted students to be modelling in groups to mimic a professional modelling team and for the educational benefits. I believed a successful holistic modelling experience is possible for students given the right prompts. In addition, while developing the teaching unit, it never occurred to me to present developing an already developed model to students and have them reproduce the model. My experiences with modelling involved actively creating models. I was always going to create a modelling experience where students where actively engaged in creating models themselves. My job as a teacher was how to create this expereince. I believed my role as the teacher was to facilitate the process including facilitating discussion around specific stages of the modelling cycle (see Chapter 2, Section 2.2.1, page 24). I used the modelling cycle to scaffold the teaching and learning experience. See Spooner (2013) and Spooner (2017) for further details of this work.

My experience as a teacher and lecturer in mathematics also has informed this study. During the twenty-two plus years I have been involved in teaching or lecturing mathematics, I have won five teaching awards and been nominated for an additional national teaching excellence award. These awards establish me as an expert mathematics educator who uses active student collaboration as a successful way to engage students with course content and learning material. I have a keen interest in successful ways to engage students in their learning and with material, skills and behaviours to be learnt. I believe learning is an active process that students need to be involved with. It is also my opinion that a rich learning experience contains a variety of approaches. No one learner has the same starting point. A lecturer needs to provide a variety of approaches to capture circumstances for learning for their audience.

My experience as a teacher, student and member of a mathematical modelling team makes me conscious of wanting to create learning experiences that are authentic in context and in regard to learning, with learning experiences that are engaging and accessible to a student. I see these types of learning experiences as an important part of teaching, valuing the richness they can bring to mathematics education.

My ideal way of teaching modelling would be to take a genuine real-world problem and have students attempt to model this. My ideal learning experience for a student is a hands-on experience where the students are actively participating and actively engaged in the 'activity of modelling', as it is a process. I think about modelling as a process not an end product. I need to be conscious in my research that, to me, learning is about active engagement with material and content. I need to be open to other signs that student learning has occurred, including perceived learning processes that lecturers have for their students.

1.6 Rationale for methodology

As already mentioned, this study uses the qualitative methodology Interpretive Description. Interpretive Description was chosen as it analyses participant experiences and uses these findings to inform and advance practitioner practice (Thorne, 2008). For this study, this refers to exploring student experiences with mathematical modelling and how these experiences can add to and better inform lecturer practices. Interpretive

Description is a methodology that is flexible enough to employ a range of tools for data collection and analysis and accommodates the use of multiple data sources (Hunt, 2009; Thorne, 2008; Thorne, Kirkham, & O'Flynn-Magee, 2004). The methodology was chosen to investigate the research question “What can we learn from tertiary student experiences to inform the teaching of mathematical modelling in the future?” Interpretive Description seeks to reveal shared patterns, themes and differences of participants’ experiences and perspectives that sit beneath the surface of the data that a practitioner may not ordinarily see and use these to inform teaching practice (Thorne et al., 1997; Thorne et al., 2004; Hunt, 2009), making it a good fit for the study. The methodology is flexible enough that along with adding to qualitative knowledge of the student experience to the field, it also contributes to the body of knowledge regarding the lecturer’s role, producing teaching implications for practice. Further details of the methodology and the journey for choosing this methodology and how it was utilised are provided in Chapter 4, page 125.

1.7 Research aims and study design

This study aims: to provide insight into the student experience to inform and advance tertiary teaching practices for mathematical modelling; to contribute to the general discussion on how to provide rich student learning experiences in mathematical modelling; and to inform the general discussion for students to be able to contribute to society through mathematical modelling.

The rationale for the study design is: to understand more about the student experience while learning to model and use these new insights to enhance teaching approaches for mathematical modelling; to gain a deeper understanding of the lecturer’s role in the student experience; and to understand the nuances of student experiences and what they might mean for teaching practice. In line with Interpretive Description methodology, this research does not intend to generate a theory, rather an understanding of the commonalities, differences and nuances of the student experience while they are involved in mathematically modelling.

As previously mentioned, this study involves the students and lecturers from three different New Zealand tertiary mathematical modelling courses. Three lecturers and twenty student participants participated in this study. Semi-structured interviews are used as the primary data source and are supplemented by observations of the modelling activities and some teaching activities. Student interviews allowed me as the researcher to gain understanding into the students' experiential knowledge of the modelling course and the modelling activities, from their perspective. Lecturer interviews provided background into how the course experiences had been created and a lecturer's views on teaching modelling. Observations allowed me to observe first-hand the different components of the courses. The main purpose of the observations was to triangulate the interview findings.

1.8 Limitations

Within this research there are some limitations. First, the diversity of perspectives is limited to those students that chose to participate and so can only be representative of the participants. Second, as already alluded to, Interpretive Description can only illuminate experiences, and not theorise. Third, the decision to use Interpretive Description was made after data collection occurred. While it meant that rich data on the student experience had occurred, data collection was not influenced by researcher insights gained from early data collection and analysis, as suggested by Thorne (2008).

1.9 Structure of the thesis

Within this chapter I have provided an overview of this research topic: what is the tertiary student experience while learning to mathematically model. This is an experiential study, originating from my desire to contribute to developing supportive teaching practices for mathematical modelling. It is designed to explore student experiences to better inform teaching practices and reflects my interest in the topic as a mathematical modeller and as a mathematics lecturer and educator.

Chapter two, Literature review, positions the study within the current literature for teaching and learning mathematical modelling. The chapter provides an overview of what mathematical modelling is, potential learning theories for mathematical modelling, teaching approaches for mathematical modelling, student experiences of mathematical modelling, and an overview of methodologies used within mathematical modelling education research.

Chapter three, Background context of study, provides the context of the study. This study involved the students and lecturers from three different New Zealand tertiary mathematical modelling courses. This chapter provides the details and modelling activity of each course.

Chapter four, Methodology and methods, provides the journey and rationale in choosing the final methodology, Interpretive Description. The chapter provides an overview of Interpretive Description including its epistemological and ontological underpinnings. The chapter will discuss the methods for data collection and analysis.

Chapters five, six, seven and eight, Findings, explore the ways students appeared to be experiencing engaging with mathematical modelling in their courses. The findings are presented in a narrative format and use quotes to illustrate the interpretation of the data.

Chapter nine, Discussion and teaching implications, discusses the findings with the current literature and uses these discussions to present teaching implications. This chapter identifies the study limitations and provides suggestions for future research on this subject.

Finally, chapter ten, Conclusion, summarises the key outcomes, from discussing the findings with literature, and teaching implications.

Chapter 2. Literature review

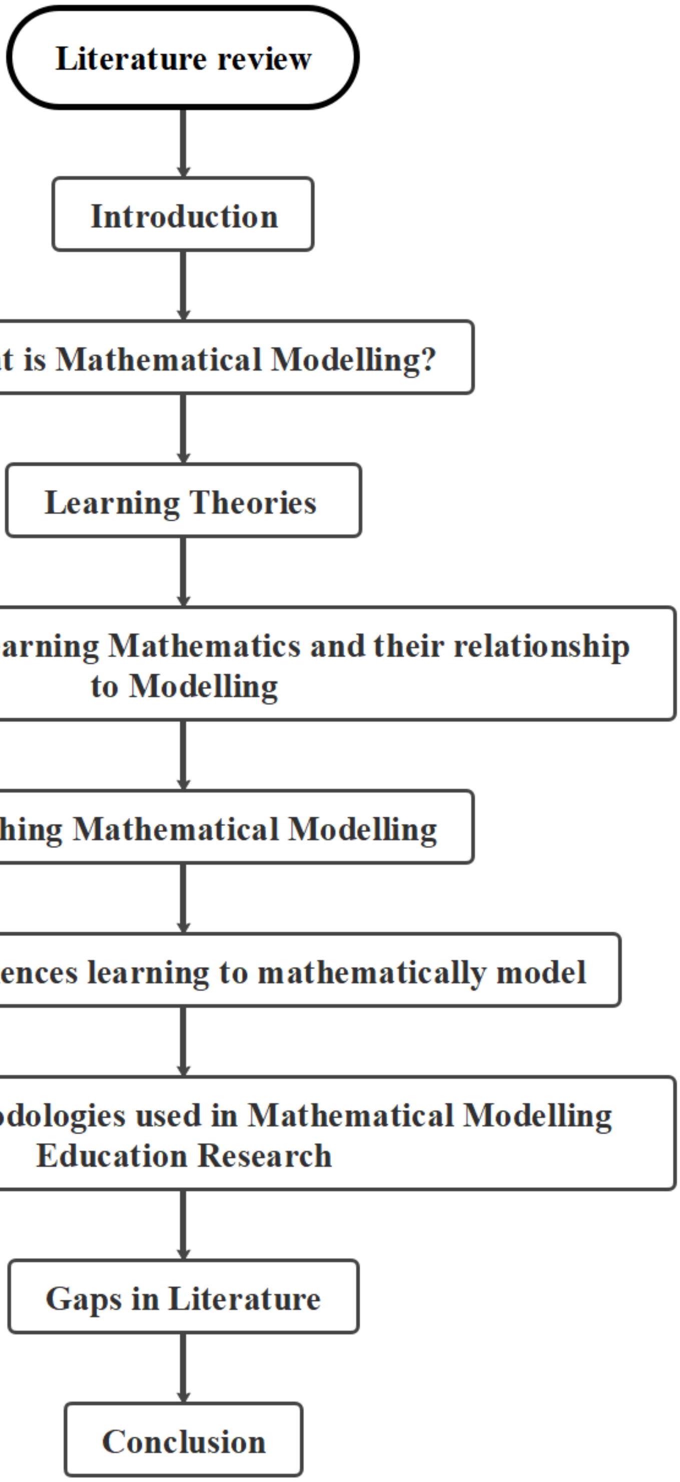


Figure 2. Overview of literature review

2.1 Introduction

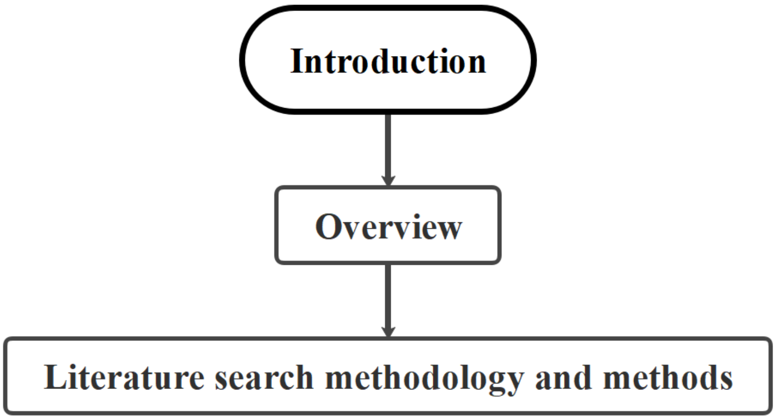


Figure 3. Introduction section of literature review

2.1.1 Overview

The main purpose of this review is to ground this study within the existing knowledge of what is known about student experiences of mathematical modelling and lecturers' teaching approaches. Consistent with Interpretive Description (Thorne, 2008), this review provided scaffolding for this doctoral research by identifying the current understanding and challenges associated with students learning to mathematically model and the teaching of mathematical modelling.

In this chapter, I will present the literature search methodology used, discuss what mathematical modelling is, present the modelling cycle, consider relevant theories of learning for modelling, reflect on conditions for learning mathematics and their relationship to modelling, review literature for teaching modelling including the lecturer's role, and examine research into the student experience with modelling including obstacles from a student point of view and potential reasons for these. Following this, a brief review of methodologies used within the mathematical modelling education literature review of the study will be presented. Figure 4 shows the scope of the literature for this review.

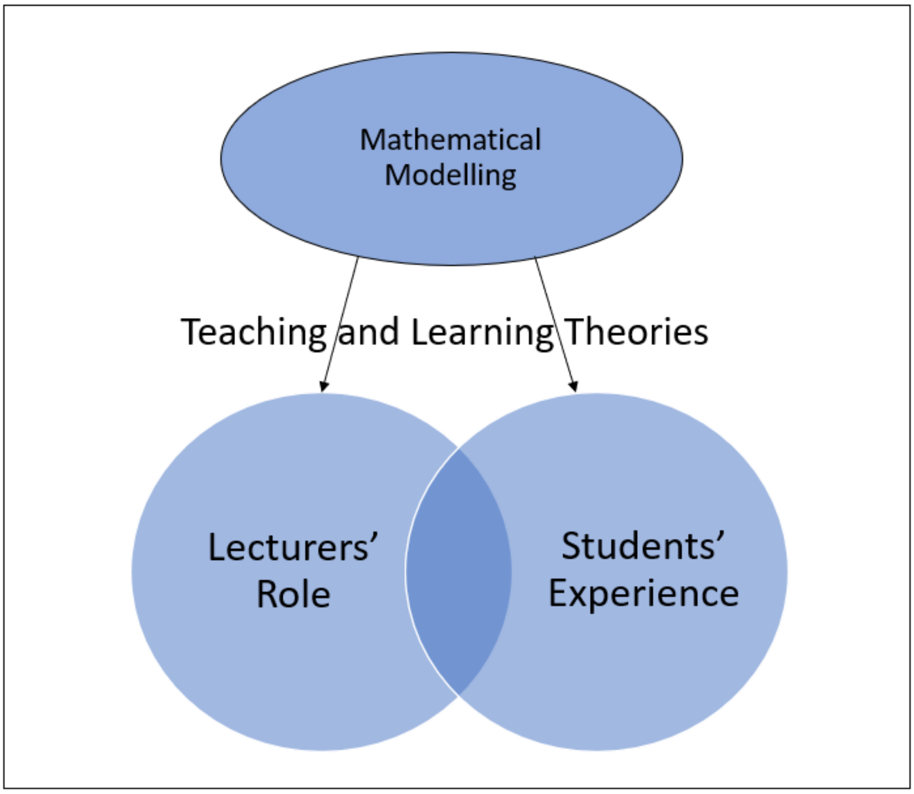


Figure 4. Scope of the literature review

2.1.2 Literature search methodology and methods

Interpretive Description, the methodology selected for use in this research (see Methodology chapter), cautions against over-reliance on the use of "keywords" as the exclusive approach to identify relevant research. Thorne (2008) argues that keywords reflect "standard" understandings and conceptualisations within a field. It is suggested that authors may use keywords common to their discipline to ensure they come up in database searches, but these might not reflect what is new and different about their research products. Therefore, this methodology encourages researchers to develop a wide repertoire of search resources when conducting a review (Thorne, 2008; 2016). Accordingly, a multi-pronged approach was used in selecting the literature to review for this chapter. This approach is detailed below.

2.1.2.1 Search strategy

As a search strategy, I started with a general review of the literature and then moved to a more systematic approach using inclusion criteria.

Initially, I conducted literature searches by looking at key publications for articles on the teaching and learning of mathematical modelling both at tertiary and secondary levels. I read articles and books that resonated and found additional literature by examining references and searching publications that had cited key material. Initially, I did not approach this as a systematic literature review, but rather as a literature review as and when needed and with particular reference to publications worked on throughout the thesis (see page x for a list of reference publications achieved throughout the thesis period). Key publications in the area of mathematical modelling were also searched. These included: the International Community of Teachers of Mathematical Modelling and Applications (ICTMA) book series; and Teaching Mathematics and its Applications international journal; and PRIMUS journal for research in mathematics education.

As I prepared for the formal write-up of this chapter, I started systematically searching databases and publications to find the latest research, conscious of performing the literature search to produce non-standard understandings and conceptualisations within the field. To help with this, I used a multi-pronged approach, as above, in addition to a more systematic search.

For the last search, I restricted the search to the last ten years. This was to ensure I was adding the most contemporary research to the earlier influential studies already identified through the multi-pronged approach described above.

Multiple Scopus literature searches were carried out. Some searches yielded results, and some searches produced no results. For example, the search terms "mathematical modelling AND tertiary AND student AND lecturer AND experience OR perception OR perceive OR perspective OR relationship OR participation OR attitude AND learning OR interaction OR dynamic” returned nothing. Whereas, “student AND experience AND "mathematical modelling" yielded appropriate results when limited to the subject areas “MATH” OR “ENGI” OR “PHYS”.

For narrowing down the literature for tertiary courses with modelling activities, a Scopus database search was conducted using "mathematical modelling" AND "activities" AND "course" "undergraduate" OR “tertiary” OR “College” OR “first year”. Scopus literature searches were limited to the subject areas of mathematics and engineering.

Inclusion criteria:

- Focussed on experiences of students or lecturers with mathematical modelling at tertiary level.
- Sought the perspective of either the lecturer or the student.
- Restricted to subject areas of mathematical modelling, mathematics, engineering and physics.

Due to the limited studies within the tertiary sector, articles were also included from the secondary sector.

Each of the articles included was read in its entirety and the article was summarised in an Excel spreadsheet and a Word document. In the spreadsheet, I extracted details of the study including: topic; education level; country; research question; methodology; key findings; and conclusion. In the Word document, a summary of the significant concepts covered, additional details of the key findings, my reflections and criticisms of the article were kept.

After reading and recording details of the articles in Excel and Word, articles were stored in NVivo and coded according to key topic areas. NVivo allowed for easier handling of the literature for writing up the literature review.

Consistent with the intent of this doctoral research to offer practical insights into teaching and learning mathematical modelling, I centred my methodological critique on the quality aspects of the research that are specifically concerned with applying findings to practice, in this case teaching practice.

The secondary purpose of this review was to examine what methodological approaches and methods have been used in qualitative research in this area, as well as to examine the kinds of understandings and insights generated by using such approaches.

2.2 What is mathematical modelling [2]

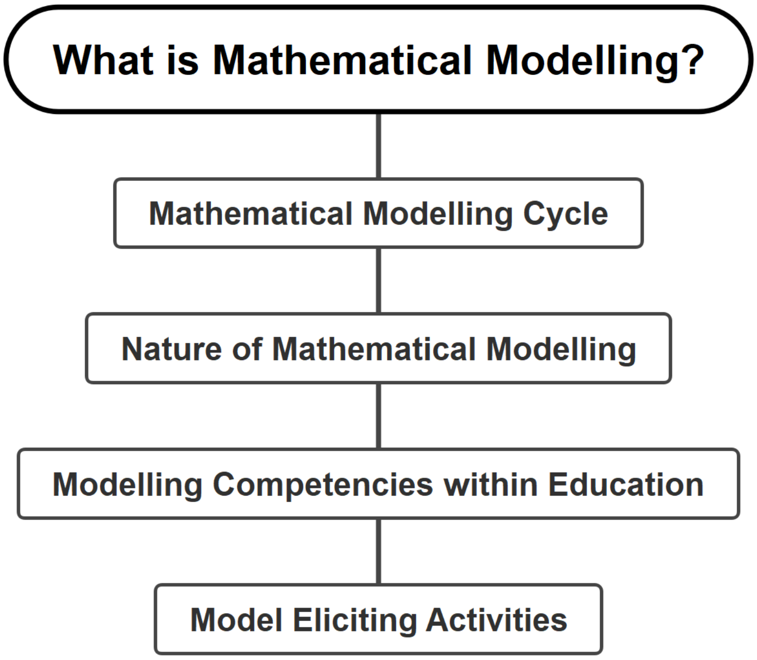


Figure 5. What is mathematical modelling

[2] This section includes re-presentation of material published in Spooner (2017), Spooner (2021), and Spooner (2022).

Mathematical modelling is concerned with taking a real-world situation and expressing it as a mathematical system (Gershenfeld & Gershenfeld, 1999). It has been used to make sense of the world since the beginning of mathematical activity (Spooner, 2013). Modelling is process orientated with the objective being to find mathematics that makes sense of the situation (Stillman, Brown, Galbraith, & Ng, 2016). A formed mathematical model is a description of the behaviour of real devices, objects and situations written in the language of mathematics (Bender, 2000; Dym, 2004; Wake, 1997). As Dr. Katie Fowler of Clarkson University explains:

> A mathematical model is a tool that uses mathematics to represent a real-world situation to facilitate quantification, analyses, predictions, and gaining insight. The process of mathematical modelling is the creation of such a tool and often requires significant creativity and brainstorming, but not necessarily complicated mathematics.
> (Garfunkel & Montgomery, 2019, p. 112).

The broad stages of an authentic mathematical modelling experience are: forming a modelling group; establishing a shared understanding of the problem; undertaking necessary research of the context and what is known; defining a mathematical direction for the model; identifying the essential aspects of the situation; mathematically interpreting the essential aspects of the situation for the model; constructing equations or other mathematical representations of the model; critiquing, modifying and refining the model mathematically and otherwise; applying the model; repeating the cycle to improve the model; and reporting on the model (Bender, 2000; Dym, 2004; Tam, 2011; Treilibs et al., 1980). Mathematical modelling is a process. Modelling is different to a 'mathematical model' and is more than an 'application of mathematics' (Garfunkel & Montgomery, 2019).

However, it can be unclear what mathematical modelling is in education. Modelling differs from applications in that it is more open and complex. Applications tend to be object orientated, involving examples of contexts where the mathematics to apply is predetermined, and some can use artificial contexts (Stillman et al., 2016). To further complicate matters, the term modelling is used in different ways in education. Stillman, Brown and Galbraith (2008) classified some approaches being currently coined as mathematical modelling. These are the use of contextualised examples, curve fitting, modelling as a vehicle and modelling as content. As originally coined by Julie and Mudaly (2007), modelling as a vehicle is using modelling to teach curriculum mathematics and modelling as content is modelling for real world problem solving. All of the approaches, except for modelling as content, overlooked the complete mathematical modelling process, ignored the contextual background when generating a solution, and the mathematics to be used in the solution was predetermined. All of these ignored behaviours are part of an authentic mathematical modelling experience (Stillman et al., 2016). In contrast, modelling as content better reflects the principles of realistic, authentic modelling. Here, the objective is to go through the full process of modelling, determine the mathematics needed for the solution and ensure that the solution is located within the context of the problem (Stillman et al., 2008; 2016). Teaching based on this is known by researchers as "the realistic perspective on the teaching and learning of mathematical modelling" (Blomhøj, 2009, p. 3).

The realistic perspective is one of many different perspectives for teaching and learning mathematical modelling identified in research (Abassian, Safi, Bush & Bostic, 2020; Kaiser & Sriraman, 2006). The realistic perspective focusses on the importance of modelling real-life situations. This perspective is underpinned by the teaching belief that it is vital for students to work with and have their own experiences creating mathematical models of realistic situations (Blomhøj, 2009; Kaiser & Sriraman, 2006).

The realistic perspective is also known as the applied modelling perspective, illustrating its focus is on developing a solution to a problem as opposed to the development of mathematical theory (Kaiser, 2020).

2.2.1 Mathematical modelling cycle

Central to the emerging (and far from complete) theory of teaching and learning mathematical modelling from a realistic perspective is the general modelling cycle (Blum, 2011; Kaiser, Blomhøj & Sriraman, 2006). The basis of the cycle is founded on a real-life scenario. A realistic situation is the starting point for forming the mathematical model. A real situation is presented, an understanding of the context of the situation is established, the problem is defined, and the important components are identified. A model is formed using these important components and what we already know about how these components interrelate (Dym, 2004; Tam, 2011; Treilibs et al., 1980). Forming a model involves having enough information to mathematise the situation. That is, mathematical modelling is about describing a situation in useful mathematical terms (Dym, 2004; Wake, 1997). The processes of mathematical modelling are the steps taken to create a mathematical model (Treilibs et al., 1980; Wake, 2011).

There are numerous diagrams describing the mathematical modelling process (e.g. in applied mathematics: Ortlieb (2004) and Pohjolanien and Heilio (2016); and in education: Ferri (2006) and Kaiser (2017)). There are also different student and lecturer views on modelling process cycles used at tertiary level (Perrenet & Zwaneveld, 2012). The diagram presented in Figure 6 is based on my work as a mathematical modeller and a review of the cycles used in industry and education.

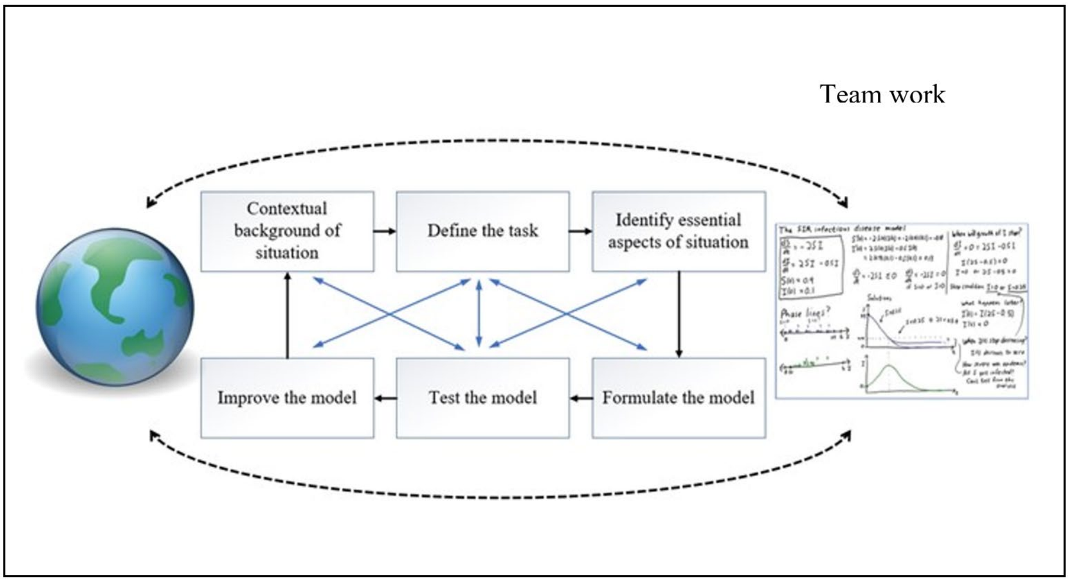


Figure 6. The process of mathematical modelling

Validating is also discussed in literature as part of the process of mathematical modelling (Blum & Leiß, 2007; Czocher, 2018; Kandasamy & Czocher, 2020) and is a form of critiquing the model. Validating occurs both throughout the modelling process and at the end of the process. In relation to the modelling process in Figure 6 validation can be viewed as the critiquing and improvement of a model that occurs throughout the modelling process. The end product of validation is a model that is usable for its purpose.

2.2.2 Nature of mathematical modelling

The nature of mathematical modelling is concerned with making connections between mathematics and the real world. Mathematical modelling is more than 'how to' approach a problem. It requires consideration of the situation and the willingness to apply and utilise mathematics in alternative ways than have been previously studied (Gershenfeld & Gershenfeld, 1999). It is a complex activity that involves initiative and creativity, including navigating your way through a process that is not always clear cut as to how to best proceed (Gershenfeld & Gershenfeld, 1999). According to Kotze (2020), modelling involves adjusting routines and procedures, the discovering of new ideas and processes, and interpreting results. Modelling involves developing solutions for open-ended questions without clear right or wrong answers (Bassanezi, 1994; Gershenfeld & Gershenfeld, 1999; Treilibs et al., 1980). It involves dealing with problems that may have many possible solutions, which can be developed in a variety of ways, and where these ways are not initially known or clear to the student (Bassanezi, 1994; Treilibs et al., 1980). Mathematical modelling is an active process that demands student involvement and generally involves working with others (Bassanezi, 1994).

2.2.3 Modelling competencies within education

As part of supporting students' involvement with the modelling process, a global learning goal for secondary and tertiary mathematical modelling teaching is to develop modelling competency (Blomhøj & Kjeldsen, 2011; Blum, 2011). An ICTMA conference (ICTMA13 in 2007) and following book publication, including rigorously peer-reviewed papers from the conference, was dedicated to 'Modelling Students' Mathematical Modelling Competencies' (Lesh, Galbraith, Haines, & Hurford, 2013). As part of supporting the development of modelling competency, research has been carried out focussing on defining and establishing competencies needed to be successful with mathematical modelling. One definition of modelling competency, as defined by Niss and Højgaard (2011), is "on the one hand, being able to analyze the foundations and properties of existing models" including assessing their validity and range. "On the other hand, competency involves being able to perform active modelling in given contexts; i.e. mathematising and applying it to situations beyond mathematics itself" (Niss & Højgaard, 2011, p. 58). Modelling competency includes overall competencies that overreach multiple aspects of the modelling cycle, and sub-competencies relating to each stage of the modelling cycle. Metacognitive modelling competencies overreach the modelling cycle (Schnapp, Vos, & Goehardt, 2011) and are necessary throughout the whole mathematical modelling process (Blum, 2011). Metacognitive modelling competencies include but are not limited to, strategies for approaching modelling, student reflection on their modelling process, knowledge of the task, and student self-regulation (Blum, 2011; Maaß, 2006; Schnapp et al., 2011). Blum (2011) believes that for metacognitive competencies to be developed, there needs to be appropriate support from the teacher. In addition, each stage of the modelling cycle can be viewed as a modelling sub-competency and has associated behaviours (Lu & Kaiser, 2022). See Lu and Kaiser (2022) and Niss and Højgaard (2019) for a current discussion regarding modelling competencies and Toews (2012) for a checklist for key competencies.

At the tertiary level, Caron and Bélair (2007) present and define communication skills, intervention skills, and evaluation skills as sub-competencies of modelling competency. Blomhøj and Kjeldsen (2011) promote reflecting internally on the modelling process and fostering the ability to reflect on the use and application of models as part of developing mathematical modelling competency. Interviews with mathematical modelling lecturers revealed that modelling competency also includes connecting mathematics to reality, looking for patterns, exploring relationships, and being comfortable that the first attempt at producing a model will not likely produce a realistic model (Drake, 2012). In earlier publications, I have suggested that lecturers should plan and determine specific learning goals for developing generic modelling competency (Spooner, 2020).

A generally agreed characteristic for developing modelling competency is to provide students with opportunities to actively participate in the modelling process (Blomhøj & Kjeldsen, 2011, 2013; Caron & Bélair, 2007). "Modelling is not a spectator sport and can only be learnt by doing" (Kaiser & Stender, 2013, p. 280), with many studies suggesting that modelling is best learnt with authentic open-ended problems (Caron, 2019). It is thought to be vital that students work with and have their own experiences in creating mathematical models of realistic situations (Blomhøj, 2009; Kaiser & Sriraman, 2006). Alongside active participation, evidence shows that mathematical modelling competencies can be developed using a combination of group work, classroom discussion, and individual work (Maaß, 2006). Open-ended modelling activities, where the model to be developed is not immediately obvious, has similar characteristics.

It is agreed that working independently is part of the modelling process, and it is desirable for students to develop "capabilities in carrying out mathematical modelling tasks independently" (Huang, 2011, p. 173). Blum (2011, 2015), Burkhardt (2006), Carducci (1996), and Pollak (2007) all acknowledge that students need to have experience independently modelling to develop modelling competency. Similarly, Huang (2011) uses Blomhøj and Jensen's (2003) work to define mathematical modelling competency as "the ability to autonomously and insightfully carry out all aspects of the mathematical modelling process in a certain context" (Huang, 2011, p. 173).

To further aid the development of modelling competencies, Blomhøj and Jensen (2003) recognised two different modelling approaches: holistic and atomistic approaches. Acorrding to Cevikbas, Kaiser, & Schukajlow (2021) :

> The holistic approach depends on a full-scale modelling process, with the students working through all phases of the modelling process. In the atomistic approach, students concentrate on selected phases of the modelling process, especially the processes of mathematizing and analysing models mathematically, because these phases are seen as especially demanding. However, the authors issued a strong plea for a

> balance between these two approaches since neither of them alone
> was seen as adequate. (p. 4).

2.2.4 Model eliciting activities

Even though this study does not focus on Model Eliciting Activities (MEAs) they are worth mentioning as some advances in modelling literature have drawn on research obtained from MEAs. MEAs are thought-revealing modelling activities designed to provide insight into students' thinking (Lesh, Hoover, Hole, Kelly, & Post, 2000; Lesh & Doerr, 2003). Thinking that can be exposed during a MEA is the quantities and variables students are thinking about, relationships they believe are important and rules they believe connect these quantities and relationships. MEAs are able to reveal thinking into parts of the formulation stage of modelling. At the same time, it is believed MEAs allow for the development of conceptual competence for modelling (Lesh & Doerr, 2003) as well as the development of new mathematical ideas for the student (Gravemeijer, 2007). There is a set of principles that define when an activity is an MEA (Lesh et al., 2000). MEAs, alongside the Kaiser (2007), Kaiser, Lederich, and Rau (2010) and Kaiser (2020) discussion looking at students modelling competencies, allow researchers to examine modelling behaviour.

2.3 Learning theories[3]

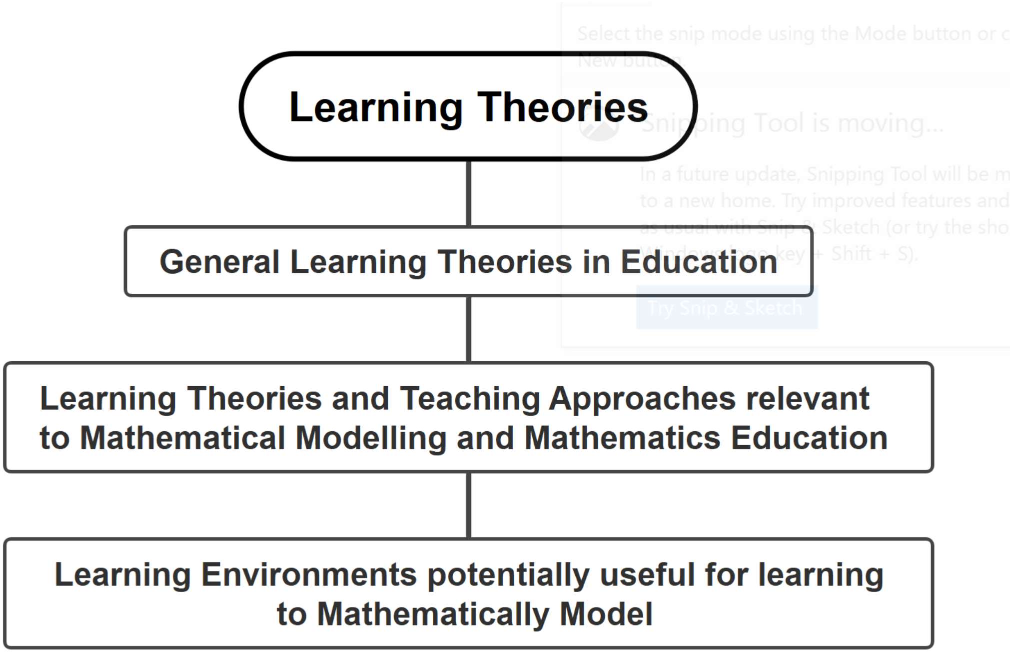


Figure 7. Learning theories

2.3.1 General learning theories in education

There are a variety of different learning theories in general education and mathematics education, ranging from the view that knowledge is something that is passively received through to something that is actively constructed by the learner (Ernest, 2010a). There are theories discussing how knowledge is constructed based on personal experience and previously created knowledge, known as constructivism (Ahmad, Sultana, & Jamil, 2020; Ernest, 2010a; Kara, 2019). There are also behaviorism learning theories concerned with learning as an observed change in behaviour due to some external stimulus (Ahmad et al., 2020; Ernest, 2010b). Within the wide spectrum of the different theories, there are different categories within a theory theme. For example, there are several different types of constructivism. Ernest (2010b) discusses the subtle differences between simple constructivism, radical constructivism, enactivism and social constructivism, all of which fall under the theme of constructivism.

[3] The section includes re-presentation of material published in Spooner (2018), Spooner (2020), and Spooner (2022).

Central to constructivism is the idea that learning is an active, autonomous construction of knowledge (Leuders, 2001). Based on Meier's (2009) discussion of Leuders (2001), a constructivist learning environment is one where:

• Students are encouraged to work independently and to think independently of their teachers and peers (independence).

• Connecting prior knowledge and experience with new knowledge is an opportunity for students (prior knowledge).

• Students are able to interact, argue, and come to an agreement (negotiation).

• Students have the opportunity to recognise that learning is a process that leads to problem-solving (student centerdness).

There is also discussion based on learning orientations as opposed to learning theories. For example, Wegerif (2002) discusses four orientations to learning – the behaviorist, constructivist, humanist and participatory. The behaviorist and constructivist take behaviorism and constructivism respectively as their orientation. A humanist perspective sees the involvement of the whole person, both feeling and cognition, as being involved in the learning. A participatory perspective takes the stance that learning happens as the learner interacts with a particular community of practice. That is, learning happens between people and the environment in which the learner is participating in.

In regard to tertiary education, one of the learning models that has been said to be useful is Kolb's Experiential Learning Cycle (Kolb & Kolb, 2017). This is a model showing the learning processes students can participate in. Kolb thinks there are four stages of learning: concrete experiences; reflective observation; abstract conceptualisation; and active experimentation. The stages all emphasise the involvement of the learner. It would be up to the lecturer to provide activities that provide opportunities to experience the different learning stages as described by Kolb. For more details on the types of activities for the different stages, please see

http://www2.le.ac.uk/departments/gradschool/gradschool/training/eresources/teaching/theories/kolb

When looking at theories of learning drawn upon for mathematical modelling, Boaler (2001) suggests the theory of situated learning as useful. Boaler (2001) defines situated theories of learning as knowledge that "emerges as a series of interactions between people and the world" (Boaler, 2001, p. 121), though she does not go into much further detail. Vygotsky and Piaget, both examples of constructivists, are also referred to for foundational theories of learning for mathematical modelling (Kaiser & Sriraman, 2006). Piaget and Vygotsky are both considered the founders of Constructivism.

Though learning theories are not teaching theories they have implications for teaching and hence should be considered (Goodchild, 2010, pp. 49-52). To better understand how learning theories may be used in tertiary mathematical modelling, the characteristics of the learning theories, behaviorism and social constructivism, and participatory learning orientation, including cognitive apprenticeship, will be presented.

2.3.1.1 Behaviorism

Behaviorism is based on the belief that behaviour can be predicted and controlled (Weegar & Pacis, 2012). Weegar and Pacis (2012) and Ernest (2010b) discuss behaviorism as learning resulting in a change in behaviour, not necessarily a change in mental schema. The educator's role is to structure the environment to produce a sought-after behavioural response. Weegar and Pacis (2012) view the educator as the focus of demonstration and interaction with students absorbing instructional routines and materials. Drill and practice exercises are examples of behaviorism. Behaviorists believe memorisation of pieces of material is needed before students can move on to higher-order learning like problem solving (Weegar & Pacis, 2012). A criticism of behaviorism is that behaviour is not disconnected from the mind or emotions.

2.3.1.2 Social constructivism

In contrast, social constructivism discusses knowledge as being constructed through social interactions and processes (Ernest, 2010b; Nesher, 2015). The educator's role is to structure and facilitate learning activities to provide an interactive experience that promotes cognitive structure and meaning (Ernest, 2010b). Social constructivism differs from simple constructivism by the inclusion of social interactions as opposed to a primarily individual focus (Goodchild, 2010). Text, language and signs play a major role in making meaning of knowledge within a social context (Ernest, 2010b; Nesher, 2015). Social constructivism is based on the principles of Vygotsky (Ernest, 2010a; Goodchild, 2010) and involves internalising mental processes and knowledge through social interactions. When knowledge has been individualised, it then goes through social processes to convert it into knowledge accepted by the community (Sriraman & Haverhals, 2010). One of the criticisms of social constructivism is that because knowledge is held as a truth by a community does not necessarily make it true (Gold, 1999, in Sriraman & Haverhals, 2010).

2.3.1.3 Participatory orientation

The participatory orientation acknowledges that learning takes place in relation to people and the environment requiring participation in communities of practice. As discussed by Wegerif (2002), the participatory orientation, like social constructivism, is based on Vygotsky. Unlike social constructivism, the participatory orientation rejects "learning as an 'interior' activity that takes place within the learner" (Ernest, 2010b, p. 54). Rather, learning is viewed as occurring through partaking in particular communities of practice, including social and conversational. For example, learning how to process insurance claims happens by working processing insurance claims in an insurance company. Participation allows for knowing how something is done rather than applying specific rules and procedures (Wegerif, 2002). The goal of the educator is to provide opportunities to know and "establish communities of practice in which conversation and participation can occur" (Ernest, 2010b, p. 54).

2.3.1.4 Cognitive apprenticeship

Cognitive apprenticeship could be viewed as a branch of participatory orientation where learning takes place through interactions with experts, with the end goal being to complete an authentic task (Collins & Kapur, 2006; Wedelin, Adawi, Jahan, & Anderson, 2015). Students learning about and processing information in the way specialists do is central to cognitive apprenticeship. Cognitive Apprenticeship focusses on developing cognitive and metacognitive skills through actively participating in an authentic learning environment (Collins, Brown, & Newman, 1989). Teachers in such learning environments move from providing information to guiding, scaffolding, and solving problems. Learning is more self-directed, and students are given more opportunities to explore (Collins, Brown, & Newman, 1988).

According to Collins and Kapur (2006), there are six principles for ways teachers can operate within a cognitive apprenticeship environment: modelling, coaching, scaffolding, articulation, reflection and exploration. Modelling involves students observing teachers performing a task, while teachers make explicit what they are thinking, doing and why. Coaching involves teachers facilitating students while they carry out a task. The teacher acts as a 'guide on the side' to foster learning. Scaffolding refers specifically to support provided by the teacher to the learner. As learners are able to take more responsibility for the task themselves, the support reduces accordingly. Articulation is students verbalising their reasoning, thinking process and knowledge while carrying out the task. Reflection involves comparing their thinking and processes with others. Ultimately, reflection involves reflecting on their thinking and processes to take on new and enhanced ideas and approaches. Finally, exploration involves students posing and solving problems of their own. As students become more comfortable exploring, teachers slowly lessen their involvement. To summarise, the ways of cognitive apprenticeship involve explicit teaching from the teacher through to student autonomy.

In summary, there is no specific formula for designing teaching and learning environments in the cognitive apprenticeship model. Through the scaffolding of independent, self-directed learning, students can visualise expert performance, challenge their solutions, and ultimately transform themselves into experts. Within a realistic instructional context, the concept of cognitive apprenticeship emphasises the importance of students actively solving authentic problems (Young, 1993). Cognitive apprenticeship could be viewed as a useful way to view the development of the student mathematical modeller (Wedelin et al., 2015).

Before moving to discussing selected learning theories and approaches useful for mathematical modelling, I will first briefly present Piaget's theory of cognitive development.

2.3.1.5 Piaget's theory of cognitive development

Piaget's theory of cognitive development discusses equilibration, which involves students experiencing clashes with their current knowledge that can act as a catalyst for students to accommodate and assimilate new knowledge into their current knowledge base (Sevinç, 2021; Kandasamy & Czocher, 2020). According to the theory, new knowledge can be assimilated with current knowledge either by adding to or modifying an existing cognitive structure or creating a new schema to accommodate the new knowledge (Kandasamy & Czocher, 2020). Von Glasersfeld (1995) stated, "Piaget has stressed many times that the most frequent cause of accommodation (of new knowledge) is interaction, and especially linguistic interaction, with others" (Von Glasersfeld, 1995, p. 66). Piaget affirms that students discussing things is the most frequent cause of accommodation (Sevinç, 2021). In particular, any conflict arising from different views creates an imbalance, often resulting in the accommodation of new knowledge, reaching a state of equilibrium (Brown et al., 1996). Piaget's theory of cognitive development is only useful when there is cognitive conflict. It does not explain how we learn in the absence of such conflict (Kandasamy & Czocher, 2020). Confusion, perplexity, and uncertainty can be symptoms of cognitive conflict and, as such, could be used for identifying occurrences of cognitive conflict (Zaslavsky, 2005) and hence potential instances for learning.

Studies by Sevinç (2021) and Kandasamy and Czocher (2020) illustrate how Piaget's cognitive development relates to mathematical modelling. Sevinç (2021) found that modelling involves social interactions, which contributes to model construction and thus student cognitive development. Kandasamy and Czocher's (2020) findings may help inform how learning in mathematical modelling may be enabled through cognitive conflict that arises as students work through constructing a model. The study "suggests a strong link between model construction and Piagetian explanations of knowledge construction" (Kandasamy & Czocher, 2020, p. 8).

2.3.2 Learning theories and teaching approaches relevant to mathematical modelling and mathematics education

I will now discuss Realistic Mathematics Education theory, the use of Constructivism while learning Modelling, and Active Learning, a form of learning based on Constructivism. These theories are useful when thinking about teaching and learning mathematical modelling.

2.3.2.1 Realistic mathematics education theory

Realistic Mathematics Education (RME) is about giving students a realistic mathematical experience based on applications that use real contexts (De Lange, 1996). Mathematics is considered a human activity involving solving problems through organising mathematical problems and mathematising and hence is best learnt through doing (Gjesteland & Vos, 2019; Van den Heuvel-Panhuizen, 2003). The beginnings of RME began in the 1970s in the Netherlands. RME has its origins in Freudenthal's view of mathematics as a human activity (Gravemeijer, 1994). A key characteristic of RME is that the experience is experientially real for students, and the focus is on the process of mathematisation. It should be mathematics as an activity as opposed to studying the end results of a mathematician's work (Gravemeijer, 1999; Gravemeijer & van Eerde, 2009). Learners create relationships between mathematics and reality, a characteristic of the scientific-humanistic approach to teaching modelling (Kaiser, 2020) and develop knowledge through experience (Gravemeijer & van Eerde, 2009).

Within RME, there are the processes of horizontal mathematisation and vertical mathematisation. Horizontal mathematisation is moving from reality to mathematics and involves describing a context problem in mathematical terms. Vertical mathematisation is mathematising one's own mathematical activity and involves formalising the mathematics to describe the problem (Gravemeijer & van Eerde, 2009). Barnes and Venter (2008) propose that a RME approach to mathematics in higher education should be adopted, with Le Roux (2011) suggesting a to-and-fro approach between horizontal and vertical mathematisation being more favourable than a one-way movement between the two, as suggested above. RME can be described by means of the following five characteristics (Treffers, 1987): the use of contexts; the use of models; the use of students' own productions and constructions; the interactive character of the teaching process; the intertwinement of various learning strands. Modelling fits with the philosophy of RME. Modelling takes an actual situation and explores the situation with the goal being to represent the situation mathematically. That is, to create a mathematical representation of the situation, thus mathematics comes out of the experience.

2.3.2.1.1 Using contexts to teach mathematics [4]

Using contexts to teach mathematics is a key component of RME (Van den Heuvel-Panhuizen, 2003). Contexts used within a RME framework should be experientially real for students (Gravemaijer, 1999). This means contexts do not need to be real contexts. Rather, that contexts are relatable to students (Gravemaijer; 1999; Van den Heuvel-Panhuizen, 2003). According to Gravemaijer (1999) and Van den Heuvel-Panhuizen (2003), contexts in RME are not limited to everyday life or the real world and can also include formal mathematics and fictional situations. Applying mathematics in different contexts is considered a way to deepen mathematical knowledge. As the mathematics is applied in context, meaning and understanding of the mathematics and its related concepts deepens (Blomhøj, 2019; Winkel, 2016), demonstrating a depth and breadth of knowledge about the mathematics is gained through its use in context. Freudenthal (1973), whose work, as already mentioned, forms the basis of RME theory, encourages the use of contexts for learning mathematics. He believes that the retention of

[4] This section includes re-presentation of material published in Spooner (2022).

mathematics is only possible through the creation of multiple connections both within and outside of mathematics (Dreyfus, 1991; Freudenthal, 1973). As discussed earlier, this mathematisation within mathematics is known as vertical mathematisation, and outside of mathematics as horizontal mathematisation. An example would be to start with a concrete context and mathematise this (horizontal mathematisation) and then process the mathematics within this mathematical system (vertical mathematisation). Freudenthal suggests that mathematics can reflect reality arguing that "reality is the framework to which mathematics attaches itself" (Freudenthal, 1973, p. 77).

In line with the principles of RME, modelling can provide a way for mathematical concepts to "be anchored and given cognitive roots" (Blomhøj, 2019, p. 4) through using real-world contexts and the application of the modelling process. The use of contexts and application can deepen the understanding of the mathematics (Blomhøj, 2019) and produce more connections within the mathematics and between the mathematics and non-mathematics, thus supporting the learning of mathematics (Freudenthal, 1973). The results of a meta-analysis carried out by Sokolowski (2015) support Freudenthal (1973) and Blomhøj (2019), showing that mathematical modelling helps students understand and apply mathematics concepts.

Modelling starts and ends with context. Context, and how we approach it from a teaching and learning perspective, is therefore an important and essential consideration for teaching and learning mathematical modelling (Alsina, 2007; Stillman, 2002; Villa-Ochoa & Berrío, 2015; Wake, 2016). Carducci (1996), Stillman (2002) and de Almeida (2018) all found that becoming familiar with, and understanding, the context was needed for successful modelling. Carducci (1996) found that when students are given modelling assignments where they can relate to the context, the assignments went a lot smoother for students. However, when students had no experience with the context, they were lost as to how to begin. Carducci (1996) found he needed to provide references to the context, then students were able to "make progress right from the start" (Carducci 1996, p. 259). However, Boaler (1993b) found that when students were working on mathematics problems using contexts, for some students being familiar with the contexts made it more difficult. In the same study, Boaler (1993b) found girls tended to focus too much on contextual information.

Claims have been made that using contexts helps mathematical understanding and development in being flexible when using mathematics. These claims have been made from within the RME community (Beswick, 2011; Freudenthal, 1968) and outside the community (Gainsburg, 2008; Smith, 2016). Ainley, Pratt and Hansen (2006), Boaler (1993b) and Freudenthal (1968) all agree that "using mathematical ideas in context can facilitate the development of understanding them" (Beswick, 2011, p. 379). However, Beswick (2011) believes that these claims lack solid evidence to back them up. Similarly, Reinke's (2020) review of the literature on the usefulness of context-based mathematics curriculums found conflicting results. Some authors claim contexts can help students recall similar problems, with other authors suggesting there are complications with using contexts.

Context alone is not enough to develop mathematical understanding (Beswick, 2011; Brown, 2017). Brown (2017) carried out a study looking into contexts being used within a mathematical modelling setting, specifically if contexts used in this setting support the development of students' mathematical understanding. Brown (2017) concluded that while understanding the context of the situation was useful, understanding the context alone would not guarantee students developing mathematical understanding of the situation. Genuine collaboration and 'interthinking'[5], a term used by Stillman, Brown

[5] Interthinking is defined by Hunter (2012) as "pulling students into a shared communicative space" (p. 3) where during the course of solving a task within a group, they verbally discuss their thinking and reasoning about the context and mathematics.

and Geiger (2015) in relation to mathematical modelling, was also needed to develop mathematical understanding and reasoning in these circumstances (Brown, 2017). Boaler (1993b) found that when contexts were used to teach mathematics, students were not necessarily able to transfer the mathematical understanding developed in one context to another. Boaler (1993a) said that including context applications would not be enough to increase the transferability of knowledge gained into areas outside of educational settings. However, the study by Reinke (2020) found that students made use of “images developed through engaging with contextual mathematics problems” (Reinke, 2020, p. 6) in later attempted tasks.

Benefits of using contexts to learn mathematics have been suggested. However, criticisms have also been made. As already mentioned, both Brown (2017) and Beswick (2011) both recommend that additional factors need to be considered when using contexts. Contextualising problems only is not the answer. Beswick (2011) argued that task design was just as important, if not more important, than the context. Student differences can also impact on how useful using contexts is (Linchevski & Williams, 1999).

In 2011, Beswick summarised the effects of using context problems in classrooms as unpredictable due to variations among students in:

> such things as their mathematics knowledge (Boaler, 1993b; Stillman, 2004), cultural background (Masingila, Davidenko, & Prus-Wisniowska, 1996), cognitive and metacognitive strategies and life experiences (Stillman, 2004), beliefs about themselves both as mathematics learners and in relation to their peers (Planas & Civil, 2002; Sullivan et al., 2006) and aspirations for their futures and

beliefs about the range of possibilities open to them (Cobb, 2007). (Beswick, 2011, p. 384).

In mathematics, contexts can be ignored completely or overly focussed on with minimal mathematics used (Van den Heuvel-Panhuizen, 1999). Verschaffel, Greer and De Corte (2000) caution against using trivial contextual problems as it can result in students ignoring contexts and not critiquing their answers against context. Silver, Shapiro, and Deutsch (1993) and Selter (1994) showed that many students merely offer answers and answer questions that demonstrate a suspension of their everyday sense when working with contextual problems. This is thought to result from overexposure to trivial contextual problems within their schooling (Verschaffel, Greer, & De Corte, 2000). On the other hand, as already mentioned, Boaler (1993b) found that for some students, familiarity with contexts produced difficulties, particularly for girls. This was seen with some students primarily concentrating on the context, while having difficulty understanding the mathematics in relation to the context. What appears clear from the literature is that it is still unclear when, where and how using contexts will be beneficial to students and may be individual specific (Beswick, 2011). Research into the nuances of using contexts for teaching mathematics is still needed. Drawing from the literature, mathematical modelling is a potential way of addressing the use of contexts while exploring other factors that may influence and help the learning of mathematics. Using contexts alone does not necessarily help mathematical understanding, however, using contexts with mathematical modelling helps students understand and apply mathematics concepts (Sokolowski, 2015). Alsina (2007) claims that using mathematics with contexts involves creativity.

However, aside from the criticisms and the need for more research into the use of contexts for learning mathematics, contexts are important for modelling. Contexts for teaching and learning mathematics provide interest and motivation for students to engage in mathematics (Beswick, 2011). Most literature states the benefits of using context to teach mathematics, both mathematics and mathematics used in mathematical modelling, is to provide interest and motivation for students. This, in turn, improves students' attitudes to mathematics (Beswick 2011, Jurdak, 2006). Context gives students a reason to engage with the context and hence the mathematics within it (Ainley et al., 2006; Gainsburg, 2008; Smith & Morgan, 2016). Alsina (2007) says that at the tertiary level, there is evidence that students learn better context, as contexts involved with mathematical modelling have the potential to generate interest and motivation. While Quiroz, Orrego and López (2015) say students using authentic contexts in mathematical modelling started to see themselves as useful mathematical citizens.

Specific student difficulties with contexts will be discussed further in Students experiences when learning to mathematically model, Section 2.7, page 85.

2.3.2.2 Constructivism and mathematical modelling

The most common learning theory discussed in relation to learning modelling is constructivism, (see Arseven (2015), Meier (2009), Spooner (2018). According to Meier (2009), the key concepts of a constructivist learning environment, as described earlier on page 30 under General learning theories in education, Section 2.3.1, fit well with mathematical modelling. Meier (2009) says:

> Being involved in a real-life subject motivates you to work and think independently. Further, the pupils have to use their prior knowledge about mathematical and real-life contents. In order to convince others of their mathematical model and their solutions, it is essential to let the pupils argue. After that they can reflect on the whole process of solving a modelling problem as a learning process. (p. 214).

It is commonly understood that modelling is best learnt by 'doing' modelling (Blum, 2011, 2015; Boaler, 2001; Blomhøj & Kjeldsen, 2006; Caron & Bélair, 2007; Legé, 2007; Toews, 2012; Wake, 2011). Legé (2007) and Boaler (2001) show that students do better by 'doing' modelling as opposed to being shown. In addition, the more experience students have with modelling the more familiar and hence more accessible modelling can become for students with Cole, Linsenmeier, McKenna and Glucksberg (2010) showing students improved with their second attempt at modelling.

2.3.2.2.1 Active learning

Active learning is based on the theory of constructivism (Vanhorn, Ward, Weismann, Crandell, Reule, & Leonard, 2019). Put simply, active learning is students "doing things and thinking about the things they are doing" (Bonwell & Eison, 1991, p. 19). Active learning involves students engaging with material to "understand the real-life application and implications of what they study" (Vanhorn et al., 2019, p. 8). To provide further understanding of active learning, Lombardi et al. (2021) carried out a study into the ways STEM education leaders perceived active learning within their domain. Lombardi et al. (2021) concluded that active learning occurs with interactions between students, their peers and teachers as they have direct experiences with domain practices, data and models of their discipline. Lombardi et al. (2021) decided that it is doubtful that the traditional lecture alone will increase STEM understanding. The theory of constructivism incorporates the importance of activity learning, whose characteristics are similar to mathematical modelling. As already mentioned, it is thought that modelling is best learnt by doing (Blomhøj & Kjeldsen, 2006; Blum, 2011, 2015; Caron & Bélair, 2007; Toews, 2012; Wake, 2011). Blomhøj and Kjeldsen (2006) promote context-based project work as one situation for students to actively be modelling. Similarly, Toews (2012) said "developing good modeling skills is a doing activity, not a watching activity” (Toews, 2012, p. 560). This approach enables students to be thrown in the deep end to have an active experience of learning by doing.

2.4 Learning environments potentially useful for learning to mathematically model

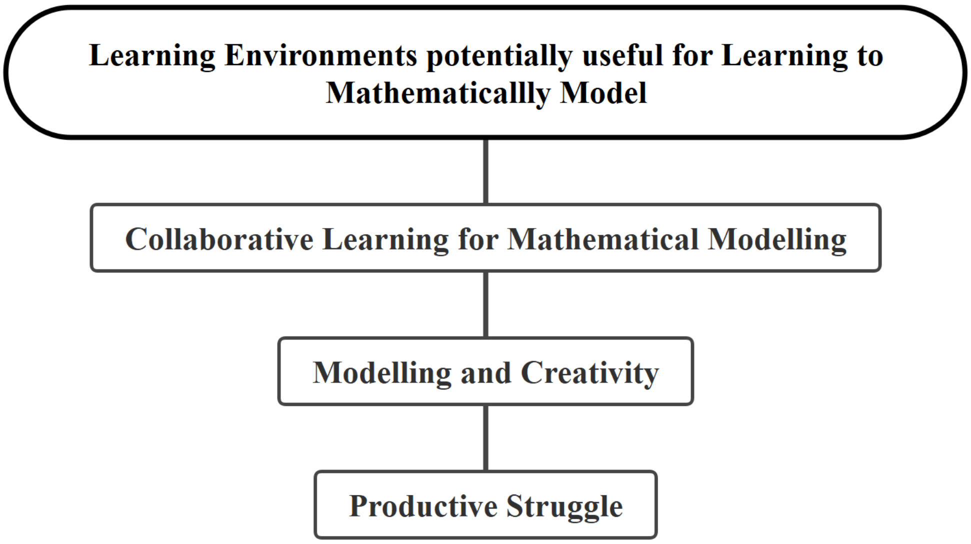


Figure 8. Learning environments for mathematical modelling

2.4.1 Collaborative learning for mathematical modelling

Aligned with social constructivism is collaborative learning. According to Vygotsky (1978), social interaction is key to developing students’ abilities (Amit & Naaman, 2014), while group work is central to an authentic mathematical modelling experience (Spooner, 2013; Toews, 2012). These characteristics of learning and modelling, therefore, make collaborative learning an essential component of a mathematical modelling experience. It is therefore not surprising that group work is commonly used in mathematical modelling education (Hernandez-Martinez & Vos, 2018).

Studies in mathematics and mathematical modelling education have shown that most students view group work favourably and as useful for learning (Sofroniou & Poutos, 2016; Hernandez-Martinez & Vos, 2018). Sofroniou and Poutos’s (2016) study showed that students valued the benefits of working in a group and were unconcerned that group work could allow some students to not pull their weight. This implies that students think the benefits of a group outweigh any possible negative dynamics of group work. Hernandez-Martinez and Vos (2018) showed students valued group work, including presenting the results of their group work to the class, and as a skill they will need in the workforce. In addition, students viewed working within a group and explaining their work in a presentation contributed to their development of mathematical understanding. Toews (2012) promotes the use of group work in modelling and presentation of work, not for their learning benefits but as fundamental skills for industry and science. Boaler (2008) believes group work helps students develop equitable relations and makes them accountable for their work.

2.4.2 Modelling and creativity

Even though there is no one way to be creative (Satyam, Savić, Cilli-Turner, El Turkey, & Karakok, 2021), mathematical creativity can be viewed as being original and novel (Runco & Jaeger, 2012; Wessels, 2017), flexible and fluent (Chamberlin & Moon, 2005; Torrance, 1974; Wessels, 2017), and useful (Neumann, 2007; Wessels, 2017). Originality can mean work new to a person (Mann, 2018; Sriraman, 2005; Sriraman, 2009) or work involving insight-based or unconventional solutions (Chamberlin & Moon, 2005; Leikin, 2009; Wessels, 2017). Chamberlin and Moon (2005) identify mathematical creativity as the ability to solve problems that are difficult or impossible to solve with conventional methods. "Mathematical creativity is generally associated with problem solving or problem posing" (Nadjafikhah, Yaftian, & Bakhshalizadeh, 2012, p. 286). A key development in mathematical creativity is the idea that the mathematics created is useful (Wessels, 2017).

Modelling involves being mathematically creative (Garfunkel & Montgomery, 2019). Research by Amit and Gilat (Gilat & Amit, 2012; Gilat & Amit, 2013; Amit & Gilat, 2013; Gilat & Amit, 2014) shows that student involvement in the modelling process fosters student creativity. Amit and Gilat (2012) suggest "that working with non-routine modelling problems, like MEAs, can simulate students' creativity in solving problems using mathematics" (Wessels, 2014, p. 402). Amit and Naaman (2014) associate creativity with challenge and relate creativity, along with high cognitive ability, perseverance and motivation, as an indicator of giftedness. Satyman et al. (2021) found that if a student could see connections between mathematics and reality and was comfortable with their understanding of mathematics, then they were more likely to be able to move forward with being creative with mathematics. Students in their study viewed approaching a problem in different ways and being persistent as part of creativity. Flexibility in thinking is a common aspect of creativity in the literature (Satyman et al., 2021; Wessels, 2017). Lofgren, Collins, Smith & Cartwright (2016) observed that modelling fictional situations allowed students to be more creative than when modelling realistic situations when looking at SIR models. An outcome of Amit and Gilat's study showed that "student engagement in modelling processes encouraged students to creatively utilise their fluent thinking skills as well as their skills in elaborating, refining, and generalising" (Wessels, 2017, p. 388).

It would make sense that with creativity being an important component of modelling, we need to be conscious of providing situations that will promote creativity and not stifle it. In fact, Lu and Kaiser (2022) have recently carried out research into the possibility of establishing creativity as a modelling competency. As educators we need to be aware that creativity is often associated with challenge (Amit & Naaman, 2014).

2.4.3 *Productive struggle*

Productive struggle is a learning strategy that is underpinned by the belief that challenge and struggle develop students' mathematical understanding (Hiebert & Grouws, 2007;

Schoenfeld, 1992) and problem solving skills (Betts & Rosenberg, 2016). Thinking, making decisions, and taking risks are all characteristics of productive struggle (Sullivan et al., 2015). Encouraging students to take risks and think outside the box can lead to innovative solutions to problems (Boaler, 2002). This reinforces the idea that risk taking supports and allows for multiple approaches to developing a mathematical solution or model. This development involves creativity and productive struggle.

However, there are also potential drawbacks to using productive struggle. For example, some students may become frustrated and discouraged when struggling with a problem, leading to a decrease in motivation and a negative attitude towards mathematics (Lamon, 2009). Additionally, students who struggle with a problem may not have the necessary background knowledge or support to find a solution, which can leave them feeling helpless and overwhelmed (Lamon, 2009). Nevertheless, students' failure to solve problems can actually help them learn, according to recent research (Kapur, 2016), while solving problems successfully does not always progress students' knowledge.

2.5 Comparisons between learning mathematics and learning mathematical modelling[6]

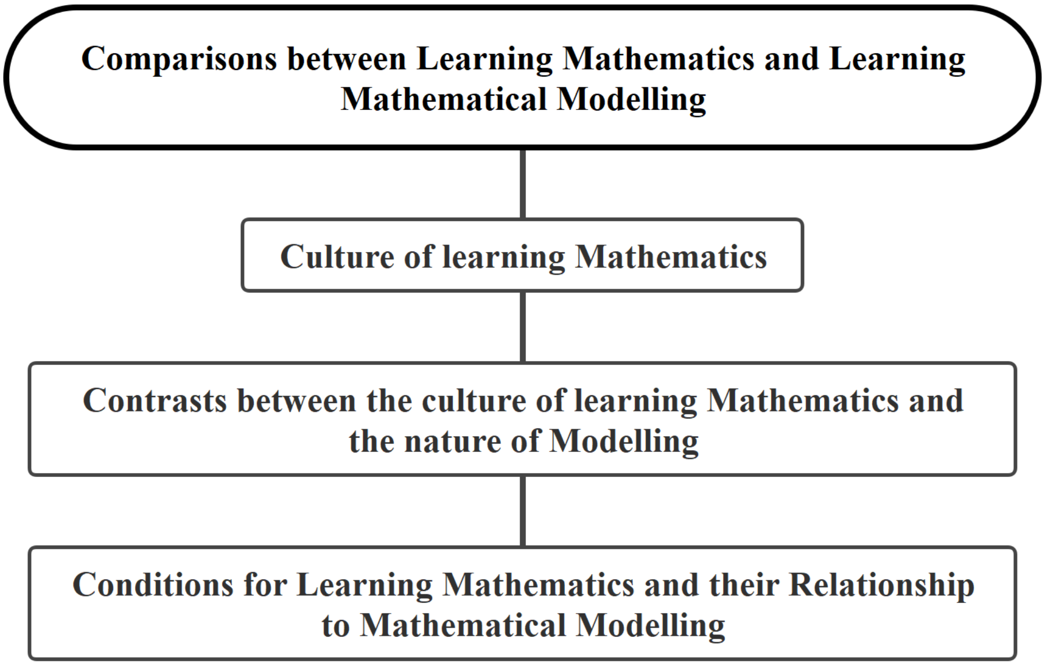


Figure 9. Comparisons between learning mathematics and learning mathematical modelling

According to Wake (2016), studying mathematical modelling is concerned more with 'how to use' mathematics as opposed to "learning mathematics as a discipline in its own right" (Wake 2016, p. 173). Toews (2012) "argued that mathematical modeling is by nature a hybrid activity, and that there are compelling reasons to approach this class as something much broader than just another mathematics class" (Toews, 2012, p. 557). The design of a modelling course "intends to make modelling explicit, in contrast to other mathematics courses where the presumption is that only 'the mathematics itself' counts" (Wake, 2016, p. 179). Similarly, Toews (2012) presents the case for modelling as not being considered as mathematics due to modelling's multi-disciplinary nature and that it draws on multiple skill sets other than just 'mathematics'.

Modelling can be a change of culture for mathematicians. Toews (2012) noted:

[6] This section includes re-presentation of material published in Spooner (2021) and Spooner (2022).

> Mathematicians are trained to appreciate rigor and formal elegance, standards which can be brought just as easily to bear on equations that describe real phenomena as equations that do not. While elegant proofs have cachet, they generally do little to support phenomenological understanding. (p. 561).

Modelling is more than proofing and stepping something out logically. It requires an understanding of the physiological nature of the problem and making connections between these and the mathematics. It is more than writing a theorem and developing a proof and is messier and often not as precise and 'tight' as a mathematical proof.

I will now present the culture of learning mathematics and its contrasts between the nature of mathematical modelling.

2.5.1 Culture of learning mathematics

Research shows that, for many students, their experience of learning mathematics has been one where the focus is on getting the 'right' answer (Boaler, Wiliam, Brown, 2000; Williams et al., 2009), following procedures they do not always understand (Solomon, 2007), and viewing the lecturer and 'information' as the authority, as well as relying on the lecturer to provide content and direction (Solomon, 2007; Yoon et al., 2011). Undergraduate mathematics has traditionally been more focussed on abstraction as opposed to application (Dreyfus, 1991) and is generally taught devoid of context (Winkel, 2016). There appears to be an embedded culture of accepting information as authority, following the lead of others while avoiding 'digging deeper' or using application to achieve conceptual understanding. In a study by Solomon (2007), 'a number of the students reported that mathematics, particularly pure mathematics, was presented as a non-negotiable finished product, as a set of rules and strategies to be learned, not constructed' (Solomon, 2007, p. 90).This approach has contributed to the generally held student view that to be good at mathematics means you can repeat things being shown, while not necessarily having any conceptual understanding of the material. Engaging, questioning material and ‘stumbling down blind alleys’ is perceived as being ‘not good at’ mathematics (Solomon, 2007).

A strong culture of needing to get things right exists, with studies by Solomon (2007) and Yoon et al. (2011) showing that most students avoided situations that could make them ‘look stupid’. The outcome of this was that some students tended to avoid asking questions during lectures (Yoon et al., 2011) and avoided working with others (Solomon, 2007) to minimise the risk of being seen as not being competent. An explanation for students not wanting to speak up or interact could be due to a “fear of public embarrassment” (Yoon et al., 2011, p. 119). However, this is not to say that all students have this experience. Ortiz-Robinson and Ellington (2009) found that students taking a course on Real Analysis (twelve to fifteen students) initially resisted working in groups. By the end of the course, students had embraced and appreciated the in-class, student-centred group work, finding it beneficial to their learning as they worked in small groups discussing and constructing proofs together.

2.5.2 Contrasts between the culture of learning mathematics and the nature of modelling

How does the student experience of learning mathematics compare to the nature of mathematical modelling? The nature of mathematical modelling is concerned with making connections between mathematics and the real world. This requires thinking about how we can apply mathematics outside of the way the discipline has been traditionally experienced by students (Solomon, 2007). The study by Hernande-Martinez (2020), which looked at the mathematical backgrounds of students and how this affected their acceptance of modelling as a practice, found that a student from a non-traditional mathematics background who was homeschooled could more readily embrace mathematics being applied to areas in real life than the student educated through a mainstream education system.

As already mentioned in discussing the nature of mathematical modelling, Section 2.2.2, page 25, mathematical modelling is more than 'how to' approach a problem. Modelling is a complex activity requiring thinking about and using mathematics in creative and often new ways (Gershenfeld & Gershenfeld, 1999). The process of mathematical modelling is active and involves teamwork and student participation (Bassanezi, 1994). It is commonly agreed that modelling is best learnt through active participation and is developed through doing (Blomhøj, 2009; Blomhøj & Kjeldsen, 2011, 2013; Caron & Bélair, 2007; Kaiser & Sriraman, 2006; Kaiser & Stender, 2013). This is a different environment than that which most students of mathematics have previously experienced.

To elaborate further, modelling requires students to use their knowledge differently (Carducci, 1996; Cole, Linsenmeier, Molina, Glucksberg, & Mckenna, 2011). There is no one textbook for the course, and "no well-defined set of techniques to be mastered" (Carducci, 1996, p. 254). Students are "asked to apply what they have learned to novel situations without any of the signposts that they have relied on to solve word problems in other courses" (Carducci, 1996, p. 254).

Findings from a case study (Spooner, 2021) I carried out as part of the investigative work of this thesis, showed the students experience with modelling involved exploration, producing an unrealistic model first time around, developing an appreciation for not being right first attempt, as well as there being no model answer, or a wrong or right way of approaching the model development. One of the key changes for students was learning that the model they produced would not be a perfect fit for the situation for which it was developed. An earlier paper I wrote based on my PhD research (Spooner, 2021) supports Carducci's (1996) observations that students initially have to struggle with their approach to realise there is no one right approach or answer and they "will not be able to find it" (Carducci, 1996, p. 261). This culture is different to that previously experienced by New Zealand school and university students for mathematics. Students are used to a mathematics culture of 'getting things right'.

Another key difference between mathematics and mathematical modelling is the emphasis on connecting mathematics and contexts. Fundamental to mathematical modelling is the establishment of a relationship between mathematics and contexts, real or otherwise. In contrast, when learning mathematics, much of the time the focus is on abstract, unconnected mathematics (Wake, 2016). This is not to say that learning abstract mathematics is not okay, as that is what learning mathematics can be (Wake, 2016). However, lecturers need to be aware of providing student learning experiences that explore and establish actual connections between mathematics and contexts when teaching modelling.

2.5.3 Conditions for learning mathematics and their relationship to mathematical modelling

Contrary to the student experience, research suggests that mathematics learning is best accessed through a culture that encourages "exploration, negotiation and ownership of knowledge" (Solomon, 2007, p. 92) with many researchers arguing that "mainstream classroom mathematics teaching excludes learners" (Solomon, 2007, p. 80). Teaching practices that are concerned with student-centred learning promote students explaining their reasoning, making sense of other students' thinking and encourage persistence with challenging problems (Yackel, Cobb, & Wood, 1991). Student understanding can be compromised when the teacher is accepted as the sole authority for knowledge, with little discussion or critique of information encouraged and no active involvement from students (Yoon et al., 2011).

As previously discussed in Section 2.3.2.1.1, page 38, applying mathematics in different contexts is also considered a way to deepen mathematical knowledge. Blomhøj (2019), Barnes and Ventor (2008), Freudenthal (1973), Gravemaijer (1999), Van den Heuvel-Panhuizen (2003), and Winkel (2016) all promote the use of contexts while learning mathematics. Meaning and understanding of the mathematics and its related concepts deepens as mathematics is applied to contexts (Ainley et al., 2006; Blomhøj, 2019; Freudenthal, 1968; Winkel, 2016), with contexts acting as an anchor for the mathematics (Blomhøj, 2019; Freudenthal,1973). However, as challenged by Beswick (2011), the benefits of using contexts to improve mathematical understanding are debatable. Beswick (2011), Boaler (1993b), and Brown (2017) contend that the use of contexts alone is insufficient to foster mathematical understanding.

The conditions for learning mathematics all have similarities to the nature of mathematical modelling. Mathematical modelling involves working together with others, making sense of each other's thinking, and persisting with challenging problems that utilise context, with students taking on a level of responsibility for their process and involvement (Spooner, 2021). Modelling can provide a way for mathematical contexts to "be anchored and given cognitive roots" (Blomhøj, 2019, p. 4) by using real-world context and the application of the modelling process. The use of context and application can deepen the understanding of the mathematics (Blomhøj, 2019) and produce more connections within the mathematics and between the mathematics and non-mathematics, thus supporting the learning of mathematics (Freudenthal, 1973).

These characteristics for learning mathematics and mathematical modelling resemble active learning (Vanhorn et al., 2019). As previously mentioned in Section 2.3.2.2.1, page 44, active learning involves students acting and reflecting on their actions (Bonwell & Eison, 1991). As students interact with one another, their teachers, and the domain practices, data, and models of their discipline, active learning occurs (Lombardi et al., 2021). However, it is doubtful that active learning takes place in the traditional lecture alone (Lombardi et al., 2021).

There has been debate around and research into the type of instruction and level of assistance that is favourable for student learning. Klahr and Nigam's (2004) study and Kirschner et al.'s (2006) literature review concluded that direct instruction produced better student results in contrast to unassisted discovery learning. Two meta-analyses carried out by Alfieri , Brooks, Aldrich and Teneenbaum (2011) determined assisted discovery learning yielded better results for students as opposed to direct instruction or unassisted discovery learning.

As an alternative to assisted discovery, Klaher (2009) proposed direct instruction initially to facilitate later discovery. Godino (2019) concluded that mathematics and science are learnt best through a combination of direct instruction, guided exploration, and independent discovery. This evidence supports assisted active learning, either through assistance provided within the discovery process or by direct instruction followed by self-discovery, as having the potential to deliver better learning outcomes than direct instruction or self-discovery on their own.

To summarise, mathematical modelling is relatively new to mathematics education. Modelling involves active learning in the form of assisted discovery, creation and collaboration. Tertiary mathematics has traditionally not been an active learning experience for students. One of the dangers of following rules and repeating what has been shown, is that these can lead to a view that mathematics is not creative, with students engaging mainly in activities that stifle creativity (Solomon, 2007). Mathematical modelling offers a different way to engage with mathematics that appears to have the potential to benefit the learning experience for students.

2.6 Teaching mathematical modelling[7]

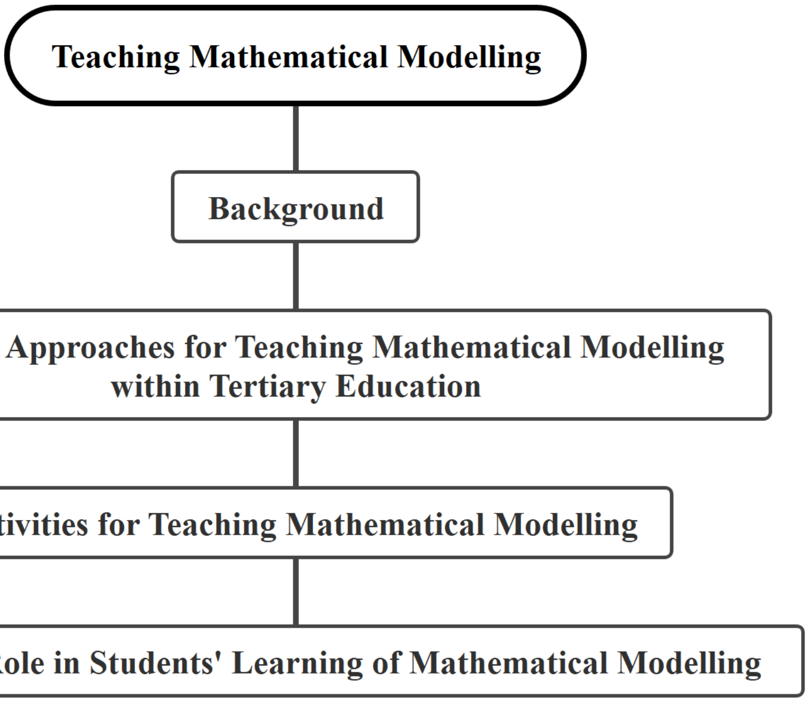


Figure 10. Teaching mathematical modelling

2.6.1 Background

In 1984, there was no widely established method of teaching mathematical modelling (Oke, 1984). This remains the same today (Blum, 2011; Kaiser, 2020). Different theoretical bases, as previously discussed in this chapter in Section 2.2, page 21 and Section 2.3.2, page 37, approaches and types of activities have been developing over the last forty years, some of these based on Realistic Mathematics Education (RME). Principles for Model Eliciting Activities (MEAs), a type of student modelling activity emerging from RME, have been established. Even so "it is still disputed how to integrate mathematical modelling into the teaching and learning processes" (Kaiser, 2020, p. 554). There is no one common method for teaching modelling, though this is not to say that one collective method is desirable.

[7] This section includes re-presentation of material published in Spooner (2020) and Spooner (2021).

There is debate whether students need to get to know how models work or be actively involved in creating models. It was suggested early on that students need both learning about how models work as well as being actively involved in creating models (Burkhardt, 2006; Pollak, 2007). Kapur (1982) said, “knowing mathematical models is necessary but not sufficient condition for learning mathematical modelling” (Kapur, 1982, p. 189). According to Pollak (2007), the big question was whether knowing models or creating modelling should be experienced first. Legé (2005a) showed that it does not matter if you get students to know 'models' first or to create models first. Legé (2005a) found that while creating models first does help students to use sub-models and links to organise their work, recognise characteristics of the contextual situation, and identify model limitations, “both approaches are viable” (Legé, 2005a, p. 95) for developing students’ mathematical modelling competencies. According to Legé (2005a, 2007), the key to both approaches was the teacher’s role, believing the teacher needs to act as a guide instead of an expert modelling the answers.

Guidance is needed as learning to model contains an independent component (Blum, 2015; Huang, 2012; Kaiser & Stender, 2013; Stillman, 2011). Students need to have some experience independently modelling to develop modelling competency (Blum, 2011, 2015; Burkhardt, 2006; Carducci, 1996; Pollak, 2007; Spooner, 2020). It is the consensus that while students are creating models, there should be a balance between instructor guidance and student autonomy (Blum, 2011, 2015; Kaiser, 2020; Schukajlow & Blum, 2023).

However, little is known about how lecturers and teachers should effectively manage balancing between guidance and independent work. Wedelin and Adwai’s (2014) study noted that it varied from student to student how much they learn by working independently and how much they learn from their teachers, implying that it is likely dependent on a variety of different factors other than just the lecturer or teacher. However, we do know that the role of the teacher has not been researched sufficiently yet (Kaiser, 2020). With most work focussed on secondary studies, even less is known about the role of the lecturer. We know that it is key not to provide students with all the answers (Carducci, 1996). Similarly, Durandt et al.'s (2021) study involving first-year tertiary engineering students, showed that directed teaching was not supportive for developing modelling abilities. With there being limited studies investigating how instructors effectively manage balancing between guidance and independent work, this leads us to the questions. What are the characteristics of providing balance between minimal teacher guidance and maximum student independence? How do we, as instructors, effectively do this?

Most studies on instructor guidance and student autonomy focus on interventions from the instructor with most of the studies focussing on secondary school students. There is acknowledgement that there is no one-size-fits-all for instructors' intervention (Kaiser & Stender, 2013; Stillman, 2011), with interventions needing to be tailored to both the student and the situation to be modelled. At the secondary school level, research is focussed on teachers being available during modelling to provide interventions where and when necessary (Kaiser & Stender, 2013; Stillman, 2011). Teachers being available suggests a potential belief that students are not capable of true autonomous modelling and or that true autonomous modelling is just not possible. The use of interventions could also mean a minimisation of student struggle, even though we know that periods of feeling helpless or insecure are part of the modelling process (Kaiser & Stender, 2013). While researchers are in agreement that there should be a balance between instructor guidance and student autonomy (Blum, 2011, 2015; Kaiser, 2020; Schukajlow & Blum, 2023), from reviewing the literature, little is known about what that balance looks like and what factors need to be taken into consideration before an instructor intervenes. How much should we let students struggle before we intervene? Does a teacher need to be present during a modelling activity, or are there other ways the teacher can support from the sidelines that do not require the teacher to be present?

With the assumption that the instructor is present and available during the student modelling experience, Blum (2015), Durnadt et al. (2021), Kaiser & Stender (2013), Kaiser (2020), and Stillman (2011) focus on instructor interventions to help move students forward with the process when they encounter difficulties. Research that has been carried out focusses on intervention strategies that teachers can use in the moment (Kaiser & Stender, 2013; Stillman, et al., 2013; Stender & Kasier, 2015; Stender, 2018). The main result from teacher intervention studies identified asking a student to explain out loud what they had done so far as being a helpful intervention (Kaiser & Stender, 2013). This is helpful on two accounts: one, so a teacher gets access to student thinking and then can see possible ways of providing appropriate guidance and assistance (Stillman et al., 2013); two, students 'collecting their thoughts' to communicate what they have done so far. Table 1 looks at the pros and cons of different possible teacher interventions that have been researched in secondary schools. Instructors providing help to students during modelling activities maintains a level of dependence on the instructor.

Table 1. Secondary school teacher interventions for modelling

Researcher	**Intervention**	**Pros**	**Cons**
Kaiser & Stender (2013) Stender & Kaiser (2015)	Talk out loud.	Instructor gets access to student thinking to decide on appropriate guidance and assistance. Gets students to 'collect' their thoughts.	Instructor needs to be present. Instructor needs to have the right knowledge to provide useful intervention.
Stillman, Brown, & Galbraith (2013)	Interventions for reflective learning. This intervention would include prompts for recognising, and hence rectifying, the application of incorrect or incomplete knowledge.	Intervention challenges students to look at their current working and how they may or may not need to change their approach.	Instructor needs to be skilled at this. More understanding and research is needed in this area to overcome robust blockages with teacher intervention.
Stillman, Brown, & Galbraith (2013) Durandt (2018) Stender & Kaiser (2015)	Modelling cycle.	Modelling cycle used as a scaffold, as a metacognitive tool.	
Stender, Krosanke, & Kaiser (2017)	Interventions as diagnostic tools.		Instructors needs to give appropriate assistance so as not to discourage students.

Researcher	Intervention	Pros	Cons
Stillman (2011)	Metacognitive teacher interventions.		Instructor must monitor individuals or groups progress in order to intervene strategically when necessary. How a teacher intervenes (uses meta-metacognition) can either promote or stifle modelling.
Stender (2018)	Strategic teacher interventions based on heuristic strategies.	‘Explain your work’ used to identify appropriate intervention.	Intervention needs to be given at the right time to have impact.
Stender & Kaiser (2015)	Present the status of your work.	Instructor gets access to student thinking to then be able to decide on appropriate guidance and assistance. Gets students to ‘collect’ their thoughts, promoting student metacognition.	Interventions need to be given based on diagnosis of students’ work. Interventions are responsive in nature, meaning they were adaptive and ‘in the moment’.
Schukajlow, Kolter, & Blum (2015)	Strategies incorporated in a ‘solution plan’ using adaptive generic prompts for students for each stage of the modelling process.	Could be used independently by students though may need further work.	

The need to balance between instructor guidance and student autonomy for modelling activities (Kaiser, 2020), places the role of the lecturer and teacher as one of a facilitator for these activities (Carducci, 1996). According to Blum (2011, 2015), a teacher needs to be able to give appropriate support, balancing this between minimal teacher guidance and maximum student independence. Carducci (1996) says it is important to keep students on track where the biggest challenge for the lecturer is “to keep quiet [and] let the students develop their own models” (Carducci, 1996, p. 259). Literature promotes providing an experience where students have a partial, if not full, independent experience of mathematical modelling (Carducci, 1996, Spooner, 2020). The key appears to be in lecturers providing enough support such that students are able to make progress, without removing genuine student involvement in the activity.

One way lecturers and teachers are able to provide a balance between instructor guidance and student independence is to provide some form of scaffolding. Durandt (2018) stated scaffolding “acts as a way of cognitively structuring what students [are] doing” (Durandt, 2018, p. 11), acting as a guide. It provides organisation, guidance and meaning to the experience (Seel, 2012). While Kaiser (2020) said scaffolding can be “described as a metaphor for tailored and temporary support that teachers offer students to help them solve a task that they would otherwise not be able to perform” (Kaiser, 2020, p. 560).

Scaffolding approaches have developed out of constructivist teaching and learning methods (Kaiser & Stender, 2013). One such approach is discovery learning. Alfieri et al. (2011) carried out two meta-analyses that concluded assisted discovery learning had better outcomes for students compared to unassisted discovery learning or direct instruction. Optimal approaches for assisted discovery learning included guided tasks with internal scaffolding, “tasks requiring learners to explain their ideas” (Alfieri et al., 2011, p. 12), and worked examples (Sweller, Kirschner, & Clark, 2007). Klaher (2009) recommends using direct instruction initially to facilitate later discovery as an alternative approach for assisted discovery.

According to Kaiser (2020), it is rare for scaffolding to be used in secondary classroom practices. Kaiser found that for secondary teachers, instead of providing scaffolding for students "most teachers provided immediate support or even favoured their own solution" (Kaiser, 2020, p. 560). Even so, forms of scaffolding have been implemented with secondary school students. These include heuristic strategies (Stender, 2019), generic solution plans (Schukajlow et al., 2015), and tutor intervention (Kaiser & Stender, 2013). Heuristic strategies are strategies that enable a person to discover or learn something themselves (Blatchford, Russell, & Webster, 2012). Examples of heuristic strategies include: simplify as much as possible, use a simulation, draw a figure, explain and discuss your thinking with someone else and asking, have you seen a similar problem before? (Kaiser & Stender, 2013; Stender, 2019). A study carried out by Stender (2019) concluded "heuristic strategies might be very helpful supporting the students but to apply them in concrete situations is not easy" (Stender, 2019, p. 210). Schukajlow et al. (2015) carried out a study involving German ninth-grade students, which showed the use of a solution plan may have contributed to students' improvement in modelling competency, but a strong conclusion could not be drawn. However, students who used a solution plan reported more frequently in the post-test "about planning, rehearsal, elaboration and organising strategies while solving modelling problems" (Schukajlow et al., 2015, p. 1241).

There is evidence that forms of scaffolding are being used at tertiary level for mathematical modelling projects. Scaffolding approaches include: externally supplied heuristic strategies (Caron & Bélair, 2007), using similar strategies to those used at secondary school; checkpoints (Caron & Bélair, 2007), and consultations between lecturers and students (Blomhøj & Kjeldsen, 2013; Duan, Wang, & Wu, 2020). In addition, Blomhøj and Kjeldsen (2006) recommend a scaffolded approach with students eventually taking responsibility for the entire modelling process.

The lecturer's role as a facilitator in promoting independent student work includes time management (Carducci, 1996) and feedback (Carducci, 1996; Lyon & Magana, 2020). Time management is needed for long-term projects (Carducci, 1996). Feedback is an important part of supporting students' modelling processes (Carducci, 1996; Lyon & Magana, 2020). Findings from Lyon and Magana (2020) found that as modelling activities are occurring, feedback by a lecturer to a student needs to be both positive and needed. Lyon and Magana (2020) found that positive feedback helped students clear up misconceptions, however, negative feedback discouraged students from persevering with developing solutions.

Despite research into student difficulties (Chaachoua & Saglam, 2006; Crouch & Haines, 2004; Klymchuk & Zverkova, 2001; Klymchuk et al., 2010; Palm, 2007; Soon, Lioe, & McInnes, 2011; Stillman, 2004), little research addresses how to minimise and overcome student difficulties when a student group is working independently of the instructor. There is a gap in the literature examining what happens when students become stuck and lost and there is no instructor readily available to provide guidance. This study, as seen in the findings page 170, will contribute to the general literature by showing some behaviours students utilise when they are stuck.

2.6.2 Different approaches for teaching mathematical modelling within tertiary education

Opportunities exist at tertiary institutions to experience mathematical modelling. These opportunities range from modelling activities within existing mathematics courses through to full modelling courses focussed on providing student experience with a holistic process of mathematical modelling (Bliss, Galluzzo, Kavanagh, & Skufa, 2019). For undergraduate modelling courses and/or modelling components, there is a range of various approaches suggested or implemented. These include but are not limited to: thought experiments (Bliss et al., 2019); case studies (Duan et al., 2020; Fu & Xie, 2013; Heilio, 2011); simulated work-placement experiences (Alpers, 2011); guided modelling projects (Blomhøj & Kjeldsen, 2013); popular culture contexts (Lofgren, 2016; Lofgren et al., 2016); open-ended modelling projects (Bliss et al., 2019; Carducci, 1996; Caron & Bélair, 2007; Lyon & Magana, 2020; Spooner, 2021). Bliss et al. (2019) suggests the use of thought experiments – activities that allow students to participate with some features of the modelling process while not involving creating a model from beginning to end. Heilio (2011) suggested the use of case studies to establish links between the real world and mathematics and to demonstrate the range and complexity of different model types. Fu and Xie (2013) used student-relevant problems where first-year students thought about the problem themselves, followed by lecturer-led class discussion of student-generated ideas and concluded with the presentation of different well-established models for the problem. A final year course at a German university used industry-type problems to simulate tasks highly likely to be encountered in an engineering workplace (Alpers, 2011). These tasks are carefully designed to enable students to apply and understand models learnt in class in a simulated real-life problem. Supplementary to a modelling first-year course in Denmark, a guided modelling project was used to develop mathematical concepts as well as modelling processes (Blomhøj & Kjeldsen, 2013). Lofgren et al. (2016) use popular-culture contexts to remove some of the hurdles students can experience with modelling. Duan et al. (2020), Blomhøj and Kjeldsen (2013), and Caron and Bélair (2007) studies, centre around the use of various open-ended modelling activities.

Literature discusses some conditions that are beneficial for a mathematical modelling course. These include, but are not limited to: allowing 'time' for modelling (Lyon & Magana, 2020); using a spiral curriculum (Lyon & Magana, 2020); frequently alluding to modelling process (Caron & Bélair, 2007); different types of support, including support for independent student model creation (Carducci, 1996; Spooner, 2021); providing opportunities for active student engagement and exploration within activities (Spooner, 2022; Toews, 2012; Wake, 2011); and working in groups (Spooner, 2022; Toews, 2012). The results from my Spooner (2020) study showed that for a productive student experience, modelling should be undertaken in groups as opposed to individually, that opportunities to make connections between reality and mathematics are provided, and that students are encouraged to be open-minded and think differently about mathematics and its potential.

It is also important for students to learn to critically compare and examine different models to begin to understand that models are unique for different situations. Toews (2012) believes that it is better for students to have an experience that enables students to compare models than learning to use a few models in-depth and proposes using ill-defined problems as a way to achieve this. Whereas Lofgren (2016) discusses the danger of students misinterpreting the use of standard models for all situations and how the characteristics of each situation must be taken into account and the standard model modified or further developed or disregarded as necessary, thus supporting Toews (2012) belief that an exposure to multiple different modelling situations is better than exposure to a few. Lofgren's (2016) and Toews' (2012) work suggests that the need for the lecturer to role model choosing a foundation model, including showing how to modify it, is critical to the development of modelling skills. It is also crucial for the lecturer to provide students with their own experiences of deciding which standard (foundation) models to apply to a situation and consequentially adjust these as necessary. By doing this “the student begins to get a sense for both how free the modeler is and also how important it is to know something about the system being modeled” (Toews, 2012, p. 558).

Finally, as already mentioned in Section 2.2.3, page 26, providing students with opportunities to actively engage in modelling is generally considered a characteristic of developing modelling competency (Blomhøj & Kjeldsen, 2011, 2013; Caron & Bélair, 2007). Kaiser and Stender (2013), Toews (2012) and Wake (2011) believe “modelling is not a spectator sport and can only be learnt by doing” (Kaiser & Stender, 2013, p. 280). Pollak (2007) encourages ‘messiness’ in a modelling learning experience, encouraging this as a way for students to begin to understand the art of modelling.

2.6.3 Activities for teaching mathematical modelling

Though there is a range of activities that can be used to teach and expose students to the art of modelling, I will discuss in detail the literature on open-ended modelling activities, case studies and the use of popular culture, the three activities used in the courses investigated in this study.

2.6.3.1 Open-ended modelling activities

Open-ended modelling activities are one activity used within courses (Bliss et al., 2019; Blomhøj & Kjeldsen, 2013; Carducci, 1996; Caron & Bélair, 2007; Duan et al., 2020; Lyon & Magana, 2020) and for modelling competitions (Garfunkel, Niss & Brown, 2021) that provide opportunities for students to have a holistic experience of mathematical modelling. As discussed previously, Pollak (2007) promotes the use of ill-defined and non-directed situations that allow students to make decisions to create models themselves. Many studies suggest that modelling is best learnt with authentic open-ended problems (Caron, 2019). It is thought to be vital that students work with, and have their own experiences of creating, mathematical models of realistic situations

(Blomhøj, 2009; Kaiser & Sriraman, 2006). I have previously argued, in reporting findings from my PhD, that open-ended modelling projects provide students an opportunity to independently model, using techniques and procedures while allowing students to take ownership of their own modelling processes (Spooner, 2020; Spooner, 2022), saying:

> The inclusion in the course of the modelling project day creates an opportunity for students to have an autonomous experience of modelling. The anticipated student learning is that they gain an appreciation of the process and the need for techniques, development of group work skills and familiarisation with real-world modelling. (Spooner, 2020, p. 487).

Blomhøj and Kjeldsen (2013), Caron and Bélair (2007), Soon et al. (2011), Hanies, Crouch and Fitzharris (2003) all promote and support the use of open-ended modelling activities for learning to mathematically model.

Open-ended modelling activities can provide opportunities for students to work independently of the lecturer in small groups (Maaß, 2006). Furthermore, as I argued in Spooner (2021), students working in groups explaining, sharing, and discussing their ideas with each other, assisted each other in their discovery of the modelling process (Alfieri et al., 2011; Kaiser & Stender, 2013; Maaß, 2006).

There are ways lecturers can help students with open-ended modelling activities (Caron & Bélair, 2007; Bliss et al., 2019). Caron and Bélair (2007) recommended that students will be more successful with open-ended projects when the modelling process is explicitly taught and frequently alluded to throughout a course, along with spending time discussing the purpose of the model. Bliss et al. (2019) recommend that the modelling process is presented early in semester, well before the modelling project is undertaken. In addition, Bliss et al. (2019) propose that lecturers provide background contextual information for the situation to be modelled, the foundation model to base the new model on, assumptions to use while developing the model, examples of previous project reports and assessment criteria check list.

To provide guidance for students, support is given, either internally or externally, with open-ended mathematical activities. Support ranges from externally supplied heuristic strategies (Caron & Bélair, 2007), that is, strategies that enable a person to discover or learn something themselves (Radford, Bosanquet, Webster, & Blatchford, 2015), to consultations between lecturers and students (Blomhøj & Kjeldsen, 2013; Duan et al., 2020). As already mentioned, specific examples of heuristic strategies include: simplify as much as possible, use a simulation, draw and use diagrams, explain and discuss your thinking with someone else (Kaiser & Stender, 2013; Stender, 2019), thinking aloud (Ramachandran, Huang, Gartland, & Scassellati, 2018) and asking, have you seen a similar problem before? The support provides a scaffold for the activities, which is a structure or framework for students to refer to (Durandt, 2018) as they progress through the modelling process. When modelling projects are undertaken over a period of time Caron and Bélair (2007) and Blomhøj and Kjeldsen (2013) suggest a scaffolded approach through "guided instruction in the form of checkpoints (Caron & Bélair, 2007) or mandatory course time lecturer consultations (Blomhøj & Kjeldsen, 2013).

2.6.3.1.1 Advantages of open-ended modelling projects

One of the main advantages of open-ended modelling projects is to offer opportunities for students to experience the full process of mathematical modelling (Bliss et al., 2019). These projects give students opportunities for ownership and agency as they discover and develop their own modelling process (Bliss et al., 2019), while allowing students to experience modelling as a creative activity (Bliss et al., 2019). Though using open-ended projects can initially have students feeling out of their comfort zone (Bliss et al., 2019), they allow for students to experience modelling as a creative activity and give student ownership and agency for both using the modelling process and developing solutions (Bliss et al., 2019). In addition, open-ended modelling projects provide opportunities to develop students' modelling competencies.

2.6.3.1.2 Disadvantages of open-ended modelling projects

One potential disadvantage for open-ended modelling projects is "students often struggle in recognizing when and how to apply the mathematical analysis encountered in prior coursework to their particular design solutions" (Cole, Linsenmeier, Miller & Glucksberg, 2012, p. 25.1428.1). However, experiencing uncertainty may not necessarily be a negative thing. As discussed in Section 2.4.3, page 47, productive struggle can develop students mathematical understanding and enable creativity while mathematically modelling.

2.6.3.2 Case studies

Case studies are used in courses to provide students an experience of mathematical modelling (Duan et al., 2020; Fu & Xie, 2013; Heilio, 2011; Kapur, 1982). They can be used to allow students to become familiar with a model or families of models (Fu & Xie, 2013; Heilio, 2011) and as a foundation to further develop a model or create their own models from (Duan et al., 2020; Lyon & Magana, 2020). As previously discussed, Pollak (2007) and Burkhardt (2006) promote the use of examining already formed models as one activity students should be involved with when learning to model. Lyon and Magana (2020) recommend adapting and extending existing models to remove some of the difficulty associated with modelling, while still giving an opportunity to develop modelling skills. Duan et al. (2020) believe that students need to be actively involved with choosing the case study. Giving students choice promotes student ownership of the problem and hence more potential for active engagement (Duan et al., 2020). Duan et al. (2020) also recommend that students work with the instructor to simplify the case study problem to enable them to solve it. Case studies can be used to promote links between the real world and mathematics (Heilio, 2011), allow students to thoroughly understand the problem (Duan et al., 2020), give students access to similarities and differences between problems (Wedelin & Adawi, 2015) and show variety and complexity of models (Heilio, 2011).

Like guided open-ended modelling projects, the lecturer's role when using case studies is that of a facilitator. The lecturer acts as a guide, focussing the students in on the main issues of the problem and give suggestions as they progress through the process (Duan et al., 2020).

When case studies are used for mathematical modelling, students can be asked to make presentations of the models (Toews, 2012). This allows for students to practice communicating their work to a general audience while explaining the technical issues relevant to their case study (Toews, 2012). Toews (2012) asks the student audience to ask questions of their peers after presentations to promote audience engagement with the presentations. By seeing many talks, students are exposed to details of unfamiliar material (Toews, 2012) and see the diversity of different model families and types (Heilio, 2011). A study by Kaur (2009) involving secondary school students' perceptions of their mathematics teachers, revealed that students viewed teachers using "student work/group presentations to give feedback to individuals or the whole class" as a characteristic of a good lesson for students (Kaur, 2009, p. 345). The study did not go into detail as to why students perceived this as a feature of good teaching.

Case studies are one way of students getting to know and experience mathematical modelling. However, within the literature, little else is known about using case studies for learning mathematical modelling, with most of the literature focussed on using case studies for nursing education.

2.6.3.3 Popular culture

Developing models for popular culture contexts is another way students can experience mathematical modelling (Lofgren, 2016; Lofgren et al., 2016). Popular culture is a way in which students can relate mathematics to a context outside of mathematics (Greenwald & Nestler, 2004; Goff & Greenwald, 2007). With mathematical modelling being concerned with connecting mathematics to contexts, it makes sense that popular contexts are also used as a way for students to experience mathematical modelling (Lofgren, 2016; Lofgren et al., 2016). The most common use of popular contexts in mathematics is the area of infectious diseases, with Lofgren being the main author writing in this area. Lofgren (2016) promotes the use of zombie fiction and movies as a way for students to explore mathematical models for diseases and see how standard models can be adjusted and modified to fit the situation. Using zombie models gives students a fictional situation to model where they are potentially familiar with the context, thus removing any obstacles associated with context specific knowledge(Lofgren, 2016). A fictional situation encourages exploration due to removing the pressure of a correct pre-determined answer (Lofgren, 2016). Lofgren (2016) and Lofgren et al., (2016) find that using popular culture allows students to have an experience of modelling without getting too bogged down. The use of popular culture allows students to have an experience of modelling and, as such, fits into the realistic modelling perspective.

2.6.4 Lecturers' role in students' learning of mathematical modelling

It has been established the lecturer or teacher is an important variable in the learning of mathematical modelling (Blum and Ließ, 2007; Blum, 2011; Cole et al., 2011; Crouch & Haines, 2004; Durandt et al., 2020; Lipowosky, 2006; Niss, 2001; Soon et al., 2011). As early as 1997, Verschaffel et al. (2000) conducted a study which revealed that instructors are somewhat responsible for pupils' inabilities to activate relevant real-world knowledge when dealing with word problems for modelling real-life scenarios. As mentioned earlier, not enough is yet known regarding the impact of the teacher at secondary school (Kaiser, 2020) and the influence of the lecturer in tertiary courses (Jaworski, 2016). Durandt et al. (2021) showed that when two lecturers used the same teaching approach for modelling, students' learning gains were noticeably different, implying that the lecturer variable is important for tertiary modelling. From the work that has been done, in this section I will discuss in more detail what is currently known pertaining to the role of the lecturer for mathematical modelling, including lessons we can draw from the research into teaching modelling at secondary school.

Modelling is a complex activity, requiring lecturers and teachers to consciously think about the role they play in enabling students to have access to successfully developing modelling skills. Lecturers play a key role in designing the student experience, with different lecturers having different approaches to creating a mathematical modelling experience. The role of the lecturer includes: choosing and structuring activities to use (Carducci, 1996; Lyon & Magana, 2020); teaching approaches for developing students' mathematical modelling behaviours and modelling competencies (Blum, 2011, 2015; Lyon & Magana, 2020; Spooner, 2021) including exposing students to strategies that can be used (Legé, 2005a, 2005b; Soon et al., 2011; Spooner, 2013; Stillman et al., 2013); providing appropriate support while offering an independent student experience

(Blomhøj & Kjeldsen, 2006; Blum, 2011, 2015; Carducci, 1996) including acting as a facilitator and guide (Carducci, 1996); and removing barriers for students.

2.6.4.1 Choosing and structuring modelling activities to use

The modelling activities and modelling situations the lecturer and teacher choose to use is an important part in creating a learning experience. Ideally, though not necessary, these activities and situations should involve contexts students are familiar with (Carducci, 1996; Fu & Xie, 2013), involve a variety of examples and contexts (Blum, 2011, 2015) and have more than one solution path (Blum, 2011, 2015; Spooner, 2020).

A key stage in the mathematical modelling process is to familiarise and gain an understanding of the context of the situation to be modelled (Dym, 2004; Kadijevich, 2009; Tam, 2011; Treilibs et al., 1980). By providing students with contexts they are already familiar with potentially enables students to be able to focus on other aspects of the modelling process (Spooner, 2013), can facilitate motivation for engagement with the modelling (Kaiser et al., 2010; Treilibs et al., 1980) and allows students to bring their own understanding and knowledge of the problem to the situation (Spooner, 2013).

Real-life context can provide motivation (Kaiser et al., 2010; Treilibs et al., 1980) and reduce some of the barriers associated with modelling (Spooner, 2013). Even so, there is evidence that this is not the case for all students. There is research showing familiar contexts that are part of students' life experiences can have a positive effect on engagement (Kaiser & Schwarz, 2006). However, other research suggests that over-familiar contexts can have a negative effect (Boaler, 1993b; Busse, 2011). Spooner (2013) found that most students enjoyed the modelling experience best when they were familiar and confident with the context, though some students also enjoyed exploring the context of an unfamiliar modelling problem.

Exposing students to a range of modelling examples and using activities that have more than one way of being solved is an important part of the lecturer's role. According to Blum (2011, 2015), the use of a broad variety of examples and contexts is necessary to make transfer of situations and domains easier for students, while, using activities that have no predetermined solution pathway to follow allows students to experience the 'openness' of modelling (Blum, 2011, 2015; Spooner, 2020).

2.6.4.2 Teaching approaches for developing students' mathematical modelling behaviours and modelling competencies

Teaching approaches for developing student behaviours for mathematical modelling, including educating them about skills, techniques, strategies, process, and thinking, while advancing student competencies in modelling, are all an important part of both the lecturer and teacher's role (Blomhøj & Kjeldsen, 2013; Blum 2011, 2015; Kaiser, 2014; Lyon & Magana, 202; Spooner, 2020, 2021). One of the roles of the lecturer is to help students build a mathematical modelling toolkit. This includes: modelling techniques and tools (Spooner, 2020) and strategies (Blum & Leiß, 2007; Legé, 2005a, 2005b; Soon et al., 2011; Spooner, 2013; Stillman et al., 2013) and knowledge of the modelling process (Caron & Bélair, 2007; Stender & Kaiser, 2015; Wake, 2011). I will briefly discuss modelling techniques and tools, and knowledge of the modelling process before I move to discuss specific teaching strategies. Strategies for modelling that students acquire will be implied within Specific teaching strategies, Section 2.6.4.2.3, page 77 and discussed more fully in Student experiences while learning to mathematically model, Section 2.7, page 85.

2.6.4.2.1 Modelling techniques and tools

Modelling techniques and tools, as well as modelling behaviours and thinking, are needed for students to fully participate in a modelling experience (Wu, Wang & Duan, 2015). However, a modelling course should be more about the art of modelling rather than any specific mathematical technique (Kapur, 1982; Soon et al., 2011; Toews, 2012; Wu et al., 2015). Bliss et al. (2019) clearly communicated that courses at the undergraduate level should focus on the modelling process.

Modelling techniques, as described by the lecturer participant in my Spooner (2020) study, are mathematical techniques, methods and tools. The difference between mathematical techniques and modelling are mathematical techniques are how to get the equations and how to solve those equations. However, modelling is the use of those techniques for solving real-world problems and to feed into the modelling process. The application of techniques is at the heart of forming a model. Without techniques to draw on, students would flounder to create models. The lecturer goes on to say, specific tools mentioned were application of proportionality relationships, dimensional analysis, optimisation, rates of change, physical laws including Newton's second law and curve fitting. For students to have a full experience of mathematical modelling, a course should contain the context of the modelling process and how these modelling techniques feed into the process. In order for students to consider when and how to use modelling techniques and tools, students need to have knowledge of the modelling process.

2.6.4.2.2 Knowledge of the modelling process

Students acquiring knowledge of the modelling process is also an important skill. As already alluded to, the lecturer participant in Spooner's (2020) study expressed modelling strategies and "techniques on their own is not modelling" (Spooner, 2020, p. 484), saying modelling techniques need to be "fed into the modelling process" (Spooner, 2020, p. 484). Research from Caron and Bélair (2007) further supports this, promoting frequently alluding to the modelling process throughout a modelling course. Cole et al. (2011) found that in order to see an impact on students' use of modelling in their own projects, specific instruction in the mathematical modelling process was needed. Knowledge of the modelling process includes exposure to modelling behaviours (Lyon & Magana, 2020; Spooner, 2020) including promoting and accepting multiple solutions (Blum, 2011, 2015; Spooner, 2021). Cole et al. (2011) found that specific instruction in mathematical modelling was required before an impact was seen on the use of modelling in students' design projects.

Modelling is learnt by doing, including studying the modelling process (Blomhøj & Kjeldsen, 2006; Blum, 2011, 2015; Caron & Bélair, 2007; Stender & Kaiser, 2015; Wake, 2011). Many studies examine students following a given modelling process combined with scaffolding (for example: Blum, 2015; Schukajlow, Kolter, & Blum, 2015), while there appears to be a gap in the literature on studies into how students develop their own modelling process.

2.6.4.2.3 Specific teaching strategies

There have been different strategies trialled and researched for use by teachers for teaching modelling. The majority of these have been at the secondary level. I will present literature from both tertiary and secondary studies so that the lessons from secondary studies can be drawn on. The teaching strategies I will present include: discussion cycles; reflection on strategies; strategic interventions; simplifying the situation; and role modelling behaviours and strategies.

2.6.4.2.3.1 Discussion cycles

Discussion cycles are recommended for use both with secondary and tertiary education. Discussion cycles have been used successfully in secondary classes, as demonstrated by the work of Legé (2005b) and Spooner (2013). Legé (2005b) called the approach "modelling discussion" "in which groups (or the entire class) are engaged in a conversation about what they know, what they might need to know and how they might proceed forward from where they currently are" (Legé, 2005b, pp. 31-32). Facilitator questions are used to stimulate conversation and engage the students in the modelling activity. I took a similar approach with my work with secondary students. See Spooner (2013) for more details. Soon et al. (2011) discuss using a scaffolded discussion cycle approach based on "modelling discussions", as proposed by Legé (2005b) as a way to build tertiary students "confidence to tackle complicated problems" (Soon et al., 2011, p. 1028). However, it could be debated that such an approach could lower the cognitive demand for students too much. Soon et al. (2011) remind us of being conscious of not removing too much of the struggle for students as they work through a modelling experience.

2.6.4.2.3.2 Reflection on strategies

Research suggests that students taking the time to reflect and think through how they will approach modelling a situation is beneficial. Soon et al. (2011) and Stillman et al. (2013) both concluded there was a need to train students to explore and reflect on modelling guidelines given and to train students to resist jumping straight in. Taking the time to explore and reflect could have the potential to enrich students' experience of the modelling process and help them to own and develop their own internal systematic approaches to modelling. At present, there is no firm research evidence to support this. Spooner (2013) used a set of reflective questions that proved successful in enabling secondary school students to identify the essential aspects of the situation. This demonstrated that the use of reflective questioning could potentially be effective in enabling students to work through stages of the mathematical modelling process. This needs to be investigated further to incorporate the formulation stage of the modelling process and to determine what impact the use of guidelines has on students' development of future modelling approaches.

2.6.4.2.3.3 Strategic interventions

As discussed earlier under Teaching mathematical modelling, Section 2.6.1, page 56, investigations into teacher interventions and their impact on secondary school students have been conducted (Blum, 2011; Blum & Leib, 2007; Kaiser & Stender, 2013). In both the DISUM studies (Blum, 2011) and work by Kaiser and Stender (2013), the role of the teacher is to provide interventions where appropriate during a modelling task. These interventions are verbal prompts given at appropriate times by the teacher. Teacher interventions can be viewed as a form of scaffolding. Examples of interventions used are "imagine the real situation clearly, make a sketch of what is still missing, does this result fit to the real situation" (Blum, 2011, p. 23). The DISUM studies show that strategic interventions are useful in a modelling context. Kaiser & Stender (2013) identified asking students to show the state of their work as a powerful intervention. In a tertiary course, strategic interventions could be given in a lecture as a list of prompts and/or a checklist for students to use independently. However, research is needed in this area.

2.6.4.2.3.4 Simplifying the situation

Literature on lecturers using simplifying the situation as a teaching strategy is limited. However, Lyon and Magana (2020) recommend that the instructor role models simplifications of the situation. Firstly, to increase the potential for students to have access to a modelling experience. Secondly, using simplifications role models the behaviour of professional engineers and modellers. Engineers must make simplifying assumptions even in the professional world (Gainsburg, 2006). These assumptions are based on their background knowledge and experience (Gainsburg, 2006).

2.6.4.2.3.5 Role modelling behaviours and strategies

Mathematical modelling involves working through an undetermined solution path. The DISUM studies noticed that if there was an absence of stimulation of strategies by the teacher while teaching modelling, there generally was also an absence of student strategies in approaching the modelling task (Blum & Leiß, 2007). This highlights the importance of providing and promoting the use of strategies by the teacher/lecturer. As I argue in Spooner (2021), there is a need for the lecturer to expose the modelling behaviours of uncertainty and unfamiliarity and provide additional strategies for dealing with these in lectures and tutorials. This would involve a change of culture not only for students but also for lecturers, that is, moving from a culture of the lecturer presenting pre-meditated solutions to one of rolemodelling strategies for dealing with uncertainty. Strategies could include exploration, heuristic strategies, and not producing a perfect model to fit the situation. It is interesting to note that Kapur (1982) suggested it can be more exciting for students if a teacher creates a model unfamiliar to him/her, though there is a risk of wasting time if he/she does.

To my knowledge, there appears to be no research into the effect of the teacher or lecturer formally role modelling formulating a model as a teaching strategy, at either secondary and tertiary levels. Nor does there appear to be research into formal teachings of strategies for formulating a mathematical model through role modelling these by the teacher. Research at the secondary level appears to be focussed on specific strategies to help the process. When formal teaching is being conducted on strategies used by educators, no research appears to have been done on how students are assimilating this teaching, that is, new strategies and tools together with their already existing ones. What is clear is that strategies for modelling need further investigation to firmly establish their benefits and situations in which they will be most useful.

2.6.4.3 Providing appropriate support for an independent modelling experience

It appears important that teachers and lecturers provide support to students while offering an independent experience (Blomhøj & Kjeldsen, 2006; Blum, 2011, 2015;

Carducci, 1996). As I argue in Spooner (2020), whatever form support takes, the support has the ability to support individual student modelling routes. The lecturer and teacher's role in providing support includes setting up appropriate scaffolding that allows students to experience modelling behaviours (Blomhøj & Kjeldsen, 2006; Blum, 2011, 2015; Carducci ,1996), acting as a facilitator as opposed to an instructor (Carducci, 1996) and providing suitable feedback (Carducci, 1996; Lyon & Magana, 2020).

As already mentioned, it is currently the consensus for there to be a balance between instructor guidance and student autonomy for modelling activities (Kaiser, 2020). This places the role of the lecturer and teacher as one of a facilitator for activities (Carducci, 1996). According to Blum (2011, 2015) a teacher needs to be able to give appropriate support, balancing this between minimal teacher guidance and maximum student independence. Carducci (1996) says it is important to keep students on track with the biggest challenge for the lecturer being "to keep quiet [and] let the students develop their own models" (Carducci, 1996, p. 259). Literature promotes providing an experience where students have a part, if not full, independent experience of mathematical modelling (Burghes, 1984; Carducci, 1996; Spooner, 2020). The key appears to be in lecturers providing enough support such that students are able to make progress, without removing genuine student involvement in the activity.

As already discussed in Section 2.6.1, page 56, scaffolding the experience is one of the common ways teachers and lecturers enable and provide student support for modelling (Blomhøj & Kjeldsen, 2006; Blum, 2011, 2015). With the use of scaffolding to promote an independent modelling experience, the lecturer's role moves from one of lecturing to facilitator and guide, and includes time management for long-term projects (Carducci, 1996). Feedback is part of the facilitator role with Carducci (1996) and Lyon and

Magana (2020) both emphasising its importance. Findings from Lyon and Magana (2020) found that as modelling activities are occurring, feedback by a lecturer to student needs to be both positive and needed. Positive feedback was utilised better than negative feedback (Lyon & Magana, 2020).

2.6.4.4 Removing barriers for students

Part of teaching students mathematical modelling behaviours is making the modelling experience accessible for students. Generally, this involves the lecturer being conscious of barriers for students and ways these can be removed. Barriers identified for students are: inability to see the links between the mathematics they learn and the real world (Blum, 2011; Klymchuk et al., 2010; Soon et al., 2011); lack of strategies (Blum, 2011; Blum & Leiß, 2007; Maul & Berry, 2001; Soon et al., 2011; Stillman, 2004; Stillman et al., 2013); challenges with the context of the situation (Blum & Leiß, 2007; Kadijevich, 2008; Klymchuk et al., 2010; Stillman, 2004); translating from the real world to mathematics (Blum, 2011; Chaachoua & Saglam, 2006; Clement, Lochhead, & Monk, 1981; Crouch & Haines, 2004; Domínguez, Garza & Zavala, 2015; Ikeda & Stephens, 2001; Kadijevich, 2008; Klymchuk et al., 2010; Klymchuk & Zverkova, 2001; Palm, 2007; Rowland & Jovanoski, 2004; Rowland, 2006; Soon et al., 2011; Stillman, 2004); specific difficulties with the formulation stage (Blum, 2011; Klymchuk et al., 2010; Soon et al., 2011; Spooner, 2013; Stillman, 2004; Stillman et al., 2013); and difficulties with metacognitive modelling strategies (Maull & Berry, 2001; Soon et al., 2011; Stillman et al., 2013). As already mentioned, in order to give students access to a modelling experience, Lyon and Magana (2020) suggest "simplifications may be necessary by the instructor" and "providing additional background knowledge to students, if content is too difficult" (Lyon & Magana, 2020, p.113). More detail regarding these barriers will be discussed in Student experiences while learning to mathematically model, Section 2.7, page 85.

Even though these barriers to modelling are evident "little research has been done on understanding these difficulties" (Soon et al, 2011, p. 1025). A literature search in 2016 confirms there was still little research into explaining student obstacles. In 2023, another literature search revealed a small number of studies on understanding aspects of student difficulties with the modelling cycle (see Jankvist and Niss (2020), Hankeln (2020) and Zulkarnaen (2021)). However, no studies were located at the tertiary level. With studies understanding student difficulties only recently being undertaken, it would follow that even less research has been done on teaching methods that address these difficulties. While there have been suggestions for effective teaching methods more research is needed in this area, particularly at the tertiary level.

What is known is that studies revealing students' lack of modelling strategies (Blum, 2011; Blum & Leiß, 2007; Maull & Berry, 2001; Soon et al., 2011; Stillman, 2004; Stillman et al., 2010) highlight the need to train students to explore, develop and reflect on strategies and guidelines for formulating models. Taking the time to develop and reflect on a plan of attack appears to be an important part of successfully starting the process of mathematical modelling (Maull & Berry, 2001; Soon et al., 2011; Stillman et al., 2013).

The large number of studies identifying student difficulties with formulating models implies that appropriate guidance and guidelines are needed in this area. With the use of the right teaching aids some of these difficulties maybe removed. This brings to mind the question, what are the appropriate teaching aids to provide to students to progress with the model formulation, without removing the productive challenge associated with learning new skills.

Stillman et al.'s (2013) study highlighted the need to train students in metacognitive modelling competencies. In particular, to reflect on strategies and guidelines given, resisting the urge to rush in, and to be open to review and revise their approach taken. The Soon et al. (2011) and Maull and Berry (2001) tertiary studies also supported the findings of Stillman et al. (2013). Similairly, Jankvist and Niss (2020) results show, in a Danish context, students have difficulty anticipating and implementing anticipated actions needed to move through the pre-mathematisation of a modelling task, supporting the need for development of metacognitive strategies to enable students to begin modelling.

There has been some work conducted to try and identify teacher competencies at secondary school level (Blum, 2011, 2015). These mainly focus on the teacher as the guide for a student-centred approach to modelling and include teacher knowledge of interventions and ability to diagnose student difficulties and mistakes. At tertiary level, the lecturer occupies a more instructive position, guiding from the front with limited audience interaction. How do these competencies identified for secondary school teachers transfer to the tertiary lecturer? Is a student-centred approach possible at tertiary level? What would a list of lecturer competencies for mathematical modelling look like? Would they be different or similar to a list for secondary school teachers? How much does a mathematical modelling lecturer know about student difficulties? How is this knowledge incorporated into lectures? These questions seem to be unanswered at present.

In conclusion, given the centrality of the teacher and lecturer to mathematical modelling it would make sense to investigate teaching methods that are effective for modelling. There is research into identifying student difficulties when formulating a model but limited research into teaching approaches that remove specific difficulties, particularly at tertiary level. There is a lack of research focussed on assisting students to choose or create appropriate formulae and teaching strategies to enable students to produce links between known relationships and equations to produce a model. It has been suggested that scaffolding approaches lead by the educator i.e. discussion cycles and interventions can be beneficial as can training students in metacognitive activities such as process reflection. These recommended teaching strategies (discussion cycles, teacher interventions, reflections) are generic in nature and do not focus in on specific difficulties within the modelling cycle.

2.7 Student experiences while learning to mathematically model[8]

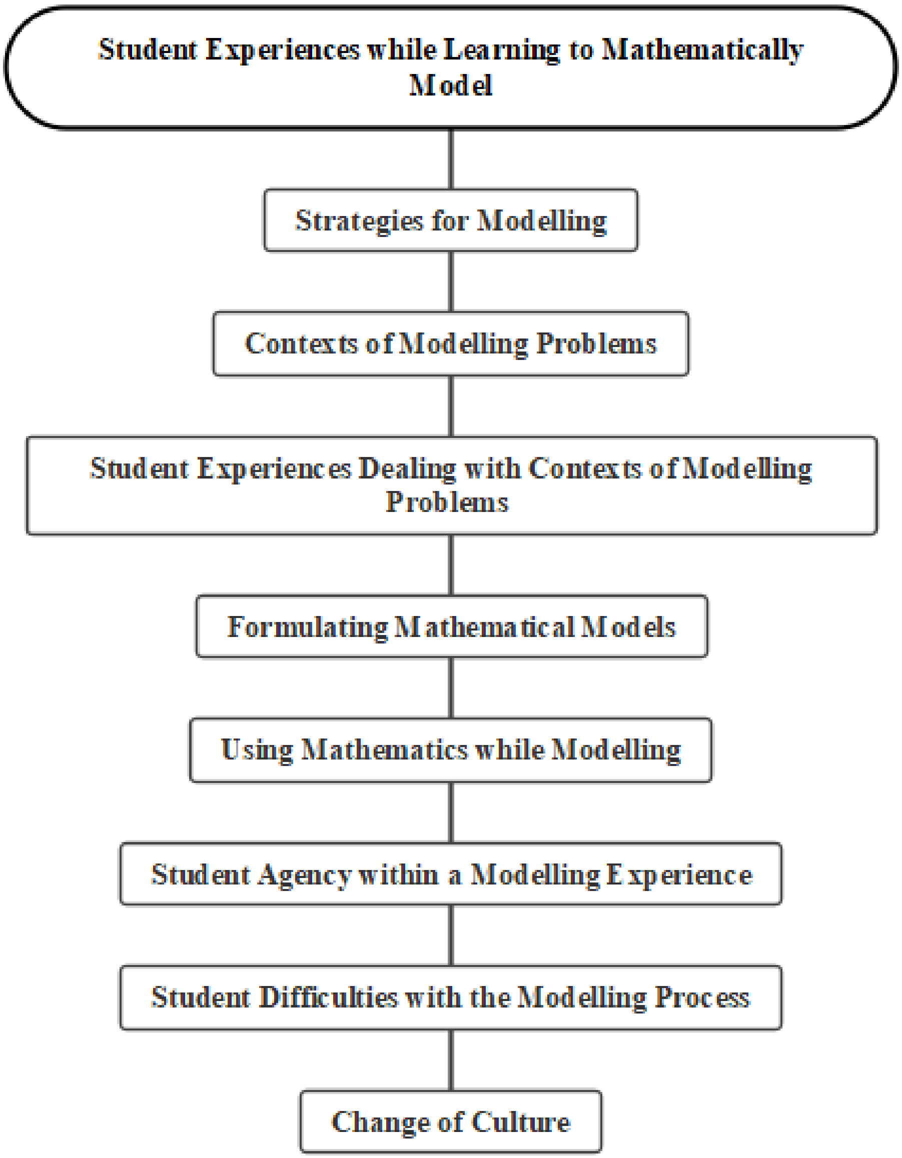


Figure 11. Students' experiences while learning to mathematically model

[8] This section includes re-presentation of material published in Spooner (2021).

It is not surprising that research shows that students do not automatically know how to model (Cole et al., 2010; Duan et al., 2020). Studies show that both instruction and allowing students to participate in modelling is important for developing students modelling competencies (Cole et al., 2010; de Almeida, 2018; Wake, 2011). It has been clearly established that a balance between both instruction and student independent participation in modelling is needed (Kaiser, 2020). De Almeida's (2018) study revealed that students "made significant progress by means of a balance between teacher guidance and students' independence" (de Almedia, 2018, p. 19). Neither activity alone will generate a truly successful experience for students. Discussed earlier is the literature on instruction focussed on the lecturer's role in students learning to mathematically model. Here I will discuss the current literature on students' experiences learning to mathematically model. Research has demonstrated that students have difficulties with modelling, in particular developing a mathematical model (Chaachoua & Saglam, 2006; Crouch & Haines, 2004; Klymchuk & Zverkova, 2001; Klymchuk et al., 2010; Palm, 2007; Soon et al., 2011; Stillman, 2004). I will discuss literature on both the difficulties and the successful experiences in regard to: strategies students use when modelling including metacognitive strategies; student experiences with contexts of modelling problems, the formulation stage of modelling; using mathematics while modelling; the role student agency plays in a student modelling experience; student difficulties with the modelling process; and change of culture students experience. I will present literature from tertiary studies, as well as draw from research done at the secondary school level.

2.7.1 Strategies for modelling

Literature on student experiences with mathematical modelling shows that in general students initially display a lack of modelling strategies. Primarily students simply do not know where to start (Soon et al., 2011). Students lack strategies for formulating models

(Blum, 2011; Soon et al., 2011), have difficulties of organisation and generally do not know how to go about forming a plan of attack to build the model (Stillman, 2004). Observations from the DISUM studies (Blum, 2011; Blum & Leiß, 2007) show that students do not consciously use strategies for formulating models. Stillman's (2004) study on strategies employed by upper secondary school students revealed organisational difficulties formulating a plan of attack. A study by Soon et al. (2011) showed that first-year tertiary students did not easily adopt a system or methodology for formulating a model even when given guidelines. However, in Spooner (2021) I argued that students used planning as a strategy for moving forward. These studies highlight the need to train students to explore, develop and reflect on strategies and guidelines for formulating models. Taking the time to develop and reflect on a plan of attack appears to be an important part of successfully starting the process of mathematical modelling (Maull & Berry, 2001; Soon et al., 2011; Stillman et al., 2013).

Despite the lack of strategies students use for modelling, it seems the main strategy students use when modelling is simplification. Results of Lyon and Magana's (2020) systematic literature review showed "students typically use strategies that involve simplifications of the problem statement and forthcoming solution" (Lyon & Magana, 2020, p. 112). As I argue in Spooner (2021), students' discovery of the need to 'keep it simple' in order to make progress is considered a major turning point in students discovering their own modelling procress. Students' use of simplification as a strategy is consistent with behaviours of professional mathematical modellers (Lyon & Magana, 2020; Wake, 2011). However, simplifications need to be thoughtfully made in order to avoid pitfalls later in the modelling process. Throughout the modelling process, even engineers can find it difficult to decide what simplifying assumptions they should make and the anticipated later effects of their decided simplifications (Gainsburg, 2006).

2.7.1.1 Metacognitive modelling strategies

Metacognitive strategies, skills, and competencies are needed in addition to general strategies for modelling. Metacognitive modelling competencies overreach the modelling cycle (Schapp, Vos & Goehardt, 2011) and are necessary throughout the whole mathematical modelling process (Blum, 2011). Metacognitive modelling competencies include, but are not limited to, strategies for approaching modelling, student reflection on their modelling process, knowledge of task, and student self-regulation (Blum, 2011; Maaß, 2006; Schaap et al., 2011).

The study by Stillman et al. (2010) demonstrates both the importance and potential deficiencies of metacognitive mathematical modelling competencies in students. The study identified potential metacognitive mathematical modelling deficiencies of year nine students carrying out a modelling task. The study involved twenty-one students (14-15 year olds). It identified two different types of blockages, from a teaching perspective, which occurred during the formation stage of the modelling cycle. Blockages that were caused by “lack of reflection, or incorrect or incomplete knowledge” and “those involving the need to revise schemas” (Stillman et al., 2013, p. 395).

One group made errors in the formulation process because they rushed into the problem and did not take the time to reflect on it first. Even though there was scaffolding provided, in the form of reflective questions, they felt confident and rushed into mathematising the situation and consequently made mistakes. The formulation errors become apparent to the students during the critiquing stage of the modelling cycle. That is when the results produced by their developed model were not what they expected.

Another group in the study had difficulty mathematising the situation despite using the reflective questions. This was due to a lack of critical thinking and flexibility in thinking, manifesting itself as the absence of questioning the appropriateness of the concept they were applying to form the model. Stillman et al. (2013) stated:

> These students persisted in attempting to assimilate, rather than accommodate, (Piaget, 1950) new contradictory information to their chosen structure, and resisted the necessity to consider the structure itself as deficient – the students engaged in cognitive dissonance (Atherton, 2003; Festinger, 1957) which prevented them from activating procedures to unblock their progress. (p. 395).

The students did not change direction with their model even though contradictory information presented itself. They wanted the information to fit into their thinking, not to change their thinking to accommodate the presenting information.

The Soon et al. (2011) and Maull and Berry (2001) tertiary studies also supported the findings of Stillman et al. (2013). The Soon et al. (2011) study showed that students had limited patience and wanted to jump straight in. The students in the Maull and Berry (2001) study jumped straight into trying to formulate the model before looking for any simplifications and assumptions that could be made, thus complicating the process.

The Maull and Berry (2001), Soon et al. (2011) and Stillman et al. (2013) studies highlighted the need to train students in metacognitive modelling competencies. In particular, to reflect on strategies and guidelines given, resisting the urge to rush in and to be open to review and revise their approach taken. Work has begun on exploring the possibilities for developing tools to measure metacognitive modelling competencies (Vorhölter, 2017), reinforcing their importance for modelling.

2.7.2 Contexts of modelling problems

The context a modelling problem is situated in, is an integral part of a mathematical model. I will now present of research on: student experiences understanding the context of the situation; the use of real-world contexts in modelling to motivate students; and how contextual problems can influence students' mathematical understanding.

2.7.2.1 Understanding the context of the situation

The first stage of carrying out a plan for formulating a model is to understand the context of the situation to be modelled, making student understanding of the context of the problem situation a necessary stage of the process of modelling (de Almedia, 2018). Carducci (1996) found that when students could relate to the context of the problem, in general they were able to find ways to start to approach the problem. Whereas, when students had no experience with the context they were lost as to where to begin. Carducci (1996) suggests directing students to resources for the context of the situation that still require students to do the research themselves, as one approach to addressing the need for students to be familiar with the context of the problem.

Studies have shown that students can have difficulties understanding the context of situations, including the contextual language used to describe a situation (Blum & Leiß, 2007; Kadijevich, 2008; Klymchuk et al., 2010; Stillman, 2004). Kadijevich (2008) found that when year one tertiary student modellers were not familiar with the context, they made inappropriate assumptions concerning some of the important variables in the situation. Stillman's (2004) study observed that secondary students can ignore the context entirely, not taking into account any real-world knowledge. Klymchuk et al. (2010) found that just under half of the tertiary students in their study did not understand the wording of the problem. These challenges, when present, inhibit students from starting a modelling activity (Kadijevich. 2008).

2.7.2.2 Real-world context to motivate students

Using context to demonstrate mathematics as relevant is often an argument given for providing motivation for students to engage with relationships between contexts and mathematics. However, the connections between learning about the importance of mathematics and motivation is not simple. Loch and Lamborn (2016) found that first-year undergraduate engineering students either engaged with the use of mathematics in engineering or were frightened by its complexity. Similarly, Kember, Ho and Hong (2008) and Black et al. (2010) showed that while context showing relevance of mathematics could motivate students, it could also demotivate students if they did not view the context as relevant to themselves. Demonstrating relevance can be counterproductive to motivation if students are not confident with the level of mathematics or do not see the context or mathematics as relevant to them.

2.7.2.3 Contextual problems impact on students mathematical understanding

Research suggests that the use of contextual problems can have an impact on students' mathematical understanding. Sijmkens, Scheerlinck, De Cook and Deprez (2022) carried out a study using contexts for learning differential equations. Interestingly, they found that using contextual problems had a significant effect on students conceptual understanding and procedural knowledge of differential equations for the experimental group of first-year engineering students. Further testing revealed that using contexts removed difficulties students were known to experience with learning differential equations. Sijmkens et al. (2022) remarked:

> The conclusion that the procedural knowledge of the students in the experimental group also improved significantly is against our expectations. Arslan (2010) concluded that an improved conceptual

knowledge of DEs also led to equal knowledge of procedures, but not vice versa. (p. 12).

Even though more research may be needed, there is an implicit implication that this improvement in both procedural and conceptual knowledge may be the result of students using contexts to relate the mathematics and contexts to each other. As this study is only in mathematics, even though it is key to understand how differential equations connect with modelling, it may also imply how when students start working with contexts and mathematics, they may need to start off simply, avoiding messy and complicated modelling.

Studies by Nguyen and Rebello (2011) and Bollen, Van Kampen and De Cock (2015) suggest students have difficulties seeing connections between using mathematics and solving contextual problems. In both studies students, did not use any of the calculus taught while solving physics problems. Bollen et al. (2015) concluded traditional physics and calculus instruction alone is not enough for students to naturally apply the mathematics in a contextually based subject like physics. These difficulties may be caused by students' lack of experience applying calculus to word problems, particularly within physics (Quimson, 2021). “This is also perceived by the students and was revealed in other studies (Cui, 2006; Nguyen et al., 2011)” (Quimson, 2021, p. 45).

2.7.3 Formulating mathematical models

Once the context is understood and the problem to be modelled clearly defined, the next stage is to formulate the model. Formulating the model can be referred to as ‘mathematising the situation’ and involves seeing links between contexts and mathematics, recognising essential aspects of the situation to be modelled and mathematising the situation. It is thought that “the creation of the mathematical model” is “perhaps the most difficult stage of modelling” (Soon et al., 2011, p. 1028). This is true for both lecturers and students (Oke & Bajpai, 1982). Oke and Bajpai (1982) said undergraduate and postgraduate students both initially find formulating models difficult, with little difference in their ability between the two groups.

Studies have shown students have difficulties in connecting real-life contexts with mathematics (Blum, 2011; Crouch & Haines, 2004; Galbraith & Stillman, 2006; Ikeda & Stephens, 2001; Kadijevich, 2008; Klymchuk & Zverkova, 2001; Soon et al., 2011). Many do not know how to convert between words and mathematics (Clement, Lochhead & Monk, 1981; Klymchuk et al., 2010; Soon et al., 2011). Blum (2011), Klymchuk et al. (2010) and Soon et al. (2011**)** found one of the main difficulties students encounter is the "inability to see the links between the mathematics" they "learn and the real world" (Soon et al., 2011, p. 1024). In addition, studies conducted by Boaler (2001), Crouch and Haines (2004), and Lesh and Harel (2003) showed that students taught traditionally, that is, not using mathematical modelling as the main teaching strategy, could not see the links between mathematics and the real world.

Lesh and Harel (2003), de Almedia (2018), Cole et al. (2010) and Oke and Bajpai (1982) identified aspects of modelling that may help students make connections between the context and appropriate mathematics. Lesh and Harel (2003) stated it was the iterative aspect of the modelling process that enables students to realise connections between physical and mathematical representations. This format allowed students to comprehend sophisticated mathematical constructs much better than if they had been subjected to traditional lectures or textbooks. De Almedia (2018) says connecting mathematics to reality requires students to be able to make "judgements about what tools to use" (de Almeida, 2018, p. 28). Not only must the mathematics make sense mathematically, it must also adequately reflect the real-world situation it represents (Carrejo & Marshall, 2007). This requires students to be versatile in their use of mathematics (de Almeida, 2018). However, due to the traditional approach to teaching mathematics in classrooms, research has found that students find these aspects of mathematical modelling very challenging (Lammi & Denson, 2017).

In conjunction with understanding the context of the situation, students need "to understand the process of getting from the real situation to the mathematics" (de Almeida, 2018, p. 28). Cole et al.'s (2010) work suggested that drawing a sketch of the situation may help students identify key aspects of the situation, breaking the problem down into its basic elements. While Oke and Bajpai (1982) say relationships between variables can often be depicted using sketch graphs. Cole et al.'s (2010) work showed that students who used sketches were "more likely to generate mathematical relationships" (Cole et al., 2010, p. 15) and viewed identifying the problem's basic elements as important for their progress. In addition, de Almeida (2018) showed students needed to idealise the situation and make assumptions as part of their process for model creation.

As I argue in an earlier publication, students learning how to recognise when a model foundation would be useful is key to being able to develop a useful model (Spooner, 2021). Understanding the context is seen as helpful for identifying the important variables and possible relationships (Oke & Bajpai,1982) and hence recognising useful model foundations. As mentioned earlier, sketching graphs can be useful in identifying relationships between variables (Oke & Bajpai, 1982). It has been suggested that students gifted in physics may intuitively find it easier to identify key variables for the model, though more evidence is needed for this (Oke & Bajpai, 1982). In the study by Soon et al 2011, it was thought that student difficulties for formulating models were attributed to students relying on memory work instead of applying new knowledge to the situation. This suggests that while memory work may be an appropriate strategy for routine problems, a more creative approach is needed. According to Oke and Bajpai (1982), the formulation stage of modelling appears to be the most creative phase of modelling.

One of the central aspects for translating between real-life contexts and mathematics is the ability to be able to see links between physical processes, physics and mathematics. These links between physical process, already known formula, physics and mathematics are key to model formation (Soon et al, 2011). Studies involving tertiary students have demonstrated that students do not tend to link between physical processes, physics and mathematics (Chaachoua & Saglam, 2006; Domínguez et al., 2015; Rowland & Jovanoski, 2004; Rowland, 2006). Chaachoua and Saglam (2006) carried out a study investigating linking physics and mathematics**.** The study showed that first-year university students have weak links between mathematics and physical situations. Two questions were asked. The first question required mathematical procedural skills to be applied to a routine mathematics problem. The second question, to interpret the solutions from the first question in a real situation. Forty-six students attempted the first question and out of these, only eight attempted the second question. Of the eight who attempted the second question, none went through the process of moving from a real situation to a mathematical model. The study showed that students carry out mathematical operations in isolation from applications. The absence of relations between mathematics and other situations within other disciplines, namely physics in this instance, shows that students are not developing transverse competencies. These skills are essential to model formation (Soon et al., 2011).

In order for students to become independent modellers, we need them to engage in many processes, but mathematising is the most crucial one (Grigoraş, Gracia & Halverscheid, 2011). Several researchers have focussed their attention on the mathematisation stage of mathematical modelling by students (de Almeida, 2018; Gellert & Jablonka, 2007; Niss, 2010; Stillman et al., 2015; Schaap et al., 2011). As defined by Gellert and Jablonka (2007), mathematisation is the process which renders something "more mathematical than it was before" (de Almeida, 2018, p. 21). For mathematical modelling, this interprets as moving from the real world to mathematics. In order to achieve this, mathematical knowledge and extra mathematical knowledge is needed (de Almeida, 2018). What is missing from this discussion is that contextual knowledge is also needed to enable this process to occur.

A study by de Almeida (2018) with first year tertiary students with no experience of modelling found that the first model a group of students develop is limited. After some practice, the majority of students appear to be able to identify variables relevant to the situation to be modelled. The main difficulty students have is deciding which of these variables are key variables and figuring out how they are related (Cole et al., 2011; Oke & Bajpai,1982). Similarly, Soon et al.'s (2011) study, involving novice first-year engineering students, showed they had difficulty identifying variables and equations needed to form models.

First attempts at modelling are never normally straightforward and successful (Cole et al., 2010; Spooner, 2021). A known modelling behaviour of practising modellers is that their first attempt at developing a model will not be their last (Drakes, 2012). Cole et al. (2010) found that students were more likely to propose non-mathematical approaches in their first attempt at developing a model. In an earlier publication, I observed students discovered that initial work that did not necessarily produce a viable model was not wasted (Spooner, 2021). All students in the study experienced and learnt to expect an unrealistic model first time around and observed that there was no one ideal model for the situation. These experiences reflect the nuances and complexities of modelling (Cole et al., 2010). Soon et al. (2011), Rowland and Jovanoski (2004), Rowland (2006), and Boaler (2001) all suggest one of the main reasons students have difficulties is there is a general lack of previous experience and exposure to non-routine problems.

As students have repeated experiences with modelling their knowledge of the modelling process increases. Cole et al. (2010) found that there was an increase, even though small, "in the number of students proposing a mathematical approach to their scenario on the second iteration" (Cole et al., 2010, p. 15.812.15). Soon et al. (2011) proposed that it was students' lack of experience with mathematical modelling that meant students had difficulties with formulating models and hence the more opportunities the students get to model the better they should become. However, Cole et al. (2011) found that it was not until students had had direct instruction in mathematical modelling, that is going "step by step through modelling in a particular scenario" (Cole et al., 2011, p. 17), did they see an improvement in modelling from students independently modelling in groups, saying that "explicit teaching about modeling had considerable impact" (Cole et al., 2011, p. 17). De Almedia (2018) concluded that this knowledge of the modelling process, along with mathematical knowledge is key for successful 'mathematisation' for modelling.

In summary, more research is needed into strategies for modelling, solutions for student challenges understanding context and understanding the complexities of mathematising the situation to help students start and navigate the process of mathematical modelling.

2.7.3.1 Specific difficulties with the formulation stage

Some specific difficulties identified within the formulation stage include, but not limited to:

- difficulty recalling appropriate formulae and procedures (Klymchuk et al., 2010; Stillman, 2004).
- difficulty choosing or creating appropriate formulas (Oke & Bajpai, 1982; Soon et al., 2011; Stillman et al., 2013).
- inability assigning correct units to variables (Soon et al., 2011; Stillman et al., 2013).
- difficulties making assumptions (Blum, 2011).
- being able to identify underlying relationships between variables but not being able to produce an equation that links them (Alona Ben-Tal & Robert Mckinnon, conversation Massey University, 2011; Oke & Bajpai,1982; Soon et al., 2011; Spooner, 2013).
- difficulties using formula (Klymchuk et al., 2010).
- difficulty identifying and determining variables, parameters, equations and procedures needed for the model (Cole et al., 2010; Cole et al., 2011; Soon et al., 2011).

Examples of difficulties include difficulties for both secondary and tertiary students. For example, students in the Soon et al. (2011) study, involving first-year engineering students, had difficulty identifying equations and variables needed to form models. As already mentioned, it was thought that these students were relying on memory work, whereas a more creative approach was needed. Students in the Stillman et al. (2013) study, involving fourteen to fifteen year olds, had trouble in the formulation stage, shown by assigning incorrect units to the formulae. Other studies, at both tertiary and secondary levels, have shown that students in general have trouble identifying and recalling formulas and procedures that may be relevant to the situation (Klymchuk et al., 2010; Stillman, 2004).

There is some evidence that when students can identify potential relationships and equations between variables, they can not necessarily produce an equation that links them (Soon et al., 2011). It is apparent from Soon et al.'s (2011) first-year tertiary study that when important variables had been identified, some students were unable to form equations relating the important quantities. They are unclear about how to formulate the types of applicable relationships. Students find forming an equation hard (Soon et al, 2011). The work carried out at secondary school by Spooner (2013) also supports Soon et al.'s (2011) findings. In Spooner (2013), secondary school students were given an activity that took them through the process of mathematical modelling. With appropriate guidance, students were able to identify the important aspects of the situation but had difficulty taking these aspects and forming a model. This could have been due to student inexperience in this area or lack of teacher knowledge with regard to effectively guiding the formulation stage of the process.

It is interesting to note that lecturers of a third-year mathematical modelling course have communicated the opposite findings. In the experience of Alona Ben-Tal and Robert Mckinnon, (conversation Massey University, 2011) they observed that if tertiary students are given the essential aspects of the model, they are generally able to form a model. They thought that one of the hardest stages of the modelling process at tertiary level, and the one done least well, is identifying the essential aspects of the situation (Alona Ben-Tal & Robert Mckinnon, conversation Massey University, 2011).

There are several difficulties students have with identifying essential aspects of the situation. The study by Klymchuk at al. (2010) supports Alona Ben-Tal and Robert Mckinnon's (conversation Massey University, 2011) assertion that tertiary students have trouble identifying the important aspects of a situation. Klymchuk at al. (2010) showed that when students were presented with a calculus application problem, where the answer was given and they had to show how to obtain the answer, students found it difficult to identify the unknown variables relevant to the problem. The most common statement from the interviews of this study was "I was confused because the result was given" (Klymchuk et al., 2010, p. 85). Similarly, Soon et al. (2011) and Kadijevich (2008), both tertiary studies, showed that many students had difficulty classifying variables. That is, identifying known and unknown variables, otherwise known as identifying the important aspects of the situation. The DISUM studies (Blum & Leiß, 2007) showed that secondary school students also had difficulty identifying the important aspects of the situation. An investigation into the difference in outcomes between the above studies and Spooner (2013), where secondary students were able to identify the essential situational aspects, may illuminate the factors that influence student thinking processes. Did the structure of the activity or the specific teaching approach influence the outcome of this case? What was different in Spooner (2013) compared to the experience of Alona Ben-Tal and Robert Mckinnon (2011), Klymchuk et al., (2010) and the approach taken in DISUM studies? What variations in the practice of Alona Ben-Tal and Robert Mckinnon (2011) and Spooner (2013) made the formulating of the variables into a model successful at tertiary, in the case of Alona Ben-Tal and Robert Mckinnon (2011), and challenging at secondary in the case of Spooner (2013)? In Spooner's (2013) work, even though she was working with younger students, a set of reflective questions was developed to identify and classify the variables involved in a real situation. This set of guidelines was successful for secondary school students.

It could be inferred from the results above that with the use of the right teaching aids, identifying and classifying the essential aspects of a situation is also possible both at the secondary and tertiary level. However, the evidence suggests that further strategies should continue to be developed. Identifying variables in tertiary appears to be a recurring difficulty and was common across problems used in research (Alona Ben-Tal & Robert Mckinnon, 2011; Soon et al., 2011). Soon et al. (2011) believe that not being able to identify and classify the important variables of a situation is precisely why students are unable to start on formulating a model.

2.7.4 Using mathematics while modelling

At some stage of the modelling process, students need to explore and identify the mathematics they will use to create the model (de Almeida, 2018). It is seen that students can both draw on their previous mathematics knowledge and experience to model (de Almeida, 2018) and can create models that use mathematics they are unfamiliar with (Carducci, 1996; de Almedia, 2018). In reference to the latter, students can use the unfamiliar mathematics encountered as an opportunity to learn new mathematical techniques (Carducci, 1996). Students also learn mathematical ideas from peers while modelling in groups (de Almeida, 2018). Using mathematics in modelling requires mathematical 'correctness' in terms of the mathematics itself and in regard to the situation the mathematics represents (Carrejo & Marshall, 2007). Students need to be versatile and flexible enough "to move between concepts combining concepts and procedures" (de Almedia, 2018, p. 27). De Almedia (2018) found the student groups used different mathematical pathways even when they were using mathematics for similar situations. According to Galbraith (2012), Grigoraş et al. (2011) and Pollak (2011) mathematical approaches taken by students reveal the respective student group's understandings and interests.

2.7.5 Student agency within a modelling experience

According to Carducci (1996), Duan et al. (2020), Grigoraş et al. (2011), and Pollak (2011), and others, student agency where students take responsibility, make choices and have the freedom to be creative is a central part of the student modelling experience. Carducci (1996) found that modelling assignments work well when students are given the opportunity to make choices themselves, with students enjoying the resulting freedom this creates. Duan et al. (2020) discusses how by giving students responsibility they are actively involved in developing their own modelling competencies. Caron (2019) and Maaß (2006) both recommend students working independently of the lecturer (an aspect of student agency) as one of the conditions for developing modelling competencies.

In an earlier publication, I suggested students can experience periods of uncertainty when they are given responsibility for their modelling process (Spooner, 2021). Providing students with ways to alleviate this uncertainty, through providing ways students can verify their solution process may help to reassure students. However, Zaslavsky (2005) found tasks that were particularly useful in terms of meaningful learning were all associated with elements of uncertainty. This implies that a level of uncertainty can be beneficial for student learning.

2.7.6 Student difficulties with the modelling process

Students have difficulty starting the modelling process due to lack of modelling strategies, not understanding the context of situations to be modelled, a general lack of knowledge and inexperience in abstraction and translating from the real world into the world of mathematics (Crouch & Haines, 2004; Palm, 2007; Soon et al., 2011) and a general inability to mathematise a situation (Stillman, 2004**)**. There are specific difficulties within the formulation stage. Metacognitive competencies are needed for successful modelling. Due to the nature of modelling, difficulties influence each other and do not usually occur in isolation (Kadijevich, 2008). Difficulties can relate to getting started as well as to the whole process. Table 2 summaries the student difficulties in mathematical modelling identified from the literature review.

It should be noted that some difficulties in one stage of the process are also reasons for difficulties in later stages of the development of the model. For example, one challenge in modelling is understanding the context. The fact that a student cannot understand the context can also be the reason they are unable to identify essential aspects of the situation and hence find the modelling problematic (Kadijevich, 2008).

Table 2. Student difficulties with the modelling process

Difficulty	Level	Researcher
Lack of strategies		
Lack of organisation, including plan of attack	Secondary	Stillman
Lack of methodology	Tertiary Secondary	Soon Blum, Blum & Leiß, Stillman
Lack of internal systematic approaches to modelling	Tertiary Secondary	Soon Blum, Blum & Leiß, Stillman
Challenges with context		
Understanding the context	Tertiary	Kadijevich, Klymchuk
Challenges with contextual language	Tertiary Secondary	Klymchuk Stillman
Ignoring context	Secondary	Stillman
Mathematising the situation (General		
Converting situation into mathematics	Tertiary Secondary	Soon, Klymchuk, Clement Palm, Crouch & Haines
Weak links between physical process and mathematics	Tertiary	Chaachoua, Domínguez, Rowland
Unable to see connections between real life and mathematics	Tertiary Secondary	Kadijevich, Klymchuk & Zverkova, Soon Blum, Crouch & Haines, Stillman

Difficulty	Level	Researcher
Specific formulation difficulties		
Assigning units to formulae	Tertiary Secondary	Soon Stillman
Identifying useful equations and formulae	Tertiary Secondary	Klymchuk, Soon Stillman
Identifying variables	Tertiary	Soon
Unfamiliar with concept of having unknown and known variables	Tertiary	Soon
Difficulties classifying variables as implicit, independent or dependent	Tertiary	Klymchuk, Soon
Difficulty recalling relevant formula and procedures	Tertiary Secondary	Klymchuk Stillman
Difficulties using formula	Tertiary	Klymchuk
Unable to make assumptions	Secondary	Blum
Unable to form equations relating important variables	Tertiary Secondary	Clement, Klymchuk, Soon Spooner
Know potential relationships but unable to form equation	Tertiary Secondary	Soon Spooner
Difficulty identifying essential aspects of situation	Tertiary Secondary	Soon Blum & Leiß
Metacognitive modelling strategies		
Lack of reflection on process	Tertiary Secondary	Soon, Maull & Berry Stillman
Lack of critical thinking	Secondary	Stillman
Incomplete knowledge	Secondary	Stillman

Table 2 suggests that a number of difficulties are common for both secondary and tertiary students. Generally, students, regardless of level of education, have a lack of methodology and systematic approaches to guide them in formulating a mathematical model. There are challenges with understanding the context of the problem situations. If the context is understood there are difficulties in converting the situation into mathematics. Difficulties identified for both secondary and tertiary include problems recalling relevant formula and procedures, identifying equations and formula appropriate for the situation, and forming equations that relate important variables.

Table 2 also shows difficulties that are specific to either tertiary or secondary level. This is because either the difficulty is educational-level specific, or research has not yet been carried out in that area. For example, one would expect it to be more likely that secondary students would have difficulties due to incomplete knowledge of the modelling process (Stillman, 2010). However, this difficulty could apply to a tertiary student depending on how much previous exposure the student has had to the modelling process (Soon et al., 2011).

Why do students generally have a deficit of metacognitive competencies including a lack of strategies, challenges with contexts, trouble translating between reality and mathematics, problems identifying important aspects of the situations and choosing or creating formula appropriate for the situation? Potential explanations for these difficulties will be presented in the following section.

2.7.6.1 Reasons for student difficulties

Reasons are lacking in the literature as to why students experience difficulties in mathematical modelling. When reasons are discussed, they are anecdotal explanations that reflect the holistic nature of mathematical modelling. Reasons suggested are the complexity of the process of mathematical modelling, the lack of previous exposure to non-routine problems and the implications of this, teacher delivery including lack of stimulation of strategies from the teacher and teaching delivery not supporting development of links between mathematics and the real world.

2.7.6.1.1 Complexity of process of modelling

The process of modelling is discussed in What is mathematical modelling, Section 2.2, page 21. An example is given in Figure 6. Modelling is an inherently complex process. It involves cross-referencing between the stages of the process and is not linear in nature. Complex cognitive activity is needed for modelling (Blum, 2011). Lack of strategies and approaches to navigating this complex cognitive exercise are given as reasons for many of the observed difficulties students have with formulating a model (Blum, 2011).

2.7.6.1.2 Lack of previous experience with modelling

There is a general lack of previous experience and exposure to non-routine problems (Boaler, 2001; Rowland & Jovanoski, 2004; Rowland, 2006; Soon et al., 2011). A non-routine problem is one where the procedure or technique to apply is not known to begin with. They are problems that are not typical or straightforward. Mathematical modelling tasks are an example of this. Most school based mathematics, for various reasons, focusses on routine problems with little time given to non-routine problems. There is very little time dedicated to teaching and delivery of the process mathematical modelling. When students arrive at university this is often the first time they are asked to look at non-typical problems involving modelling.

Soon et al.'s (2011) study investigated understanding student difficulties in learning mathematical modelling with first-year engineering students in Singapore. These engineering students tend to first encounter mathematical modelling in their first year of study. Soon et al (2011) found that students encountered problems with forming mathematical models as they had had no previous exposure to non-routine problems and open mathematical modelling problems within their previous schooling, though this struggle could have also been compounded by difficulties of the secondary/tertiary transition. Studies conducted by Rowland and Jovanoski (2004), Rowland (2006), Haines, Crouch and Fitzharris (2003) and Haines and Crouch (2001) also found that first-year students new to modelling with a lack of exposure to non-typical problems experienced difficulties in formulating models. The study by Klymchuk and Zverkova (2001) involving university students from nine different countries showed they found it difficult to move between the real world and the mathematical world because of lack of experience with real-world application tasks.

Lack of exposure to non-routine problems means that students do not get the opportunity to develop systematic approaches to modelling. When they are presented with a mathematical modelling task, they do not know how to navigate through it. This lack of exposure creates a lack of confidence for a student when they first encounter a mathematical modelling experience. This lack of confidence, even if they were able to start the process, prevents them from completing the task (Soon et al., 2011).

Soon et al.'s (2011) study also revealed that difficulties students had with making connections between real-life contexts and mathematical representations were due to "students lack of systematic approaches in attempting problems in modelling contexts" (Soon et al, 2011, p. 1036) and lack of systematic approaches and scaffolding to tackle their individual areas of difficulties. They concluded that "students rely heavily on memory instead of true understanding of the concepts" (Soon et al., 2011, p. 1035). The study by Stillman (2004) also showed there are difficulties of organisation of approach. Student's ability to form a plan of attack to build the model was underdeveloped (Stillman, 2004). The DISUM studies highlighted students ignoring context and using familiar strategies not necessarily appropriate for the situation (Blum, 2011). These obstacles arise from a culture of solving routine problems with routine solutions. Mathematical modelling does not fit this culture and requires a different set of skills.

In summary, lack of exposure to mathematical modelling means students can arrive at university with a lack of internal systematic approaches to modelling (Soon et al., 2011, Stillman et al., 2013) and a lack of knowledge and experience in abstracting and translating from the real world into the mathematical world (Crouch & Haines, 2004; Palm, 2007; Soon et al., 2011).

Why is there a lack of exposure to non-routine problems? One of the apparent reasons is curriculum and time restraints at lower educational levels. Mathematical modelling needs to be given a bigger voice and amount of time in the secondary and even primary curriculum in most countries. There is also a lack of professional development for secondary teachers in the teaching of mathematical modelling, a general lack of teacher confidence in its delivery, teacher practice that does not encourage exploration and teachers' own mathematical beliefs (Armstrong & Bajpai, 1988; Burkhardt, 2006; Julie, 2002; Maaß, 2005). This helps to explain some of the reasons why students arrive at tertiary level with a lack of exposure to mathematical modelling. Still to be explored is what type of exposure students are getting to mathematical modelling once they arrive at tertiary level and what impact this exposure has on students experiencing difficulties at this level.

2.7.6.1.3 Lack of teacher knowledge about difficulties

Presented in the previous sections is a snapshot of the research to date on student difficulties and the potential reasons behind them. It is interesting to note that in 1988 the SINUS project (Blum & Leiß, 2007) revealed a paucity of secondary teacher knowledge about student difficulties, lack of teacher knowledge of the procedures students use for modelling and a shortage of corresponding research. In 2007, the DISUM studies revealed that there was still a lack of secondary teacher knowledge about student difficulties and the procedures students use for mathematical modelling (Blum & Leiß, 2007). Do secondary teachers and tertiary mathematical modelling lecturers share a similar lack of understanding with regard to students' procedures and difficulties? Currently this question appears to be unanswered. At a general level, there is a need to educate mathematics and modelling educators on the difficulties students face when attempting modelling tasks coupled with increasing their knowledge about how students approach these tasks.

Traditional teaching methods involving lack of exposure to non-routine problems and lack of stimulation of strategies by the teacher means that students lack the systematic approaches and internal guidelines necessary to approach a mathematical modelling task. In both secondary and tertiary education, the development of systematic approaches for students and the provision of catalysts to develop internal schema of students appropriate for mathematical modelling need to be part of a modelling course at any level of education (Crouch & Haines, 2004; Stillman et al., 2013). There is a need to provide stronger experiences in translating, abstracting and traversing from the real world to the mathematical one and back to the real world again (Crouch & Haines, 2004; Klymchuk et a1., 2010; Stillman et al., 2013). There is a need to build a culture of applying knowledge to new situations as opposed to routine situations. Teaching and learning styles need to focus on these activities (Soon et al., 2011).

Other more general reasons for student difficulties are a general lack of teacher knowledge of student difficulties with mathematical modelling (Blum & Leib, 2007) and the question of how teachers should best support student modelling practice.

In summary, overall, little research has been done into understanding why student difficulties occur (Soon et al., 2011). The only solid reason revealed by the research seems to be a student's lack of systematic approaches towards modelling (Soon et al., 2011; Stillman, 2004). The DISUM studies suggest the lack of student strategies being attributed to absence of stimulation of strategies by the teacher (Blum & Leib 2007). A generic anecdotal reason for the difficulties is attributed to lack of exposure to non-routine problems in earlier education. Reliance on memory instead of applying knowledge to the situation is given in studies by Soon et al. (2011) and Klymchuk et al. (2010) as anecdotal evidence for challenges in creating or using formula. Students appear to rely on routine procedures for solving problems rather than creating new approaches necessary for modelling.

Other potential reasons that may be useful to explore are:

- Are there any cognitive challenges that are contributing to student difficulties?
- What type of exposure are students getting to mathematical modelling once they arrive at tertiary level?
- What impact does this exposure have on students experiencing difficulties at the tertiary level?
- What type of guidance for teaching delivery is being used or provided by the instructor?
- Does the teacher delivery of mathematical modelling contribute to or remove student blockages?
- Does the structure of a modelling activity have an impact on the student's capability to experience the process of mathematical modelling?

- Are the difficulties more mathematical in nature rather than about process or teacher delivery? (Soon et al., 2011).

Research has clearly shown that students find modelling difficult, implying that it is difficult to teach. Research has been undertaken into different ways to teach modelling and there is not one set way. This thesis contributes to this discussion by looking at the student experiences with mathematical modelling across different courses to begin to understand the student experience and therefore further ideas as to how we might support this.

2.7.7 Change of culture

Mathematical modelling is experienced as a change of culture for mathematics students (Cole et al., 2011; Hernandez-Martinez, 2020; Kotze, 2020; Spooner, 2021, 2022). Modelling requires students to use and think about mathematics in different ways than they have done in the past. Students apply prior knowledge of mathematics in new ways than previously (Cole et al., 2011). According to Toews (2012), modelling is more than proofing and stepping something out logically. It requires understanding of the physiological nature of the problem and making connections between these and the mathematics. It is more than writing a theorem and developing a proof and is messier and more often not as precise and 'tight' as a mathematical proof.

2.8 Review of methodologies used for all studies[9]

One of the contributions of this thesis is a methodological contribution. This is the first time a study has been undertaken using Interpretive Description methodology in mathematical modelling education and mathematics education. I will provide a brief overview of the methodologies that have been used in the field of mathematical

[9] This section includes re-presentation of material published in Spooner (2022).

modelling education. This will be followed by discussing what an Interpretive Description methodology can contribute to the field.

After a review of research done within mathematical modelling at both the tertiary and secondary level I found that while there are some studies that use established methodologies (for example: case studies (Brown, 2013; Brown, 2015a; Hernandez-Martinez-Vos, 2018); grounded theory (Brown, 2015a; Brown, 2015b; Rosa, Orey, & de Sousa Mesquita, 2021); teaching experiments using design research (Ng et al., 2015), the methodologies for the majority of research consist of a use of a variety of methods for data collection and analysis (for example: Blum & Leiß, 2007; Caron & Bélair, 2007; Czocher, 2018). This applies to whether the study is a qualitative, quantitative or mixed method study. In addition, for tertiary studies, there are many teaching practice papers (for example: Duan et al., 2020; Lofgren et al., 2016; Toews, 2012; Wu et al., 2015) and discussion papers (Bassanezi, 1994; Li, 2013). This is consistent with secondary studies. There are general literature reviews on teaching mathematical modelling that incorporate all levels of education (for example: Stillman et al., 2016; Blomhøj, 2009; Kaiser, 2020; Abassian et al., 2020). However, due to a large number of studies having been conducted in the secondary space, the majority of the papers reviewed focus on secondary studies. In addition, many researchers use the modelling cycle as their theoretical foundation, though they do not declare why and do not justify their ontological position.

When reviewing research described as qualitative, as already mentioned, there were limited papers citing qualitative methodologies. Most of these papers used deductive analysis looking for pre-determined aspects (for example: Czocher, 2018; Lu & Kaiser, 2022). When interpretation is being used, most interpretation appears to be through a theory or framework as opposed to interpretative analysis from a purely qualitative perspective. For example, Hernanadez-Martinez (2020) uses narrative analysis and Bourdieu's theory of practice as the lens to interpret and explain the study's results. Thorne (2008) would challenge the use of pre-determined aspects and the use of a theory framework to conduct analysis, as a truly qualitative study. Thorne (2008) states, in reference to description studies, "qualitative description builds findings based on inductive reasoning, while quantitative description builds them by deducing" (Thorne, 2008, p. 48).

While studies take their findings and use them to suggest recommendations for modelling courses (Caron & Bélair, 2007) or discussion papers on different teaching approaches (Bassanezi, 1994; Li, 2013), the suggestions are suggestions and often lack research backing them. What is missing from the literature is a methodological qualitative study that aims to understand the nuances of student experiences and use these to inform future teaching practice.

This study uses Interpretive Description methodology (Thorne, 2008) to explore how students' experience of learning mathematical modelling may differ from learning mathematics and how this might impact teaching practices. While studies influenced by qualitative methodologies have been carried out in mathematical modelling education (Ferri, 2017; Hansen, 2021; Mhakure & Jakobsen, 2021; Rosa & Orey, 2017), this is the first time an Interpretive Description methodology study has taken place. Qualitative methodologies such as grounded theory, phenomenology and ethnography are primarily focussed on theorising. As an alternative, qualitative Interpretive Description methodology offers a theoretically flexible way to address complex experiential questions while generating implications for applied practice (Thorne, 2008).

Using Interpretive Description methodology for the first time in this field, (though not in education) will contribute to methodological development and increase qualitative research in the field to help develop a better understanding of what is going on for students and produce research informed teaching implications. Most studies at tertiary have the researcher as the lecturer/course creator researching into their own practice. For this study, I am an outsider looking in. Currently, there is no comprehensive study into tertiary mathematical modelling practices within New Zealand.

2.9 Gaps in literature

There is a gap in research focussed on student learning in mathematical modelling studies, with most of the studies reporting on student perceptions of modelling and motivation to model (Lyon & Magana, 2020). More qualitative studies are needed to obtain a better understanding of students' experiences while learning to mathematically model as discussed at the ICTMA 2019 conference. Similarly, Soon et al. (2011) and a more recent review of the literature, show more research is needed into understanding student difficulties while modelling. To gain a deeper understanding of student experiences while learning to mathematically model, including difficulties that students may encounter in the process, has the potential to inform teaching practices to better address students learning for mathematical modelling.

Literature suggests there should be a balance between instructor guidance and student autonomy (Blum, 2011, 2015; Kaiser, 2020). There are gaps in literature regarding the dynamics of the balance between guidance provided by a teacher or lecturer and students working independently. We know not enough is known regarding the impact of the teacher (Kaiser, 2020) and the influence of the lecturer (Jaworski, 2016). Little is known regarding what happens when a student starts to become stuck and lost and there is no instructor readily available to provide guidance. To obtain a deeper understanding of the lecturer's role in promoting student autonomy, and difficulties students encounter when working independently, will potentially help inform teaching practices to support developing students mathematical modelling competencies.

With reference to using contexts to support mathematical understanding, it is still unclear where and how using context will be beneficial to students (Brown, 2017; Beswick, 2011). There are implications this may be individual specific (Beswick, 2011). It seems research into the nuances of using contexts for teaching mathematics is still needed. Mathematical modelling is one way of addressing the use of contexts while exploring other factors that may influence and help the learning of mathematics.

Studies have been undertaken that are focussed on connecting mathematics with physics and physical processes (Chaachoua & Saglam ,2006; Domínguez et al., 2015; Nguyen & Rebello, 2011; Quimson, 2021; Rowland & Jovanoski, 2004; Rowland, 2006). Studies are needed that examine the connections between using mathematics within a modelling context and how experience with physics can influence this.

In summary, more studies focussed on student learning are needed (Lyon & Magana, 2020), in particular the nuances of student experiences with modelling as presented at the conclusion of the ICTMA 2019 conference. More studies are also needed into the role lecturers play in the student experience of modelling, the use of contexts to support mathematical understanding, and the relationship between student experience with physics and the influence this might have on using mathematics within a modelling context.

2.10 Conclusion

Research work has been conducted on challenges experienced by students engaged in mathematical modelling. In general students find modelling hard. They simply do not know where to start (Soon et al., 2011; Stillman, 2004). There is a lack of knowledge of how to behave as a modeller (Soon et al., 2011; Stillman et al., 2013). Gaps are apparent in students' abilities to link physical processes, physics and mathematics (Chaachoua & Saglam, 2006; Rowland & Jovanoski, 2004; Rowland, 2006). Students struggle when choosing or creating appropriate formulas and using accurate units associated with these formulas (Soon et al., 2011; Stillman et al., 2013). Difficulties identifying important variables are observed at tertiary level (Alona Ben-Tal & Robert Mckinnon, conversation Massey University 2011, Klymchuk et al., 2010; Soon et al., 2011) and students can not necessarily produce an equation that links identified variables (Soon et al., 2011; Spooner, 2013). Work is underway on recognising and measuring metacognitive competencies needed for modelling (Blum, 2011; Maaß, 2006; Schnapp et al., 2011; Stillman et al., 2013; Vorhölter, 2017).

Overall, there has been little research into understanding why students have problems with mathematical modelling. Lack of student strategies and systematic approaches towards modelling were identified by Soon et al. (2011) and Stillman (2004) as one reason for difficulty. Lack of exposure to non-routine problems is cited as anecdotal evidence in the literature, with further research needed to confirm this. This suggests modelling is a change of culture for students.

The lecturer and teacher's role is key to the learning of mathematical modelling. From what is known, the literature suggests that a generic teaching approach to address student difficulties with modelling would include: mathematical modelling as a course's main teaching approach (Domínguez et al., 2015; Soon et al., 2011); reflective guidelines for the different stages of modelling to encourage students to reflect on what they are doing (Soon et al., 2011; Spooner, 2013; Stillman et al., 2013); and the use of some form of scaffolding for students (Blum & Lieb, 2007; Blum, 2011; Caron &

Bélair, 2007; Kaiser & Stender, 2013; Legé, 2005b; Schukajlow, Kolter, & Blum, 2015; Spooner, 2013).

Having mathematical modelling as a courses' main teaching approach is to develop student understanding and knowledge of essential connections between physical processes and mathematics (Domínguez et al., 2015) and to provide the necessary exposure to mathematical modelling (Soon et al., 2011).

Using reflective guidelines for the different stages of modelling to encourage students to reflect on what they are doing (Soon et al., 2011; Spooner, 2013; Stillman et al., 2013) enables students to be more successful at forming models. At this stage there has been no research into the possibility that the use of reflective guidelines might encourage students to develop their own systematic approach to modelling.

Use of scaffolding in the form of the generic modelling cycle (Caron & Bélair, 2007) and discussion cycles (Legé, 2005b; Spooner, 2013) or strategic interventions (Blum & Leiß, 2007; Blum, 2011; Kaiser & Stender, 2013) and solution plans (Schukajlow, Kolter, & Blum, 2015) can help students navigate the modelling process.

There appears to be no available literature that focusses on targeted teaching strategies to address specific student difficulties with formulating a model.

An important point to consider when discussing challenges associated with the aptitudes needed for modelling, both mathematical and process, is that they are broad and cannot be predicted. A modeller does not start out with all the answers. A modeller ventures into the unknown, both in terms of mathematics and context, and needs the confidence to do so. Thus, it is important to address the issue of self-assurance and how to develop it in a potential modeller. The ability to explore and experiment with new and existing tools requires a certain degree of confidence. Is this a matter to be addressed solely by the education system, are these skills developed outside of the formal education system, or are they developed by a combination of both?

This research project relates to existing knowledge and literature by building on work already conducted on the teaching and learning of mathematical modelling. Most research focusses on identifying student difficulties in varying situations and/or teaching interventions. Little research has been done into the student experience to find reasons why difficulties occur and how teaching approaches can minimise these. The literature has identified the teacher as a major player in the learning of mathematical modelling and a lack of student strategies as one of the main student difficulties with mathematical modelling. It is acknowledged that a balance between lecturer guidance and student autonomy is needed to develop student modellers. However, little is known about what the dynamics of that balance looks like. This research project aims to investigate the student experience with modelling to identify the role the lecturer plays in the student modelling experience. This study aims to contribute to the gaps in literature concerning understanding the student experience to inform teaching practices. It is anticipated that this will add to the literature further insight into the origin of student difficulties. In addition, it is projected that the development of guidelines for the effective teaching of mathematical modelling is added to from this research.

Chapter 3. Background context for study [10]

The research question "What can we learn from tertiary student experiences to inform the teaching of mathematical modelling in the future?" and research sub-question "What is the student learning experience while participating in a mathematical modelling activity?" was established for this study. This study involved the students and lecturers from three different New Zealand tertiary mathematical modelling courses. This study focusses on the students experience of a mathematical modelling activity used within each course. The modelling activities for all courses involved students working collaboratively as a group independent of the lecturer. The details of each course and the modelling activity used are as follows:

3.1 Course one

Course one was a first-year summer school university mathematical modelling course designed for engineering students. The five-week course consisted of two-hour classes (a combination of lecture and tutorial), four times a week. Following an assessed pre-modelling project tutorial, students in groups of four undertook an assessed, open-ended modelling activity that formed 5% of their grade. Students were allocated seven hours to work on the one-day activity. *A question applied on the day was: How high can you jump from a building without causing injury?* The lecturer selected this question as it was ill-defined and open, enabling students ample opportunity to engage in, and independently develop, a group-directed modelling process. After working through the modelling process with the question, students submitted a written report outlining their process and showing the model produced. All student groups produced unique models in the form of a function or set of functions. The lecturer is considered an expert lecturer, having received five teaching excellence awards. The lecturer is focussed on

[10] This section contains re-presentation of material published in Spooner (2021) and Spooner (2022).

developing modelling competencies within a holistic experience of modelling, that is, one that includes the full process of modelling (Haines et al. 2003). The lecturer's approach to provide a mathematical modelling experience for students included the following learning goals: Develop techniques for modelling; Acquire knowledge of the process of mathematical modelling; Engage in independent and critical thinking; Become familiar with skills for effective group/teamwork; Understand the criteria for the assessment of modelling project; Relate mathematics to genuine contexts; Participate in the modelling process as a group, independent of the lecturer; Communicate thought processes. See Spooner (2020) for additional details. As part of the development of techniques for modelling, a generic modelling cycle was introduced during lectures and tutorials. This consisted of: Clarify the problem, list and classify the factors, make assumptions, formulate the problem statement, formulate the model, solve the model, interpret the model, and compare with reality. The lecturer utilised this, along with a "keep it simple" principle when role modelling the modelling problems and in assessment solutions.

Groups of four students each were selected by the lecturer for the modelling activity. The lecturer conscientiously selected groups such that there was a balance of academic abilities and gender within a group. The students were put in these groups for the pre-modelling activity tutorial to allow students to get to know each other's strengths and weakness before the activity and to explore issues around how to effectively work in a group before encountering the modelling activity.

3.2 Course two

The context of course two was a third-year university mathematical modelling course occurring over a semester for mathematics students. The course consisted of two-hour classes (a combination of lecture and tutorial) two times a week, case study presentations, assignments and an examination. The case study presentations consisted of presenting a mathematical model in three stages over a period of three weeks, contributing to 10% of students' final grade. The first stage involved presentation of the model including defining the terms of the model. The second stage was presenting the mathematical workings of the model. The third stage was presenting the student modification of the model and its mathematical working. Student groups prepared their presentations between each stage, with approximately a week between presentations. The third stage presentation was considered to be when the students had an opportunity to engage intimately with the modelling process. There was an assumption by the lecturer that the modelling process was implicitly experienced during stages one and two in preparation for stage three. *One case study concerned the introduction of cactus to Australia and how the cactus was being controlled by a moth.* The model was based on a system of differential equations. The case study contained background information about the case, the model, and some but not all of the model's mathematical working. Preparation and presentation of case studies were worked on in student groups consisting of either two or three students.

Student groups were formed through students being presented the different case studies available to work on during one class. The students submitted their first and second choice of case study to work on. The lecturer then formed groups to balance out numbers and, where possible, skill sets in each group, based on the students' choices. Groups of three students each and one group of two students were formed.

The lecturer's learning goals for students included: Understand the modelling process; Develop mathematical techniques for modelling; Engage in independent and critical thinking; Become familiar with skills for working in a group; Participate in the modelling process as a group, independent of the lecturer; Communicate thought processes.

3.3 Course three

Course three was a first-year university mathematical modelling course occurring over a semester for students from a range of different disciplines. Disciplines included mathematics, computer science, oceanography, geography and computational modelling. The course consisted of three one-hour lectures and one one-hour tutorial a week (group work took place within lectures), assignments, midterm test and an exam. The modelling cycle was introduced with the lecturer developing simple models using known mathematics. In the two lessons before the modelling activity, the lecturer delivered lectures primarily focussed on the infectious disease SIR model, the model students would use as the base for model development during the activity. The non-assessed modelling activity took place during one lecture, using fictional scenarios based around zombie movies as situations for students to collaboratively create a model to represent the movie situations. *One of the situations based on the movie Night of the Living Dead was: 'A radioactive leak at a nearby military base leads to dead bodies becoming reanimated as zombies. The zombies attack and kill humans and can only be stopped by decapitation or by burning.'* The students were given one lecture to collaboratively create a model to represent their given situation. The final ten minutes of the lecture were dedicated to student groups presenting what they had created. Student groups of five to six students were selected by students receiving a type of sweet as they entered the lecture theatre. All students who received a brand A sweet formed one group, brand B sweets formed another group, et cetera. Students were given preparation before the modelling activity in a generic infectious disease model. The lecturer role modelled how to manipulate the model to fit different situations.

The lecturer’s learning goals for students included: Develop mathematical techniques and knowledge for modelling; to build confidence to attempt the modelling process; Participate in the modelling process as a group, independent of the lecturer; Engage in independent and critical thinking; Communicate thought process.

3.4 Mathematical modelling activities - further details

Table 3 gives a summary of the characteristics of each course’s modelling activity and focus on the modelling cycle.

Table 3. Characteristics of course modelling activities

Course	Modelling Activity used	Students' responsibility for decisions	Presentations of models to class	Assessment weighting	Focus on modelling cycle	Size of modelling group	Time spent on activity
Course one	Open-ended	All decisions, including foundation model to use	No	5%	explicit	4	One full day assessment activity
Course two	Case Studies	Modification of model only	Yes, three presentations	10%	implicit	2-3	Self-determined over 1 week
Course three	Popular Culture	All decisions except for foundation model	Yes, one presentation	0%	explicit	5-6	1 hour during lecture

Chapter 4. Methodology and methods

This chapter outlines the methodology and methods that underpin this study of how tertiary students can experience learning mathematical modelling. First, I will give a brief overview of the journey in deciding on the methodology. Following this, an overview of the Interpretive Description methodology and its core principles is given. The methods for data collection, analysis, and the steps taken to ensure rigour are then detailed.

4.1 Deciding on a methodology

Before deciding on my final methodological approach, I considered several different options. I initially started work on exploring lecturers' hypothetical learning trajectories (Gravemeijer, 1999; Simon, 1995) and how these played out within student experiences of mathematical modelling – see Spooner (2020) for published work from this initial exploration. After several discussions with one of my supervisors, it was decided to investigate different qualitative methodologies to see what they could offer regarding gaining insight into the student experience with mathematical modelling. From this exploration I looked deeper into using phenomenology to explore students lived experiences of mathematical modelling. As I was primarily interested in how students' experiences with learning modelling could better inform our teaching practices, I realised that I needed a methodology that could add more than just an understanding of students' lived experiences and was more flexible and adaptable to allow me to capture issues associated with student learning, collective experiences and also had the potential to inform teaching practice. I then sought advice from qualitative methodology experts from within my university. In conjunction with these experts, it was decided that an Interpretive Description methodology (Thorne, 2008) would provide a robust methodology based on exploring participants' experiences to inform practitioner practices. It was flexible enough to accommodate the work that had been already done. Not only could Interpretive Description provide insight into student learning experiences, but the findings would also provide implications for teaching practice. The decision to use Interpretive Description occurred after data collection was complete. This late decision influenced how and what I was able to present in this final thesis. My earlier exploration into phenomenology has influenced the use of Interpretive Description and primarily focussed on interview data to look at the student experience.

It is fitting that the methodology I finally settled on was qualitative. While attending the state of affairs discussion session at the 2019 International Community of Teachers of Mathematical Modelling and Applications (ICTMA) conference held in Hong Kong, it was identified that more qualitative studies were needed to better understand the student experience with mathematical modelling. This validated the extensive investigation into qualitative studies that I entered into before settling on my final methodology.

4.2 Interpretive Description: Methodological overview[11]

Interpretive Description is a methodology designed to explore participants' experiences within their natural context. Interpretive Description aims to use these experiences to add to existing and emerging bodies of knowledge to better inform and advance practice (Hunt, 2009; Thorne, 2008). Interpretive Description considers practice-based questions to generate findings which can be applied to and advance practice. As I discuss in Spooner (2022), an Interpretive Description methodology is used to explore some of the ways students' experience learning mathematical modelling, and how this might impact teaching practices. While studies influenced by qualitative methodologies have been carried out in mathematical modelling education (Ferri, 2017; Hansen, 2021; Mhakure & Jakobsen, 2021; Rosa & Orey, 2017), this is the first time an Interpretive Description

[11] This section contains re-presentation of material published in Spooner (2022).

methodology study has taken place. According to Thorne (2008), Interpretive Description offers a theoretically flexible way to address complex experiential questions while generating implications for applied practice as an alternative to traditional methodologies such as Grounded Theory, Phenomenology and Ethnography.

Thorne identified that there was a need for quality qualitative research to address practice-based questions that does not fit traditional qualitative methodologies. Interpretive Description methodology emerged as a way to address this need. As a methodology, it aims to reveal shared patterns, themes and differences in participants' experiences and perspectives that sit benefit the surface of the data that a practitioner may not ordinarily see (Thorne et al., 1997; Thorne, 2004; Hunt, 2009). A key outcome of Interpretive Description is practitioner implications that are realised from the contextualised interpretation of the phenomenon in a way that is useful for practitioners (Hunt, 2009; Thorne, 2008; Thorne et al., 1997; Thorne et al., 2004). Interpretive Description is a flexible, yet rigorous methodology (Thorne, 2008). Interpretive Description is set apart from other qualitative methodologies as it does not aim to generate a theory as in grounded theory or capture the essence of the phenomenon as one would in phenomenology.

4.3 Ontological and epistemological positioning of Interpretive Description

Interpretive Description has constructivist and naturalistic orientations as philosophical underpinnings (Hunt, 2009; Thorne et al., 2004; Thorne, 2008).

The key "epistemological foundations" (Thorne, 2008, p. 74) as described by Thorne (2008) and Hunt (2009) are:

a) Insight into a phenomenon is fundamentally obtained from people's subjective and experiential knowledge of the subject under investigation, however, it is not possible to attain a full or objective knowledge (Hunt, 2009).
b) People's experiences are not bounded by time and assumes individuals' previous experiences underpin their current perceptions and experiences.
c) Individual's realities are socially and experientially based. Knowledge is based on these realities and therefore socially constructed. Therefore, there are multiple and potentially contradictory realities of a phenomenon that may exist. As such, both similarities and differences across people's experiences of the phenomenon are of interest.
d) Researcher and participants co-construct knowledge. They individually do this through their interpretation of a situation, and together through the interview process. The researcher actively constructs knowledge through the analysis process, and as such, reflexivity is critical.
e) A priori theoretical understanding cannot incorporate multiple realities and does not acknowledge the inseparable relationship between the researcher and participant. Therefore, findings are constructed from the data, encompass multiple realities and are considered within their context. Prior coding is not suitable for generating findings for such a study (Thorne, 2008; Thorne et al., 2004).

A critical aspect of Interpretive Description methodology is the acknowledgement of the practical and theoretical knowledge of the researcher. The researcher comes to the research with a position and with practice-informed questions that inform the theoretical scaffold of the study. The researcher's understanding from reviewing the literature and from their own practice informs the study design, analysis and how the findings are shared (Hunt, 2009; Thorne, 2008). In the case of this current study, and as part of researcher reflexivity, my researcher assumptions can be found in Positioning the researcher, Chapter 1, Section 1.5, page 7 and within the Literature Review, Chapter 2, pages 16 to 121. Raw data should be used to provid e evidence of the researchers' decisions about data interpretations. More detail is given on the use of quotes in the writing of the findings in Section 4.10 of this chapter.

Researcher reflexivity is an important part of the researcher's positioning within the theoretical scaffolding of the study. Researcher reflexivity acknowledged my role as the researcher in the research. Research reflexivity occurred through acknowledgement of my previous experience and assumptions in Section 1.4, the use of field notes and memoing during, and immediately after, the interview and observation processes, within data analysis via notes and analytical memos, and during supervision to reflect on the ongoing analysis. As demonstrated in Section 1.4 , my own positioning has significantly influenced how I have engaged in this research process. An example of how this has impacted on my interpretations is through the questioning of what I was seeing in the data. How was my positioning influencing what I was seeing in the data? Was what I was seeing justified? Or, am I not seeing everything?

Interpretive Description allows the researcher to employ a range of tools for data collection and analysis (Hunt, 2009; Thorne, 2008; Thorne et al., 2004). Multiple data sources can be used for Interpretive Description, though often interviews with individual participants are used as the primary data sources (Hunt, 2009). Triangulation of the data can be provided through multiple data sources increasing the trustworthiness of the findings generated (Hunt, 2009). The data collection and analysis tools used for this study will be described in Section 4.8, page 132 and Section 4.9, page 135 respectively of this chapter.

4.4 Sampling

A combination of purposive and judgement sampling (Sharma, 2017) was used to identify three tertiary mathematical modelling courses occurring in New Zealand. The courses were selected as they all focussed on the process of mathematical modelling and mathematical modelling techniques, as opposed to simply mathematical modelling techniques, and the courses provided the first opportunity for a student to study mathematical modelling at the course provider's institution. All three courses were slightly different in nature, providing three alternative scenarios for teaching mathematical modelling. Three courses were selected to try and capture a breadth and depth of experiences of modelling courses and to establish a level of rigour with the findings. I invited the lecturers and students from the purposive selected courses to participate in the study. Lecturers and students both self-selected to be part of the study. New Zealand mathematical modelling courses were selected, because New Zealand is my country of residence as the researcher. Selecting New Zealand Universities also allowed for investigating the current New Zealand student experience for students taking a tertiary mathematical modelling course for the first time.

4.5 Inclusion criteria for the study

The inclusion criteria for the study were:

a) First mathematical modelling course a student would take at the University and either be:

b) the lecturer of the tertiary mathematical modelling course

   or

c) a student of a participating lecturer of the tertiary mathematical modelling course.

4.6 Recruitment

Tertiary mathematical modelling courses occurring at New Zealand Universities were first identified. Three university mathematical modelling courses were selected based on the sampling criteria previously detailed. The lecturers of these courses were approached by email to ask if they would be interested in participating in the study. All three lecturers accepted the invitation to participate in the study. A follow-up meeting was then held with the lecturers. During this meeting the proposed methodology and research plan was discussed. If there were aspects the lecturer was uncomfortable with, or were not suitable for their course, these were discussed, and the method of data collection was modified as needed for each lecturer and their course. For example, lecturer two was not comfortable with the students being asked to consent to being videoed while they presented their models. Therefore, this aspect of the original research plan was not conducted for course two students.

Once courses had started, as the researcher, I visited the different courses. At the end of a lecture or tutorial when the lecturer was not present, I invited students to participate in the study. Contact details of interested potential student participants were then collected by me. Those students who came forward were then sent an invitation to participate and a participant consent form (see Appendix A for Participant information sheet and Appendix B for Consent form). There were no exclusion criteria for interested potential student participants, meaning no interested student of a participating lecturer was excluded from participating in the study. After repeating this process with the three courses a total of twenty student participants consented to participate in the study.

In total, three lecturer participants and twenty student participants participated in this study. For qualitative research, the sample size depends on the uses of the data collected and its intended purpose (Sandelowski, 1995). For Interpretive Description, the aim is to capture a breadth and depth of participant experiences and "can be conducted on samples of almost any size" (Thorne, 2004, p. 94). The smaller number of participants for this study provided an opportunity to conduct in-depth interviews across the three courses and multiple interviews with some participants. This allowed for in-depth details of these participants' experiences, much like case studies, to be obtained. Ideally, having more student participants would have added to the breadth of the study, though too many more would have restricted any in-depth analysis.

4.7 Ethics

This study received ethics approval 16/387 from the Auckland University of Technology Ethics Committee. Participation in the study was entirely voluntary with all participants having the right to withdraw should they wish to. The identity of participants has been protected by using either a pseudonym, as is the case for student participants, or a number for lecturer participants. See Appendix A and B for copies of the Participant information and Consent forms for lecturers and students.

4.8 Data collection

The primary data source for this study was in-depth semi-structured interviews, supplemented by observations of the modelling activities and some teaching activities. Please see Appendix C for Interview guides and Appendix D for Observation protocols. Interviews were conducted with two groups of participants. The first group was the lecturers of each modelling course. The lecturer interviews were conducted to understand how each lecturer created student learning experiences for mathematical modelling, including their goals and planned learning activities. Lecturer interviews for all lecturers occurred before each course and at the conclusion of each course. With course one occurring over summer school, the pre- and post-course interviews were the only interviews conducted for lecturer one. For lecturer two, additional short five-minute interviews were conducted before key activities. Course three was conducted at a university that required travel away from home. Additional interviews occurred with lecturer three on these visits.

The second group of interviews were student interviews with students partaking in the modelling courses. The main focus of the student interviews was to primarily gain insight into the learning experience in relation to the mathematical modelling activity specific to each course. The interviews also served to gain an understanding of the overall student learning experience from taking the course. One interview was conducted per student as soon as was practically possible after the completion of the course mathematical modelling activity to capture as accurately as possible students' recall of their experience of the activity. As course one was an intense summer school course, this was the only student interview conducted. The interview timing was approximately one week after the end of the course, equating to approximately two weeks after participation in the lecturer-identified mathematical activity. The timing of the course one interview occurred once the course had finished so as to not encroach on students' available study time. For courses two and three, an interview focussing on the students' experience of the mathematical modelling activity occurred one week after completion of the activity. More than one interview was conducted for course two and three student participants. This was because course two and course three were both semester courses with more opportunities to interview students. Multiple interviews at differing stages of each course gave students time to reflect and develop their thinking as well as provided an opportunity for me, as the researcher, to obtain a deeper understanding of the complex student experience for learning to mathematically model.

For course one:

- One student interview was conducted approximately one week after the end of the course.

For course two:

- Two interviews per student participant were conducted.
- Interview one was conducted mid-course approximately one week after the completion of the modelling activity.
- Interview two was conducted approximately two weeks after the end of the course.

For course three:

- Interview one occurred a third of the way through the course.
- Interview two occurred approximately one week after the completion of the modelling activity.
- Interview three occurred approximately three weeks after the completion of the course and occurred via Zoom.

Due to the location of course three requiring planned away-from-home travel, the number of interviews differed per student participant ranging from one interview to three interviews. The differing number of interviews was due to the wish to capitalise on my visit and dependent on students' availability for the interviews. All course three student participants participated in interview two.

Collection of student demographic data collected was limited and occurred at the end of interview one for each student participant. Two demographic questions were asked: Is this the first time you have taken this paper? Which degree are you currently undertaking?

Students responses showed it was the first time taking the paper for all but two students. These two students were repeat students. However this was the first time participating in the modelling activity within the course. While I did collect data on which degree students were taking, this has not been reported here as this would likely identify the course, programme and university which would be a breach of ethics in regards to confidentiality. However, the indicative qualifications students were taking included Engineering, Mathematics, and Natural and Environmental Sciences. From the interviews it was established that there was a range of mathematical backgrounds amongst students. One student had potentially had experience with an open-ended modelling activity prior to attending University. While the modelling activities occurred in groups, data on the groups and their composition was not recorded due to not all group members consenting to participating in the study.

4.9 Data analysis

Data analysis in Interpretive Description is inductive in nature (Hunt, 2009; Thorne et al., 2004), and can use multiple methods (Hunt, 2009; Thorne, 2008; Thorne et al., 2004). Interpretive Description recommends an iterative approach to data analysis, with a focus on concurrent data collection and interpretation, and constant comparison of data pieces (Hunt, 2009; Thorne et al., 2004. Thorne, 2008). Reflexive thematic analysis (Braun, Clarke, Hayfield & Terry, 2019; Terry, Hayfield, Clarke & Braun, 2017), a method consistent with these methodology characteristics due to its iterative and inductive approach, was used to transform the data into themes. In reflexive thematic analysis, themes are conceptualised “as reflecting a pattern of shared meaning organized around a core concept or idea, a central organizing concept” (Braun et al., 2019, p.845). Data analysis occurred concurrently with data collection, was iterative in nature, and involved constant comparison between data pieces, as outlined above all strongly recommended characteristics of an Interpretive Description study (Hunt, 2009; Thorne, 2008).

4.9.1 Reflexive thematic data analysis

To gain insight into the students learning experience of the course, modelling activity, and the lecturer's role in the student experience, a method based on reflexive thematic analysis (Braun et al., 2019, Terry et al., 2017) was used. Data analysis focussed primarily on the student learning experience created from the modelling activities for each course, as well as the general student experience of the course. Details of the modelling activities for each course can be found in Background context for study, Chapter 3, page 119.

Thematic analysis is a flexible method "that can be conducted within various ontological frameworks" (Terry et al., 2017, p. 21). The ontological framework then determines the epistemological underpinnings of the meanings created from the data. A critical realist/contextualist ontological framework was used to underpin the use of reflexive thematic analysis for this study, meaning "people's words provide access to their particular version of reality" (Terry at al., 2017, p. 21). A critical realist/ contextualist perspective acknowledges that explicit access to reality is not possible and an interpretation of reality is achieved through the participants and researcher's interpretive process of reality. A critical realist/contextualist approach allows the researcher to reach an understanding of participants' subjective experiences in relation to the phenomenon (Terry et al., 2017). Thematic analysis seeks to find patterns and similarities within the data to inform this understanding. Reflexive thematic analysis involves inductive coding and is an active, iterative and flexible analysis process. Inductive, iterative and flexible analysis are essential and strongly recommended characteristics of an Interpretive Description methodology (Hunt, 2009; Thorne, 2008).

The general phases of reflexive thematic analysis are:

1. Familiarisation with the data.
2. Systematic generation of codes for the data set.
3. Constructing themes.
4. Reviewing themes.
5. Defining themes.
6. Producing the report.

While I am presenting the stages of analysis as though they are discrete stages, in reality there was a recursive process of moving between the stages to progress the analysis.

4.9.1.1 Phase 1: Familiarisation with the data.

The first engagement with the data was through my role as the interviewer. As soon as possible after each interview I wrote a summary of the interview capturing the essence of the students' experience. I then transcribed the interviews adding insight observations to the interview summaries. A final listen on completion of the transcription was carried out to ensure accuracy of transcription and familiarisation of the data. Once student interviews had been transcribed for each course, student interview summaries and interviews for the course were looked at together to gain insight into what may be going on for the students. This initial familiarisation with the data was ongoing, occurring congruently and at completion of the data collection for each course. See Appendix E for an example of an Interview summary.

4.9.1.2 Phase 2: Systematic generation of codes for the data set.

Coding is a way of reducing the data down to enable organisation and synthetisation across the data sets. Inductive coding, with codes derived from the data (Braun & Clarke, 2006), was initially generated from the data set. This is an active, iterative and flexible process where applicable data within the data set is labelled "with a few words or a … phrase that captures the meaning of that data segment to the researcher" (Terry, Hayfield, Clarke & Braun, 2017, p. 26). Inductive coding can involve semantic coding and/or latent coding. Semantic coding captures the explicit meaning of the data and tends to "stay at the surface of the data" (Braun et al., 2019, p. 853). Latent coding captures deeper underlying meaning of the data and is conceptual in nature (Terry et al., 2017; Braun et al., 2019). "I did get a chance to hear different ideas and try and come up with different ways to do ours" was coded as *more than one way* and is an example of semantic coding. For the data extract, "not having the pressure of having to get it right", a deeper meaning is implied and hence was coded within the latent code *freedom to explore.* The generation of codes was focussed on an interpretive orientation to make sense of what is going on within the data rather than classifying things into categories.

NVivo was used as a software to store the codes. Initially this proved to be problematic. As a new user of NVivo, I fell into the trap of getting bogged down in the detail, over-coding as opposed to first coding broadly and then reviewing later as needed (Thorne, 2008). As I matured in my coding ability, letting my "reason, intelligence, and inductive thinking" (Thorne, 2008, p. 144) drive the analysis, and as I learnt to use NVivo as a tool for sorting and saving data, the initial problems were minimised.

4.9.1.3 Phase 3: Constructing themes.

To construct themes the generated codes for course one was initially examined for significant features and patterns across the data set. The data for courses one and two were then combined and the process repeated. On completion of data collection for course three, the data for all three courses was combined and the additional data was used to confirm and /or challenge the features and patterns already identified within the data set. The research question "What can we learn from tertiary student experiences to inform the teaching of mathematical modelling in the future?" and research sub-question "What is the student learning experience while participating in a mathematical modelling activity?" and the data-probing question "How are students navigating and discovering the mathematical modelling process?", which had been identified through earlier examination and explorationof the data, were then used to examine the codes. All codes were relooked at to see if they related to these questions. Those that did not were discarded. Initial exploration into how all the codes fitted together was conducted through writing all the codes on Post Its and then clustering the Post It codes into potential themes. For example, 'producing unrealistic solutions', 'finding modelling complex', 'going down rabbit holes', and 'getting stuck' were all grouped under the potential theme 'not a straightforward process'. Regrouping and clustering of the Post It codes was repeated until key features of the data relating to the student experience and discovery of the mathematical modelling process were identified. Further investigation was then undertaken to gain further insight into what was happening for the students. From this exploration, it was anticipated that themes would be developed from these key features. Draft mind maps were created to initially explore possible clusters and relationships between codes (see appendix F) for theme development. A centralising organising concept was looked for in the development of themes.

4.9.1.4 Phase 4 and 5: Reviewing and defining themes

A table of candidate themes, including potential codes within themes, was created to help define the themes, including their boundaries and how each theme might fit with others and "tell an overall story of the data" (Terry et al. 2017 p. 28). (See appendix G). Each candidate theme was then checked to see if it truly did capture the meaning of the coded data segments and if it could be defined around a centralising organising concept. Some codes and themes were discarded, some incorporated into another theme, and some themes completely reshuffled until there was coherency across the themes and their data extracts and the story the themes told about the data. For example, 'grappling with uncertainty', 'more than a mathematics course' and 'out of my comfort zone' were collapsed to be subthemes of *pushing our boundaries*. 'Finding ways forward' was re-examined and split into *moving forward on our own* and *being moved forward*. This reviewing of the themes was central to the credibility of the analysis (Terry et al., 2017) and focussed on reflection, rigour, and a systematic and thorough approach for greater depth of engagement with the data rather than focussing on coding accuracy.

4.9.1.5 Phase 6: Producing the report

The final write up of the data analysis involves a final testing and refining of the themes. Producing the report (for this thesis, the findings and discussion chapters) involves knitting together the data, analysis, literature and research question (Terry et al., 2017; Braun et al., 2019). This stage is viewed as the final stage of analysis.

4.9.2 Memoing

Memo writing was used to support the data analysis and occurred throughout the data collection and analysis. Writing captured my thoughts as the researcher, critiques of my process and the emerging analysis. The act of writing itself was part of the analysis helping to uncover hidden meaning in the data (Thorne, 2008). Constant comparison was also part of the memo-writing process, allowing me to capture similarities and differences across the dataset within the emerging themes.

Writing memos contributed to the rigour of the study. In conjunction with the above, writing allowed me to be aware of my assumptions and beliefs, with the objective to remain as open as possible to new experiences of teaching and learning mathematical modelling that were different to my own. In particular, I continually thought about, in conjunction with writing and reflecting, the actions of my own modelling lecturers and how this impacted me. As a result, I was able to keep an open mind while reflecting on what lecturer participants did and how this might affect their students' experience. I stepped back and looked at student responses to what the lecturer was doing and separated this from my own experience. I needed to remember that even though I will have a feel for what their experience could potentially be, this may not necessarily be the case. I needed to remain open to the experiences of the students, that were alternative to my own.

4.9.3 Verifying analysis

It was important that I had moved beyond description, formulated a useful coherent narrative and established what the analysis implied for teaching practice. Meetings with the advisor and supervisory team were a crucial part of this process. Regular meetings throughout the data analysis process helped ensure that the analysis was logical and grounded in the data. During these meetings data interpretations, potential narratives and teaching practice implications were debated and challenged or affirmed with the aim of supporting my process with the emergent analysis.

4.10 The use of quotes in writing of findings

Once the themes were fully constructed from the data analysis, the findings then needed to be written to communicate my interpretation and sense making, as the researcher, of the data (Thorne, 2008). As I discussed in Spooner (2022), quotes chosen from the raw data were used to best give the reader access to these interpretations (Corden & Sainsbury, 2006; Thorne, Kirkham & O'Flynn-Magee, 2004). It is important to note here that in qualitative research and in particular Interpretive Description quotes and "examples are used to illustrate the data's story as opposed to 'proving' it (Thorne, 2008)" (Spooner, 2022, p. 7). The use of quotes chosen from the raw data contributes to demonstrating rigour within this study.

4.11 Rigour

Multiple approaches to rigour were utilised in alignment with the principles of rigour as described by Thorne (2008). Providing a detailed account of the methods for data collection and the process of analysis makes explicit the research process followed. Sampling participants from three different modelling courses and, where possible, completed multiple interviews per participant, captured a breadth and depth of experiences. The use of quotes chosen from raw data illustrates the findings are grounded in the data. The use of memoing shows the reflective and critical approach undertaken for the analysis and interpretation of the raw data. The iterative reviewing and redefining of candidate themes ensured the themes, coded data, raw data and research question work well together.

Regular discussion regarding the emerging conceptualisation of the data analysis occurred with the supervisory team and external experts in the disciplinary field. Preliminary findings were presented at national and international conferences. Publications of these findings underwent rigorous peer or blind peer review (Spooner, 2017, 2019, 2020, 2021, 2022, 2023, 2023).

4.12 Methodology for discussion

The findings of an Interpretive Description study are not complete until they have been discussed and interpreted with literature. Discussing and interpreting the findings with literature grounds the findings in their relevant discipline. As advised by Thorne (2008), before starting to write the discussion chapter, I set aside time to identify the major themes in the findings that deserved further attention in the discussion chapter. In order to do this, I first reread each findings chapter, making bullet point notes of the key findings and central conclusions of each chapter. I wrote these bullet points into summary paragraphs for each finding chapter. I then read these summary paragraphs out loud to colleagues, asking what they thought the main message of the findings was. I also asked myself what was new to me from the findings that I did not know before or now knew in a different way. I used my colleagues' insights, as well as my own reflections, to identify the main themes for further discussion with the current literature.

The following four chapters communicate the findings of the data. In a separate chapter the findings are discussed with current literature and teaching implications clearly stated.

Chapter 5. Findings: Overarching theme, pushing our boundaries

5.1 Introduction to overarching student experience

Pushing our boundaries was the overall student experience while mathematically modelling. This experience consisted of students *moving forward on their own* while working within a group independently of the lecturer and of *being moved forward* through utilising and taking up support provided by the lecturer and guidance from resources. Whether students were moving *forward on their own* or *being moved forward*, they were not always certain how to proceed and moved through states of *being uncertain* and *having some direction*. There were tensions between moving *forward on their own*, *being moved forward, being uncertain,* and *having some direction.* Navigating these tensions could potentially affect students' development as mathematical modellers. Navigating between these tensions involves working collaboratively within student groups and interacting with guidance provided. Guidance that supports student exploration to promote independent student work is needed. This guidance needs to enable productive struggle without overwhelming students.

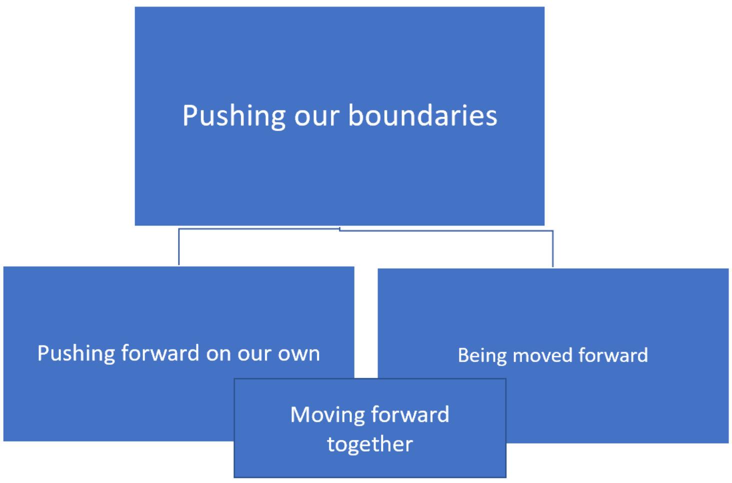


Figure 12. Thematic findings of the student experience of mathematical modelling

This chapter will first introduce the overarching experience of *pushing our boundaries.* I will then detail two of the primary experiences of moving forward, including how students found direction while working on their own, and how they moved forward through utilising advice, in the next two chapters. Finally, in the following chapter I will look at the experience of how students moved forward as a collective, transitioned being uncertain and finding direction and considerations for lecturer guidance that supports independent student work.

5.2 Pushing our boundaries

'To do more than I would normally do, going further than I have been told or shown.'

Pushing our boundaries is the overarching experience for students while mathematically modelling. All students have experiences that are new, unfamiliar, and outside of their normal dealings with mathematics. They enter and explore situations where the process is new and often creative, along with encountering new mathematics as they progress. This produces a situation where the way forward is *not always clear* for the student. In order to navigate this uncertainty, students find themselves moving between *pushing forward on their own* and *being moved forward* through guidance external to themselves. The tensions between working and exploring things independently and the guidance received from the lecturer appears to be core to the students' experience of learning to mathematically model.

I will begin by first discussing the reasons and sources of *pushing our boundaries*.

5.2.1 Reasons for pushing our boundaries

Pushing our boundaries occurred as students being involved in experiences that were new and unfamiliar to them, including navigating into areas they had never encountered before. These new experiences may involve embracing a new process, new mathematics, and new ways of thinking about and behaving mathematically.

Students had to provide workings that were over and above what they had been given in resources. One student described how the workings required them to extend their thinking. “Actually, we did some practice problems but it’s not like that. The practice problem like in the exam paper, you need to model and identify factors but was shorter questions and easier” (Sue, course 1). This suggests that, for Sue, the workings of the practice problem could not simply be transferred across to the activity. She needed to apply, modify, and expand on the original workings to fit the situation her group was working on. Students’ need to extend their knowledge was described as requiring “big leaps” (Ngaire, course 2) in new procedural and mathematical knowledge over and above that which was provided in their resources. Students found these “big leaps” (Ngaire, course 2) hard and felt “thrown in the deep end” (Rhys, course 2). Don (course 1) related this to “going into something completely in the dark”, finding the lack of clarity “nerve racking”. Students had to learn to *back themselves* to find ways forward with their work.

The lack of clarity in how to proceed was an unfamiliar situation for students who were used to step-by-step workings when working with mathematics. Even though there was material to refer to, students were required to ‘step into the unknown’ and try out new ways of working things. They needed to have a go without having certainty that it was the ‘right’ way to do things. Reference materials did not provide all the answers, as noted in observations. “Sophie’s group found an equation on the internet to base their model on. Sophie and Shirley engage with the model to see if it can be manipulated and modified to fit their unique situation” (descriptive field note, course 1 observation). Students needed to attempt workings that were new to them and more advanced than provided in resources. For instance, Nicola’s group discovered the “second fixed point gave a complicated solution to a quadratic equation. “This is the first time these students have independently attempted to solve something this complicated” (descriptive field note, course 2 observation).

$$“f_2 = r_2 H\left(1 - \frac{JH}{V}\right) = 0$$

$$So\ H = 0\ or\ r_2\left(1 - \frac{JH}{V}\right) = 0\ which\ gives\ V = JH$$

$$When\ H = 0\ gives\ V = 0\ or\ V = K$$

$$When\ V = JH\ and\ assume\ H \neq 0$$

$$\mathrm{H} = -\frac{r_1 JD - r_1 KJ + c_1 K \pm \sqrt{2KDr_1^2 J^2 + K^2 J^2 r_1^2 + 2KDc_1 r_1 J - 2K^2 c_1 r_1 J + D^2 J^2 r_1^2 + K^2 c_1^2}}{2r_1 J^2}$$

$$V = -\frac{r_1 JD - r_1 KJ + c_1 K \pm \sqrt{c_1^2 K^2 + K^2 J^2 r_1^2 - 2K^2 c_1 r_1 J + 4KD{r_1}^2 J + 2KDc_1 r_1 J - 2KDr_1^2 J^2 + D^2 r_1^2 J^2}}{2r_1 J^2}"$$

(Nicola, presentation workings).

Working through the analysis themselves extended their thinking, enabling Nicola’s group “to understand the [nature] fixed points” even though they had originally found it “ridiculous”.

Similarly, Luke’s group used a standard model, adding new material to the model without being provided with clear direction. “Once we assigned the variables to SIR, we had to come up with another variable, for the people that had been bitten and then they weren't infected, or they didn't die, but they came back as a zombie” (Luke, course 3). Independent work provided students the opportunity to push their boundaries. To modify and explore the models meant that working independently and using a level of self-initiated creativity and discovery was required of these students, regardless of the type of reference material provided. The situation pushed them to start to develop their own mathematical modelling process alongside learning and applying new mathematics.

5.2.2 Sources of pushing our boundaries

What appears to be one of the main reasons for *pushing our boundaries* existing is the experience was outside the normal range of mathematical activity for most students. Ngaire (course 2) said, “I've never had to put, um, a practical situation into a model, so creating a model for the situation. Never done that before.” Even when students had been exposed to mathematical modelling previously, they still found it tricky. Sophie (course 1) and Rhys (course 2) both describing it as “challenging”. From my experience, this is because modelling is a creative act that involves a unique application of mathematics for each situation. This implies that no matter what level of experience someone has with modelling, the nature of the real world and modelling means it is most likely you will be dealing with a situation that will have its own unique features. This uniqueness illustrates that regardless of the level of involvement someone has had with modelling, there will always be some level of uncertainty within a group of students.

Further reasons the experience was outside of students’ normal range of activity involved embracing new behaviours and ways of thinking including: working independently; relating and applying mathematics to contexts; using mathematics as part of a creative process; and there being no one ‘right answer’.

5.2.2.1 Everything new and unfamiliar

The newness of each modelling situation required students to embrace new behaviours and ways of thinking. Students in both courses experienced feeling ‘thrown in the deep end’ having to “learn how to think differently and approach problems differently to how you’re used to, to what you’re used to” (Rhys, course 2). This illustrates that the nature of mathematical modelling is foreign to students. Mathematical modelling’ s complex nature combined with students’ minimal or no previous exposure to it, make it difficult to know exactly how to best structure an activity that makes this change in thinking easily accessible.

New behaviours experienced by students included working independently of the lecturer without step-by-step instructions, relating and applying mathematics to the context, being creative in your application of mathematics to context, and not having one 'right' way of doing things. These new behaviours challenged students to adopt a new way of thinking and behaving for a mathematics student.

5.2.2.2 Working independently

Having to work independently of the lecturer was part of the activity for students in all courses. To help move forward at these times, all students initially looked at resources the lecturer had provided but found these did not provide the answers they were looking for. Stella discussed that while they were formulating their model they "found it hard because we can't find any sources from the course book. There was not much resource at all. The standard models didn't fit" (Stella, course 1). Nicola's group had a similar experience. "So, we actually went to, we were stuck and we were like, what the hell does the textbook say? The textbook doesn't say anything. The textbook was exactly what [the lecturer] gave us. There was nothing extra in the textbook. So, we were like, gosh, we have to work out this stupid ridiculous quadratic ourselves" (Nicola, course 2).

First looking for resources to guide them illustrates students wanted a more passive experience with worked solutions readily available to them. They did not initially respond well to being responsible for the solution themselves. This was most likely because their previous experience with mathematics had been with solution pathways being provided for them. By withholding details of these pathways, the lecturer was actively creating a situation where students needed to engage in finding ways forward for themselves. This experience of having to explore and navigate on their own appears to be relatively new and foreign to the students with “we were left to learn and work that out by ourselves” (Sophie, course 1) being a common comment from students.

What was different between the courses was the level of independent work required. By default, course one students were required to work fully independent of the lecturer, course two students had the lecturer partially available to them, and course three students’ experience was a mixture of course one and course two’s experience. From interviews with lecturer one, it was clear the lecturer expected the students to get their ‘hands dirty’ saying students “are not dealing with a toy little problem that you can nut out in five minutes…. I am not going to be there to help them out” (Lecturer 1). In contrast, lecturer two thought modelling was too hard for students to do on their own and expressed not wanting students to get “bogged down in the process” (interview summary, Lecturer 2). While lecturer three viewed students as “capable of developing a model with appropriate guidelines” (Lecturer 3). These differences highlight differing lecturer beliefs regarding the amount of support and guidance, balanced with independent work, required to learn mathematical modelling. These different approaches also allow us to glimpse at the views of the lecturers concerning whether they view modelling as something that is possible without ongoing expert advice.

5.2.2.3 Relating and applying mathematics to the context

Relating and applying mathematics to reality appeared to be a fairly new concept for all students. For example, Maria (course 2) stated that “when you're in school, no one understands how to apply math. You don't actually get to see how it's applied.” This seems to indicate for students that connecting mathematics and the real world was not a standard practice in general and not something they had experienced at school.

How students responded to this new practice differed across the courses and the participants. Steph (course 1) had a positive experience. She discussed how having a context to apply the mathematics to reality brought the mathematics alive for her, made it a lot easier to understand the mathematics and enabled her to see how she might use it. This may explain why she had struggled with this paper previously when it had not been so context based. Steph (course 1) stated:

> If I didn't do summer school and I just did 211 then we're just doing like differentiation and like, you know they're just equations but putting it into context, like oh, so this is how you model like a car moving up and down. It's way more interesting doing it that way.

Steph highlights that it was the context of "the real-life examples" that gave her something to relate the mathematics to, thus reducing the abstract thinking required. Steph's different experiences with both the semester and summer school version of the paper illustrates the potential using contexts with mathematics has. Steph's experiences suggest she obtained a depth and breadth of knowledge about the mathematics that may not have been accessible to her otherwise.

Rhys (course 2) also had a relatively positive experience. He found that the skills he had learnt in a previous nonrelated contextual based tertiary mathematics paper helped reduce some of the cognitive load involved with relating mathematics to contexts. Rhys (course 2) commented:

> Although the actual solutions and problems that we were doing are a lot harder than anything I've come across in that physics paper, I was still kind of able to work through it … all the skills specific to this I'd already learnt.

This tends to suggest that having historically been exposed to some of the skill set needed for the modelling activity meant that, even though it was still unclear at times, Rhys was able to navigate his way through the thinking needed for connecting the mathematics of his group's model to its relevant context.

In contrast, some students struggled with relating mathematics to context. Coming from a background where mathematics had traditionally been taught separately from context, the experience of needing to think about how to apply mathematics to reality was foreign to Natalia (course 2). She struggled with this process, finding it difficult to connect the mathematics with the context of the case study. Having English as a second language amplified this experience for her further. "This paper is quite hard for language. It is quite hard for me … it's not just the mathematics, it's the whole things" (Natalia, course 2). Ngaire (course 2), who was more comfortable working with the context compared to working with the mathematics, found having to think about context with mathematics added to her confusion. She became overwhelmed with the different aspects of reality having "too many things to think about", finding it hard to simplify the situation. The collective student experiences indicate that creating situations where students start to think about mathematics and context together could minimise the associated discord experienced by some students. However, it seems that there needs to be some thought about how these situations are constructed.

When students entered with complementary physics knowledge, it appeared to support their engagement with mathematical modelling. Those students who had a strong physics background seemed to embrace the newness of relating mathematics to context, finding it easier than those that did not. Course one participants, regardless of whether they had a physics background or not, all mentioned in their interviews that having this background made the modelling experience easier. Students who did not have a physics background further mentioned physics as an important attribute for successful modelling. Likewise, course 2 students who had a physics background, found the modelling challenging but could embrace the applied nature of the experience more readily than their fellow students. As physics[12] is an applied subject, that relies heavily on using mathematics to understand the real world, students with physics backgrounds already had some experience with this. Students who did not have a physics background were able to recognise their fellow students drawing on previous experiences applying mathematics to concrete examples, illustrating this as an attribute of modelling behaviour. This demonstrates the need for creating experiences for future modelling students that enables students to think about connections between mathematics and the real world early in their education.

As already starting to show, the student experience for relating mathematics to context was different depending on whether students were doing course one or course two. Overall, course one students did not really discuss relating mathematics to context, suggesting that they had accepted the need for doing this as part of the modelling process. In contrast, course two students did mention issues relating to context, suggesting that integrating the context involved some level of complexity for them. As mentioned, Natalia (course 2) and Ngaire (course 2) struggled with adopting this practice. Rhys (course 2) discussed how he perceived his previous experience helped in this area. These experiences appear to imply that the total immersion experience of course one students, and their previous experience, made it easier for students to accept and embrace relating mathematics to context.

The way the activities were set up contributed to the different experiences in relating mathematics to context. While students across both courses realised that relating mathematics to context was part of the process, the need to fully engage in this

[12] Though mathematical modelling is more than physics, due to its applied nature it is considered a part of Physics.

experience differed across the courses. Exploration appeared critical for supporting students in moving forward with the mathematical modelling. Course one students needed to fully engage with the context in order to recognise key elements that could be important for their future model. Sophie (course 1) said “I think though it didn’t give us the right answer … it gave us more details about the question”. The exploration of the context enabled her group, as it did others, to identify components of the situation in proposed approaches for the model development saying “at least it made us feel like we knew what he is talking about you know …” when referring to an approach proposed within her group. In contrast, course two students were provided the context and key elements reducing the need to explore the context. However, this did not minimise their need to understand the context. As can be seen from the experience of Natalia and Ngaire, the need to know how to manage the context was still important. When they came to modify their models, they struggled with identifying what was and was not important for their situation. Creating a situation where students need to explore the contextual background forces students to fully engage with it, understand it and start to develop strategies for identifying what is and is not important within the context of the situation. Handing the key elements of the model to students did not necessarily help students with their process. By doing this, it could have denied course two students the opportunity to engage with the context to such a degree that it would inform further workings needed as they worked through improving their given models. In comparison, individual engagement with the context by course one students meant they could appreciate contributions from their fellow group members. This created a situation for potentially recognising when proposed possible models could be worth further development. These experiences illustrate that strategies for engaging with the context either need to be provided for students or situations created where they can appreciate the need to explore the situation enough to be able to recognise key elements of the situation.

5.2.2.4 Using mathematics as part of a creative process

Students were used to mathematics being about getting things right, having limited solution pathways and repeating something that someone else had already created. They were not used to mathematics being a creative activity. This was seen through students wanting to be provided with 'how' to do something and being provided with 'the answers', not knowing what to do when coming across new mathematics and focussing on getting a 'right' answer.

As students approached the creative side of the modelling process, formulating and modifying models, they wanted specific guidance. When formulating their models, both Sophie and Stella first went looking for resources to follow. Sophie commented "we found it hard because we can't find any sources from the course book" (Sophie, course 1). While Stella said "it was difficult. There was not much resource at all." (Stella, course 1). Ngaire (course 2) remarked "we did not have the step-by-step is this correct or not", making the modification of her group's model "a lot harder." These responses show students were not used to thinking things through for themselves and found it challenging. This creative nature of modelling meant students often experienced unanticipated events. This included coming across unfamiliar mathematics and navigating multiple possible approaches for solution pathways.

While students were working through producing their models, they came across mathematics either they had not seen before or had not previously mastered. Sue's group produced some mathematics they "haven't studied" and made little sense to them. "Like it's too difficult to solve. When we put it in wolfram alpha, the result is something this complex chaos thing that we can't solve" (Sue, course 1). Steph's group could see the usefulness of learning the new mathematics they encountered saying even though “we didn’t learn the math [in the course], but it was still applicable to what we were doing” (Steph, course 1). Similarly, Natalia (course 2) discussed how “[the lecturer] gave us hints” for the mathematics that had not been covered in the course and worked through these ‘hints’ within her group.

These comments illustrate that students need to go outside of their comfort zone and seek out and learn new mathematics that may be needed for the situation. Being prepared to engage with new mathematics is important as while students may have a good mathematical tool kit, it is not always possible to predict all the mathematics that could be useful to model a situation. It is likely providing ways to approach new mathematics with help build students confidence to learn new material as the need arises. This seems to be an important key skill to develop for mathematical modelling.

Nicola and Rhys’s group (course 2) used mathematics creatively when they modified the model of controlling the introduced prickly pear cactus with an introduced Argentinian moth. In order to create a modified model, the group individually carried out research, then brainstormed together, to identify potential variables to add to the model. They identified temperature was a key missing variable from this process. A temperature function was selected, modified and then incorporated into their model. This creative process produced a new modified model. Nicola said “we wanted to change it [our model]. So, we tooted around with it and tried to see how the input of our new function would work. Initially it was a sine, but we changed it to a cos graph” (Nicola, course 2). See page 173 for the modification of the model.

Students across the courses dealt with being confronted with new mathematics differently. As already seen, course one students either could see the applicability of the new mathematics and pursued it, or when the mathematics became too difficult, they simply gave up. This suggests that learning mathematics students perceived to be difficult could be too much of a cognitive load on top of navigating through the modelling process. This cognitive overload could have been because it was the first time the majority of students were experiencing the modelling process. In contrast, course two students had the lecturer to ask advice of or provide steps forward. Natalia was very clear that it was the lecturer’s hint that enabled her to find the equilibrium points of her group’s model, producing the “analysis by herself after getting hints for direction to take from [the lecturer]” (interview summary, Natalia course 2). While Ngaire discussed how the lecturer’s guidance helped her learn “how the trace and determinant can determine positive and negative eigenvalues. We had to pull out a part [of the phase portrait] so that was really good. So, I'm quite happy that we pulled that apart ourselves after been given the guidance on where to start” (Ngaire, course 2). Having access to the lecturer enabled this group of students to independently navigate through the new mathematics and experience it from beginning to end, thus minimising the cognitive load for these students. The lecturer support helped to minimise feeling overwhelmed and ‘in the dark’ concerning the mathematics.

In contrast, course three students did not specifically discuss using new mathematics during the modelling activity in their interviews. This was most likely because the lecturer taught the mathematical model and the mathematics related to the model before the course three modelling activity. This left students to focus on translating and modifying the model to the situation they were modelling. Instead, course three students talked about a level of mathematics needed for the paper. Hope and Hannah initially struggled with the mathematics with Hannah (course 3) saying “it felt like the paper was for maths whizzes”. Hope discussed how she felt there was an expected background knowledge that not everyone had when they started the paper (interview summary, Hope course 3). This assumed mathematics knowledge was most likely because the lecturer presumed students had covered the mathematics at secondary school. Lucas (course 3) said “I found the first two modules very easy, I had done the mathematics at school”. Hope and Hannah’s different experience and the misalignment with the lecturer’s assumption suggest a variety of mathematics backgrounds coming through the New Zealand education system.

The different experiences across the courses regarding how students dealt with new mathematics appears to be related to the scope of new mathematics encountered by students. For course one, due to the open-ended nature of the activity, the new mathematics students came across was not limited. The lecturer was not present and could not give hints. If the lecturer was present, the range of new mathematics students encountered to provide hints for, could also be overwhelming for the lecturer. For course two, the new mathematics was contained within the activity. With the lecturer present, and possible mathematics to be used is potentially already known, hints are more easily given to students. The different experiences suggest that if the focus of the modelling activity is on learning the process of modelling, it is encouraged that students draw on mathematics that is already known to them, so not to overwhelm students. If the focus is on learning new mathematics, it is suggested to have the lecturer present to provide hints.

To add further complication for students, decisions needed to be made concerning how to proceed. All student groups needed to weigh up and explore different options. Taking responsibility for the process appeared to be new for most students and seemed to be influenced by searching for a ‘right’ way to do things. Ngaire commented, “there was so many of if this happens then this would happen, but if that happened then this would happen. And so, um, yeah, it was quite hard trying to make sure that we got it absolutely right.” (Ngaire, course 2). Steph remarked, “in the process we came up with three different solutions to the same problem but each one, we weren't quite there" (Steph, course 1). Don recalled "one of our team members was like, oh let's just flip things around and get a right answer" and discussed how he intuitively knew "that the procedure to get that right answer wasn't right" (Don, course 2). One student's (Rhys, course 2) thinking concerning the possibility of different answers was more developed, though the student was still primarily concerned with finding "the best option" while exploring different alternatives. "When it comes to making a modification and discussing limitations, that's something that's completely open to discussion about what's the best method and what's the best option and you've got so many choices" (Rhys, course 2). These examples show that overall students were focussed on attaining a 'right' solution, illustrating they were used to the mathematics culture of 'model' answers. Most students were yet to grasp that modelling was not about producing a textbook answer. They appeared to be heavily influenced by thinking that mathematics is about getting a right or wrong answer. This illustrates that the experience was a new way of using mathematics for the students.

The structure of the activities appears to have influenced how readily students adopted the creative side of modelling. As already alluded to, due to the nature of the modelling activities, course one students soon realised that they needed to overcome looking for a right answer and needed to be creative and explore different options. Having the modelling activity being a low stakes assessment also helped encourage these creative behaviours with Don commenting "not having the pressure of having to get it right, gives you like more room to come up with your own procedure" (Don, course 1). The teaching of the model and its mathematics beforehand for course three allowed students to create new mathematics by translating and modifying the taught model to their modelling situation. When asked what the lecturer did that enabled his group to produce the model, Leo's (course 3) reply was "the SIR model. We basically took that model and changed it. This was the previous knowledge we were relying on. It prepared us.” Other students commented similarly to Leo.

In contrast, not all of course two students were able to fully embrace the creative side of the modelling. The need to learn new mathematics while working through the modelling activity, appears to have stifled the creative aspect needed for students who found the new mathematics challenging. They appeared to be reliant on guidance and this restricted the development of the model to be in line with the model given in their resources. Ngaire (course 2) illustrated her reliance on guidance by saying:

> Looking at the case study was good because it meant that you knew if you were right or wrong, so you can still do the working out by yourself, or within the group, but there is like that level of like, oh yeah, okay, I'm on the right track and so you can keep going. And then if it's wrong, you know, it's wrong at each point, which is nice.

For students to move beyond what they have been used to, an openness to a new way of using and doing mathematics needs to be cultivated within students. This is because the situation requires them to use mathematics in a creative way as opposed to reproducing already developed mathematics. Rhys commented to be successful with this new way of using mathematics involves “kind of adjusting my way of thinking and stuff and trying to re-learn how to answer questions differently” (Rhys, course 2). “It’s … not something that … comes naturally to people who have been studying mathematics” (Rhys, course 2).

5.2.2.5 Experiencing struggle

As students worked independently, relating mathematics to reality and using mathematics as part of a creative process, they experienced struggle. As already seen struggle occurred when prescriptive resources were unavailable. Students also experienced struggle when they did not feel fully prepared for the modelling experience and while grappling with how to simplify their situation in order to move forward. Students needed to learn how to navigate struggle in order to move forward.

Students did not feel fully prepared for the experience, with most students having times where they felt overwhelmed. One student, Sue, described the experience as "just too intense" (Sue, course 1). While Rhys (course 2) commented that he felt "thrown in the deep end" with the "the different way of thinking that's required for mathematical modelling". Ngaire said "we learnt a few in class but then we'd jump to these ones that we'd actually have to really work at" (Ngaire, course 2). Sophie (course 1) commented they had been given standard models in class but none of them fitted the actual situation. Even though course three students felt prepared via the lecturer providing the foundation model to use students expressed there were still a lot of things to think about. Hazel (course 3) remarked:

> Which humans would kill some sort of vampires. But like, would the vampires kill the humans or just try and infect them to turn them back? And did the feral vampires that died and then came back to life, if you killed them again would they come back again?

These comments illustrate students did not think lectures fully prepared them for the experience, and suggest students had to grapple with the resources they had been given, really think about the situations they were modelling, make decisions, and apply themselves to find approaches that would work for their current situation. They needed to grapple with the 'one size does not fit all" concept that is unique to modelling and different to their previous experiences with mathematics. In order to move forward students needed to be resourceful and extend themselves, by both digging within themselves and seeking external resources.

Most students grappled with how to simplify their situation so they could better understand what to do next. Hazel (course 3), Ngaire (course 2), Stella (course 1) and Sue (course 1) all commented that there were too many things to think about. Stella (course 1) specifically mentioned "it was too complicated". While Sue said, "the truth was we'd already made it really simple" (Sue, course 1). These comments illustrate that these students grappled with both coming to terms with the process and how to simplify the situation to make sense of what they needed to do next. Being able to identify what is important for the situation and being able to have an idea of your process, is an important step in moving forward with model development.

Struggle was experienced differently depending on which course you were doing. Struggle was either a central part of the activity or the level of struggle minimised. The difference in the activity set-up produced a different focus on struggle and hence a different experience for students. For course one students, struggle was at the forefront of the activity. While, for course two struggle was initially minimised. For course three struggle was an underlying characteristic of the student experience.

Course one students were actively grappling with struggle to figure out how to move forward. With the lecturer not immediately available, this was the only way they were going to proceed. For course one students, they needed to think out how to put their model together, including deciding on a workable base model, the pieces to use to make up the model, and similar to course two and three, negotiate everyone's different ideas. Exploration and going down dead ends were a key part of this process. As seen through Sophie's (course 1) experience, out of this students learnt to recognise what was going to be useful and what was not. Steph (course 1) and Stella (course 1) mentioned how the final versions of their models were based on or contained parts of their first attempts at developing the model. Steph (course 1), after several attempts at developing a model, felt confident that the final model her group settled on was worth spending time on as they had “got the basic idea from the first model”. Steph’s (course 1) group used “half of what [they] did at the start” in combination with further attempts to produce their final version of the model. These comments illustrate the initial struggle involved with exploring an unfamiliar situation and how this struggle helped students develop confidence and direction with their process. This demonstrates the importance of exploration as key for finding a way forward with model development.

Course three students also were actively grappling with struggle to figure out how to move forward. Struggle with choosing the base model was removed for these students. Decisions still needed to be made as to how to proceed with model development. Hope (course 3) said her group was “struggling at some point. They were disagreeing, like half the group saying one thing and the other half was like, "nah, maybe it's like this. Kinda thing, you know?” Hannah (course 3) mentioned her group also “had a lot of debate” when they were grappling with how to proceed. Course three student experiences shows struggle was an underlying component of their experience and was navigated through discussion. It is highly likely that working as a group was one of the factors needed for students to make progress with the model development.

In contrast, struggle was initially minimised for course two students. With the initial directive approach of course two’s activity, struggle became secondary to working through the material. They were able to move forward using the case study to provide direction for the model. Even with direction provided, there were still things that were not clear for students that produced challenges. Nicola (course 2) remarked:

> We had a little difficulty understanding the model, in particular one of the variables. And we had quite a hard time understanding exactly what they meant with this variable and how it affected the model. So in that case we had quite a hard time. And then [the lecturer] kind of helped us figure it out but it took us a while.

Ngaire’ s (course 2) group also experienced struggle, with Ngaire saying “we did have to work it out as well, give more information than the textbook gave. That was challenging. [The lecturer] helped point us in the right direction”. Nicola and Ngaire’ s group’s experiences illustrate the lecturer was needed to help resolve their difficulties and thus continue to move forward. This was most likely as the models and aspects of their mathematics were new for students.

Course two students found it a big ‘jump’ to move from the directed to self-directed part of the activity. When it came to the self-directed component of course two’s activity, while all students made progress, not all students managed to produce a reasonable modification of their model. Nicola and Ngaire both discussed needing several examples to internalise new processes. Nicola commented “so to move forward from here I’d have to go back and learn by myself the entire process again from the basics” (Nicola, course 2). Ngaire said “it’s not enough to know the process. It’s more individual to the case studies rather than a-a um underlying process” (Ngaire, course 2). While Rhys remarked that his biggest challenge was adjusting to this new way of thinking. These student comments show that more process examples and engagement with these examples could have helped students to get to know this new process and minimise their struggles with understanding it. In the initial part of the activity, many decisions regarding process had been made for students. Not providing students with space for supported exploration experience could have inhibited students fully grasping the concepts involved with the process of mathematical modelling. This could illustrate that exploration, along with frequently alluding to the modelling process during lectures, is a key component for developing mathematical modelling skills. Even though the struggle was reduced for students in course two, it could still play a vital role for students. The key could be in finding the balance between struggle and direction.

How struggle influenced finding ways forward will be explored more in detail in Transitioning between uncertainty and having direction, Chapter 8, Section 8.2, page 239.

In summary, *pushing our boundaries* occurred for students due to modelling being a new activity, requiring students to deal with different, unfamiliar situations, skills and processes outside of the scope of what students had previously experienced. Modelling was clearly a change of culture for students, being outside the range of their normal mathematical activity. Modelling involved periods of independent application and navigation of mathematics and the modelling process for students. All lecturers had different viewpoints on how accessible and possible an independent modelling experience is for students. Even so, all students were required to work without clear guidance for differing periods of time. Students initially resisted this, being used to being shown what to do each step of the way while studying mathematics. Generally, students were not used to using mathematics with contexts, including the need for understanding the context of the situation and finding ways to relate the context to mathematics. The exception was for students who had previous physics (an applied subject) experience. These students were more comfortable relating mathematics to real-world contexts than their peers. For all students, using mathematics as part of a creative process was a novel experience. A certain amount of confidence, self-initiated creativity, discovery and application was needed for students to grapple and make progress with the modelling. All of these things contributed to pushing students boundaries.

Out of struggle, students developed behaviours in order to move forward. There were two different states of moving forward. Students were either *moving forward on their own* as a student group independent of the lecturer or were *being moved forward* through advice or guidance provided by the lecturer. I will now explore students *moving forward on our own* independent of the lecturer.

Chapter 6. Findings: Subtheme one, moving forward on our own

6.1 Moving forward on our own

In order to make progress, students needed to learn how to move forward on their own. While they still had each other for support, students needed to work through mathematics and parts of the modelling process without the help of the lecturer. Moving forward involved taking responsibility, both individually and as a group, to actively find ways forward. Working independently of the lecturer provided opportunities for learning by doing, giving students a chance to push themselves to figure out how to proceed. All students were provided opportunities to work things out autonomously and experienced moments of both being uncertain and having direction. Taking responsibility and struggle were part of students navigating moving forward on their own.

I will first discuss students taking responsibility for working independently, followed by student coping behaviours developed while working on their own.

Part of working autonomously for students is taking responsibility for developing your own solution. Don commented "you had to, rather than just figure out, like how to solve the model, you gotta also know, you've gotta see if you can figure out why your model isn't working" (Don, course 1). Ngaire found taking responsibility "challenging", saying "we were a lot less confident with it because it wasn't just reciting from the thing [case study], it wasn't like the answers were there in the textbook. There were workings of the model, but we did have to work it out as well and provide more information than what the textbook did" (Ngaire, course 2). These examples illustrate students had experiences where they were 'on their own'. The absence of the lecturer provided space for students to 'have a go' and learn to *back themselves*. As expressed by Don and Ngaire, students did not always find working independently easy. Even so, it was important for the lecturer to not provide all the answers to enable the students to engage with the situation and think independently. This promoted learning by doing, giving students an opportunity to push themselves and to experience independence. Working independently could be a very important part of the process for learning to mathematically model. It enables students to be 'thrown in the deep end' to have an active experience of grappling with the material. Having to work on their own enabled students to experience the modelling process as a mathematical modeller would by not having all the answers immediately available and to potentially develop resilient behaviours to persevere when doing mathematics 'gets tough'.

Lecturers were consciously creating opportunities for students to work independently, to persist to find a way forward, whether it was with the mathematics or the full process of modelling. As seen in lecturer interviews, all lecturers believed students needed to experience modelling themselves to begin to comprehend the process. Lecturer one was so convinced that independent work was important that one of his student learning objectives was for students to have a go at "figuring things out". Lecturer two stated independent work gave students "an opportunity of getting to deepen their own understanding about how things work". While lecturer three thought that "it might not be until they have their own project … and then they come back to what they've seen that they can then actually absorb it for the first time". These lecturer beliefs allowed students to experience some of the nature of mathematical modelling itself. That is, the way is not always clear and there is not a 'one size fits all' answer when it comes to modelling. By providing students space to work things out by themselves allowed students to start to experience and grasp this important aspect of the nature of mathematical modelling. This helped students to experience one of the necessary behaviours for mathematical modelling: to work through things without clear direction and back yourself, regardless of the outcome, moving forward. The active encouragement of working independently illustrates that the lecturers believed that working independently was an important part of the modelling process and the learning process.

These student and lecturer examples also show that students moving forward on their own was not straightforward and involved more than working through some mathematics. Navigating the new 'process', including the mathematics involved, was not simple and did not always go to plan. Not producing a usable model the first time around was a common experience for course one and three students. Sue's group initially produced a model for the size of a crater that gave an unrealistic result, saying "the result we get I remember was 8 something times 10 to the power of negative 13. It just means it would create some dust [not a crater]" (Sue, course 1). While Hannah expressed getting started with the modelling process "was a bit hard working it out in the beginning" (Hannah, course 3). Course two students described grappling with the process, with Ngaire saying it was "a lot harder because we did not have the step-by-step is this correct or not? So, we didn't know which way to go, which made it a lot harder." (Ngaire, case 2). Exploration, going down dead ends and a cyclic process was part of these students' modelling processes and was needed in order for students to progress.

Exploration was a key student behaviour used by students to navigate the modelling process. Exploration was enabled for course one and three students because the lecturer was unavailable. The aspects of the structure of the different activities also promoted exploration. For course one students, this was through the open-ended ill-defined nature of the activity. For course two and three students, it was through being only given the foundation model to base their own models on. As already seen, students were used to being provided with all the answers. Part of a mathematical modellers process is not having all the answers immediately available, making exploration an important characteristic of the student experience. Needing to apply yourself and seek help outside of yourself in order to move forward is an important behaviour of a mathematical modeller. I will now discuss the coping behaviours students developed while independently modelling.

6.1.1 Coping behaviours

As students learnt to take responsibility as they worked independently, they developed and displayed behaviours that helped them progress with the activity. These behaviours included *backing themselves, being open-minded, actively seeking reassurance* and *working together.*

6.1.1.1 Backing ourselves

Due to needing to self-direct, all students were presented with opportunities to learn to back themselves, either as a collective or individually, in order to move forward. Learning to back themselves was integral to the student experience and primarily involved making decisions.

Decision making was a key component of the experience for all students. Decisions were made concerning how to develop and/or modify the foundation models and regarding how to progress with the mathematics to use for their models. Making decisions included how to proceed, sourcing information, taking risks and making sense of conflicting information.

6.1.1.1.1 How to proceed

Decisions were needed to be made in order to move forward. For Ngaire, whose foundation model was provided in the activity, the way was not clear. “There was so many of if this happens then this would happen, but if that happened then this would happen” (Ngaire, course 2). For Sue, once her group had decided on their foundation model the work had just begun. “Even though you knew the base equation, it had so many things to think about” (Sue, course 1). Shelly commented “we went down a lot of rabbit holes and then had to come back up” (Shelly, course 1). These experiences show modelling for students is a complicated process, involving deciding what to do next. Making decisions included exploring and evaluating different possible factors relating to the situation and approaches to be taken. These student comments reveal that students were given the responsibility for their process, needing to think through and make decisions and work through the results of these decisions. Being given responsibility enabled students to discover their own pathway for their mathematical modelling process and navigating new mathematics. This also resulted in students having responsibility for their own learning process. Working through the specifics for their situation provided valuable experiences with key components of the process. These experiences created memories of possible approaches and strategies so that students would have them available for use with any future modelling activity.

Examples of making decisions concerning how to proceed with the mathematics to use with their models were seen across all courses. Depending on the course, decisions were made involving which mathematics to use, how to navigate forward with new mathematics and how to represent and add new variables and functions into a model. For course one students, these mathematical decisions involved going down a lot of dead ends and abandoning things as they just “got too hard” (Sue, course 1). Shelly said her group “started going in loops because it just kinda wasn’t really working” (Shelly, course 1). While Sue's group "came up with u t something [they] haven't studied" so changed their approach suggesting that students abandoned approaches that produced mathematics outside of their current knowledge range. These examples show course one students tended to learn new mathematics that was based on mathematics they were already acquainted with, ditching any approaches that took them too far out of their comfort zone. Choosing to use familiar mathematics, instead of persevering with more difficult material, could be because students were experiencing a high cognitive load already with deciding on which mathematical approach to use. The focus for these students became the decision making for choosing which mathematics to use, as opposed to learning new mathematics. Learning new mathematics, on top of decision making in a short time period was just too much. This implies the modelling experience should either be primarily focussed on process or on learning new mathematics.

In contrast to course one, course two and three students were given the mathematical basis to work with, meaning they did not have to make initial decisions on the mathematical approach to take. Instead, their decisions were primarily focussed on how to mathematically include and represent new variables and/or functions into their models. Nicola's group referred to using external information to help them make decisions. It was the background research they had conducted individually and then discussed together that gave them the confidence to move forward with their approach. Nicola commented "we were discussing different limitations. I was like, why don't we just do temperature that the moth best survives at, and then [my group member] was like, Oh I read something about that" Nicola (course 2). This suggests that it was the intersection of information group members identified individually as relevant that groups chose to further investigate, thus providing direction for the group to proceed. The following is Nicola's groups modified model. The highlighted extracts are the modification of the model based on the study by Legaspi and Legaspi (2007).

“Modification

We will amend the system of equations by including a function of temperature over time ($f(T)$) to the equation for the change of moth population over time.

$$f_1 = \frac{dV}{dt} = r_1 V\left(1 - \frac{V}{K}\right) - c_1 H\left(\frac{V}{V + D}\right)$$

$$f_2 = \frac{dH}{dt} = f(T) r_2 H\left(1 - \frac{JH}{V}\right)$$

Where $Model\ of\ temperature f(T) = \begin{cases} 1, & 299.15K \leq T \leq 307.15K \\ \frac{10}{127}T - 22.5551, & T < 299.15K \\ \frac{-1}{16}T + 20.1969, & T > 307.15K \end{cases}$

Where T expressed with respect to time (t) is $T = 5\cos(2\pi t) + 294.15$”

(Nicola & Rhys presentation 2, course 2).

Nicola mentioned “it was an easy amend to the model. We didn't actually have to write like a whole another, just like touch it up. We put another actual equation into the model. We had three equations for the modification of the model” (Nicola, course 2). Being able to locate a similar model eased some of the cognitive conflict regarding making decisions on how to proceed and thus gave students access to being able to modify their model.

For Hannah, Hope and Luke’s course three groups, decisions regarding how to proceed with the mathematics for the model were shared by the group but appeared to be led by the more confident students. Hope and Luke both mentioned ‘taking-a-backseat’ when the mathematics became unfamiliar to them “when they were adding the carrying capacity and stuff like that, that's where I just don't really understand” (Hope, course 3). Both Hope and Luke mentioned that they had not gone over the material or fully grasped the material before beginning the modelling activity. It appeared that due to very specific preparation given by the lecturer on the foundation model to be used, students who had prepared beforehand found moving forward with these decisions relatively easy and tended to lead the modelling. Hannah illustrates being prepared saying "we had been given all the information that we needed, so we had already learnt about the SIR model, and he'd shown us how to manipulate it" (Hannah, course 3).

What was it about the structure of the activities that created these different experiences? As seen above, the way the activities were set up and the preparation provided contributed to the different experiences in making decisions as to how to proceed. For course one's activity, total immersion experience produced a high cognitive load resulting in students focussing on process as opposed to learning new mathematics. The construction of the course two activity aimed to minimise the cognitive load for process, allowing students to focus on learning and applying mathematics in a new way. While course three reduced the cognitive load even further with decision making by providing the foundation model and role modelling how to modify it to the students before they attempted the activity. However, more time was needed for some students to fully engage with the mathematics needed to develop the model. The different experiences of the students illustrate a modelling experience should focus either on process or on learning new mathematics, as opposed to both. While course one students tended to abandon the mathematics when it became too hard, and course three students either took a backseat or lead the group's mathematical workings, course two students persevered with grappling to comprehend and master any new mathematics. Like course three, course two students did not have a choice about the mathematics they could use. They were given the type of mathematics to use and had to persevere with comprehending and mastering it. Some groups sought guidance from the lecturer while others were able to independently navigate their way forward with it. Natalia (course 2) discussed how when her group came across new mathematics the lecturer "gave us hints". Ngaire, Natalia's partner, said the lecturer gave them "guidance on where to start. After that we could pull the mathematics apart ourselves. I was happy" (Ngaire, course 2). In contrast, Nicola's group persevered and mastered the new mathematics themselves. Nicola (course 2) said:

> We were stuck, and we were like, what the hell does the textbook say. The textbook doesn't say anything. The textbook was exactly what [the lecturer] gave us. There was nothing extra in the textbook. And so we were like, gosh, we have to work out this stupid ridiculous quadratic. And, so we just, we did it by parts. It was really slow going, but eventually we were able to work it out.

These examples show that course two students appeared more readily able to learn new mathematics. This was most likely because the decision making prior to encountering new mathematics had been minimised due to the structure of their activity. It could also have been that these students had stronger mathematical backgrounds than course one students and the lecturer was available to students if needed.

Moving forward with the mathematics appears to involve persistence and some sort of resilience. Lecturers can make it easier by providing foundation models and being available for advice. One danger in doing so could be that by removing the opportunity for students to have the opportunity to make the decisions about what mathematics to use, the lecturer could unknowingly deny students a learning experience in how to recognise when a particular foundation model, or piece of mathematics, would be useful or not. If the lecturer was going to provide this type of experience, they may need to be prepared to value process as much as model development. At the end of the day, what the lecturer wants the student to experience, or values as important, will dictate the experience of the student i.e. experience of process or developing a mathematically advanced model.

6.1.1.1.2 Sourcing information

Decisions around how to source information and what information would be useful needed to be made. The student experience for this involved how to decide if this information was going to be useful or not and navigating tensions concerning who should be responsible for sourcing this information, the students or the lecturer.

Some students found it difficult to know what information was needed. Drew's group really struggled to "research the problem and wrap [their] heads around the problem" (Drew, course 1). While David (course 1) found it difficult to know how to decide what information was going to be useful to use. David questioned how reliable the information his group found was. He personally did not feel confident to discern when this information was trustworthy or not. He had held a view that lecturers and authors of textbooks were the authorities when it came to information, believing that learning is about receiving information from these sources. "I take an exercise book and I take it as truth you know, so I can't fill my own answers in because that's why I am here to receive knowledge from others [the experts]" (David, course 1). For David, to be put in a situation where his group needed to source and decide when information was useful, made him uncomfortable and he struggled. This was particularly apparent when discussing using the internet for research. "Probably the next two hours or so we spent just looking on the internet for relevant citations … and data that we wanted with appropriate references … you know from decent websites ideally. That was very hard" (David, course 1). For Drew's and David's groups, not knowing how to manage sourcing information meant they struggled with really getting to know the context of the problem, and hence couldn't efficiently decide how to scope the context of the problem and got stuck here.

As already seen, for some groups the intersection of information that group members identified individually as relevant, influenced decisions on whether this information was going to be useful. This was also seen when groups who had spent time getting to know the context of the problem were more readily able to recognise when information would be useful and reliable. As already seen in the section on relating and applying mathematics to context, course one students used findings from their earlier exploration of the context to help determine when something would be useful or not, and appeared to be able to determine when the found information would be of value. Course two students found that when they had spent time further exploring the context of their case studies, things went smoothly when deciding what was going to be useful to include in modifying the model. When students in either course tried to bypass this step or found it difficult to access information to understand the context, they progressed slowly as they had not been able to determine key elements of the situation which hindered their progress moving forward. Whether students willingly accepted responsibility for finding information or not, all students were actively seeking pieces of the puzzle. It appears ways of determining credibility of information and an appreciation of the importance of understanding the context of the situation are needed. Trusting that the information you are using will be useful, as well as understanding the context, are important to reduce struggle in moving forward.

Comparing the experiences of students between the courses reveals there were tensions between who was responsible for providing any missing information. While course three students didn't appear to need any additional information as the lecturer had provided this earlier, course one and two students needed to source additional information. Course one initially resisted taking responsibility for additional information themselves. Due to the set-up of the activity they quickly came to realise they needed to find and source material themselves. However, for course two, students expected gaps in knowledge to be filled by the lecturer. Why this might appear to be obvious for both courses, with course one working independently of the lecturer and course two having the lecturer available, what actually happened for course two was the lecturer only provided hints and signposts. This meant course two students also needed to source missing information and worked through how and if they would use it. With students' previous experiences with prior mathematics courses and the lecturer being accessible, this created an expectation from some of the course two students that the missing information would be provided by the lecturer.

All lecturers purposely withholding step-by-step workings seems to suggest the lecturers were wanting students to start developing skills for independently sourcing and making decisions about information needed to move forward. As stated before, opportunities to work independently are important, as being able to actively seek out information and figure out what to do with it is an important part of mathematical modelling behaviour.

6.1.1.1.3 Taking risks

Backing yourself for students also meant taking risks through speaking up, sharing your work and ideas. This involved students putting forward ideas for scrutiny and, for course two students, standing up and presenting work in front of the class. Taking risks involved being vulnerable for a lot of students.

Feeling vulnerable was a common experience for students. Students expressed different feelings when it came to sharing their ideas and work, either within their groups or to the whole class. It was observed "several students were apprehensive and nervous about presenting their work to the class" (descriptive field note, course 2 observation). Natalia (course 2) was so nervous about presenting she did not remember any of the feedback the lecturer gave her and was only conscious of "phew it is over, now I can relax" (Natalia, course 2). In contrast, Nicola (course 2) was not fazed presenting, saying "I had presented before so I never had to worry about it". The reason for students' nervousness could be because they did not feel confident about the content they were presenting, were not used to presenting in front of others or, as Natalia (course 2) expressed, due to giving the presentation "in English, as if you use your own language it does not matter" (Natalia, course 2).

What was common for a lot of students was the more confident they were in what they were presenting or talking about, the least vulnerable they felt. Students felt the most vulnerable when they were the least prepared and unsure of the content they were working with or presenting. Hope (course 3) and Luke (course 3) both expressed wanting to be in groups where everyone was of a similar ability. Luke viewed this type of group as "a safe zone" (interview summary, Luke course 3). Similarly Hope (course 3) voiced she "did not want to ask questions as she was concerned about looking 'dumb' in front of others" (interview summary, Hope course 3). This shows that Luke and Hope felt vulnerable within their mixed ability groups. This was most likely because Luke and Hope both were not confident with the mathematics needed for the development of their groups models and hence did not feel fully prepared for the situation.

Interestingly, students in the course one total immersion experience mentioned feeling uncertain as opposed to vulnerable. Course one students accepted that each other did not individually know all the answers. According to Sue (course 1) her group pooled their ideas together. This created a situation were initially no one's ideas were considered better or worse than anyone else's. This placed students more on an even playing field with each other. This suggests that for students to feel like it is okay not to have all the answers, the nature of the modelling activity is important.

Depending on the nature of the activity, preparing students in the mathematics needed for the activity, preparing students with how to give presentations, and grouping students in similar abilities could reduce some of the inadequacies, real or perceived, felt by students. However, these solutions could be counterproductive. Being prepared may help students to confidently speak up. Being too prepared may also stifle students' creativity. The more experience they have working with open-ended modelling activities may make students more aware that they do not have to know all the answers all the time, and not knowing all the answers allows them to be more creative. Being able to have the confidence to share ideas and thoughts with the group is important as it makes it possible to navigate and find ways forward with model development.

Creating situations where everyone feels they can contribute appears to be an important part of mathematical modelling, and a solution for minimising student vulnerability. Having access to and being open to different perspectives helps individuals push past their current thinking of the situation to gain a deeper insight into the situation. This is an important attribute for modelling, allowing a student to think outside of their 'normal' to gain insight into possible ways to approach the situation and being able to embrace the complexities of trying to capture a real-world situation.

6.1.1.1.4 Making sense of conflicting information

Students encountered a lot of situations that had the potential for cognitive conflict. These situations were either from within their working groups concerning different possible approaches, getting ridiculous and/or unrealistic answers, the material provided, or standard models not fitting the situation. While Ngaire discussed how when watching other groups present their models gave her exposure to “different ideas” and helped her to “come up with different ways to do ours as well” (Ngaire, course 2), Shelly talked about how everyone in her group was “quite different in their opinions on how to solve it” (Shelly, course 1). These examples show how the experience was more than what it appeared, having many levels of complexity and depth. It was not straightforward, with lots of different aspects and possibilities to consider.

Multiple perspectives produced cognitive conflict, providing situations where students needed to consider readjusting their thinking or adding onto their current thinking about how to approach things. By the lecturer not providing all the answers, enabled students to experience this cognitive conflict. Comments from lecturer interviews showed that lecturers wanted students to be critical and think deeply. Lecturer one expressed “I really want them [students] to engage their brains, [figure out] what do I put in, what do I leave”. Lecturer two wanted the students to be “discussing and criticising the models and their approaches”. While lecturer three wanted students to independently change and modify given models. These comments illustrate the lecturers’ beliefs that it was important to fully engage with the material so that multiple options were considered, and decisions were made. This created situations where students needed to back themselves concerning ways to move forward.

Backing ourselves is an important behaviour for students to be experiencing in mathematical modelling education. Providing opportunities for students to take responsibility helps students to develop resilience and confidence to step out on their own and take risks. This is an important part of the modelling behaviour. Students need to step out into the unknown and be creative as they relate mathematics to a real-world situation. As alluded to, as students navigated through making decisions, they needed to be open-minded, considering others’ way of seeing things and how these alternative perspectives can help move things forward. This leads us into exploring *being open-minded* as another behaviour students adopted to move forward.

6.1.1.2 Being open-minded[13]

Being open-minded involved being open to new thinking regarding behaviours and processes of mathematical modelling. Due to the complexity and nature of these new behaviours and processes, it becomes imperative for students to be willing to listen to and consider, and at times embrace, other people's ideas and approaches to enable them to move forward. Steph (course 1) summed this up when she said, "it was really good to have other people's opinions to help you along there cos you could get stuck and just look at it in one way." Don (course 1) commented "there were points where we stopped working on the project for a bit, just to discuss a concept that we hadn't heard before". On seeing other people's proposed approaches, Nicola remarked "you never thought that it could be done that way, like it gave you ideas" (Nicola, course 2). Similar to other student comments, Hannah (course 3), Stella (course 1) and Shelly (course 1) all mentioned that a substantial amount of their respective group's time was spent discussing, critiquing, and revisiting the best way to approach things for their group. These commentaries illustrate students needed each other to help look at their approaches in a different way and demonstrating *being open-minded* is necessary in order to make progress.

6.1.1.2.1 A new way of thinking

All students discussed the experience as being about a new way of thinking and doing mathematics. New things included relating mathematics to a context and using new thinking and behaviours to navigate through the process that used mathematics.

[13] This section includes re-presentation of material published in Spooner (2022).

Relating mathematics to context was an integral component of the experience. Students across the courses had varying degrees of experience with relating mathematics to context. Students who had previous experience with physics tended to be more comfortable with applying mathematics to the real world. As discussed by Freddy (course 1), having a physics background made you aware that relationships exist between mathematics and the physical world "giving you a place to start". Among the students that had little or no experience with relating mathematics there were conflicting experiences. Some students struggled with relating mathematics to context. This was the case for Natalia (course 2) and Ngaire (course 2). Coming from a background where mathematics had traditionally been taught separately from context, Natalia found the experience completely foreign to her. English as a second language seemed to amplify this difficulty for her. "This paper is quite hard for language. It is quite hard for me … it's not just the mathematics, It's the whole things" (Natalia, course 2). Ngaire (course 2) found thinking about context with mathematics added to her confusion and she became overwhelmed with "too many things to think about", finding it hard to simplify the situation. Steph (course 1) had a different experience. She expressed appreciating relating mathematics to context saying it really "brought mathematics alive" for her and made it easier to understand the mathematics.

These examples illustrate that students who could see how mathematics and reality could be related coped better with the modelling than those that did not. Reasons students may have found seeing the connections between mathematics and reality difficult could be due to them having minimal previous experiences using mathematics with context and not knowing how to identify the important parts of the context, whereby they start to simplify the situation to make connecting it to the mathematics easier. Steph's experience of having the mathematics brought alive illustrates there is potential for relating context to mathematics to give mathematics some relevant meaning for students. The experiences of Natalia and Ngaire suggests this needs to be managed somehow to make the connections between mathematics and reality easier for students. This is important because relating mathematics to context is a central aspect of mathematical modelling.

6.1.1.2.2 Being open to other people's thinking and ideas

All students in both courses experienced that others perceived things in different ways. All students saw the benefit of each other's contribution. What differed was the level of conflict produced from differing perspectives across the courses and how each individual group dealt with any conflict that arose. There seemed to be more potential for conflict with course one students. David's (course 1) group spent a lot of time arguing about their different perspectives, although they could recognise the benefit of conflict, with David remarking "yeah you know, arguing's good as well." Instead of arguing when they disagreed about things, Sue's (course 1) group pooled everyone's contributions saying, "we just put them all together… we put all these important factors together into an equation". Hope (course 3) recalled how her group was "disagreeing, like half the group saying one thing and the other half was like" and managed to resolve the conflict through "asking each other questions" about their ideas and "going backwards and forwards" discussing the different approaches. Hazel (course 3) mentioned a similar experience for her group. This illustrates that even though the groups had different ways of dealing with conflict they still saw the value in other group members' contributions. By sharing different viewpoints, students' options to consider for moving forward increased.

In comparison, course two students appeared to experience minimal conflict within their groups. They actively welcomed each other's input and acknowledged that they would not get as far without each other. Rhys (course 2) said "being able to work with lots of people … helps … um … cos lots of different ideas coming from other people helped me and our group prepare in a way where we could kind of cover the whole thing really well."

One reason course two students had a more passive experience considering different perspectives in contrast to the other courses was most likely due to having access to the case study. The case study set the general direction of the model. The need to negotiate different ideas and approaches was minimised to become predominately about which mathematical technique should be applied and how to improve the model. Deferring to the stronger mathematicians for mathematical techniques naturally happened within the groups. For instance, in Maria's group two members worked through the mathematics on the whiteboard. Maria then went home and worked through it, learning as she went, even finding "a few mistakes" in the workings. In contrast, for course one the whole process had to be negotiated and this fell on the whole group to contribute to brainstorming different approaches and perspectives so they knew what could be possible.

To be able to make the most of what their fellow course members offered, each student needed to be flexible and receptive to ways of doing things that were not the same as their own. The need to be open and receptive appeared to be accepted by all students. Initial exploration and work done during the early stages of the course one and two activities appeared to be central to students adopting others approaches. Sophie's (course 1) group initially brainstormed important factors. This enabled them to recognise these factors within, and hence consider, different approaches suggested by group members. Natalia (course 2) was able to comprehend the similarities of her group's approach in other groups' presentation and how she could incorporate the differences in approach within her own group's approach. This illustrates how the initial preparation work provided students with a platform to explore each other's ideas and approaches. This was needed because no one student had a clear idea of how to move forward. For students being unclear was due to the uniqueness of the situation being modelled, the nature of mathematical modelling, and the newness of the process and, in most cases, the mathematics. The situation created by the lecturers meant that everyone was on a similar playing field, needing to be open to releasing any preconceived ideas should the need arise and be willing to consider others' input in order to move forward. Being able to gain the additional insight others bring to the situation seems to be important in enabling students to navigate their way forward with their own process.

Constructive conflict, when managed properly, can help others see things in a different light. In order to produce useful models, good insight into the situation is useful and can be made possible by evaluating and discussing different viewpoints others can bring to the situation. Being able to be open to others different perspectives seems to be integral in moving forward with group-based modelling. It could be argued that far more can be achieved by modelling within a group making this is an important mathematical modelling skill.

6.1.1.3 Actively seeking reassurance

Students having ways to check they were heading in the right direction appeared to be important to students for pushing forward on their own. All students mentioned things that gave them reassurance they were on track. Students discussed using reference material, tools provided by the lecturer and seeing the work of others as some of the ways they were encouraged that they were progressing favourably. These things were mentioned by participants as things that eased their uncertainty with what they were doing.

As already seen, students looked at reference material in the hope it would guide them through the process and the mathematics needed for their unique situations. Reference material provided the basis for the modelling in all courses. Course one students quickly discovered the reference material did not hold all the answers and they would need to work independently. Stella (course 1) illustrates this when she said, “the standard models didn’t fit”.

Whereas course two and three students were able initially to use reference material provided by the lecturer as a guide. Ngaire (course 2) expressed having the case study was good as it helped her to double check her workings. It appeared to be important to Ngaire to know that she was on the right track and have some reference guide. All course two student groups used the case study as a ‘checkpoint’ for their workings. Having something to refer to illustrates that the case study was an important part in contributing to students’ reassurance. Ngaire (course 2) went on to describe working through the workings provided in the case study as “self-evaluating” and provided opportunity to learn by engaging with the material “instead of you just copying down notes off the board” illustrating that the experience was more than just being given the information.

Course three’s use of reference material did not provide reassurance as such. Rather, being given the SIR foundation model gave certainty with the direction of the model. This resulted in reducing some cognitive conflict and helped students feel it was possible to develop the model. Leo (course 3) expressed this, saying he felt prepared having the SIR model and his group “took the model and changed it”. This suggests having the foundation model made it possible for student groups to develop a model suitable for their particular modelling scenario.

Another way students tried to ease their uncertainty regarding the direction they were taking was through using tools provided by the lecturer. Using the tools to provide reassurance was common across course one and two, illustrating that regardless of the amount of structure within, or focus of, the activity there was a level of uncertainty involved for students.

Don (course 1) and Shelly (course 1) both mentioned using dimensional analysis to enable them to keep heading in a productive direction with their model development. The use of dimensional analysis enabled them to think through different aspects of their model. Without this tool they could have become stuck, going round and round in circles. Don (course 1) said:

> At one point when we got stuck, thinking oh what's going on why's this not working. [The lecture] taught us this thing about dimensional consistency. So, we checked that, and we were like, ah the dimensions aren't right. So where are the dimensions not right? It's not right in this part. What's wrong with that part of the equation? Oh, it's not the correct drag force equation So we went and found the correct drag force equation.

With having the case study available as a guide, only one course two student mentioned using tools to check their working. Rhys (course 2) mentioned using a visual representation of the solution (phase portraits) to critique his workings. Phase portraits had been briefly introduced in lectures, though Rhys was the only student confident enough to use them. This was most likely because he had been introduced to phase portraits in his physics paper, meaning he had time to understand them and add them to his toolbox of skills. Ngaire (course 2) and Nicola (course 2) both had commented that they had "only been given a couple of examples" and 'had no idea" how to use them.

This suggests that they had not had enough time or engagement with phase portraits to internalise them as part of their toolbox. The idea of a phase portrait was unfamiliar for course two students and needed more previous engagement to uptake this as a useful tool. In contrast, dimensions had been introduced at high school, so all course one students were familiar with these. This meant learning about dimensional analysis may have been new but was based on students' prior knowledge. These findings suggest that exposure to tools should happen at least a semester before they are going to be used by students so they can be internalised as part of an independent 'checking process'.

Interesting to note, course three students did not specifically mention drawing on tools, other than the foundation model that was given by the lecturer. This could suggest that having the foundation model and already being familiar with the type of mathematics to use for the model, enabled students to confidently create a model.

How students used tools provided by the lecturer will be discussed further within Lecturer support, Chapter 7, Section 7.1.1, page 199.

As already alluded to earlier, reassurance can come from exploration that initially does not feel productive. Exploring that feels like you are going round in circles enables information about the situation to be gathered, so that you can later on recognise features of a possible solution and hence provide reassurance you are on the right track. As mentioned earlier by Sophie (course 1) the results of initial exploratory working "didn't give us the right answer, but it gave us more details and understanding about the questions." Sophie's comment illustrates the importance of exploration to build understanding of the situation and thus develop students' abilities to confidently determine solution paths.

Students felt reassured when they saw what others were doing. Seeing other student approaches has the potential to provide reassurance that there is no one set answer and encouraged students to explore or adopt alternative solution pathways. Drew (course 1) said, “I remember I went over to another group at some point, and they were doing something else. But I didn’t feel bad that we were doing our thing, cos you never feel like, ooa that’s the right way to do it.” Hazel (course 3) said, “so it would've been interesting to see how two different groups approach to the same [situation].” The reassurance provided from seeing different solutions also exposed student to potential or alternative solution pathways and possible pitfalls. Rhys (course 2) mentioned that while “watching the other presentations, we were running through ours in our head.” Rhys (course 2) further said watching other people present their solutions gave him a sense that they were all struggling with similar issues and things “could have gone a whole lot more wrong” than they did. These comments all indicate students saw the value in learning from how other groups approached their problems and also provided the reassurance that everyone’s solution would be unique, thus encouraging students to adopt the creative nature of modelling.

The active effort involved with knowing you are on the right track differed between the courses. Course one students had to actively apply themselves to navigate the direction they were taking, learning out of this experience how to recognise when things were going well or when they may need to readjust their direction. Course three students, also had to actively grapple with how to put together the model, though to a lesser extent than course one students. In contrast this active grappling with the direction was minimised for course two students, instead relying on workings to guide them. The more directive approach by lecturer two utilised the mathematics culture students were familiar with to give students access to a mathematics model. In the process, struggle was minimised but students may have missed out on learning how to navigate the modelling process.

Figuring out how to balance reassurance with uncertainty is a key consideration for students' modelling experience. Experiencing uncertainty is an important thing for students to encounter as even for an experienced modeller, each situation is unique and will always involve unknown elements. Learning how to navigate this becomes key for the development of a mathematical modeller. This suggests that ways to build students' confidence while they grapple with uncertainty are needed as students determine their process with modelling.

As seen earlier, students focus on being right and the above discussion illustrates how students actively need some way to reassure themselves that they are right. This suggests that the culture of being in the dark, exploration, and playing around is not something they are familiar with. This again could be because of the preconceived views that mathematics is focussed on right and wrong answers. However, in reality the application of mathematics is more than right or wrong. It is less procedural and more about sense making. Interestingly, the lecturers had different views on this. Lecturers were either more focussed on process, or on mathematical procedures.

6.1.1.4 **Working together**

Underpinning the coping behaviours of *backing ourselves* and *being open-minded* was navigating how to successfully work together. The data shows that in order to move forward students needed to work together in a combined effort. It became clear to all students that they needed to collaborate, consider what each other could bring to the situation and as already seen, where beneficial adopt ideas that were not their own. Collaboration involved sharing information and thoughts, discussion and communication and independent work.

6.1.1.4.1 Sharing information and thoughts

Collaboration involved sharing information and thoughts. Outcomes of sharing information and thoughts were gaining new ideas, going further together including learning from each other, and sorting out misconceptions.

Nearly all students mentioned gaining new information or ways of seeing things from their peers. Rhys said there were “lots of different ideas coming from other people. These helped me and our group prepare in a way where we could cover the whole thing really well" (Rhys, course 2). All students discussed how if they were stuck there was someone else in the group that could help them out. Don remarked “if someone didn’t know something, someone else almost always did” (Don, course 1). For Luke, “it was during the modelling activity the ‘penny dropped’ as to what modelling is. Luke discussed watching his group needing to apply the model to a situation as opposed to fitting numbers into a model” (interview summary, Luke course 3). Drew mentioned that by working with others he learnt “different ways to tackle the problem” (Drew, course 1) other than his own. Steph found other group members reminded her of skills she had learnt “but didn’t realise we should or could use [these] to apply to this” (Steph, course 1). While Rhys said, “other people in my group shared stuff that I would usually overlook” (Rhys, course 2). These experiences illustrate the collaborative nature of the experience and how everyone, as a collective, made gains from working together. Lecturer three further illustrates this saying:

> That’s an advantage of, a small group. You can teach somebody something that you know, is a great way of, you know, sorting it out in your own head. At the same time students who don’t know can learn from ones that are following. This thing of mutual recognition - I’m not actually standing on the side-line.

There is an implication that the activities used allowed students to engage with the modelling at their current mathematical ability, as well as enabling learning from others. This appears to be an intentional aspect of the modelling activity as alluded to above. However, these experiences suggest that the modelling may have only been possible because it was undertaken in a group.

Students perceived they were able to progress further than they would if they were working by themselves. Don (course 1) said "it is really, really, helpful, it saves time". Natalia (course 2) and her partner Ngaire (course 2) both discussed how they both had different skill sets that complemented each other. They both believed that without the other group members it would have been very difficult to make the progress that they did make. This experience was similar to other students with all students discussing how learning and deepening their understanding of modelling was a positive outcome from working with others. "I think working as a group in that sense was so much more helpful than doing it on your own because you don't understand" (Steph, course 1). "I could listen and see what the others were doing, and I could learn that way as well" (Hope, course 3). These comments imply that more was able to be done with the help of others than what could have been achieved alone, and meaningful learning was achievable while working together.

While students were working together there were instances where they were able to sort out each other's misconceptions. Don (course 1) and Shelly (course 1) both discussed how the sharing of ideas and discussions of these ideas allowed students to address previous misunderstanding they may have. This meant that the lecturers' creation of situations where the students had space to discuss and share their own understanding about concepts, produced some cognitive conflict. This cognitive conflict allowed students to readjust their thinking as needed. This is an important component of any learning experience.

6.1.1.4.2 Discussion and communication

Sharing information and ideas involved discussion and good communication. All students identified discussion as key for moving forward with model development. Groups actively engaged in discussion to determine the direction to take with their model. David's group "spent the first hour just talking with each other, with the group really thinking about how we were going to answer it. What the question was about. It was really just really talking and brainstorming" (David, course 1). Hannah (course 3) remarked how "there was a lot of debate about what to do with the variables" to help them decide what to include in their model. Maria specifically identified that coming together and discussing helped her group work through the modification. "So, the modification wasn't so bad cos we had talked within the class, and we assisted each other" (Maria, course 2). Shelly (course 1) went as far as to say how talking about different approaches revealed information that could be useful with the modelling saying others "brought a different opinion into it". In contrast, Ngaire said "we didn't get very far, there wasn't a lot of communication" (Ngaire, course 2). These comments illustrate that discussion was necessary to explore and decide on ways forward. They suggest students working individually would find it difficult to determine a direction forward with their model development. Taking time to bounce ideas off each other helps move the experience forward. As already seen, talking things out loud help people to see things they would not normally see. This is important because thinking outside the square is needed to create a model. Creativity is not straightforward and requires considering something in a new way. This may be best accessed through talking things through with someone else, and as already alluded to, in the process being open to new ideas and ways of doing things.

Safe, open and honest discussion was important for the group dynamic. Don identified discussion as a key behaviour for successful group work commenting that "stopping and talking I know from past experience that that's effective for bringing a group closer together" (Don, course 1). Shelly (course 1), who was in the same group as Don, said we were "really good at bouncing ideas off one another, and yeah communicating and stuff. And if they[someone] didn't know something they piped up" (Shelly, course 1). This suggests that there was a level of safety within her group that enabled group members to honestly share their current knowledge base. This may suggest that group members conscious of good group dynamics are able to promote and actively encourage good discussion and communication.

6.1.1.4.3 Independent work

To get the most out of working together students needed to 'do their bit'. Whether it was researching, taking time out to think about things or making sure they were coming together to discuss things, each student needed to individually contribute.

Taking time out to independently reflect on the work, or do pieces of the work, was a strategy used by several groups. For Nicola and Rhys's (course 2) group, it was really important group members did some of the work independently before coming together to discuss their findings. "Before we all met together, we decided to do the research individually" (Nicola, course 2). Most of the course one groups spent time individually thinking about possible approaches and then shared their ideas. Taking individual time to reflect and work enabled students to gain background knowledge that could be used to inform their group brainstorming and discussion sessions. This became important when it came to choosing a foundation for their model or modifying their model as it allowed group members to recognise potential useful ideas for model development. This is turn enabled the groups to make a confident decision on how to move forward.

Natalia and Ngaire also first worked individually on approaches to modify their model, but due to limited time were unable to get together and discuss their findings and thoughts. As seen from presentation observations, in contrast to Nicola and Rhys's group, they were unsuccessful at fully working through a modification for their model. "This group listed limitations of the model without really thinking about how they could be addressed. A simple modification was tentatively suggested, though no mathematical workings, analysis or implications of the modification were given. [The lecturer] was clearly disappointed" (descriptive field notes, course 2 observation). Ngaire commented "so there wasn't a lot of communication at all for the third presentation, both of us needed to be on the same page and understand the same things." Clearly lack of assimilation of what each member of the group had done, led to no confident direction being undertaken for their modification. Students talking through their thinking and ideas, can help make sense, affirm individual thinking and find common ground. As seen from Nicola and Rhys's group common ground found between different group members was used as the basis for proceeding with the development of their model modification. Having time to sit together, share, discuss, look for differences and similarities between each other's individual work is an important part of the process for being creative in the development of the students' model.

There were some groups that approached things differently. Don and Shelly's (course 1) group did not do any preparation beforehand and went straight into discussing the context of the problem. What made this group's approach successful was that everyone was willing to participate and to honestly contribute to the discussion. As previously illustrated, this was seen through their willingness to openly discuss their current understandings and clear up misconceptions.

6.1.1.5 Going further together as a group

Working together pushed each other to go further than they might have done by themselves suggesting that as a group, students are *going further together*. Ngaire commented on how "the presentation had to get done to both our standards, so the standard was at least as good as the best person, you know?" (Ngaire, course 2). For Rhys (course 2) working in a group made him accountable, remarking:

> Working in a group means that you kinda of have, when you're preparing, you're preparing with other people and you've gotta be accountable to the whole group for your work. Um so that's, that was, that meant that I probably prepared a little more than I would if I worked on my own.

This shows that Rhys felt a sense of responsibility to his group. These comments suggest that students pushed each other on when working together. Working with others helps drive the experience further than it may go when working alone. This means group work using modelling has the potential to push boundaries for students. By lecturers providing unfamiliar situations for students to work through provides opportunities for students to become resourceful, accept responsibility and start to develop independent work habits.

Pushing forward on their own was made possible through the interactions of group members. With modelling being a complex activity and unfamiliar for the majority of students, differing points of view and minds to help out meant it was possible to navigate forward. When students were asked whether or not they thought the experience would be possible if they had undertaken it individually some students said "yes but it would have taken a lot longer" while others clearly identified the contributions that would not have been possible if they had worked sepearately on the problem. The interwoven nature and roles students played within the experience will be discussed in more detail in Going further as a collective, Chapter 8, page 231.

In summary, students developing student agency appears to be central to students *moving forward on their own*. Spending some time working independently of the lecturer appears to be a key component of the experience for all students. Accepting responsibility for working through the problem is key in moving forward both individually and as a group. It appears not having all the answers and grappling with the process is a necessary part of the process.

Even though this uncertainty involved struggle, most students developed behaviours that enabled them to progress. These coping behaviours included: students taking action including backing themselves; being open-minded to others' inputs; and finding ways to feel reassured they were progressing in a productive way. All students discovered modelling was not a straightforward activity. Discussion and sharing of ideas were necessary to explore and find ways forward. Successful discussion involved students sharing their own ideas as well as hearing and discussing other people's ideas. Both spending time individually reflecting on the work and working within a group appear to be beneficial and crucial for moving forward.

Finding ways forward with the mathematics was not always easy. Students struggled with new mathematics and tended to abandon approaches that used unfamiliar mathematics unless the new mathematics was built on material they were already comfortable with, and/or the lecturer had provided hints or resources to help navigate the new mathematics. Persistence, resilience, and confidence in themselves, appear to be additional behaviours students used to persevere with their modelling situations that involved learning and applying new mathematics.

Chapter 7. Findings: Subtheme two, being moved forward

7.1 Being moved forward

"*Being moved forward*" means students progressing forward through using advice external to themselves. Students being moved forward occurred when they utilised: support provided by the lecturer, guidance within resources, and contributions from other group members. As Teodoro (2002) stated, "learning takes place in a community of practice where students learn both from their own effort and from external guidance" (Teodoro, 2002, p. 13).

7.1.1 Lecturer support

Lecturers played a key role in moving students forward, even though they provided space to enable students to work independently of them. The lecturers designed and delivered their instruction such that it included opportunities for them to *role model mathematical modelling behaviours, provide resources and promote independence* for students. Through the student interviews these behaviours were seen within two main areas of support: scaffolding and coaching style.

I will discuss these areas of support by first describing how the lecturers provided this support, illustrating how the behaviours above were apparent, followed by what opportunities this provided for the student experience, how students engaged with the support, and what outcomes this produced for the student.

7.1.1.1 Scaffolding

Scaffolding acts as a guide, offering a way for students to cognitively structure what they are doing (Durandt, 2020). A scaffold provides organisation, guidance and meaning to the experience (Seel, 2012).

With the aim of promoting student independence with modelling, lecturers provided scaffolding for students through exposure in lectures to modelling behaviours and skills, and providing resources. Scaffolding provided by the lecturer offered a way for students to experience mathematical modelling. One student observed that "[Lecturer 1] would always be talking theoretically in class about how this all works, so getting to actually apply those things he told us about, like to get an actual solution was real cool" (Don, course 1). Another student said they had been given "the basic SIR model. We started with that and then we figured out what steps, between susceptible to removed, would be different" (Hazel, course 3). The direction provided from the lecturers enabled students to create new models and modify or recreate already established models. Even though the scaffolding was implied and did not provide step-by-step instructions, it gave direction, enabling students to work independently of the lecturer. This was important because working independently allows for student creativity. Being creative appears to be an important component of modelling competency.

The focus the scaffolding placed on the modelling process was different across the courses. The guidance provided either placed the modelling process as a central part of the scaffold, or as an abstraction from the scaffold. This created a situation where students were either primarily focussing on the generic modelling process or on the details of the models. This meant that, depending on the course, the modelling process was either explicitly or implicitly provided. Both course one and two students had specific explicit exposure to the generic modelling process at the beginning of the course. For course three, students the modelling process was initially implicitly introduced through the lecturer role modelling the process. After this initial exposure, lecturer two's focus on the scaffold changed to be implicit. These differences in scaffold focus were further supported by the lecturers' different role modelling styles and resources provided. I next discuss the student experience around these differences.

Course one students had a generic modelling process clearly available for them throughout the course. After the initial exposure to the modelling process, students had continued, repeated exposure to this process. “I try to drum it in with diagrams that I repeat over and over till they’ve got this little picture in their heads of moving from the real world to the model” (Lecturer 1). This demonstrates the lecturer believed it was important for them to have the process available as a scaffold, to see the different ways it could be applied and how its application could differ in different circumstances.

Steph (course 1) commented that the modelling process had been shown by the lecturer during lectures enough times that it had become “really engrained after a while”. Don (course 1) had a similar experience, saying that he had “done the process on so many past papers so many times” that by the time they came to the modelling activity day the process component “was really easy”. This illustrates that through preparing for assessments and attending lectures, the process had become internalised for these students. If for some reason students had not managed to internalise the process before the modelling activity, the modelling process was also detailed as part of the activity marking scheme that all students had access to (Appendix H). This further illustrates the importance the lecturer placed on having the process available as a scaffold. When asked what the most useful information from the course material was, a common reply was, it was the “four or five steps of the problem-solving cycle [modelling cycle] we were given that right at the start of the course” (Shelly, course 1). Shelly further said, “having that [modelling cycle] in the background throughout the course I think was really helpful”. While Drew (course 1) commented that “we definitely implemented what we’d done earlier in the course.” Don (course 1) went as far as to imply that because modelling was complex, a scaffold was needed to make sense of the situation, commenting:

> You can access everything and like, it's up to you to bring it together in a way that makes sense, but that's where the mathematic modelling process is helpful, have some order, order all your thoughts, like a real structured way. (Spooner, 2021, p. 589).

These student comments suggest that having explicit exposure to the generic modelling process provided a cognitive structure to manage the complexity of the modelling process. Having a way to manage this complexity, enabled students the freedom to create unique models, assisting them in putting their models together.

Students used the generic modelling process to move them forward with their own process. Even though they had a generic modelling process they needed to discover sub-process within the generic cycle. As part of the 'listing and clarifying the factors' stage of the process, Steph's group brainstormed different factors. **"**We got two pieces of paper, and we just wrote down all the factors on one piece of paper. We all went around and [added factors] like mass and height and things like that and [listed] assumptions as well" (Steph, course 1). Another group also used brainstorming to further 'clarify the problem' as part of the group's iterative modelling process:

> We started by clarifying the scenario and considering factors as well as, stating assumptions that led to our precise problem statement. We classified the sentence 'how big would the resulting crater be' and defined it as the resulting diameter of the crater. (Student written report AHLJ, course 1).

Sophie and Stella's group went through the stages of "clarifying the problem, listing and classifying the factors, making assumptions" and then needed to consider different approaches in order to “formulate the model”. This enabled them to produce a model. Sophie and Stella’s model is given below.

$$D=0.07Cf(g_s/g)^{1/6}(W\ p_c/p_M)^{1/3.4}$$

D = Crater Diameter
Cf = Crater Collapse Factor (Cf = 1.3)
g_s = Gravitational Acceleration at the surface of Moon
g = Acceleration at the surface of the CubeSat (g = ge)
W = Kinetic Energy of the impacting body
P_C= Density of the CubeSat
P_M = Density of the moon

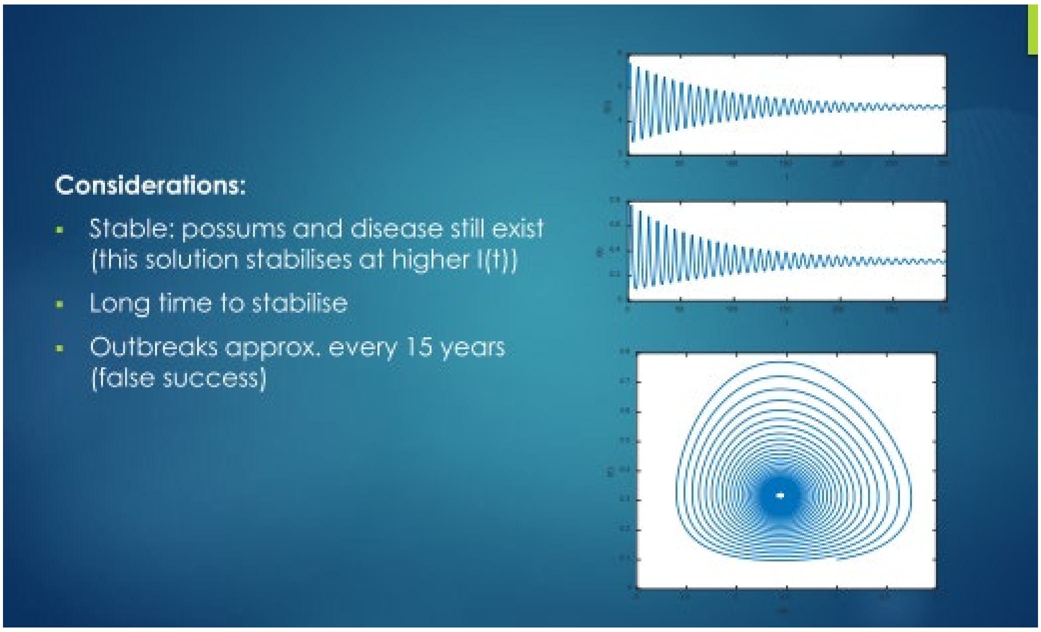
Substitute all values into the diameter formula:
$$D=0.07*1.3*(1.45/1.62)^{1/6}*(4.70*10^{-7}*1.17/3.34)^{1/3.4}$$
$$=9.20*10^{-4}m$$

(Sophie & Stella WZ student written report, course 1).

These examples show even though the students had a generic modelling process they needed to discover sub-processes within these in order to move through the generic process of modelling. The generic modelling process provided the ‘structure’ to enable them to do this. The lecturer frequently alluding to the modelling process in lectures, and making this process explicit, meant that the students had the process available. It appears discovery of the sub-processes involved discussion and purposeful consideration of the situation to inform different stages of the generic modelling process. It appears this process was cyclic in nature, involving cycling backwards and forwards, cross-referencing and sorting information.

For course two, after the initial exposure to the generic modelling process as a scaffold, the modelling process was presented more implicitly to students. While preparing to present their case studies, students were given an opportunity to become acquainted with the modelling process via osmosis. The lecturer provided the case studies to act as a scaffold for students to initially reproduce the model and believed via the case study students would implicitly experience the modelling process. “The third part, students are critiquing the modelling cycle, participating in the modelling cycle. They are now in a position where they are on a lot more firmer footing” (Lecturer 2).

When it came to implicitly experiencing the modelling process, students had differing experiences. For Ngaire (course 2), it was during preparing for the second presentation of the case study that she started to gain some insight into the modelling process.“Once I hit the second presentation and actually trying to look at the graphs of the solutions and being like, oh, what does this mean? That was where it came back to that process” (Ngaire, course 2). As Ngaire started to translate her mathematical workings back to the situation she drew on the modelling process to help her interpret what the mathematics meant in relation to the situation. Even though she did not have to create a model from scratch, needing to explain what the mathematics meant in terms of the situation enabled her to experience translating backwards and forward from the real world to mathematics. This is an important step in the modelling process, as this connection between the real world and mathematics is central to mathematical modelling.



(Ngaire & Natalia Presentation 2, course 2).

For Natalia (course 2) she was not sure how to modify the model but gave it a go anyway. She did not fully realise that there was no right or wrong answer, suggesting she had not grasped the concept that modelling was a creative activity. *"*So, we try to write two modifications, but I don't know it's right or wrong. Even after [the lecturer's] feedback, we don't know if it's right or wrong" (Natalia, course 2). This suggests for Natalia that the experience was so foreign for her that explicit process guidance for modifying the model was needed to overcome the difficulties she was experiencing with this new idea of 'modelling'.

For Nicola (course 2) and Rhys (course 2), their group understood their model well enough that they could grasp the relevant parts of the modelling process, in order to modify their given model. "So, we did follow the maths, like the modelling process. We went to the real world and then we went to the mathematical model, but that wasn't just a one way process. We went back and forth quite a bit" (Nicola, course 2). This suggests the members of this group were able to experience the iterative nature of the modelling process.

Why did students have differing experiences with the modelling process? It appears that students connected into the experience at the level they were at. As seen earlier, Ngaire (course 2) was comfortable thinking about reality and hence could see the connections between mathematics. This was Natalia's (course 2) first experience with forming connections between reality and mathematics and hence she struggled. However, Rhys (course 2) had previous experience with modelling through his physics course which could have contributed to his group's success. Neither Natalia nor Ngaire had any previous experience with physics. Their group was composed of two students instead of three, with one member not having a confident command of English. Combining these things with a limited time doing additional research, they were not able to manage to pool their individual strengths together enough in order to move beyond both of their weaknesses to confidently complete a modification of the model. In contrast, Nicola and Rhys's group set time aside for additional research, everyone had a good command of English and there was previous experience in their group. This appears to have enabled the group to rise above their individual weaknesses and draw on each other's strengths to be successful in accessing the modelling process to enable a successful modification of their case study model to occur. They set aside additional time for research, realising the way forward was not going to be given to them via step-by-step answers and they needed to take some responsibility for the process. This enabled them to recognise when something was going to be useful for modifying the model. These student experiences illustrate that through the work they had done so far, they grasped the modelling process requires effort and it pushed their current boundaries and mathematical thinking. As a substitute for scaffolding, students used their peers and internal processes but the time taken may have been slower, as the earlier comments show.

To make internalising the modelling process easier, students expressed it would have helped if they had more time and explicit exposure to the full modelling process before attempting the modelling activity. When discussing how her group modified their model, Nicola (course 2) said "there was really not much time to get to grips with it". She recalled "in class [the lecturer] only went through one example of the entire process". For Ngaire (course 2), when asked what would have made things easier, she replied "a lot more case studies … it's [one case study] not enough to know the process. It's more individual to the case study rather than an underlying process." This illustrates that both students comprehended that the modelling process was more than a set of steps to be followed and was not something to be fully comprehended after a single exposure. Students wanting more exposures of how the process was applied to different situations were at the beginning of their understanding of the complexities of the modelling process and its unique application to different situations. Even though the lecturer was most likely indirectly role modelling the modelling process during lectures, this was not enough for these students. They needed further explicit concrete examples.

For course two it was the case study and the lecturer themselves that was the scaffold. I will discuss the use of the case study as the scaffold here but discuss the lecturer as a scaffold under Lecturer's coaching styles, Section 7.1.1.2, page 210.

Course two students worked independently, and at times in unfamiliar territory, using the case study to provide direction and as mentioned earlier under coping behaviours, reassurance they were on the right track. As Ngaire (course 2) reported:

> Looking at the case study was good because it meant that you knew if you were right or wrong, so you can still do the like working out by yourself, or within the group, but there is like that level of oh yeah, okay, I'm on the right track and so you can keep going. And then if it's wrong, you know, it's wrong at each point, which is nice.

By providing the model foundation, and some of its working, the need to make choices on how best to approach forming the model, and any associated struggle, was removed for students in course two. This approach suggests it was important for the lecturer to give students access to a scaffold that enabled students to get to know a model, work through its mathematics, while implicitly experiencing the modelling process. Recreating the model, opposed to creating it from scratch, enabled this to occur. The lecturer's focus appears to be on minimising some of the struggle for students. "We hardly look at creating a conceptual model, because to create a conception model requires a really good understanding of the problem you're working on" (Lecturer 2). This shows that the lecturer believes modelling requires specialist contextual knowledge and is doubtful that students have this. It also suggests that the lecturer perceives that modelling is 'bigger than the student' and they are not yet ready for this experience. The cost of removing the struggle associated with creating a model from scratch meant students did not get to experience first-hand the full modelling process.

Even though the struggle around the choice of the base model to use for the students was removed for the students, course two students still got to experience independent work from the use of the case study. Ngaire mentioned how the "textbooks skip heaps of working and the stuff that you need, the stuff the lecturer does when [the lecturer] fills in the gaps" (Ngaire, course 2). This shows that the case study was not like a lecturer role modelling the full process and working for the students. With the lecturer only providing hints forward meant the students had to grapple with the mathematics of the model and its different representations to 'fill in the gaps' themselves as Ngaire points out. Having to present the model to an audience appeared to be the catalyst for students to really engage with the material. "Regarding presenting the model, I found that you did have to understand it to present it" (Ngaire, course 2). This illustrates the recreation stage of the modelling activity was more than reproducing the given mathematics. It involved students being self-resourceful and engaging with the material to achieve understanding.

As well as having the case study as a scaffold, course two students also had the lecturer available to provide scaffolding, as mentioned above, this will be discussed later.

Course three students experienced the modelling process implicitly through being role modelled "building a model from scratch in lectures" (interview summary, Lecturer 3). As seen by Luke and Hannah, this initial exposure to the modelling process was not clear for students. Luke said, "I've progressed heaps from the first two or three weeks where, um, I was just not really having a clue what was going on" (Luke, course 3). It was not until the students had engaged with the modelling activities within the course did the modelling process start making more sense for students. This was seen from Hannah who found it was assignment two where she had to grapple with the modelling process herself did things ‘fall into place’ concerning the modelling process. For Luke it was during the zombie activity via the help of his group members that he started to grasp what the modelling process was about.

It appeared the lecturer was aware that students may not have grasped the modelling process either from his initial delivery or activities leading up to zombie modelling activity. To allow for this, the lecturer explicitly taught the foundation model before the zombie modelling activity to remove some of the struggle associated with the modelling process. This approach supports the lecturer’s beliefs expressed in interviews. The belief that the lecturer “wasn’t convinced of the value of open-ended projects at stage 1” and that “conceptual modelling was difficult” (interview summary, Lecturer 3). However, in contrast to lecturer 2, the lecturer “believes that first-year tertiary students are capable of developing a model with appropriate guidelines” (interview summary, Lecturer 3).

The different experiences students had with accessing and utilising the implicitly or explicitly available modelling process suggest~~s~~ that even though this is a new experience for the majority of students, something is needed to help organise the thought process for the creative component of the modelling. When the modelling processes was frequently alluded and referred to, it made the process navigable. For those students for whom this was a new experience and did not have any reference points either within their group, or explicitly available, a clear structure or support within their student groups is needed. The course two lecturer did provide opportunities for students to do additional research, but this was not taken up by everyone due to time pressures on students. This implies that having time set aside to specifically focus on the modelling process is needed. With the full day available, course one students engaged with the modelling process and, in fact, could not move forward without doing so.

Even though course two and three who did not have the modelling process alluded to as much as course one, some students were familiar enough with it to apply it. What seemed to cause difficulties was when students were not used to relating mathematics to context, and when groups did not contain members who were confident with relating mathematics to context, or the mathematics required for the particular foundation model that was being used for creating the model.

One thing common across all the courses is that regardless of the way the modelling process scaffold was introduced, all students experienced it as a change of culture at some stage of the course. Some students experienced this earlier than others.

7.1.1.2 Lecturer's coaching style

Coaching styles involved role modelling behaviours, providing resources and promoting independence, as already indicated. While it may not initially be apparent, all coaching styles were focussed on promoting a level of student independence, while developing mathematical modelling skills. One of the main skills promoted within all coaching styles was 'keep it simple'.

7.1.1.2.1 Promoting independence

What was similar in lecturers' coaching styles was that lecturers promoted independence as a key behaviour for students to develop. This was seen through students needing to figure out how to apply the modelling process, new mathematics skills and techniques to their situations.

As already seen, lecturers provided a generic modelling process that did not provide specific details for the students' individual situations. When it came to students using this process, either to create or modify a model, students had to work out the specifics themselves and 'how to' apply this process to their situation. This was common for all courses. Rhys (course 2) felt he had been "thrown in the deep end" and had to "to learn how to approach these questions" in a way that was a lot different to what he was used to. Stella (course 1) said "we had a lot of discussion with each other about how to figure out what might be the best way or the approach to take." Similarly, Hazel's (course 3) group discovered how to progress by discussing and sharing their ideas. "We had to talk about, like, people dying that then become infected. We all did our own and then came back together as a group to kind of compare" (Hazel, course 3). The need for students to discover how to navigate the process illustrates the lecturers believed that 'how to' apply the process is learnt through doing. It shows students did not have a standard recipe for their situation, only a set of guidelines to help the process which requires a level of independent work to get to know the process. As there was more than one way of doing things, with no detailed plan to follow, students had to be creative, acknowledging modelling as a creative activity. These examples also reinforce the importance of individual thinking time and discussion in the modelling process.

An alternative explanation for lecturers leaving students to learn how to apply the process themselves, could be that some of the lecturers believed that the process is difficult to teach and should be left to the 'experts'. As already referred to, lecturers two and three believed that students coming up with conceptual models is "too big" for the student. The level of expertise lecturer two believed was needed for modelling was illustrated by the comment:

> I don't think you can learn in one semester to model every problem in the world because as I said every problem you have to study, understand it really well before you can write down before you can come up with a conceptual model and sometime before you can write down the equations.

This belief meant students did not get to experience creating a model from scratch. In contrast, lecturer one believed that students were capable and needed to "have a go" at the full modelling process in order to comprehend it and hence start to develop expertise.

Regardless of lecturers' beliefs about what is and is not possible for students, skills taught in lectures were independently used by students during the modelling activity. There was an implication that autonomously applying these in the modelling activities would encourage getting to know these skills, begin the process of mastering them, and allow students to become conscious of how they contribute to the modelling process.

Before we look in detail at the different coaching styles lecturers used, I will present the findings on 'keep it simple', one of the main skills promoted by lecturers as important for modelling. I will present how lecturers promoted 'keep it simple', student difficulties with 'keep it simple' and how, regardless of what the lecturer does, students need to experience modelling in order to comprehend the importance of 'keep it simple'.

7.1.1.2.2 Keep it simple

One of the main skills lecturers promoted, both advertently and subtly, was 'keep it simple'. This was seen by lecturers providing 'keep it simple' either as an acronym, reducing the complexity of examples for students to work with, and/or using simple examples in lectures. One lecturer actively promoted ‘keep it simple’ through continually mentioning and role modelling the acronym in lecturers and including it in the modelling activity instructions. Other lecturers chose examples that drew on the students’ background knowledge to make the modelling examples more accessible to students. Lecturer two drew on examples as close to the students’ own field of knowledge as possible stating “I chose study cases that are relatively easy. That can, you know, most of the study cases were population dynamics. They didn't involve a lot of knowledge of chemistry” (Lecturer 2). Lecturer three used simple modelling examples in lectures that utlisied mathematics students were already familiar with. This was seen both in lecture observations and lecturer interviews with the lecturer saying “the mathematical concepts, keep them quite simple. They should already have them in their back pocket” (Lecturer 3). Using mathematics that students were familiar with, removed mathematics as a difficulty for the student and, hence, kept the modelling process simpler for students. These examples show that lecturers were consciously aware of the complexities of modelling for students and the need to somehow minimise these while teaching. ‘Keep it simple’ gave students a way to access the modelling experience. Though as already seen, if examples and role modelling are kept too simple, students can miss out on grasping the true complexity of modelling. This aside, as seen from behaviours of mathematical modellers, ‘keep it simple’ is also an important part of the modelling process.

Even though lecturers had different ways of approaching ‘keep it simple’, all lecturers acknowledged that once students were dealing with models that had been made as simple as possible, this did not mean working with the model would then become straightforward. All lecturers perceived that without ‘keeping it simple’ things would be just too complicated to proceed.

All lecturers perceived a key student difficulty was how to keep things simple enough to translate between the real world and the mathematical world. Lecturer one commented:

> You can hopefully write a nice simple sentence that encapsulates what the heck you are trying to do. We teach them that process, but even once they get to that simple sentence even translating that into an equation is often, you know, tough.

Lecturer two discussed how "students should do a lot of translating of word problems to equations and back again first, to learn what the model means" (interview summary, Lecturer 2). Lecturer two went on further to say, "this is the first step in learning to model!".

Lecturer three believed:

> The first exposure you want them to have from going from the real world to the mathematical world, it needs to be quite simple, and it needs to be something that they're quite familiar with in regard to the concepts that they are going to need to use mathematically. That mathematical jump has got to be quite simple.

These comments show that simplifying a situation in order to translate it into mathematics is complex and is not necessarily easy to teach. Lecturer one's comment suggests that progress could be up to chance. In contrast, lecturer two and three offer solutions, suggesting becoming familiar with going backwards and forwards between the real world and mathematical world should be the first step in modelling. When this translation is attempted, it should be done using mathematics that students have already mastered. The need for starting simply and in familiar mathematics territory could be important as students have been used to a mainly abstract pure mathematics education.

For the majority of students, previous mathematics education had been focussed more on learning and using abstracted mathematics as opposed to making connection with the real world.

Despite the lecturer's use of 'keep it simple' either mindfully or subtly during lectures, it appears that students learnt and experienced 'keep it simple' by doing. While there were not any specific student tools for 'keep it simple', other than reminders and role modelling from lecturers that this was key, students soon discovered 'keep it simple' was needed to move forward with the process. Many students acknowledged that 'keep it simple' was a key behaviour in moving forward with the modelling process and was a technique used by all groups. Without 'keeping it simple' it appeared impossible to move forward. Shelly (course 1) specifically said "not overthinking things … otherwise you'd just around and around and spiral off somewhere" communicating that simplicity was key to maintaining direction and momentum for her group's process. When asked what helped with the modelling activity, Luke (course 3) stated "just simplifying things. It makes it quite easy" showing he recognised keeping it simple was needed in order to make progress. Don (course 1) similarly commented how they needed to "keep it as easy as possible" in order to move forward. It is apparent that it was not until students were involved in the modelling activity themselves did they discover, understand and appreciate what 'keep it simple' meant and how it would help them with the process of mathematical modelling. It seems that self-discovery is one of the factors involved with 'keep it simple' and hence learning to mathematically model. Course one groups tended to overcomplicate things at first and discovered they "really had to simplify things" (David, course 1). David (course 1) said "we can't deal with all of it, you're gonna make some assumptions, we've gotta focus in on the most simplest way to get this up and running to start with." Course three students did not mention 'keep it simple' as such, but as was seen from interview comments, they consciously made decisions about how simple or complicated they were going to make their model. Hannah's group illustrate the need to set boundaries on the scope of their model with Hannah (course 3) commenting "there was a lot of debate about which variables that we should add in because there was so many, especially for our one. Like how complicated we should make the model and what should be what." It appears that with lecturer three spending time actively teaching the foundation model to use for the modelling actively, this minimised the cognitive load for students enabling them to discover 'keep it simple' for themselves. In contrast, lecture one threw students 'in the deep end' to grapple with discovering they needed to simplify things in order to move forward.

Course two students understood the need to simplify, although did not mention any specific examples in their interviews. However, the course two student interview comments showed students could see modelling was about reducing the complexities of the real-world situation in order make a simplified version in mathematical terms. Natalia (course 2) mentioned:

> For the real world they have lots of factors to include. They have lots of factors that we don't know, but for the modelling it's just a few factors, we don't consider other things…a simple model of the complex situation…. As I said, the real world is complex and the advantage for the modelling is to make the real world simple to understand.

Nicola discussed the complexities of keeping it simple when she considered how well their model had captured the real-world situation saying "so I think limitations would be like, if the real-life situation was too complicated. It wasn't something that could be … easily um, formulated with the formulas" (Nicola, course 2).

Even though students had difficulties with ‘keep it simple”, it appears self-discovery is an important component of learning to model and learning how to ‘keep it simple’ in order to progress forward.

7.1.1.2.3 Different coaching styles

Underpinning all lecturers’ coaching styles is promoting student independence. This was seen through the use of generic modelling cycles without specific situational details and the promotion of skills that allow students to navigate the modelling process themselves. I will now present the different coaching styles lecturers used.

The clear difference in coaching styles was that coaching styles were either external or internal, or a combination of both, to the activity. External coaching styles involved students having to independently access resources and actively apply these during the activity. Internal coaching styles involved the lecturer providing direction forward at key parts of the activity, followed by students actively applying this direction. When an internal coaching style was used, it was used in combination with an external coaching style.

External coaching styles behaved similarly to those of a gymnastics coach or tennis coach. Skills were externally practiced, behaviours role modelled by the lecturer, and students were given opportunities to access skills and behaviours through worked examples. It was lecturer one’s hope that “when they get to the modelling day, they have a range of modelling tools in their pocket. They could be using one, combination of another, who knows how it’s going to come together” (Lecturer 1). Students then had to perform independently without the coach’s input, having to draw on resources acquired and their own internal reserves. The performance was not as highly staked as a gymnastics performance, having room for students to ‘figure things out’ on the way but the essence of coaching where resources being provided before the activity occurs applies here.

Lecturer one's coaching style is an example of a coaching style external to the activity. As already discussed, part of the training for course one students was in the explicit exposure of the modelling process. In addition to this, lecturer one provided strategies for students to use during the activity. These strategies included checking dimensions used in equations were consistent, making a plan, drawing a picture and keeping it simple. These strategies enabled students to discover ways forward with their group's process.

Don (course 1), Sue (course 1) and Shelly (course 1) all mentioned checking dimensions through using dimensional analysis to check their models were proportional, as key for moving forward with their process. Using dimensional analysis enabled students to identify where they were off track, aiding them to question their working and consequently discover how to move forward. As already commented by Don (course 1):

> I remember one point when we got stuck, thinking oh what's going on? Why's this not working? Peter taught us this thing about dimensional consistency. So, we checked that, and we were like, ah the dimensions aren't right. So where are the dimensions not right? It's not right in this part. What's wrong with that part of the equation? Oh, it's not the correct drag force equation, so we went and found the correct drag force equation.

Groups further tended to take up the lecturer's advice to make a plan. It was not until Drew's group had started the process did they realise how important making a plan was. Drew (course 1) remarked:

> About half an hour in, we sort of like, oh we should probably do, like make a more formulated plan, to make sure we're getting things.
> Yeah, cos it could be very easy to think you're going places but not really doing anything for a while if we have no plan to it.

This suggests that, like 'keep it simple', the importance of making a plan was not apparent to students until they had experienced the modelling activity themselves.

Lecturers and students found drawing pictures was important for helping students clarify their thoughts and communicate their thinking. During interviews lecturer one discussed "you [the students] should be thinking about graphs, diagrams and that kind of thing when you are modelling to make it clear in your own head" (Lecturer 1). Don's (course 1) and Drew's (course 1) groups draw diagrams as part of their planning procedure. These groups found a diagram not only helped clarify their thinking but also provided a reference point for their work. Don's group found using a diagram enabled them to start making sense of the process and how things would come together. "A diagram is definitely helpful because you kinda know how everything fits together, yeah that's how a diagram is helpful for me … If you draw diagrams, you can visualise the process" (Don, course 1). Drew's group used their diagram when they were baffled as to what to do next. "It's a step-by-step process, it's always, if you get stuck the diagram would be a good place to turn back to, to look how where you go to next yeah" (Drew, course 1). This illustrates that the work is complicated. Students wanted an additional reference to generate their own scaffold for navigating the process. Drawing a diagram enabled students to have a reference point when they got confused. The diagrams aided students to start making sense of the process, helping to find ways to move forward.

Notably, for the majority of the students it was not until they had their own experience of modelling that they could see and appreciate how the strategies provided by the lecturer could be used. In some cases, it could be viewed that the students discovered these strategies themselves and then were able to recall this as something previously mentioned by the lecturer.

The experience of the students illustrates the complicated nature of the mathematical modelling process and how strategies provided by the lecturer enabled them to move forward discovering their own process, as well as how and when to use these strategies. Without the active experience of the activity, students would not have discovered and experienced for themselves the importance of these strategies. This active experience has important implications for learning and retention for students. This will be further addressed in Discussions and teaching implications, Chapter 9, page 245.

Lecturer two's coaching style was both external and internal to the activity. The lecturer's role was similar to that of a basketball coach is to its team. Like a basketball coach trains its players in skills and then advises from the sideling during a game, so the lecturer initially trained the students in mathematical modelling skills and then advised students as they were working on the activity. Initially, the lecturer role modelled mathematical modelling skills through worked examples. Once the activity began, the lecturer advised students in office hours and via feedback given immediately after their the first and second presentations (more on this later in Chapter 8, page 231). Advice was given through hints and pointers for the next stage of workings within the activity. As seen in the observations, the lecturer's hints enabled students to progress to the next stage and/or complete what they were working on to a satisfactory level. "Using the lecturer's explanation [unnamed student] and Rhys are able to discuss what was missing from their work and how they would now do it. Rhys saying, "but there was no way we were going to try that before" (descriptive field note, course 2 observation). This shows that these students felt 'out of their depth' and needed the lecturer's help to navigate the next stage. This could have been because the case study models were sophisticated models, the mathematics needed for the model was new and/or students had not had enough time to fully integrate any new mathematics into their cognitive schema. This situation created a level of dependency on the lecturer.

Lecturer dependency was seen when students 'just did not know what to do next'. Unsurprisingly, not knowing what to do next appears to occur when students lacked the necessary mathematical skills and strategies to progress. While exploring the phase portrait for their model, Rhys's group came across a new feature they had not come across before in their graph and were not quite sure how to deal with it. "[The lecturer] suggested a much better method. Instead of setting something to zero and getting a divided by zero, you set it to epsilon and take the limit as epsilon … which is a much better way" (Rhys, course 2). When Ngaire' s group had "no idea" what they were doing, the lecturer helped them get "on the right track". "Like we had this massive equation. She's like, oh, try this. And so, we worked on trying that and we actually got an answer out of it" (Ngaire, course 2). This illustrates students needed the lecturer to provide hints in order to progress. The lecturer was able to point them in the right direction. Again, this could be because the level of mathematics or strategies was too difficult, or new or students had not had enough time to integrate these things into their schema. If students had more freedom to choose the mathematics to develop their model, hence chose mathematics that they were familiar with, they could have been more comfortable 'having a go at it'. Using less sophisticated models, that utilised familiar mathematics, may have not created such a dependent relationship on the lecturer. The approach taken by the lecturer hints at possible lecturer's beliefs about modelling: there is a certain standard to modelling; and modelling is too hard for students. The lecturer could have subconsciously set up a dependent situation reflecting these beliefs. Alternatively, the lecturer might have wanted students to experience as much of the mathematics of the models as possible and viewed a dependent approach as the only way possible. Again, this supports a possible lecturer belief that modelling is hard, involves hard mathematics and requires support from an expert.

Another example of lecturer dependency was seen while observing the question-and-answer session following Rhys and Nicola's group presentation as follows: "Student question from audience: I don't understand what you mean about asymptotically stable and this stationary point? Rhys starts to explain. Lecturer interrupts and explains as she can see Rhys hasn't understood the concept fully" (descriptive field note, course 2 observation). The lecturer stepped in as [the lecturer] could see the potential for misconceptions. In this instance it was important the lecturer stepped in to clear up these misconceptions to prevent them from becoming part of students' thinking on this topic. The expert advice of the lecturer was needed. This suggests when modelling involves unfamiliar concepts it can be important to have the lecturer present to moderate discussions. Not only did the lecturer's intervention benefit the presenting students, but also the class as a whole.

Overall, by following the lecturer's advice all students picked up new skills that they had not been able to access themselves. "Getting feedback definitely changed the way we approached it. Like we learnt how to do like the, the taking the limit of epsilon yeah, absolutely" (Nicola, Course 2). Without the lecturer's help it is unclear how far these students would have progressed with the mathematics.

Students could have been used to expecting and hence being reliant on the lecturer's direction and therefore did not enter into cognitive struggle independently to find a way forward. Alternatively, they could have been genuinely stuck and unintentionally a dependent relationship between the lecturer and student was created. As seen earlier via Nicola (course 2) and Ngaire (course 2) comments, students initially resisted independent application and exploration. This suggests they were expecting direction from the lecturer. Nicola (course 2) and Ngaire (course 2) both support this conclusion having discussed being shown new material briefly before having to apply it and not feeling confident with this.

Even though a dependent relationship between the lecturer and the student was occurring, students were expected to use the lecturer's hints and worked examples to produce independent work. As well as hints, the lecturer would demonstrate potentially useful skills once, then students were expected to put these into practice. The limited help provided by the lecturer implies a belief that learning occurs through independently applying yourself over and above what you have been shown, illustrating learning is more than having skills and behaviours role modelled. The hints and singular demonstrations were not always enough for weaker students, producing mixed feelings for at least one student. Ngaire appreciated being able to actively engage with the work but also found it tricky, commenting that "you can work through it at your own pace which was really nice … though it was challenging" (Ngaire, course 2). Ngaire went on to describe the experience as challenging saying you had to do background work yourself. Without the lecturer she would not have known where to start. Natalia also had a similar experience.

While the lecturer was providing hints and skill sets for moving forward, they were also providing public feedback. There were mixed feelings concerning feedback. On one hand, students appreciated the direction provided by the lecturer. Rhys (course 2) commented:

> It's good because, we kind of try things and then come back and get feedback and I feel like that's better because then you actually know where you're at rather than working through an assignment and getting feedback after it’s complete.

Nicola (course 2) expressed the lecturer’s feedback as a way of helping and guiding her through to the end product. “So, and then you could go and you can do the second part and you can make it better, from on the feedback from the first part and it gets better and better. Theoretically, your last presentation supposed to be this” (Nicola, course 2).

On the other hand, some students felt criticised by the lecturer. Ngaire was one of these students. She did not always receive the lecturer’s feedback favourably. “I can't remember if it was the first or second, but I was actually quite offended by one of the feedbacks that [the lecturer] gave back” (Ngaire, course 2). As seen from observations, Rhys also found that he defended his position and consequently found it difficult to incorporate the lecturer’s feedback. “Lecturer providing different approach to take. Rhys repeatedly saying we did it this way because it was too complicated to work out otherwise” (descriptive field note, course 2 observation). What is interesting to note is, Rhys (course 2) thought that he was open to feedback saying:

> I’m quite a very kind of stand-up person so I don’t take very much offence to anything, even if someone, you know, telling me that I’m completely wrong. I’m more likely to take on more with it saying, perhaps defend myself if I don’t think I’m wrong but I’m not someone who kind of takes that’s that kind of thing to heart.

What was observed was different. When feedback was given by the lecturer, several times Rhys firmly defended his position and appeared to not incorporate the feedback. This demonstrates that he saw himself and his contributions as important.

How well the lecturer acknowledged the student's contribution and their own process with it, seemed to influence how readily the student accepted the feedback. Ngaire felt the lecturer's advice was insensitive for her first presentation saying, "I was so nervous about it, I had already criticised myself and to hear someone else criticise it again was like, ah, I already knew" (Ngaire, course 2). In regard to her second presentation, Ngaire (course 2) went on to say:

> I was so disappointed because I knew that after preparing for this presentation I was like, okay, we've got some good things in here. We've actually got to a solution. It was really hard to explain those solutions. And so for her to say….

In both Ngaire and Rhys's case, it was important that the lecturer acknowledged them and their contribution while giving feedback. Acknowledgement would help them take on their advice and become open to alternative ways of moving forward. Ngaire and Rhys's experience could also demonstrate that students could have resistance to taking on board information from authority figures rather than peers, making it harder to incorporate new information received this way into their cognitive schema.

In summary, there were some instances where the lecturer's presence was crucial and helped progress students to the next stage of workings and minimise misconceptions. This seemed to be most effective when students could then discuss the lecturer's feedback with their peers. Then there were instances when the lecturer's advice was not taken up by students. This seemed to occur when students perceived the advice to be a criticism of their work as opposed to a 'class misconception'.

One similarity between a basketball coach and the lecturer two's style of coaching was the lecturer provided 'in the moment' teaching. As this involves the lecturer utilising students work as teaching points, I will discuss this in more detail in the Interwoven nature of the lecturer, Section 8.1.2, page 234.

Lecturer two's coaching style changed to be 'hands-off' in the final stage of activity. Students were left to their own devices, being given an opportunity to perform on their own as they modified their models. Students had differing experiences with this. These experiences are discussed in detail in Moving forward on our own, Chapter 6, page 167.

As seen in the lecturer interviews and observations, it was the lecturer's goal and belief that by the time the lecturer's coaching style became hands-off, students would have internalised the modelling process. Lecturer two's approach centred around "students learn[ing] the process of modelling by looking in detail at case studies" (interview summary, Lecturer 2). "[Lecturer 2] believes she gives students the tools to become modellers" (descriptive field note, course 2 observation). Students did not find adjusting to a hands off approach straightforward. Rhys (course 2) mentioned "it is difficult to adapt to especially when the questions are given in such a way it's kind of … a little bit of vague, you know there's not much input to help …", showing the struggle his group experienced with the independent components. Rhys went on to further say, "cos a lot of it you are kind of thrown in the deep end and have to learn how to approach these questions which seemed a lot different from what you are used to" (Rhys, course 2). Ngaire' s group had a similar experience stating "because if we didn't understand then, we can't really find it anywhere else" (Ngaire, course 2). This illustrates that even though course two's activity had more structure, the students still felt overwhelmed and needed to embrace and learn a new way of thinking. Even as third-year university students, these students were used to explicitly being shown what to do. “They also still wanted help, maybe because that is what they are used to – 'being spoon fed' for maths even at stage 3" (comparison case 1 and case 2 memo). This implies that either modelling is far more complex to teach than currently thought and/or our culture of learning mathematics involves minimal ownership of knowledge creation for students.

Even though students struggled with things and did not get as far as the lecturer hoped students did make progress. As noted in researcher memos, students had adopted some of the behaviours of modelling, even though they may not have been able to successfully modify their model.

> “As she [Natalia] moved through the experience she developed a critiquing mindset for the model and the process and was able to ask questions of the model. This demonstrates that she was able to start self-guiding herself and had developed a sense of mathematical modelling” (student autonomy vs lecturer direction researcher memo).

Overall, the way lecturer two coached meant that students were able to progress forward through the initial stages of the activity, though it meant that they were not yet fully independent. The lecturer clearly was part of the scaffolding for the students and an integral part of the activity. The lecturer’s guidance provided clear direction for the way forward. This minimised cognitive conflict for students. When students started to work independently some struggled more than others in moving forward. This implies cognitive conflict and struggle may be necessary to enable students to internalise and develop their own process for mathematical modelling. More than promoting precision in working may be needed to encourage creativity, and when using a process to create, with mathematics. This suggests ways are needed to manage the tensions between accuracy, creativity, and process.

In summary, the lecturer, available resources, and peers all played a role in students *being moved forward.* The lecturer played a key role, having an impact on student experience. The lecturer's main role was one of a facilitator. Lecturers provided guidance while enabling space for students to work on their own. The facilitator's role comprised of providing scaffolding for the experiences and coaching for students. Scaffolding and coaching were never 'complete', that is, no clear step-by-step instructions to produce a polished finished product were given. For students, having a scaffold provided support and a form of reassurance they were progressing in a favourable direction, even though it may not always have felt like it. All lecturers taught the modelling process to scaffold the student experience. The modelling process was presented either explicitly and/or implicitly to students throughout the courses. The way the modelling process was alluded to during teaching had an impact on students' access to it. Multiple explicit exposures to the modelling process provided students with a cognitive structure to manage the complexity of the process. Having a framework to refer to opened up the possibility for students to have their 'own' experience of the process and discover sub-processes of the modelling cycle. Implicit role modelling of the process was only successful with some students. Students with weaker mathematics and no previous exposure needed more explicit and continual exposure to the modelling process. A clear support structure and support within their groups is needed for these students. Continual explicit exposure to the modelling process, committed periods of time for active modelling, having a form of scaffold available, having each other to bounce the process off, all appear to contribute to students' progress with modelling.

A range of coaching styles was used for teaching modelling. They were either external or internal to the activity. An advantage of lecturer internal coaching was to direct students to new mathematical skills and understanding and/or potentially avoid mathematical misunderstandings. Common to all lecturer coaching styles was promoting independence as a key behaviour for students, illustrating lecturer beliefs that, at least in part, modelling is learnt by doing. Students' initial response to 'independence' was to feel overwhelmed. Students used self-exploration, discussion and sharing ideas within the groups alongside strategies provided by lecturers to find ways forward. These strategies included practical tips such as 'keep it simple'. The concept of 'keep it simple' as a modelling strategy was not easy to portray by lecturers. The importance of 'keep it simple' and 'how to keep it simple' was discovered while students had their 'own' experience of modelling. Feedback was also part of the lecturers' coaching role. The findings show feedback should be delivered thoughtfully and respectfully while positively acknowledging the contributions of the student.

Different levels of dependency of the student on the lecturer was observed in the findings. Levels of dependency differed, depending on the form of scaffolding provided and coaching style used by the lecturer. Dependent coaching styles may be based on a limiting belief that a full modelling experience is not possible for a student.

Common for all students, having their own experience of the modelling process provided opportunities to 'learn' and develop understanding, even though it may have been limited understanding. Students needed their own experience of modelling before the intricacies of modelling started to make sense and begin to fall into place. Cognitive conflict produced from not having clear direction may be necessary to develop, grasp and internalise an understanding of the modelling process. The space provided by lecturers allowed students to be able to ‘create’ their own models and experience mathematics as a creative activity.

Chapter 8. Findings: Subtheme three, going further as a collective

Within 8.1, I present an analysis that integrates the analysis of data with participants. However, within 8.2 and 8.3 I present a discussion of the significance of this subtheme.

8.1 Going further together as a collective

Through the experiences of *moving forward on our own* and *being moved forward* by the lecturer, students had both their own individual and collaborative experience. What was very clear from the data was that students' experiences were interwoven and collective in nature. The impact each person had on the others learning resulted in a shared learning experience for those involved and the direction of this learning experience was set by the collective. As already discussed under Section 6.1.1 Coping behaviours, page 170 in Chapter 6 *moving forward on our own*, students within groups influenced and contributed to each other's experience. Groups within courses also had an impact on each other's experience and the lecturer's role played a part in moving students forward.

8.1.1 Impact of groups within courses

Being able to see what other groups were using as their approach had an impact on students. While opportunities for this to occur mainly occurred in course two and three, it was clear that seeing what others were doing was beneficial across all courses. Don (course 1) mentioned when seeing that other groups were using a different approach to his group's that this was 'okay'. This shows seeing what others were doing reinforced and contributed to Don having the confidence that there was no one "right way to do it" (Don, course 1). This in turn reinforced the creative nature of the modelling process.

Course two and three groups got to observe other groups' workings during the class presentations. The majority of students mentioned benefits in seeing different approaches and techniques other groups were using. The presentations provided opportunities for students to learn from each other. Natalia summarised this by saying "we do the group presentations so everybody can teach each other what they didn't know" (Natalia, course 2). For Natalia (course 2) it was not until she saw another group's presentation did she understand what modifying the model meant and how her group might approach modifying their own model. Nicola (course 2) discussed how the presentations gave her new techniques to use for their own situation commenting:

> I really enjoyed Ngaire' s one cause they had done a lot of their own learning. They found how they could tell stability from the determinants and the trace. I was like, that's cool. You never thought that it could be done that way sort of thing. Like it gave you an idea though, you could look at it this way.

These examples suggest that students being invested in the work, allowed them to identify with relevant ideas and techniques in others work. This resulted in students realising the potential for applying and using these new ideas and techniques within their own work. It was through students publicly sharing their work, and the use of modelling problems that use similar mathematics, that made this possible.

In addition to appreciating seeing other group's working for their model, Luke (course 3) discussed how useful he found the lecturer's use of questioning after their group presentation. "[The lecturer] asked us how we ran through this and what he would've done and what we missed out. And then we just sat and watched different answers and like listened to what they did. It was valuable learning" (Luke, course 3). The lecturer facilitated the groups critiquing and further work needed for the models, promoting this key skill. The lecturer giving feedback to group presentations will be discussed more in Interwoven nature of the lecturer, Section 8.1.2, page 234.

Not all students mentioned benefits from seeing how others approached the development of their models. Students did not always appreciate others' presentations unless they could see the relevance to their own work or if the presentation promoted students questioning the presenters or their own reasoning. Hope (course 3) commented when watching the other groups present "they were like talking about the capacity stuff and that wasn't what I understand about it." Seeing another group present their workings created cognitive conflict for her. Even though the cognitive conflict may not have been pleasant for Hannah, and she did not acknowledge it as a benefit at the time, the cognitive conflict created a situation for her to continue to examine her own understanding, and the understanding of others, to resolve what 'carrying capacity' meant.

Not all course three students got to present their models to the class due to time restrictions. Hannah (course 3) mentioned being disappointed that her group, along with other groups that had the same model situation, didn't have time to present saying "it would have been nice, cause someone else had ours as well, I would have liked to compare it to see what they got but." Hannah's comment echoes how if students are already invested in the workings of a problem, they can be more receptive to feedback and different ways of approaching the situation. This highlights the potential benefits for student groups to be working on the same open-ended problems, or models using similar mathematics, and sharing their solutions with the other groups.

The above examples strongly suggest that students should have opportunities to see each other's work, particularly work of other students working on the same or a similar problem to themselves.

8.1.2 Interwoven nature of the lecturer

As seen when discussing lecturer support, the lecturer played an integral role that was key to the student experience. All lecturers provided techniques, mathematical tools, role modelling and a form of scaffolding that students utilised during the modelling activity. The role of the lecturer as a resource and/or resource provider demonstrated how the lecturer is interwoven into the overall experience. The key differences in the interwoven nature of the lecturer, students and experience, was whether the lecturer support was provided external to the activity, 'in the moment' or a combination of both.

Some interactions, between the lecturer and students during the modelling activity, occurred via resources set up externally before the activity. For course one students, this 'metaphysical' way was the only way they interacted with the lecturer. Don (course 1) commented:

> That preparation tutorial I was telling you about, he gave us a different problem and told us the steps he would use to get through it. So even though he showed us those things separately, he also did show us how to work together. That's how we knew how easy it was to do the steps, the first hour was really productive, cos it was just replicating things we'd done and been shown before.

Don's comment illustrates the lecturer's role modelling of the modelling process provided students with examples of the process to "fall back on". This was important so that students did not feel "in the dark" about how to approach developing the model. Don went further on to say, "I wouldn't of known where to start if Peter hadn't taken us through how it worked beforehand." Don also said "the lecturer had taught us beforehand, some of the techniques, we knew what to do when things went wrong and we knew where to start, and things like that." These comments illustrate how the metaphysical interwoven nature of the lecturer, through resources provided, gave Don confidence through having something to refer to, to help him navigate the process.

Course two and three students also had resources provided for them external to the activity. In contrast to course one, these resources provided key stages of the modelling process already worked through (course 2) or gave the foundation model to use for their model development (course 3). As discussed earlier, these key resources minimised cognitive load for the students, though left students responsible for only part of the modelling process. Even though all lecturers supported students' modelling activity, the key difference in support provided was whether it was designed to support a student being fully responsible for the modelling process, or only partly responsible. The interactive role of the lecturer and student in this respect was different across the three courses and, as discussed, reflected a lecturer's beliefs concerning how they thought students learnt to model and what they thought was or was not possible for students.

Interactions between the lecturer, student and activity for course two and three also occurred 'in the moment'. For course two, during the activity the majority of the interactions with the student were 'in the moment'. The lecturer was personally involved in facilitating the process, providing pointers forward at key moments. This was seen in the class observations, where as a student "the lecturer offers me [the student] feedforward on what I need to work on for next time. This helps lift my confidence by giving me direction on where to go next, for the next presentation" (descriptive field note, course 2 observation). It was observed:

> The lecturer is watching and looking at students present looking for areas to offer improvement for ways to improve presenting, ways to help improve students understanding of model, next steps for students to working on for the modelling, pushing students for further

> understanding of their model and its development (descriptive field note, course 2 observation).

These observations show the lecturer's advice and prompting was key to students finding their way forward. Having the lecturer present enabled clear expectations for what to do next for the students with Nicola saying, "we were getting really clear expectations on the next presentation we were supposed to present" (Nicola, course 2). This suggests it was not necessarily clear if students would know what to do next if the lecturer was absent. As already seen, the lecturer utilised opportunities that were presented 'in the moment' to use students' work, as teaching points for the whole class. This resulted in being able to teach students new skills and concepts 'right where they were at'. That is, by having students present their work enabled the lecturer to see the current level of skills and concepts students had acquired and take them from their current level of understanding to the next level. The lecturer used students work to teach them new skills. This would have had great benefit as students were already invested and therefore more open to feedforward. Although, there was a risk that the presenting student may perceive the feedforward as criticism.

When students had their work used as examples, it made them feel good. This was the case for Natalia and Ngaire. The lecturer had given them hints to use a new technique, which they used and presented. In the following class, the lecturer then taught the intricacies of this technique. "[The lecturer] really liked your, what you did, cause she then of course she taught it the following class" (Interviewer, Natalia course 2 interview). Natalia (course 2) commented "it proved our idea correct' making her feel "confident" and "good in that class". For another group the lecturer dismantled and rebuilt the phase portrait the group presented and used it as a teaching tool for the group and the whole class. Nicola (course 2) recapped this event saying, "that's right. She did a wonderful, she basically just pulled our thing apart and showed us how to do the little bits and how it was created.” From observations of this lecture, “the class has a really good feel. It was a positive and together environment. People positively participating engaging in discussion and engaging in feedback. Feedback beneficial for all groups, even if it was feedback on another’s presentation” (descriptive field notes, course 2 observation). Using students work meant their work was valued and allowed students to achieve greater understanding of the concepts and techniques used. Using students work for teaching moments shows how the collective experience was dependent on contributions from the students as well as the lecturer. Using students work enabled the lecturer to directly target the cohort exactly where they were academically at.

Lecturer feedback given on other groups’ work had the potential to provide direction for all students. Nicola commented on how hearing the lecturer feedback for other groups’ work provided reassurance that her group were on the ‘right track’ saying, “it was good [hearing the feedback] because we were like, yeah, we did that. Yeah, we did that” (Nicola, course 2). For her own group’s feedback Nicola said “getting feedback definitely changed the way we approached it. Like we learnt how to do like the, the taking the limit of epsilon yeah, absolutely.” When asked how receptive she was to feedback for her own group’s work Nicola stated “because I understand it better. It now has meaning….” (Nicola, course 2). As already presented, from class observations it was seen “all groups had instant feedback and feedback beneficial for all groups, even if it was feedback on another’s presentation” (descriptive field notes, course 2 observation). This illustrates how more receptive students can be to feedback when they have already invested themselves into the work in question.

For course three, the interwoven nature of the lecturer involved both the metaphysical and ‘in the moment’ dynamics. Lecturer three provided resources prior to activity, as already seen with specific teaching of the foundation model to use for the activity and, like lecturer two, then provided "in the moment" teaching immediately after students presented the models to the class. As already seen, students used the foundation model to develop their models from. A common student comment was, "[the lecturer] gave us the SIR compartment tool, like how it flows and what the equations and that are and we used that to do a transference of variables" (Hope, course 3). In-the-moment teaching can be seen from Luke's description of the feedback sessions following the student presentation when, as already mentioned, Luke (course 3) said:

> Individually he went through each group on the board, then asked us how we ran through this and what he would've done and what we missed out. Um, and then we just sat and watched different answers and kinda like listened to what they did. It was valuable learning, watching what others did and the feedback.

These examples highlight the relationship between the lecturer and the student and how both the lecturer and the student contributed to developing the models and the learning from the experience. Again, this highlights how when students are actively invested in the activity the learning becomes relevant and real for them.

In summary, the experiences of all individuals, both students and lecturers are all interconnected. An individual's experience is dependent on the contributions of the collective. That is, both fellow students and lecturers played a key role and influenced the overall student experience. The experience was a dynamic one: students' experiences within a group were interwoven; groups within a course could impact on another's experience, lecturer's influence could be both 'metaphysical' and/or 'in the moment'. Contributions from others provided reassurance, ideas and skills for moving forward and for some students generated cognitive conflict. Lecturers further contributed with materials and direction and did not need to be present to play a key role.

8.2 Transitioning between uncertainty and having direction

Students shifted backwards and forwards between '*moving forward on our own'* and *"being moved forward'*. The moving between the two positions was initiated when students either felt *uncertain* or had gained some *direction*. Students grappled with *being uncertain* and *having direction*, fluctuating between the two states. These states could also exist independent of whether they were actively engaged in *moving forward on their own* or were *being moved forward* through utilising available guidance. As we have seen, the coping behaviours students used while *moving forward on their own* involved becoming self-reliant, being open-minded and looking for reassurance. Lecturers provided ways students could find reassurance, direction and structure for the experience, including skills to use and space to work through their own process.

Struggle and creativity appeared to be a common component of *moving forward on our own* and *being moved forward*. (See Figure 14). Students had to be creative in their process to enable them to produce solutions that would be appropriate for their situation. Students needed to interact with the guidance to find direction, meaning students needed to navigate the process themselves. This involved grappling with the situation and material to find direction. With ambiguity being at the heart of creativity, it follows that for students to practice the art of mathematical modelling, there needs to be room for some level of uncertainty to be experienced by students in order for students to appreciate the creative nature of the process. One way forward out of this uncertainty is to embrace the doubt and struggle that can be involved by trying different approaches. Through trying and exploring different approaches, different options became possible. This makes uncertainty and exploration key behaviours for a mathematical modelling experience that fully embraces learning to mathematically model. This is a key difference between mathematical modelling, including modifying of models the students were doing, and students' normal mathematical experiences.

See Figure 13 below for potential outcomes and responses to struggle.

Figure 13. Responses to and outcomes of struggle

8.3 Being moved forward or modelling on your own?

The amount of independent student work vs lecturer guidance could play a role in whether the students are being moved forward with modelling, learning about mathematical modelling or simultaneously being moved forward and learning about mathematical modelling. Students were either *being moved forward* or were *moving forward on their own* with the modelling. The amount of guidance vs independent work encouraged, and the number of students in the group, all influenced whether students were being moved forward or moved forward on their own. As seen from course one students, the first approach was to look for ways of being moved forward. Students' natural response was to look for handholding. To fully embrace the creative nature of modelling, situations needed to be provided that give students space to produce their own work. Activities need to be set up that provide frameworks for students to use that minimise the struggle but promote independent creative work.

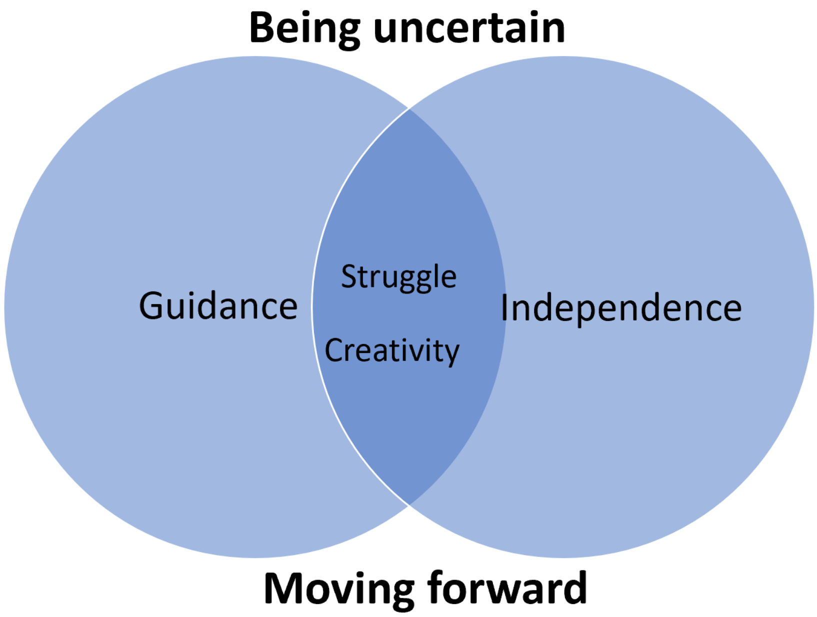


Figure 14.Transitioning between being uncertain and moving forward

As seen, factors identified that minimise student struggle and promote independent creative work are: keep it simple; collaboratively working together; utilising lecturer's advice and other students' contributions; and being open-minded. Further work is needed to look at the tensions between guidance and independence.

Enabling students to move forward in mathematical modelling looks different to how most mathematics lecturers and teachers are used to instructing and teaching. The data analysis shows that regardless of the form of guidance given by the lecturer, whether it is in-the-moment advice or external resources to refer to, guidance given needs to:

- Create space to allow for discovery and creativity.
- Allow productive struggle to develop students' resourcefulness and maximise learning opportunities.
- Provide reassurance that students are on the right direction with their development of a model.
- Provide reassurance that being uncertain is part of the mathematical modelling process.

- Enable students to manage the process without minimising its complexity.
- Contain timelines to move things forward.
- Allow for student groups to have their own unique experience of developing a model.
- Promote collaboration within student groups.
- Promote student ownership and responsibility of their process.

Preparation and guidance need to promote and develop behaviours for successful student independent work. These key student behaviours include:

- Being open-minded
- Backing ourselves
- Being resourceful
- Working collaboratively
- Being comfortable with uncertainty

The following Table 4 shows the key considerations identified for lecturer guidance and independent student work for a mathematical modelling activity:

Table 4. Key considerations for lecturer guidance and independent student work

Underlying philosophy	**Guidance**	**Independent work**	**Behaviours to develop**
Help students be equipped for their own self discovery and creativity Help students embrace being uncertain as productive (productive struggle)	Facilitatory in nature Provides Reassurance Checkpoints for advice and support Tools to check work Sharing of work to see others' approaches Role modelling of behaviours of modelling Scaffold aligned to the modelling process No detailed one final solution Freely available modelling cycle as scaffold Lecturer not to be freely available Explicit reference to modelling cycle	Collaborative in nature Exploration Discussion Debate Sharing of ideas Independent learning	Open-minded Backing ourselves Resourcefulness Collaboration Comfortable with uncertainty

As seen, if the lecturer is present during the activity and provides hints, these hints need to contain a certain level of ambiguity, while providing enough direction without providing all the answers. All guidance should promote productive struggle, to enable both learning and creative. Students productively struggling while attempting to create a model provides a platform for learning that is relevant. Learning in this way has the potential to be retained for use in future situations.

It appears to be important that the lecturer is not always freely available. It was seen that students who utilised the lecturer each time they struggled, were the ones that had most difficulty when it came to independent modelling. It appears important if lecturers are to develop confident modellers that there is a focus on developing students to become risk takers, confident to explore and try out new things, and developing students who are confident they can find and use resources to guide their modelling behaviour when needed.

In summary, learning takes place between states of ‘being uncertain’ and ‘having direction’. ‘Being uncertain’ and ‘having direction’ can occur whether students are ‘moving forward on own’ or ‘being moved forward’. Commonalities to ‘being uncertain’, ‘having direction’, ‘moving forward on own, ‘being moved forward’ are struggle and creativity.

Students working independently was a key part of the learning process. All the provided support had periods where the lecturer was ‘hands off’. Discussion and trying different approaches appear to be key for students moving between ‘being uncertain’ and ‘having direction’. Uncertainty, exploration, and ‘give it a go’ were key behaviours experienced by students.

Chapter 9. Discussion and teaching implications

Overall the results indicate that the students' experience when learning to mathematically model is one that pushed their boundaries. When I look across the previous findings' chapters, the core findings for the experience of *students having their boundaries pushed* while learning to mathematically model were: Modelling is a change of culture for students; The lecturer has a role to play; Balance between independent work and the lecturer's role is needed. In light of existing literature, I will discuss these core findings within the experience of *students having their boundaries pushed*. I will structure the discussion using the subthemes used to present the findings: *sources of pushing boundaries, moving forward on our own, being moved forward*, *going forward together as a collective* and *dynamics between independent student work and lecturer guidance*. Within these I will discuss change of culture for the student experience and what this means for the learning of mathematical modelling and mathematics. I will follow with the role the lecturer plays and the potential factors that contribute to achieving a balance between independent work and the lecturer's guidance. Teaching implications from the findings will also be highlighted.

9.1 Sources of pushing boundaries[14]

Table 5 provides a synthesis of findings and the existing literature in the form of a summary of what was known before, what was found, and how what was found supports, extends or refutes previous knowledge in reference to sources of pushing boundaries for students learning to model.

[14] This section includes re-presentation of material published in Spooner (2022).

Table 5. Sources of pushing boundaries

Sources of pushing boundaries		
What was known before	**What was found**	**How what was found supports, extends or refutes previous knowledge**
Use of mathematics with context can deepen students understanding of mathematics (Blomhøj, 2019). The use of contexts gives students something to anchor the mathematics to (Blomhøj, 2019; Dreyfus, 1991; Freudenthal, 1973).	Applying mathematics to contexts can make mathematics easier to understand. As students see connections and see its applicability this may reduce the view that mathematics is hard and inaccessible.	Contexts can give students access to mathematics by providing situations students can relate the mathematics to. This supports Blomhøj (2019), Dreyfus (1991), and Freudenthal (1973). The use of contexts has the potential to change students attitudes to mathematics in a positive way (extends previous findings).
Belief students need practical experience to be able to connect mathematical concepts to reality (Armstrong & Bajpai, 1998; Quimson, 2021).	Using mathematics with context is new for most students. More experience students have with applied mathematics subjects the easier they found it to relate mathematics to context. The nature of mathematical modelling, the lack of detailed guidance and the aspects of a total immersion experience have the potential to create a situation where students are forced to consider the links between mathematics and the real world.	The more students get to work with applying mathematics to contexts, the easier it becomes to see these connections between the two (teaching implication). Regularly apply mathematics to contexts in mathematics and modelling course to normalise using mathematics to contexts wherever possible (teaching implication).

Sources of pushing boundaries		
What was known before	**What was found**	**How what was found supports, extends or refutes previous knowledge**
A task should be as authentic as possible (Alsina, 2007; Galbraith, 1995; Kaiser-Messmer, 1993; Palm, 2007). However, contexts should be experientially real for students, not necessarily real (Gravemaijer & Doorman, 1999; Van den Heuvel-Panhuizen, 2003).	Students were able to connect mathematics to popular fictional contexts when the context was relatable.	Contexts need to be relatable not necessarily authentic. This supports and extends Gravemaijer and Doorman (1999).
More than context alone is needed for students to make connections between contexts and mathematics (Boaler,1993b; Beswick, 2011; Brown, 2017).	Students who had a total immersion experience with mathematical modelling more readily embraced relating mathematics and context. A motivator, in the form of a reason, why the mathematics is being related to the context is needed. Exploration appears to be key for connecting mathematics and context.	The total immersion experience provides space for students to independently engage in having a go at making connections between mathematics and context as opposed to avoiding it. The goal of developing a model was the motivator for making the connections. This supports and extends Boaler (1993b), Beswick (2011), and Brown 2017.
Genuine collaboration and 'interthinking' is needed when working with context within a mathematical modelling situation (Brown, 2017).	Student coping behaviours for relating mathematics to contexts while working within collaborative modelling groups includes being open-minded to others' input and backing themselves to take independent action.	To promote students' 'interthinking' and genuine collaboration through being open-minded, include training in group brainstorming, sharing ideas within group discussions within modelling courses and participation in independent group work. This supports and extends Brown (2017).
Creativity is a requirement for using mathematics with contexts (Alsina, 2007).	Space to explore fosters creativity. Exploration appears key for connecting mathematics and contexts.	Students need to be given space to explore the connections between mathematics and contexts. This supports and extends Alsina (2007).

The main reason for *pushing our boundaries* being the overall experience for students was that the modelling experience was new for students and involved using new behaviours and ways of thinking for applying mathematics to unfamiliar/novel situations. One of the key new behaviours and ways of thinking is relating mathematics and context, an essential component of modelling. The results showed that using mathematics with context was new for almost all students. Even though this experience was new for students, it was seen that relating mathematics with contexts has the potential to make mathematics relatable for students. Steph (course 1) was an example of this demonstrating that when mathematics is applied in contexts, students' internalisation of mathematical procedures and understanding of the mathematical concepts' meaning can deepen (Blomhøj, 2019). Within RME, moving backwards and forwards between mathematics and context is known as horizontal mathematisation (Gravemeijer & van Eerde, 2009). Blomhøj (2019) believes students' understanding of mathematical concepts can be critiqued and developed further through horizontal mathematisation processes involved with mathematical modelling and its use of contexts. While relating mathematics and context may not reduce cognitive load for students, the context can give access to productive cognitive conflict that may not have been accessible otherwise. While Steph's experience initially suggests making mathematics relatable has the potential to reduce cognitive conflict, it may be more likely that the use of contexts gives students something to anchor the mathematics to. This enables the mathematics to become relatable, opening up the possibilities of students examining their previous understandings, challenging them, adding to them and thus allowing further developing understanding (Blomhøj, 2019).

Historical exposure to using mathematics in context was seen in the results to help students see the connections with mathematics in reality. Rhys (course 2) is an illustration of this, supporting literature that suggests previous experience with modelling helps future modelling experiences (Cole et al., 2010; Klymchuk & Zverkova, 2001; Soon et al., 2011). Experience with using mathematics with context as beneficial for modelling skills development implies that the more students get to work with applying mathematics to contexts, the easier it becomes to see these connections between the two. This supports Armstrong and Bajpai (1998) and Quimson (2021) who believe students need practical experience to be able to connect mathematical concepts to reality. Practical experience allows for "hands-on experience for concepts to become tangible mathematical ideas … bring[ing] mathematics into the student's world (Julie, 2002; Dindyal & Kaur, 2010)" (Klymchuk & Spooner, 2020, p. 30).

It was clearly seen in the results of this study that students who had a physics background, that is, those that had previous experience with an applied mathematics subject, found relating mathematics to context a lot easier within the context of mathematical modelling. Students' experiences with physics prepared students in the thinking needed to see connections between mathematics and contexts. However, Chaachoua and Saglam (2006), Dominquez, Garza and Zavala (2015), Nguyen and Rebello (2011), Rowland and Jovanoski (2004) and Rowland (2006) studies showed students do not tend to naturally link between physics, mathematics and physical processes. Quimson (2021) found that in a physics context "students have little experience using calculus in word problems" (Quimson, 2021 p. 36). While the studies of Chaachoua and Saglam (2006), Dominquez, Garza and Zavala (2015), Nguyen and Rebello (2011), Quimson 2021, Rowland and Jovanoski (2004) and Rowland (2006) were focussed on connecting mathematics with physics and physical process, they did not examine the connections between using mathematics within a modelling context and how experience with physics can influence this. The results of this study suggest it is through the use of mathematical modelling that students begin to realise these connections between physics, mathematics and the physical processes. It is the nature of mathematical modelling, the lack of detailed guidance allowing space for exploration, and the aspects of a total immersion experience that creates a situation where students are forced to consider the links between mathematics and the real world. (The aspects of the total immersion experience will be discussed more fully shortly).

Further support for students' experiences with physics preparing their thinking for seeing connections between mathematics and contexts, is the different use of contexts across the different courses. The contexts of the problems used across the courses ranged from physical processes, population growth, to spread of infectious diseases, showing it was not that the problems necessarily utilised physics principles. More likely it is the previous experience with physics, an applied subject, had potentially primed the students to be aware of connections between physical processes and mathematics and the thinking needed to make these connections. Further research is needed into the nuances of a student's previous experience with physics and how this helps prepare them for relating mathematics and contexts. Further research is also needed into the nuances of how previous physics experience can influence making connections between mathematics and contexts within a modelling situation.

Students who had a physics background found relating mathematics to context within different contextual modelling situations easier implies that to prepare students for mathematical modelling, we need to be, at the minimum, regularly applying mathematics to contexts in our mathematics and mathematical modelling courses. This ranges from using examples involving real-world contexts through to translating word problems to mathematics, to start to normalise this way of using mathematics. It would be expected that a modelling course, by its nature, would involve applying mathematics to contexts. What is clear from the results is more opportunities are needed for students to apply mathematics to contexts outside of the modelling course. This means for teaching and lecturing, that if mathematical modelling is valued as a discipline within mathematics, including teaching and learning activities that show connections with mathematics and the real world are needed. This, in turn, will make mathematics more relatable to students, deepen mathematical understanding, normalise using mathematics with contexts and will potentially address the common student question of ‘when will I need this?’

One aspect to be wary of when using contexts with mathematics is the use of artificial contexts. Alsina (2007), Galbraith (1995), Kaiser-Messmer (1993), Palm (2007) advise that a task should be as authentic as possible. If artificial contexts are used, there is a risk that students will not see applying mathematics to contexts as a worthwhile behaviour. In the early RME writings, Gravemeijer and Doorman (1999) stated that contexts needed to be experientially real for students. Similarly, Van den Heuvel-Panhuizen (2003), states contexts in RME are not limited to everyday life or the real world but can also include formal mathematics and fictional situations. This means contexts do not need to be authentic, only that the student can relate to the context. Carducci (1996) found when students could relate to the context, the modelling went a lot smoother for students. The results of this study showed the use of popular contexts, as in the zombies used for course three, did make the context relatable for students and thus gave some ‘roots’ for the modelling to occur. This result supports Gravemeijer and Doorman (1999), Van den Heuvel-Panhuizen (2003) and Carducci (1996) showing that the contexts used need to be relatable, and not just ‘toy’ problems.

The results showed students had differing experiences relating mathematics to contexts, even though using context provided the opportunity for mathematics to be more relatable for students. Students either struggled with relating mathematics to context or took it in their stride. The results showed that students who had a total immersion experience with mathematical modelling more readily embraced relating mathematics and context. This is supportive of Blomhøj (2019), Dreyfus (1991) and Freudenthal (1973) who promote the use of contexts to give mathematics cognitive roots, giving the mathematics something to anchor itself to. The total immersion experience allows for students to truly engage with the context and mathematics, providing the space for students to explore and embrace this new way of using mathematics. It also provides the space for students to search out new mathematics necessary to develop the model, though students do not always capitalise on this opportunity. It is the situation (context) that provides the motivation to not only learn new mathematics (Rowland & Jovanoski, 2004; Lingefjärd, 2006), but also use mathematics in a new way. It is the total immersion experience that provides the space and necessity to embrace this experience as opposed to avoiding it. There is a teaching implication here that instructors need to provide students independent opportunities to explore connections between mathematics and the real world. A motivator, in the form of a reason, as to why the mathematics is being related to the context is needed. This is because the student experience is more than connecting mathematics to reality. It was the goal of developing a model that drove the students to embrace these connections. The fact that the 'modelling activity' was valued as a formal part of assessment, also provided motivation for students. It was also clear that if the assessment was not highly weighted, competition was minimised amongst students, resulting in students having more freedom to explore. The results support Boaler (1993b), Beswick (2011) and Brown (2017) that when teaching mathematics, more than context alone is needed for students to make connections between contexts and mathematics.

The results of this study showed students utilised coping behaviours to work with both the context and the modelling process to develop the mathematical model. This is similar to Brown (2017), who recommends genuine collaboration and 'interthinking' is needed when working with context within a mathematical modelling situation. Our results identified coping behaviours within this collaboration and ‘interthinking’ including being open-minded to other’s inputs and backing themselves to take independent action as they worked through relating mathematics as a group. A teaching implication to address being open-minded, is to include training in group brainstorming, sharing ideas within group discussions within modelling courses and for students to practice this in student groups. Providing opportunities for students to make their own mathematical decisions will give students a chance to back themselves.

It appears it was the independent experience that forced students to embrace these other behavioural factors associated with modelling and thus contributed to students’ progress. Progress was not necessarily easy. It appears some students who had a partial immersion experience struggled more than those students that had a total immersion experience when embracing mathematics with contexts. This suggests that specific activities or guidance focussed on relating mathematics to context would be beneficial for students only experiencing parts of the modelling process. These activities would, in turn, be beneficial to all students, regardless of a partial or total immersion modelling experience. Further research is needed into the key characteristics of the activities and guidance.

It was clear from all student experiences that students needed to understand the context in order to make progress, supporting Carducci (1996), De Almeida (2018) and Stillman’s (2002) claims that understanding the context is needed for successful modelling. Ngaire’ s experience of being overwhelmed by the context was an example of Boaler’s (1993b) findings that some students could focus too much on contextual information and that being familiar with the context already did not necessarily make the process easier. This adds further evidence to the findings that some sort of guidance around connecting contexts and mathematics, particularly in partial immersions modelling experiences, is needed for students.

Struggle was a part of the experience as seen above, with results showing that not all students had an easy time relating mathematics to context and vice versa. It is important to reduce and manage this struggle, as relating mathematics to context is integral to the nature of mathematical modelling (Alsina, 2007; Bender, 2000; Dym, 2004; Stillman, 2002; Tam, 2011; Treilibs et al., 1980; Villa-Ochoa & Berrío, 2015; Wake, 2016). As shown, one way of addressing this is to provide ample opportunities to connect mathematics with contexts. This demonstrates the need for creating experiences for future modelling students that enables students to think about connections between mathematics and the real world early in their education. Developing students' ability to make connections between mathematics and reality will also benefit the general teaching of mathematics by giving students ways to see and apply mathematical thinking in their lives. As students see connections and see their applicability, this may reduce the view that mathematics is hard and inaccessible. This implies that the use of contexts has the potential to change students' attitudes and access to mathematics in a positive way.

Making mathematics relatable by using contexts has the potential to reduce cognitive load and thus reduce abstract thinking (Blomhøj, 2019). Due to the complexities of modelling, relating mathematics to contexts may need to be structured into simpler exercises. To be able to relate mathematics to contexts in a way that is useful for modelling, students need previous experiences with this. This leads to a recommendation that using more concrete mathematics and reality examples in primary and high school education would not only benefit students access to mathematics, but also their future modelling abilities.

The student experience is supportive of the realistic mathematics education theory for learning mathematics. Utilising context for students to make connections between the real world and mathematics and then exploring mathematics within this structure makes provision for multiple connections to be made, allowing for a fuller mathematical experience and possible retention of knowledge (Freudenthal, 1973). Allowing these connections to be formed has the potential to promote more sustainable learning and a deeper understanding of mathematics that becomes internalised and adopted by students (Dreyfus, 1991). This illustrates the need for creating experiences that get students thinking about connections between mathematics and the real world early in their education. As students start to see the connections between the world and mathematics, they begin to appreciate its applicability. This may contribute to diminishing the belief that mathematics is challenging and inaccessible. This in turn would aid the development of 'seeing the world through the eyes of mathematics' which will help with mathematical modelling. There is an implication here for future work to be done on developing strategies that can be implemented by the lecturer that could help make the connections easier to see for all students.

Potential teaching implications from "sources of pushing boundaries" discussion:

- Include using authentic contexts and relating these to mathematics during lectures, both in mathematical modelling lectures and in general mathematics courses.
- Include using authentic contexts and relating these contexts to mathematics earlier in students' mathematics education.
- Ideally students to be exposed to connections between mathematics and authentic contexts both at secondary and primary school.

- Allow time for students to explore connections between mathematics and the real world in groups.
- Include tutorial sessions/workshops focussed on brainstorming in a group, sharing ideas within groups.
- Use teaching and learning activities that involve students making their own decisions and choosing solution pathways to explore.

9.2 Moving forward on our own[15]

Table 6 provides a synthesis of findings and the existing literature in the form of a summary of what was known before, what was found, and how what was found supports, extends or refutes previous knowledge in reference to students moving forward on their own while mathematically modelling.

[15] This section includes re-presentation of material published in Spooner (2021) and Spooner (2022).

Table 6. Moving forward on our own

Moving forward on our own		
What was known before	**What was found**	**How what was found supports, extends or refutes previous knowledge**
Modelling is best learnt by ‘doing’ (Blum, 2011; Blum, 2015; Boaler, 2001; Blomhøj & Kjeldsen, 2006; Caron & Bélair, 2007; Legé, 2007; Toews, 2011; Wake, 2011). Students to be actively involved with the modelling process (Bassanezi, 1994). Some independent modelling experience is needed to develop mathematical modelling competencies (Blum, 2015; Blomhøj & Kjeldsen, 2006; Carducci, 1996; Caron & Bélair, 2007; Kaiser, 2020; Stillman, 2011).	Learning to model is more than just ‘doing’ modelling and involves having an autonomous experience creating a, or part of a, model. Working independently of the lecturer was a key component of the student experience and involves taking responsibility of the student group process.	Lecturer and teacher promote students taking responsibility for their process and workings. This involves being fully responsible for making decisions on how to proceed with their solutions. This supports and extends modelling is learnt by doing (Blum, 2011, 2015; Boaler, 2001; Blomhøj & Kjeldsen, 2006; Caron & Bélair, 2007; Legé, 2007; Toews, 2011; Wake, 2011). Any guidance or intervention given to be of a facilitatory nature (teaching implication). Lecturer and teacher professional development in facilitation techniques and method for students to be responsible for their process (professional development implication). Students to work on exploratory activities without detailed guidance (teaching implication).

Moving forward on our own		
What was known before	**What was found**	**How what was found supports, extends or refutes previous knowledge**
Students have very little opportunity to work independently solving open problems during their previous mathematics education (Soon et al., 2011; Kaiser & Stender, 2013). Mathematics students are used to "following the lead" of authority figures involving "repeating things being shown" (Solomon, 2007).	Students were unaccustomed to working independently of the lecturer. Students used to following step-by-step instructions for mathematics.They were not used to applying independent thought while following instructions.	Inclusion of problem solving, including open-ended problems and investigations, in students' general mathematics education (teaching implication). Provide opportunities to work undirected, creatively, and independently in mathematics (teaching implication). Change in role of lecturer and student is a change of culture for mathematics.
Most research within ICTMA community focussed on when teacher is present to facilitate the experience (Blum, 2011; Blum, 2015; Kasier & Stender, 2013; Stillman, 2011). Students can successfully independently model during modelling competitions (Garfunkel, Niss & Brown, 2021).	Lecturer does not need to be present to facilitate for students to make progress. However, the lecturer had an impact on how the student experienced the independent component of the modelling activities. In the absence of teacher interventions, the independent total immersion experience helped students embrace applying themselves to the process.	Students are capable of independent modelling. This supports and extends Garfunkel, Niss and Brown (2021). This extends Blum (2011, 2015), Kasier and Stender (2013), and Stillman (2011). More studies needed into the student experience while independently modelling, including student preparation and support resources available to student. Provide opportunities for students to work independently of the lecturer (teaching implication).

Moving forward on our own		
What was known before	**What was found**	**How what was found supports, extends or refutes previous knowledge**
A lot of research into student difficulties (Chaachoua & Saglam, 2006; Crouch & Haines, 2004; Klymchuk & Zverkova, 2001; Klymchuk et al., 2010; Palm, 2007; Soon et al., 2011; Stillman, 2004). However, little research addresses how to minimise and overcome student difficulties when a student group is working independently of the instructor.	Working within a group makes the students more resilient to cope with struggle better. Being open and willing to consider incorporating or adopting other people's ideas was key to successfully moving forward on their own as a group.	Need to adopt a cooperative culture as opposed to an individualistic one. This extends and adds to the research for minimising student difficulties while independently modelling.
Help provided to students should continue to persevere student independence with their workings (Kaiser, 2020).	If help given is too much or guidance is too directive during the modelling activity, dependence on the lecturer can result.	Lecturers and teachers consciously provide guidance that promotes independence. This supports Kaiser (2020). Further research is needed into the boundaries and nuances between when guidance and help promote independence vs dependence.

Moving forward on our own		
What was known before	**What was found**	**How what was found supports, extends or refutes previous knowledge**
It can be common practice for mathematics students to feel unsure and uncertain of what they are doing (Stender & Kaiser, 2015).	Moments of "being uncertain" is a key behaviour experienced by students while modelling. "Being uncertain" was experienced when: working independently of the lecturer; working as a group without all the answers; and applying independent thought processes to work through developing a model or parts of a model as a group. Students wanted reassurance they were heading in the right direction, that what they were doing was modelling. Students who were exposed to the tool of dimensional analysis used this tool to provide reassurance their model was making 'mathematical sense'. While students were modelling, they were unsure that they were experiencing behaviours conducive to modelling.	Through experiencing productive struggle, students master and learn behaviours conducive to modelling, including knowledge of exploration and uncertainty as characteristics of modelling. This extends Stender and Kaiser (2015). Normalise uncertainty as okay and provide strategies to deal with uncertainty (teaching implication). Involves change of culture from lecturer presenting pre-meditated solutions to role modelling strategies for dealing with uncertainty (teaching implication). This extends Stender and Kaiser (2015). Lecturer to role model developing a model for a problem they are unfamiliar, exposing the behaviours associated with "being uncertain" as part of modelling (teaching implication).

Moving forward on our own		
What was known before	**What was found**	**How what was found supports, extends or refutes previous knowledge**
Heuristics strategies are recommended for modelling (Kasier & Stender, 2013; Stender & Kaiser, 2015; Stender, 2019). Recommendation of 'keep it simple' as a heuristic strategy for modelling (Bender, 2000; Stender, 2019; Wake, 2011).	Strategies could include exploration, heuristic strategies, and not producing a perfect model to fit the situation. Techniques and strategies provided by the lecturer need to be experienced by students in order for the students to appreciate and see their relevance.	Students need to experience modelling to realise when strategies are useful extends and supports modelling being learnt by doing (Blum, 2011, 2015; Boaler, 2001; Blomhøj & Kjeldsen, 2006; Caron & Bélair, 2007; Legé, 2007; Toews, 2011; Wake, 2011).
Explicitly teach modelling cycle and frequently allude to modelling cycle during the course (Caron & Bélair, 2007).	In addition to teaching the modelling cycle, modelling behaviours also need to be taught. Further need for the lecturer to expose the modelling behaviours of uncertainty and unfamiliarity and provide additional strategies for dealing with these in lectures and tutorials.	Lecturer to role model developing a model for a problem they are unfamiliar with (teaching implication).
Key behaviours for the experienced modeller involve being about to explore, go down dead ends and be comfortable not always immediately seeing a solution and 'just not knowing' and being able to step out and explore possibilities (Drake, 2012).	Exploration, including going down dead ends, was experienced as a key modelling behaviour by students, particularly those participating in the open-ended and popular fiction modelling activities. Students were initially not comfortable with not having a clear solution path to follow.	The less guidance and non-availability of the lecturer during the modelling activity there is, the more opportunity there is to experience the behaviours of an experienced modeller. This extends Drake (2012). Students initially being uncomfortable suggests the modelling experience is a change of culture for students.

Moving forward on our own		
What was known before	**What was found**	**How what was found supports, extends or refutes previous knowledge**
Culture needed for learning mathematics is one where students are actively involved in, both individually and with others, in creating their own knowledge (Solomon, 2007; Yackel et al.1991).	Active approach to mathematical modelling supports the culture needed for learning mathematics.	Supports students learn best how to model by doing modelling (Blum, 2011, 2015; Boaler, 2001; Blomhøj & Kjeldsen, 2006; Caron & Bélair, 2007; Legé, 2007; Toews, 2011; Wake, 2011).
Sharing and discussion of students' ideas and lecturers sharing responsibility of students' knowledge development with the student is needed (Yoon et al., 2011). Knowledge is developed, created, discovered as you proceed with your own process of mathematical modelling (Kotze, 2020).	Active approach to mathematical modelling supports students' understanding through promoting sharing and discussion of students' ideas and shares the responsibility of students' knowledge with the student.	Modelling provides an opportunity for students to be responsible for their own knowledge creation. This supports Yoon et al. (2011) and Kotze (2020).

The requirement to be *moving forward on our own* was a change of culture for students. Even though students were required to work independently of the lecturer, the lecturer still had an impact on how the students experienced the independent component of the modelling activities. In conjunction with the current literature, I will first discuss the change of culture for students while *moving forward on our own*. I will then discuss the key role the lecturer plays in students *moving forward on our own.* I will then look at how the students' process for *moving forward on our own* maps to the lecturer's processes involved with supporting students *moving forward on our own.*

9.2.1 Moving forward on our own - a change of culture for students

Working independently while dealing with mathematics was a change of culture experienced by all students. This experience was different to the experience students were used to for mathematics, that being, closely following the lead of authority figures involving repeating things being shown. Working independently pushed students' boundaries, requiring them to work and behave with mathematics in ways they had not worked before. This shift to working more independently involved a change in mathematical mindset for students. Students were unaccustomed to: working independently without clear guidance; taking responsibility for all or part of their working and learning process in mathematics; and being creative with their applications of mathematics. Working independently of the lecturer was a key component of the student experience. Students developed coping behaviours to enable them to *move forward* while working *on their own.* These coping behaviours included: taking action including backing themselves; being open-minded to others' inputs; and finding ways to feel reassured they were progressing in a productive way. This change of culture produced new experiences for all students. This new way of working, and the complexities of mathematical modelling, put everyone on an even playing field.

I will now discuss the change of culture and the lecturer's key role in students *moving forward on our own* within the current literature.

Students working together independently of the lecturer, taking responsibility for their group's process, was a key component of the student experience. It is commonly understood that modelling is best learnt by 'doing' (Blum, 2011, 2015; Boaler, 2001; Blomhøj & Kjeldsen, 2006; Caron and Bélair, 2007; Legé, 2007; Toews, 2012; Wake, 2011). The results showed that learning to model is more than just 'doing' modelling and involves having an autonomous experience creating a, or part of a, model. Blum (2015), Blomhøj and Kjeldsen (2006), Carducci (1996), Caron and Bélair (2007), Kaiser (2020), Stillman (2011) all acknowledge that some independent modelling experience is needed to develop mathematical modelling competencies. The results of this current study show that while an independent component of the modelling experience was offered to all students, students were unaccustomed to working independently of the lecturer. This is consistent with the findings of Soon et al. (2011) and Kaiser and Stender (2013), which shows students have very little opportunity to work independently solving open problems during their previous mathematics education. The results support this, showing not only can a student reach tertiary education not having experienced open problem solving, but a student can also reach their third year of undergraduate study and still not have encountered open problem solving. Independently using knowledge to solve open problems, including for mathematical modelling, is clearly a change of culture for students.

The results showed when students started working independently of the lecturer, they were initially dependent on being shown what to do, resisting taking responsibility for their process. Even though students were used to following step-by-step instructions for mathematics (Solomon, 2007), they were not used to applying independent thought while following these instructions. Modelling requires independently applying thought processes and being creative within this thought process. Literature (Blum, 2015; Durandt, 2018; Kasier & Stender, 2013; Kaiser, 2020; Stillman, 2011) focusses on teacher intervention to help secondary students navigate forward with the process. While there are valuable teacher interventions, the results of this study showed that it was the independent total immersion experience that helped students embrace applying themselves to the process.

Students needing to embrace and learn how to work independently of an instructor has clear teaching implications. Opportunities need to be given for students to work on exploratory activities without guidance. These activities need to be incorporated in our mathematical and modelling learning spaces. If an instructor is present, instructor guidance and interventions should be set up such that they do not provide a 'quick fix' for students and are of a facilitatory nature. Professional development in facilitation techniques and methods would be beneficial for instructors of mathematical modelling. The results show that there is evidence we need to train students how to work independently of us as instructors. While putting students in a total immersion situation where there was no instructor present did force students to embrace moving forward on their own, there may be other ways we can better prepare students for this experience. Instructors need to allow space for students to struggle enough to find their own way forward. At the same time instructors need to be aware of finding the balance between providing enough space for the struggle to be productive and avoiding too much struggle such that the student becomes despondent and gives up. There is an implication from the results that working within a group makes the students more resilient and cope with struggle better. This will be discussed more under "going forward together'.

The results showed that providing help during the modelling activity can create dependence on the lecturer. The first thing students did when they struggled was to look for something to show them what to do. If an intervention is provided prematurely, or contains too much detail, students' opportunity to engage in productive struggle is reduced. There is consensus in the literature that any help provided to students should continue to persevere student independence with their workings. Blum (2011, 2015), Kasier and Stender (2013) and Stillman (2011) focus on secondary studies where there is an assumption the teacher is present during students modelling activity to provide support as necessary. It has been shown that students can successfully independently model during modelling competitions (Garfunkel, Niss, & Brown, 2021) showing that it is possible both at secondary and tertiary levels to independently model. While this study did involve courses where the lecturer was present as well as courses where the lecturer was not present, a key result was the encouragement of students taking responsibility for their process. This was a key component of the student experience. The results suggest that not only should the instructor be supporting independent workings, it is for students to take responsibility for those workings, including being fully responsible for making decisions on how to proceed with their solutions.

Despite a lot of research into student difficulties, little research addresses how to minimise and overcome student difficulties when a student group is working independently of the instructor. Research that has been carried out focusses on intervention strategies that teachers can use in the moment. The main result from teacher intervention studies identified as being helpful is asking a student to explain out loud what they had done so far (Kaiser & Stender, 2013). This is helpful on two accounts: one, so a teacher gets access to student thinking and then can see possible ways of providing appropriate guidance and assistance (Stillman, 2013); two, as a metacognitive tool for students 'collecting their thoughts' to communicate what they have done so far (Ramachandran et al., 2018). Understanding what students do when no help is available could provide insights into what support structures could be beneficial for developing students' independent modelling skills.

Students all had moments where they felt unsure, were struggling and, as a group, unsure about the direction they would take during the experience of *moving forward on our own.* The results showed there were two ways feeling unsure was addressed: students developed coping behaviours; or students used tools provided by the lecturer to provide reassurance and direction. Coping behaviours included: taking action including backing themselves; being open-minded to others' inputs; and finding ways to feel reassured they were progressing in a productive way. Taking action has already been discussed. *Being open-minded,* as one of the key behaviours, will be discussed here.

Being open and willing to consider incorporating or adopting other people's ideas was key to students successfully moving forward on their own as a group. Students needed each other's contributions in order to make progress, manage different possibilities for approaching the problems and any conflicting ideas. This meant that students needed to adopt a cooperative culture as opposed to an individualistic one. This experience is in direct contrast to the student mathematical experience Solomon's (2007) work highlighted when it found that students tended to avoid working with others. The mathematical modelling experience required students to work with each other involving a willingness to consider new ways of thinking about, behaving, and approaching a problem. This practice is different to that which most students of mathematics have previously experienced (Hernande-Martinez, 2020; Kotze, 2020; Solomon, 2007; Yoon et al., 2011). In this present study, the student experience involved being flexible and open to new ways of doing things including being actively involved, relating mathematics to context, working in groups, and being open to considering and adopting other people's ideas. This was due to the nature of the modelling process being indirect and complex.

Students having moments where they felt unsure was not necessarily a change of 'learning mathematics' culture for students. It is the behaviours we are asking students to use for modelling that produce states of feeling uncertain and struggle that is a change of culture for students. These behaviours of spending time working independently of the lecturer, working as a group without all the answers, and applying independent thought processes to work through developing a model or parts of a model as a group, are all key behaviours of mathematical modelling (Carducci, 1996; Garfunkel & Montgomery, 2019; Spooner, 2022) and are necessary to master. The results suggest it is through experiencing the productive struggle that students start to master and learn behaviours conducive to modelling.

Students used tools provided by the lecturer to provide reassurance and direction. Not only did students use social characteristics as in coping behaviours, they also used mathematical techniques and strategies in order to move forward. These mathematical techniques and strategies were either provided by the lecturer or were discovered by the students. A lot of the techniques and strategies provided by the lecturer needed students to have personal experience of the modelling in order for the students to appreciate and see their relevance. One of the main techniques/strategy students used to keep moving forward was 'keep it simple'. This result supports Bender (2000), Stender (2019) and Wake (2011) recommendation of 'keep it simple' as a recommendation as a heuristic strategy for modelling. The results showed that students who were exposed to the tool of dimensional analysis used this tool to provide reassurance their model was making 'mathematical sense'.

The results of this current study showed that along with lecturers providing tools for reassurance, it would be beneficial for a lecturer to work through developing a model for a problem the lecturer was unfamiliar with. This role modelling would be beneficial for students to see that the lecturer (perceived expert) does not necessarily have a clear idea of how they will approach the problem. This, in turn, would provide reassurance for students that not having all the answers when they start to develop the model is normal. Although the lecturer will never truly be able to role model the behaviour of a novice modeller, demonstrating the process they would use to approach modelling an unfamiliar situation would at least demonstrate the behaviours of exploration and decision making crucial to the modelling process.

The results from Spooner (2021), a publication on the student experience of course one students from this current study, showed the change in working culture experience left some students displaying modelling behaviour, though unsure if what they were doing was modelling. This suggests that even though the modelling cycle was explicitly taught and frequently alluded to during the course (as recommended by Caron and Bélair 2007), there is further need for the lecturer to expose the modelling behaviours of uncertainty and unfamiliarity and provide additional strategies for dealing with these in lectures and tutorials. As I argued in Spooner (2021), it is recommended that lecturers work through problems unfamiliar to themselves during lectures in order to role model these behaviours that are integral to a modelling experience. This would involve a change of culture not only for students but also for lecturers, that is, moving from a culture of the lecturer presenting pre-meditated solutions to one of role modelling strategies for dealing with uncertainty. Strategies could include exploration, heuristic strategies, and not producing a perfect model to fit the
situation.

Along with students developing coping behaviours and using tools provided by the lecturer, students also used each other to provide reassurance and direction. Students using each other to provide reassurance and direction will be discussed more fully in *moving forward together* discussion section.

The student experience of working in groups *moving forward on our own* supports the views of Solomon (2007) and Yackel et al. (1991) concerning the culture needed for learning mathematics, that is, one that students are actively involved in, both individually and with others, in creating their own knowledge. The active approach to mathematical modelling experienced by the students supports students' understanding through promoting sharing and discussion of students' ideas and shares the responsibility of students' knowledge with the student (Yoon et al., 2011). Students collaborating, discussing and sharing ideas is an example of students using peer interaction, in particular linguistic interaction, to accommodate new understandings and learning as promoted by Piaget's theory of cognitive development (Sevinç, 2021). This interaction is important as, for modelling, not everything is known. Knowledge is developed, created, discovered as you interact within groups, discovering the process of mathematical modelling (Kotze, 2020). The results showed it is through the social collaborative interactions and being able to accommodate others' thinking that students learnt how to progess with the modelling. This finding is supported by Sevinç (2021) who found "modelling involves social interaction that cause accommodation in the models" (Sevinç, 2021, p. 82). This is a very different experience to that experienced by mathematics students (Solomon, 2007; Williams et al., 2009; Yoon et al., 2011) where knowledge can be perceived as being given by the authorities, that is lecturers and teachers (Solomon, 2007; Yoon et al., 2011).

The results of this study show that the independent component of the student learning experience is important for modelling. I will now discuss how students *moving forward on our own* can be a change of culture for lecturers.

9.2.2 Moving forward on our own – a change of culture for lecturers

As is seen from the research done in secondary schools, students moving forward on their own can be a change of culture for mathematics instructors where the focus of a teacher is always how to support students to reach the end goal. This end goal is generally reproducing an already formed piece of mathematics. Here, in modelling, our goal is to support students to reach their own goals, creating their own piece of mathematics. One of the lecturer's key roles was to prepare students before the modelling experience. Preparation included preparing students with tools and techniques, and role modelling behaviours. It was clear from the results the lecturer's role also included providing support structures to scaffold students through development of a model. This more hands-off approach is different to how most mathematics instructors have operated in the past. The results showed that all lecturers had a different belief system about was and was not possible for students. This influenced the support structures the lecturers used. Support structures used by lecturers will be discussed more fully in the 'being moved forward' section of this discussion. The results showed that when given the right support frameworks students can make independent progress and the lecturer does not have to be present to make an impact. The key appears to be to provide the right support frameworks and resources before the independent component of the modelling experience, though it is not quite as simple as this. From the results it is recommended that students who are in the process of learning how to work without instant access to lecturer guidance, that is still dependent on feedback and direction form lecturers, need support structures to be in place. These support structures need to 'wean' the student off the lecturer. The results also showed that part of preparing students to work on their own should involve the lecturer 'role modelling' modelling behaviours, in particular role modelling the behaviour of uncertainty. These results, and the lecturer's role within these, will be discussed more fully with literature under the '*being moved forward'* section of the discussion. I will now discuss the results of the lecturer role modelling the behaviour 'being uncertain' as a way to support students moving forward on their own.

In Spooner (2021), I discussed the need for lecturers to role model the behaviour of uncertainty to students during lectures as a way of providing reassurance to student's during independent modelling experiences: This paper, which focussed on the independent student experience for course one in this study, found results show that the students discovered their own process, how to overcome challenges, and had an adult-like experience, in which they were responsible for their process and therefore learning from the experience. They discovered that their first attempt of a model will not be their last, a known modelling behaviour of practising modellers (Drakes, 2012). As already mentioned, this change in working culture experience left some students unsure if what they were doing was modelling. This implies that the lecturer should explain modelling behaviours of uncertainty and unfamiliarity more and offer more strategies for handling them in lectures and tutorials. It is recommended that lecturers work through problems unfamiliar to themselves during lectures in order to role model these behaviours that are integral to a modelling experience. This would involve a change of culture not only for students but also for lecturers, that is moving from a culture of the lecturer presenting pre-meditated solutions to one of role modelling strategies for dealing with uncertainty. Strategies could include exploration, heuristic strategies, and not producing a perfect model to fit the situation.

To further add to this discussion, it is not possible for a lecturer to role model the full extent of uncertainty that a novice student experiences when first navigating their way through an independent experience. Nevertheless, the lecturer can still role model the uncertainty experienced that is part of a modeller's experience when they model a new unfamiliar situation. This situation encompasses being presented with a situation where not all the answers are known to the modeller. The key behaviours for the experienced modeller involve being able to explore, go down dead ends and be comfortable not always immediately seeing a solution and 'just not knowing', and being able to step out and explore possibilities (Drake, 2012). Though the lecturer will never have the same experience as the novice student modeller, it is still important to role model the key behaviours of not having an immediate solution, exploring and trying out different things so students witness these key modelling behaviours which are, according to Solomon (2007) and Yoon et al. (2011), in direct conflict with their previous experience of mathematics. Students witnessing these new behaviours provides reassurance to students that the struggle involved with being uncertain is part of the modelling experience. This in turn provides assurance they are going in the 'right' direction for developing their model.

In summary, the experience of *moving forward on our own* was a change of culture for students and mathematics educators in general. It involves a change in mathematical mindset and behaviours required of students. A change in division of labour between the lecturer and the student was part of the culture change (Wake, 2016). The total immersion experience appears to be one way to help students embrace the change in culture between their past experiences learning mathematics and learning to model. While working independently of the lecturer is important, the lecturer plays a key role in preparing students for the independent components of modelling activities and in the set-up of these independent modelling opportunities. The key role the lecturer plays is discussed more fully in the following section *being moved forward.*

Potential teaching implications from "moving forward on our own" discussion:

- Incorporate exploratory activities in teaching activities.
- Professional development in facilitation techniques and methods for instructors that promote independence for students.
- Lecturers provide support that encourages successful independent work without minimising struggle.
- Lecturers role model developing a model for a problem they have never previously attempted for students.

9.3 Being moved forward

Table 7 provides a synthesis of findings and the existing literature in the form of a summary of what was known before, what was found, and how what was found supports, extends or refutes previous knowledge in reference to students being moved forward in their mathematical modelling experience.

Table 7. Being moved forward

Being moved forward		
What was known before	**What was found**	**How what was found supports, extends or refutes previous knowledge**
Students engaging independently in the modelling process is beneficial for student development of modelling competency (Burkhardt, 2006; Blum, 2011, 2015; Carducci, 1996; Durand et al., 2015; Legé, 2005b, 2007). A directive teaching approach is not favourable as a teaching approach for developing modelling competency (Durandt et al., 2021). Students can struggle when modelling independently (Soon et al., 2011; Stillman et al., 2010). Students need both independent work and guidance (Blum, 2011, 2015; Durandt et al., 2021; Kasier, 2020).	Key lecturer action to promote student independence as a key behaviour for students. Students need both independent work and guidance. Guidance can be given before the activity or in person. Guidance involves preparing students beforehand, providing resources, role modelling behaviours. Important how guidance is given. Guidance needs to provide relevant support students can utilise on their own, leads to personal direction being found by students, allows students to be actively thinking and creating their own modelling process. Students can successfully model independently if all guidance is provided beforehand.	Key factors identified that contributed to the student being able to have an independent experience when the lecturer was not present included: having the modelling cycle explicitly available and frequently alluded to; sharing ideas through discussion and debate; being open-minded; activity structure allowing for student decision making. This supports and extends Blum (2015), Caron and Bélair (2007), Durandt et al. (2021), Kasier (2020). Key factors identified that appear to contribute to enabling a dependent relationship on the lecturer include: not clearly having a scaffold for the modelling process readily available for students; the lecturer being too readily available allowing students to ask the lecturer for help when stuck, as opposed to entering productive struggle to find own way forward; activity structure not allowing for student decision making. This supports and extends Blum (2011, 2015), Caron and Bélair (2007), Durandt et al. (2021), Kasier (2020).

Being moved forward		
What was known before	**What was found**	**How what was found supports, extends or refutes previous knowledge**
Explicit and frequent reference to a generic modelling cycle during teaching (Caron & Bélair, 2007). A modelling course should make modelling explicit, as opposed to only the mathematics counting (Wake, 2016).	Explicit reference to a generic modelling cycle contributes to students having an independent modelling experience. A generic modelling cycle as a form of scaffold can be used to support students' metacognitive activity.	Important lecturers are consciously making clear what they are doing and provide a 'generic step-by-step guide' that students can use to attach what they are doing to (teaching implication). Lecturers to provide a commentary of the modelling process alongside the models being shared in lectures (teaching implication).
Forms of scaffolding are used to support students modelling process (Durandt et al., 2021).	A case study can be used as a scaffold but does not necessarily promote independent student engagement with the modelling process.	Providing opportunities for students to independently modify case studies, and explicitly expose modelling process. This supports Blum (2015), Caron and Bélair (2007), Durandt et al. (2021), Kasier 2020).

Being moved forward		
What was known before	**What was found**	**How what was found supports, extends or refutes previous knowledge**
Creativity is a requirement for using mathematics with contexts (Alsina, 2007) and the process of modelling (Garfunkel & Montgomery, 2019). Mathematical creativity can be developed through student engagement in the modelling process (Amit & Gilat, 2014; Wessels, 2017). Only following rules and what has been shown stifles creativity (Solomon, 2007).	Lecturers provide opportunities for students to independently model with minimal guidance to allow opportunity for students to be creative and to engage with the process of modelling. Providing opportunities for independent student exploration could be key for students to develop modelling competency.	Use of directive resources limited students' opportunities for exploration and creativity, behaviours that may be key for mathematical modelling. This supports and expands on Durandt et al. (2021).

Being moved forward means students progressing forward through using advice external to themselves. It was seen from the results external advice came from the lecturer, available resources, and peers. The lecturer played a key role in providing external advice which impacted on student experience. Lecturers acted as facilitators, while promoting independent student experience. Facilitation through the use of scaffolding and/or coaching, either beforehand or during the activity, was provided by the lecturers. Common to all lecturer coaching styles was promoting independence as a key behaviour for students. The results showed that multiple exposures to a generic modelling cycle can act as a scaffold for students. Explicit exposure to the modelling process provides students with a cognitive structure to manage the complexity of the process. Committed periods of time for students to actively model allows students to engage with, and discover, the modelling process. If resources or advice given is too directive in nature, students could miss out on engaging with the modelling process and not fully experience working independently of the lecturer. The results showed students initially went looking for step-by-step instructions to follow and could not find any. Having no clear step-by-step instructions to produce a finished product and needing to independently engage with the support being provided was a change of culture for students. I will now discuss the role the lecturer plays in promoting students being moved forward while preserving the students' independent engagement with the modelling process.

9.3.1 Key roles lecturers play

Lecturers played a key role in students *being moved forward* with their modelling experience. The lecturer's main role was as a facilitator, promoting an independent student experience as well as providing guidance. As already discussed, results showed that independence was a key behaviour encouraged by lecturers in order to move students forward. Guidance was also needed to provide direction for students. Guidance was given through lecturers' coaching styles and various scaffolding frameworks. Guidance could take the form of resources students could independently use, either external or internalised resources, and in the case of course two, 'in the moment' direction for the next stage of modelling. It was seen that a lecturer's coaching style could promote student independence or lecturer dependence. It was important that the coaching role adopted by a lecturer would promote independence, as opposed to dependence, for maximum student gains. The way the lecturer role modelled the modelling process also had an effect on students' access to the modelling process. It was seen that explicit exposure of modelling process is needed for students. It was also found that whether a lecturer was present or not present, they played a role in the independent work of the student, demonstrating the influence that the lecturer has on the situation and illustrates the dynamic relationship between the student and lecturer.

It is firmly established in literature that the modelling is learnt by 'doing' (Blum, 2011, 2015; Boaler, 2001; Legé, 2007; Toews, 2012). Providing students with an independent experience of actively developing a model or parts of a model for the development of modelling competency is supportive of the literature (Blum, 2011, 2015; Carducci, 1996; Durandt et al., 2021, Legé, 2005b, 2007). Working independently students develop confidence as modellers, while Soon et al. (2011) and Stillman et al. (2010) acknowledge students can struggle when modelling independently. The findings indicate students need both independent work and guidance, similar to Blum (2011, 2015), Durandt et al. (2021) and Kasier (2020). The results also show the lecturer does not need to be present during the modelling activity in order to play a role in guiding students. This is different to the work of Blum (2011, 2015) and Durandt et al. (2021) who promote the lecturer and/or teacher physically being present to facilitate the modelling activity as necessary. Blum (2011, 2015) focusses on secondary students, therefore the teacher's being present may be justified from an institutional point of view. What the results showed was that it was possible for students to have an experience of mathematical modelling with the lecturer not present and could be more beneficial for students depending on the coaching style and scaffold used by the lecturer. The lecturer not being present did not mean that the lecturer was not 'helping' the students. Students taking part in the fully independent modelling activity all drew on resources provided from the lecturer before the activity, including the instructions of the activity. Resources provided allowed students to fully engage in the process of mathematical modelling. It is seen from both Blum (2011, 2015), Burkhardt (2006), Carducci (1996), Durandt et al. (2021), and Legé (2005b, 2007) that allowing students to engage independently in the modelling process is highly beneficial.

This means for teaching, the lecturer and teacher's role in students "being moved forward" is to provide relevant support that students can utilise on their own. Support should guide student behaviours that will lead to personal direction being found. Support needs to allow students to be involved in the active thinking and creating of their own modelling process. As already implied, providing support during lectures involves the lecturer both preparing students before a modelling experience with resources to draw on and role modelling of behaviours. If the lecturer is present during the activity, support involves providing hints that are limited to enabling independent student work. Guidance given needs to involve things that students can utilise 'on their own'. Preparation in modelling behaviours would also be useful for students to participate in before the modelling experience.

Key factors identified that contributed to the student being able to have an independent experience when the lecturer was not present included: having the modelling cycle explicitly available and frequently alluded to; sharing ideas through discussion and debate; being open-minded; activity structure allowing for student decision making.

Key factors identified that appear to contribute to enabling a dependent relationship on the lecturer include: not clearly having a scaffold for the modelling process readily available for students; the lecturer being too readily available, allowing students to ask the lecturer for help when stuck as opposed to entering productive struggle to find their own way forward; and the activity structure not allowing for student decision making.

The results suggest that the use of a generic modelling cycle and explicit reference to the modelling cycle (Caron & Bélair, 2007) contributes to students having an independent modelling experience. This implies that it is important lecturers are consciously making clear what they are doing and provide a 'generic step-by-step guide' that students can use to attach what they are doing to. Similar to Durandt et al.'s (2021) use of the DISUM solution plan (an example of a scaffold for modelling), course one's lecturer provided a generic modelling cycle as a form of scaffold to support students' metacognitive activity. Course two students, however, were not directed to a generic modelling cycle while doing their activity. This unavailability of a modelling cycle meant that their scaffold became the available case study workings and the lecturer. This also meant they were experiencing the modelling cycle implicitly, through the workings of the case study.

The use of the case study as a scaffold, as seen with course two, meant that students were initially dependent both on the scaffold and the lecturer. How the modelling process worked in relation to the development of the model in the case study might have been clear to the lecturer, it was not clear for all students. The results show that this dependence on the lecturer and case study for direction was not helpful for all students' development as modellers. This result is consistent with Durandt et al.'s (2021) result where it was found a directive teaching approach is not favourable as a teaching approach for developing modelling competency. Though it may not have been the lecturer's intention to maintain a dependent relationship with the students, the use of directive resources limited students' opportunities for exploration and creativity. This in turn appeared to have an effect when students later needed to work more independently; they did not appear to know what to do. Even though the lecturer provided space for students to work dissecting the model and working through the model's mathematics on their own, this was not enough for all students to develop the needed modelling skills. There is a teaching implication here that when a case study approach is used, a clear framework is needed to support that process. As already discussed, a commentary of the modelling process alongside the models being shared in lectures would help students to see the connections between the model and its development. This is important if our aim is to develop independent modellers.

As seen from the student experiences, explicit exposure of the modelling process is needed. An implication for practice is to set aside class time for students to be shown and experience the modelling process. Exposure to the modelling process should not be left as part of the things that students do 'outside of class" or as a part of the activity that can be by passed by if time pressure is too tight. With course one having the full day set aside to independently experience the modelling process meant that students engaged with the modelling process, in fact students could not move forward without doing so.

As seen from the results, opportunities for exploration and creativity are important for developing modelling skills. Creativity is a requirement for using mathematics with contexts (Alsina, 2007) and the process of modelling (Garfunkel & Montgomery, 2019) and has been suggested as a modelling competency (Lu & Kasier, 2022). This makes opportunities for students to work undirected a necessary condition for developing modelling competency. As seen from the Amit and Gilat's work (Wessels, 2017) mathematical creativity can be developed through student engagement in the modelling process. This makes opportunities for independent student exploration key for students working with a modelling process. We know from the work of Solomon (2007) that only following rules and what has been shown stifles creativity. The question is how can instructors set up a learning environment within which opportunities to be mathematically creative can occur, while also giving students some direction so the creativity can flow? This will be discussed further under Dynamics between independent student work and lecturer guidance, Section 9.5, page 300 of this chapter.

The other key question concerning the lecturer's role in students *being moved forward* is how do we as lecturers maintain the balance between promoting independence and minimising dependence while providing an experience that allows students to move forward? This will also be discussed further under Section 9.5, page 300 of this chapter.

Students working without clear direct advice or step-by-step working to follow was without doubt a change in culture for students. The results showed students initially went looking for step-by-step instructions to follow and could not find any. Having no clear step-by-step instructions to produce a finished product and needing to independently engage with the support being provided was a change of culture for students.

In summary, the lecturer plays a key role in students *being moved forward*, whether the lecturer is present during the modelling activity or not. The lecturer's role is to act as a facilitator, promoting students' independent work, and providing relevant support students can utilise on their own. The findings show the modelling cycle should be explicitly exposed and frequently alluded to during teaching. The activity set-up should allow for exploration, student decision making, and open and respectful discussion and sharing of ideas. Exploration appears to be key in enabling the creativity that is needed while working with the modelling process. This makes students having time to explore, while providing guidance that does not stifle creativity an important consideration for lecturers and teachers.

Potential teaching implications from "being moved forward" discussion:

- Provide opportunities for students to work undirected.
- Provide relevant support that students can utilise on their own.
- Provide guidance in the form of questions that places responsibility of answers on the student.
- Use a generic modelling cycle and explicitly reference the modelling cycle throughout lectures.
- Provide commentary of the modelling process alongside the models during lectures.
- Preparation in modelling behaviours before students participate in the modelling experience.
- When using a case study approach, a clear modelling framework to support students' modelling process development.

9.4 Going forward together as a collective

Table 8 provides a synthesis of findings and the existing literature in the form of a summary of what was known before, what was found, and how what was found supports, extends or refutes previous knowledge in reference to students going forward together with their peers and lecturers within the mathematical modelling experience.

Table 8. Going forward together

Going forward together		
What was known before	**What was found**	**How what was found supports, extends or refutes previous knowledge**
Students need both independent work and guidance (Blum, 2011, 2015; Durandt et al., 2021; Kasier, 2020).	Lecturer plays an important role in facilitating students to utilise resources and discover their own modelling process. Giving students responsibility for utilising modelling resources helps students discover their own modelling process. Active student involvement permits students to develop responsibility and ownership for developing knowledge and understanding of the modelling process.	Lecturer role is a facilitatory role, and the lecturer should not be available all of the time, to allow students to experience and/or self-discover the modelling process. This supports and extends Blum (2011, 2015), Durandt et al. (2021), Kasier (2020). Lecturer use of a set of facilitatory questions aligned to the modelling process, whether as part of a scaffold or in providing hints in person for moving forward (teaching implication).
Issue of division of labour between student and lecturer raised (Wake, 2016).	If lecturer support is too directive or readily available, the student can remain dependent on the lecturer for guidance.	Lecturer's role is as a facilitator (teaching implication).

Going forward together		
What was known before	**What was found**	**How what was found supports, extends or refutes previous knowledge**
Student support required differs from student to student (Wedelin & Adwai, 2014).	Students unconfident in their mathematics or language ability, and those with time-management issues had difficulty embracing the independent component of the modelling, resulting in these students remaining reliant on the lecturer for direction.	Lecturer to be conscious that unconfident mathematics and language students may struggle more than others (teaching implication). Create student working groups that have a range of mathematical and language abilities for the modelling activities (teaching implication). Training in time management for students is needed (teaching implication).

Going forward together		
What was known before	**What was found**	**How what was found supports, extends or refutes previous knowledge**
Working in a group is a key competency for the mathematical modelling (Bender, 2000; Dym, 2004; Spooner, 2017; Tam, 2011; Toews, 2012; Treilibs et al., 1980), and helps develop mathematical modelling competencies (Caron, 2019; Maaß, 2006; Toews, 2012). Students working in groups allows multiple perspectives to be shared and has the potential to increase learning outcomes (Lyon et al., 2020). Genuine collaboration while working with real-world modelling contexts deepens mathematical understanding (Brown, 2017).	Working in groups is a key component for productive mathematical modelling and helps develop mathematical modelling competencies. Students working in groups provides a way for students to negotiate the challenges of modelling. Students pooled resources and acknowledged the contributions of other group members, resulting in students progressing further than they would have individually. Working within a group can make students more resilient than when working individually and helps students to develop resilience to challenges of modelling. Group contributions enable understandings to be challenged and can clear up any confusion or uncertainty within a group.	Students had an appreciation that they needed each other to navigate the complexities of the modelling task (extends previous findings). Students perceived they would have been able to progress without each other. This supports working within a group as a behaviour for mathematical modelling and modelling development (Bender, 2000; Caron, 2019; Dym, 2004; Maaß, 2006; Spooner, 2017; Tam, 2011; Toews, 2012; Treilibs et al., 1980). Working in groups contributes to the collective student understanding. This supports and extends Brown (2017) and Lyon et al. (2020).

Going forward together		
What was known before	**What was found**	**How what was found supports, extends or refutes previous knowledge**
Students initially resisted working in groups, but later embraced and appreciated group work, finding it beneficial to their learning (Ortiz-Robinson & Ellington, 2009).	Students embraced working with a group from the beginning of the modelling activity. No student mentioned preferring to work alone. Students pooled resources and acknowledged the contributions of other group members, resulting in students progressing further than they would have individually.	This extends Ortiz-Robinson and Ellington (2009), and supports Bender (2000), Caron (2019), Dym (2004), Maaß (2006), Spooner (2017), Tam (2011), Toews (2012), Treilibs et al. (1980), showing students perceive that working within a group is necessary while mathematically modelling.

Going forward together		
What was known before	**What was found**	**How what was found supports, extends or refutes previous knowledge**
For modelling, students find it useful to to work in a group, formally present their workings with others, finding explaining work to others develops mathematical understanding (Hernandez-Martinez &Vos, 2018). Having to formally present workings makes students responsible for the work they are doing (Boaler, 2008). Presenting workings to others contributes to skills needed in future work situations (Hernandez-Martinez-Vos, 2018).	Formal sharing of students working is important for helping students to further develop and understand their models. Benefits of students sharing their work include: becoming aware that there is no one way of approaching the model; developing conceptual meaning for modelling; sharing techniques for approaches; engaging in cognitive conflict to critique and compare approaches. Students presenting their work enables students to collect their thoughts through explaining their work to someone else. Having to formally share workings makes students responsible for the work they are doing. Students sharing workings has the potential to produce cognitive conflict for some students that is not easily resolved and can take time to resolve.	Students sharing their working allows all students to benefit from each other's progress while providing an opportunity for students to organise, collect and challenge their thoughts. This extends Hernandez-Martinez &Vos (2018). Presenting within a group makes students responsible for being actively involved supports Boaler (2008). Further research into benefits of student groups to work on the same open-ended problem and share their solutions with other groups. Further research into students working on the same open-ended problem to sharing their working at key stages of the modelling process. It is important for the lecturer to be mindful of the potential cognitive conflict sharing different workings can produce for some students. Further research into how to best support students to resolve cognitive conflict and find it different to assimilate different approaches.

Going forward together		
What was known before	**What was found**	**How what was found supports, extends or refutes previous knowledge**
Role of the lecturer is to provide feedback (Carducci, 1996; Lyon & Magana, 2020). Students regard teachers who provide individual or whole-class feedback after student presentations as a characteristic of a good lesson (Kaur, 2009). Positive feedback is utilised better than negative feedback (Lyon & Magana, 2020). Feedback given can either encourage or discourage students (Lyon & Magana, 2020; Stender et al., 2017) or promote or stifle modelling (Stillman, 2011). Promote students "talking out loud" their workings to enable teachers to assess what interventions would be helpful (Kaiser & Stender, 2013; Stender, 2018; Stender, 2019).	When students are invested in the working of a problem their receptiveness to feedback and different ways of approaching the situation increases. Students appreciated the direction provided by the lecturer's feedback. However, some students felt criticised by the lecturer. Important lecturer acknowledges the student and their contribution while giving feedback. Students presenting their work to the cohort enabled the lecturer to see the current level of skills and concepts students had acquired.	Students seeing other's presentations and receiving feedback on their own work increases their understanding of modelling. This supports the value of formal presentations (Hernandez-Martinez &Vos, 2018) and the lecturer providing feedback (Carducci, 1996; Kaur, 2009; Lyon & Magana, 2020). Lecturer needs to be aware to make the student feel valued while giving feedback. This extends Carducci (1996), Kaur (2009), and Lyon and Magana (2020). Lecturer feedback following student presentations supports Kaiser and Stender (2013), Stender (2018), and Stender (2019)

Going forward together		
What was known before	**What was found**	**How what was found supports, extends or refutes previous knowledge**
Useful instructor interventions require the instructor to be skilled at providing the right assistance (Kaiser & Stender, 2013; Stillman, Brown & Galbriath, 2013; Stender, Kromanke & Kaiser, 2017; Stillman, 2011).	If lecturer advice is too directive or the lecturer is too available, it can promote dependence on the lecturer.	Advice for students needs to be of a facilitatory nature that allows students to play an active part in thinking through and resolving any cognitive conflict. Advice should avoid step-by-step instructions for students to follow.

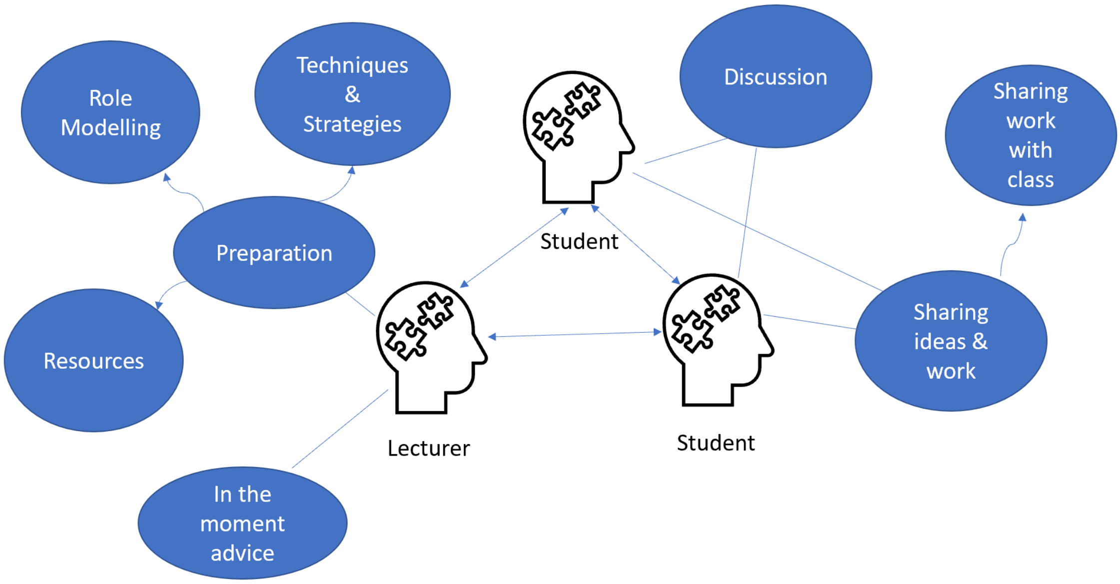


Figure 15. Going further together

The results show that students went further together. The experience was possible because of the lecturer contributions. The student and lecturer experiences were interwoven and contributed to individual students' experiences. The lecturer contributed with direction for students, whether present in person or by preparing students beforehand. The sharing of students' working was seen to be important for helping students to further develop and understand their models. I will now discuss these results in relation to the current literature identifying when these results were changes of culture for the student and/or lecturer and the key role the lecturer played in relation to the students.

We have already seen that students draw on lecturer resources when they perceive they need direction, regardless of whether the lecturer is present or not. If the lecturer is present, the direction students take can be further influenced by the lecturer. Results showed that students working independently of the lecturer engaged with the lecturer by recalling the lecturer's role modelling of process and techniques taught for modelling. This was also true for students who had access to the lecturer. The difference was when the lecturer was present, the lecturer gave clear expectations of what to do next. There is a danger that if students have not been given enough space for independent exploration, direction from the lecturer could limit independent student thought. In Spooner (2021), it was shown that giving students responsibility for utilising modelling resources helps students discover their own modelling process. This is in line with Bassanezi (1994) who advocates for students to be actively involved with the modelling process. For students to ultise resources and discover their own modelling process this study found the lecturer plays an important role. As mentioned by Wake (2016), the issue is on how the division of labour is shared between the lecturer and the student to promote student agency and student learning. The results of this study show that if the lecturer provides resources that are too directive, the student can remain dependent on the lecturer and miss out on experiencing/discovering the underlying process of the modelling. There is a teaching implication that support given from the lecturer should be of a facilitatory nature. If the lecturer is present, a set of facilitatory questions should be used. When the lecturer is not present, the use of a set of facilitatory questions aligned to the modelling process should be used. Facilitatory questions would be in the form of a list of prompts and questions that promote independent student exploration relevant to each stage of the modelling process.

The results also show that if the lecturer is too readily available, independent student thought can be minimised, resulting in the student maintaining dependence on the lecturer for guidance. This result shows that providing time for students to explore modelling where the lecturer is not available for advice and guidance is important to allow students to actively engage in the modelling experience. This active involvement permits students opportunities to develop responsibility and ownership for developing knowledge and understanding of the modelling process.

We know that support required for students differs from student to student (Wedelin & Adwai, 2014). The results show less confident students in either their mathematics ability or the language medium, and students with time management issues, had difficulties embracing the independent component of the modelling. This resulted in these students remaining reliant on the lecturer for direction. There is a clear need to support these students. One way of providing support could be to use student working groups that have a range of mathematical and language abilities for the modelling activities. If this is not possible, then the lecturer needs to be conscious that these students may struggle more than others and provide appropriate support as needed.

Working within a group produces a collective experience, with students individually impacting and contributing to the experience of others'. As mentioned, the collective student experience determines the direction that students take and, therefore, influences the student learning. Students pooled resources resulting in students progressing further than they would have individually. It was found that for a productive student experience, modelling should be undertaken in groups as opposed to individually and working within a group can make students more resilient than when working individually. Similar to Bender (2000), Dym (2004), Spooner (2017), Tam (2011) and Treilibs et al. (1980), this result shows working in groups is a key component for productive mathematical modelling and hence develops mathematical modelling competencies (Caron, 2019; Maaß, 2006; Toews, 2012).

Ortiz-Robinson and Ellington (2009) found that students initially resisted working in groups. This result was not seen from this study. This is most likely because students accepted working in a group as part of the experience and, due to the complexity of the modelling task, readily accepted group members as resources to enable progress to be made. None of the students mentioned preferred to work alone. All students commented on contributions from other students, acknowledging they saw each other as beneficial to the experience. Ortiz-Robinson and Ellington (2009) found that, by the end of their course, students had embraced and appreciated the in-class, student-centred group work, finding it beneficial to their learning. All students in this present study acknowledged the contributions of the other group members, illustrating that they do not perceive they would have been able to progress without each other. Like Toews (2012), this result provides further evidence to support being able to work in a group as a key competency of the process of mathematical modelling.

Students working in groups provides a way for students to negotiate the challenges of modelling (Spooner, 2022), and hence develop resilience. Similar to Brown (2017) who found genuine collaboration while working with real-world modelling contexts deepens mathematical understanding, it was found group contributions enable understandings to be challenged and can clear up any confusion or uncertainty within a group. This in turn contributes to the collective student understanding, supporting the idea that working within groups increases learning outcomes (Lyon et al., 2020).

The results showed that students sharing their working with the wider cohort was beneficial. The benefits include students: becoming aware that there is no one way of approaching the model; developing conceptual meaning for modelling; sharing techniques for approaches; and engaging in cognitive conflict to critique and compare approaches. These benefits are seen when students were working on the same open-ended problem and when students were working with different situations using the same base model.

Students sharing their work through presenting their work to the class is a powerful way for all students to benefit from each other's progress. As seen by Hernandez-Martinez and Vos (2018), students find it useful to be able to work in a group and present their workings with others. Students presenting their work enabled students to explain their working. The results of this study showed sharing of work also contributes to the collective understanding, enabling learning from others. While students find explaining work to others develops mathematical understanding (Hernandez-Martinez &Vos, 2018) this study shows it can also deepen their conceptual understanding of modelling. Through seeing each other's work, students were able to see firsthand the different approaches that produce unique models to solve a problem. This enforces the idea there are multiple ways to model a situation. Watching others present their work increases students' toolbox of skills, techniques and strategies that can be used. Students sharing different ways and approaches to the modelling situation helped students accommodate new ways for approaching the situations. As students witnessed other approaches to similar problems, they were able to engage in abstractive reflection (Kandasamy & Czocher, 2020) and further develop their cognitive thinking about modelling approaches and processes. This was not necessarily a direct conflict to what they were doing but used the work they had already invested in the problem to be able to recognise how they might further develop or improve their work. Congruent with Boaler (2008), having to share workings with others also makes students responsible for the work they were doing. Presenting workings also contributes to skills needed in future work situations (Hernandez-Martinez & Vos 2018).

Students intuitively saw the value in sharing peer workings for the same problem. For those students in courses that shared their work with the whole cohort, the results showed that students are disappointed when they do not get to see peer workings of the problem they are working on. This result shows when students are invested in the working of a problem their receptiveness to feedback and different ways of approaching the situation increases. While it is important for students' modelling development to share their workings and see the working of others, it is also important for students to see alternative workings, other than their own, for the problem they are currently working on. This highlights the potential learning opportunities for student groups to work on the same open-ended problem and share their solutions with other groups. Following on from this, it would also be potentially beneficial for students working on the same open-ended problem to share their working at key stages of the modelling process. The lecturer role modelling different models that could be produced for a situation and sharing different situations using the same foundation model would also increase students' understanding that there is no 'one size fits all' solution for a situation. For some students, the cognitive conflict that can be produced from seeing others' workings and different approaches may not always be easily resolved and may take time to resolve. For these students, it would be beneficial to provide support and/or ways to work through this cognitive conflict. Further research, including what this support would look like, is needed in this area.

Students presenting their work to the lecturer enabled the lecturer to see the current level of skills and concepts students had acquired. This opportunity to assess students' current development allows lecturers to decide on the prompts needed to enable the student to move to the next level of workings and understanding of their model and modelling process. This result is consistent with the work of Kaiser and Stender (2013), Stender (2018) and Stender (2019) who promote students "talking out loud" their workings to enable teachers to assess what interventions would be helpful. Research shows that in order for instructor interventions to be useful the instructor needs to be skilled at providing the right assistance (Kaiser & Stender, 2013; Stillman, Brown & Galbriath, 2013; Stender, Kromanke & Kaiser, 2017; Stillman, 2011). Literature suggests advice given can either encourage or discourage students (Stender et al., 2017) or can promote or stifle modelling (Stillman, 2011). While the lecturer participants in this study are all experts in their fields, the results show that if the advice given is too directive or the lecturer is too available, it can promote dependence on the lecturer. This dependence inhibited students' ability to independently model. This was particularly evident if the student is not mathematically confident or unconfident with the language medium being used for the activity. As already discussed, advice for students needs to be of a facilitatory nature that allows students to have an active part in thinking through and resolving any cognitive conflict. The lecturer making themselves unavailable for parts of the workings, and withholding directive advice, would encourage students to embrace independent thought and genuine collaboration.

An additional benefit of students explaining and discussing their thinking with someone is it enables students to 'collect' their thoughts. As seen in the results students need to collect their thoughts while preparing for formal presentations to the class. As an alternative way to fostering individual students to 'collect' their thoughts, it was seen when students take time away from the group they individually think about approaches and then are able to share approaches within their groups. This approach helped students to contribute to the direction of the model and allowed for discussion and debate around different possible approaches for the group to take.

This study shows students sharing their working allows all students to benefit from each other's progress while providing an opportunity for students to organise, collect and challenge their thoughts. It is important for the lecturer to be mindful of the potential cognitive conflict sharing different workings can produce for some students. If the lecturer is present during students sharing of work, it is important the lecturer provides facilitatory advice, avoiding step-by-step instructions for students to follow, and is conscious of uplifting the student while giving feedback.

In summary, student and lecturer experiences were interwoven and contributed to the individual and collective student experience. Along with the lecturer having an impact on student experience, students within groups, and groups within classes, impacted and contributed to the overall experience of others. Students working in groups developed resilience, perceiving they were able to progress further than working individually. Students formally presenting modelling activity benefitted by encouraging all group members to be actively involved and initiating collection of their thoughts to produce a clear presentation. Students observing other groups' modelling activity benefitted by being reassured of their own modelling experience, along with being exposed to alternative ways of thinking about and approaching their model development. Formal presentations allowed lecturers to provide feedback that benefitted both the presenting group members and the wider cohort. Individuals within the class cohort, both students and lecturers, contributed to the collective experience of their group and the class. The dynamics between the student and lecturer will be discussed deeper in the next section *dynamics between independent student work and lecturer guidance.*

<u>Potential teaching implications from "going forward together" discussion:</u>

- Students carry out modelling within student groups.
- Students to share their workings for models developed.
- When students are working on the same open-ended problem and/or situations using the same base model, students to share their work as final formal presentations.
- When students are working on the same open-ended problem and/or situations using the same base model, students to share their work at differing stages of the modelling process.
- Support given needs to help students that need it while promoting students' own discovery of solutions and processes.

- Facilitatory advice given regardless of whether the lecturer is present or not present during the modelling activity.
- Guidance given through the use of a set of facilitatory questions aligned to the modelling process should be used.
- Lecturers to role model different models that could be produced for the same situation.
- Lecturers to share different situations that can be modelled using the same foundation model.
- Lecturers to use students' own work as teaching points and advice for steps forward with the process.
- Support resource containing a list of prompts and questions that promote independent student exploration. This resource could be aligned to the modelling process.
- Students to take time away from the group where they individually think about approaches and then share these within their groups.
- Lecturer to be unavailable for advice for different sections of workings. Lecturers to be available 'after the moment' for any advice given.

9.5 Dynamics between independent student work and lecturer guidance

Table 9 provides a synthesis of findings and the existing literature in the form of a summary of what was known before, what was found, and how what was found supports, extends or refutes previous knowledge in reference to the balance between students working independently and receiving guidance while mathematical modelling.

Table 9. Independent work vs guidance

Independent work vs Guidance		
What was known before	**What was found**	**How what was found supports, extends or refutes what was known before**
A balance of independent and teacher-guided work is needed for learning to mathematically model (Kaiser, 2020). Directive teaching approaches are not favourable for developing modelling skills and competencies (Durandt et al., 2021). Active engagement in the process promotes longer retention through the experiential nature of the learning (Specht & Sandlin, 1991).	Guidance provided by lecturer needs to facilitate self-discovery, creativity and productive struggle, guidance needs to be non-prescriptive. Students and instructors need to solve the problem together, with the instructor playing a 'behind the scenes role'. Guidance should provide reassurance to students who are experiencing modelling, and confirm they are on track with model development. Key lecturer behaviours to enable students independently moving forward include: providing students with material to refer to; using facilitatory questions/prompts aligned with the modelling process; creating environments for students to share their thoughts and workings. Flexible and facilitatory guidance allows students to 'create' their own outcomes, providing students with their own unique experiences for modelling.	Guidance being non-prescriptive to promote self-discovery, creativity and allow for productive struggle supports and extends Durandt et al. (2021) and Kaiser (2020). Lecturer behaviours that enable students independently moving forward extends the work of Kaiser (2020) and Durandt et al., 2021).

Independent work vs Guidance		
What was known before	**What was found**	**How what was found supports, extends or refutes what was known before**
Research focusses on student approaches to modelling activity being underpinned by a modelling cycle (Abassian et al., 2020; Ferri, 2006; Kaiser, 2020).	The use of the modelling cycle to scaffold the experience should be flexible enough to allow students self-discovery and ownership of their own process.	Self-discovery could be considered a component of developing modelling competency. Further research and discussion is needed.
As a learning process, students should reflect on the whole process of solving a modelling problem (Meier, 2009). Reflecting on the modelling process is considered a metacognitive modelling competency (Blum, 2011; Maaß, 2006; Schnapp et al., 2011).	Reflection and consequential internalisation of process has the potential to support development of metacognitive strategies that are needed for modelling.	Encouraging students to keep a log of their process and then reflect on this would aid student learning and self-discovery and ownership of the modelling process (teaching implication).

Little is known from literature about how to help students navigate the uncertainty involved in modelling, other than removing the uncertainty. Creativity involves uncertainty and challenge (Amit & Naaman 2014).	Student moving forward on their own involves moving from being uncertain to having direction. Experiencing struggle, exploring, being creative and becoming self-reliant are all behaviours involved with students finding direction. Any support given by the lecturer neededs to promote these behaviours. Productive struggle was a key component of the student learning experience. Students were unaccustomed to uncertainty being part of the process and uncertainty being viewed as a necessary characteristic in model development. An outcome of struggle associated with uncertainty is student ownership through experience and self-discovery of how to navigate the modelling process. Guidance including scaffolding for assisted self-discovery should allow students to go down blind alleys and learn what does and does not work as they develop their model. Assisting students to self-discover the modelling process is important for deeper learning of the modelling process.	Lecturer support should enable exploration, struggle, creativity, self-discovery and becoming self-reliant (teaching implication). A supported experience that allows students to get their hands dirty, go down blind alleys and explore (teaching implication). Difficulties associated with uncertainty need to be managed, while remaining part of the experience to foster creativity. This extends on (Amit & Naaman, 2014). Lecturers need to be mindful not to fully remove difficulties for students (teaching implication). Develop a culture where struggle involved with uncertainty is seen as a productive attribute within mathematical modelling and mathematics education. Self-discovery could be considered a component of developing modelling competency. Further research and discussion is needed.

Independent work vs Guidance		
What was known before	**What was found**	**How what was found supports, extends or refutes what was known before**
Mathematical modelling is experienced as a change of culture for mathematics students (Cole et al., 2011; Hernandez-Martinez, 2020; Kotze, 2020)	Change in working-culture experience left some students displaying modelling behaviour, though unsure if what they were doing was modelling. Recommended that lecturers work through problems unfamiliar to themselves during lectures in order to role model these behaviours that are integral to a modelling experience, including uncertainty.	Mathematical modelling is more than mathematics. This extends Cole et al. (2011), Hernandez-Martinez (2020) and Kotze (2020). Lecturers working through unfamiliar problems has the potential to role model the modelling behaviour of uncertainty and unfamiliarity. This extends the work of Cole et al. (2011), Hernandez-Martinez (2020) and Kotze (2020). Lecturers working through unfamiliar problems as a way of teaching students is a change of culture for lecturers (extends previous findings).
Student support and guidance given through mandatory course-time lecturer consultations (Blomhøj & Kjeldsen, 2013).	Lecturer provided ‘in the moment’ guidance on student-presented work to the whole cohort.	Lecturer witnessing student-presented work allows for relevant guidance to be given, supporting Blomhøj and Kjeldsen (2013) and Hernandez-Martinez and Vos (2018). Collective student experience occurs when feedback provided in student presentations, extending the work of Blomhøj and Kjeldsen (2013), Hernandez-Martinez and Vos (2018), and Kaiser and Stender (2013).

The results show that students working independently is an important behaviour for mathematical modelling, though guidance is also necessary to allow for a successful student experience. Guidance given needs to allow students room for self-discovery, being creative and participating in productive struggle. Guidance should provide reassurance to students that what they are experiencing aligns with the nature of modelling and develops self-confidence that they are on track with developing their model. Guidance should promote the key characteristics of student independent work identified in the results. These characteristics are: collaborative in nature; involves exploration; sharing of ideas through discussion and debate; and opportunities for learning. Guidance and the activity structure should also encourage student behaviours of being open-minded, backing themselves, being resourceful and being comfortable with uncertainty. A change of culture, where struggle is seen as a productive attribute needs to be developed within the discipline.

Guidance given needs to allow students room for self-discovery, allow for creativity and enable productive struggle. To facilitate self-discovery, creativity and productive struggle, guidance needs to be non-prescriptive. This finding is supportive of Durandt et al. (2021) who found directive teaching approaches are not favourable for developing modelling skills and competencies. Research focusses on student approaches to modelling activity being underpinned by a modelling cycle (Abassian et al., 2020, Ferri, 2006; Kaiser, 2020). While having a modelling cycle to scaffold the learning experience is important, the use of the modelling cycle should also aim to allow students self-discovery and ownership of their own process. Assisting students to self-discover the modelling process is important as it has the potential for deeper learning through active engagement of the student with the modelling process (Spooner, 2022). This active engagement, in turn, promotes longer retention by students through the experiential nature of the learning (Specht & Sandlin, 1991). Assisted self-discovery through scaffolding and/or alternative supporting guidance needs to be provided as a skeleton framework that allows students to go down blind alleys and learn what does and does not work as they develop their model. Encouraging students to keep a log of their process and then reflect on this would aid student learning and self-discovery and ownership of the modelling process. Students logging and reflecting on their process is supported by Meier (2009) who encourages students reflecting “on the whole process of solving a modelling problem as a learning process.” (Meier, 2009, p. 214). This reflection and consequential internalisation of process has the potential to support development of metacognitive strategies that are needed for modelling.

Creativity is essential for working with the modelling process (Garfunkel & Montgomery, 2019; Spooner, 2022) and when using mathematics with contexts (Alsina, 2007). Creativity involves uncertainty and challenge (Amit & Naaman, 2014). Flexible and facilitatory guidance allows students to ‘create’ their own outcomes, providing students with their own unique experiences for modelling. Amit and Naaman (2014) found creativity is associated with challenge for a lot of students. Amit and Naaman’s (2014) finding may go some way to explaining why research has shown students find mathematical modelling difficult.

Productive struggle was a key component of the student learning experience. While we know that students find modelling difficult, with a lot of research focussed on identifying student difficulties with modelling (Chaachoua & Saglam, 2006; Crouch & Haines, 2004; Klymchuk & Zverkova, 2001; Klymchuk, et al., 2010; Palm, 2007; Soon et al., 2011; Stillman, 2004), we must be careful not to fully remove these difficulties. It is important that difficulties associated with uncertainty remain part of the experience to allow for opportunities for students to be creative (Amit & Naaman, 2014). It is crucial to understand the nature of student difficulties to gain insight into how to best support students to navigate them. It is also important to know how to leverage these difficulties for student learning. Students grappling with difficulties to determine how to navigate the difficulty is a potentially powerful learning experience. As already alluded to, what is needed is a supported experience that allows students to get their hands dirty, and go down blind alleys and explore. Blomhøj and Kjeldsen (2013) suggest mandatory course-time lecturer consultations as one way of supporting the student experience. The results of this present study show getting 'in the moment' guidance on student-presented work was also a way of providing mentorship. Guidance given in this situation benefitted the students sharing their own work and the class audience, allowing for a collective student experience. Providing 'in the moment' guidance in this way takes Kaiser and Stender's (2013) 'talk out loud' strategy one step further so that it benefits the whole cohort as opposed to just the individual or a small group of students.

The results show guidance should provide reassurance to students that what they are experiencing aligns with the nature of modelling as well as confirms they are on track with developing their model. The results showed that students wanted reassurance they were on the right track. Being on track with model development also includes being confident that what they are experiencing is consistent with the modelling process. Part of mathematical modelling is experiencing moments of uncertainty including being unsure of the direction to proceed and the final answer or product. Students were unaccustomed to uncertainty being part of the process and uncertainty being viewed as a necessary characteristic in model development. As already mentioned, role modelling of the behaviours of uncertainty and dealing with unfamailairty when modelling is needed from the lecturer. A change of culture where struggle involved with uncertainty is seen as a productive attribute needs to be developed within mathematical modelling education and mathematics education in general. A change in attitude towards being uncertain would allow for developing behaviours and tools to deal with uncertainty.

Equipping students to be comfortable with uncertainty will help mathematical modelling students navigate the modelling process and help general mathematics students to develop confidence. Providing ways to embrace uncertainty as a productive behaviour will remove the perception that being good at mathematics means knowing all the answers all of the time. This in turn will help the development of mathematical modellers.

Little is known from literature about how to help students navigate uncertainty other than removing the uncertainty. What is seen from literature is when students have difficulties, research has focussed on ways teachers and lecturers can remove these difficulties (Alona Ben-Tal & Robert Mckinnon, conversation Massey University, 2011; Crouch & Haines, 2004; Kasier & Stender, 2013; Klymchuk et al., 2010; Stender & Kasier, 2015; Stender, Krosanke & Kaiser, 2017; Stillman, 2011; Stillman et al., 2013). Is removing difficulties the answer? When should lecturers step in with a 'solution plan' or give advice? Any advice or scaffolding for solutions needs to maximise students' involvement in navigating through any difficulties they are experiencing (student ownership). We need lecturer and teacher solutions that keep the student active and involved in finding their own solution through grappling with their difficulties. An outcome of this struggle is student ownership through experience and self-discovery of how to navigate the modelling process. Students and instructors need to solve the problem together, with the instructor playing a "behind the scenes role'. An example of this can be seen through the use of a set of reflective questions.

There is an interplay between students *moving forward on our own* and *being moved forward.* The lecturer plays a key role in the balance between independent students' work and students using guidance. The key question concerning the lecturer's role in students *being moved forward* is how do we as lecturers maintain the balance between promoting independence and minimising dependence while providing an experience that allows students to move forward? The key characteristics of lecturer behaviours is to provide support while prompting students to work independently. Key lecturer behaviours that keep students independently moving forward include: providing students with material to refer to; using facilitatory questions/prompts aligned with the modelling process; and creating environments for students to share their thoughts and workings.

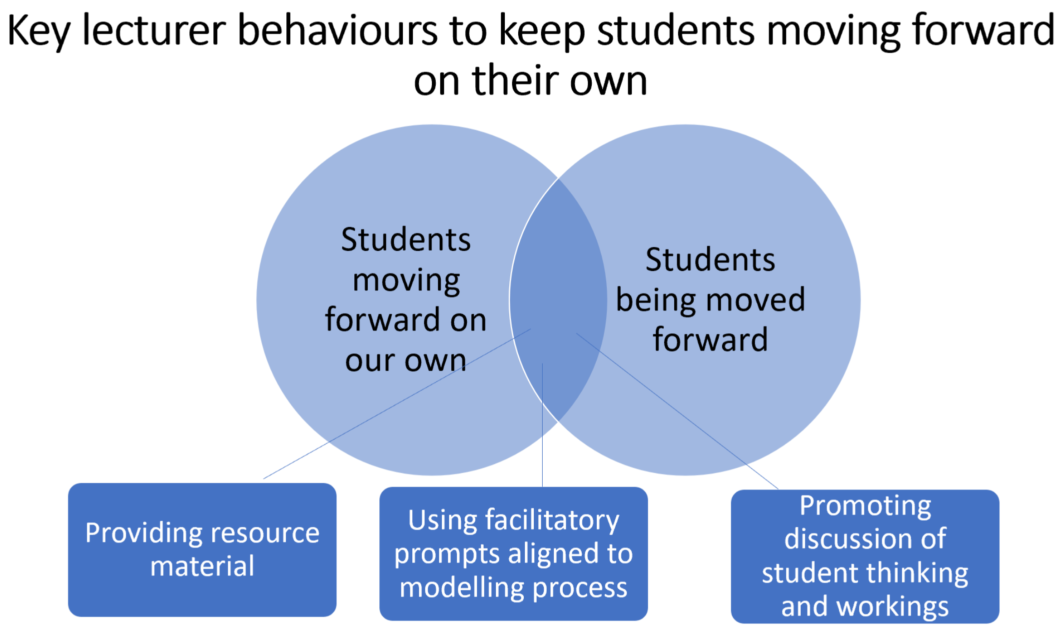


Figure 16. Students moving forward.

Students moving forward on their own involves moving from being uncertain to having direction. Experiencing struggle, exploring, being creative and becoming self-relaint are all behaviours involved with students finding direction. Any support given by the lecturer neededs to promote these behaviours.

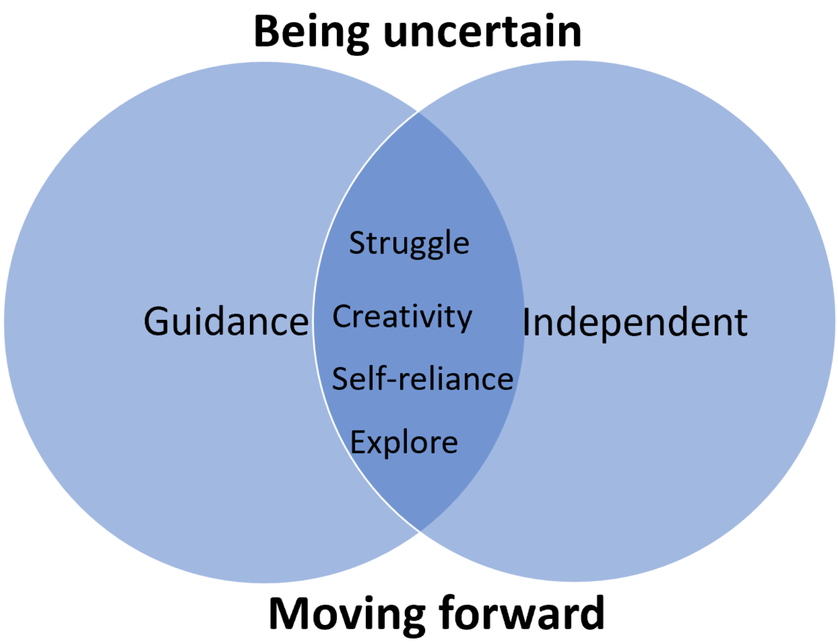


Figure 17. Going between being uncertain to moving forward.

Potential teaching implications from "Independence vs Guidance" discussion:

- Guidance non-prescriptive to allow for creativity.
- Lecturer to role mode behaviours of uncertainty and unfamiliarity.
- Lecturer to provide tools to deal with uncertainty including exploration, heuristic strategies, producing imperfect models for the situation.

To conclude, this study found mathematical modelling was a change of culture for students. This change of culture involved a change in mathematical mindset for students. One of the main components of this change in mindset is the application of mathematics to the real world.

Literature shows clearly that a balance of independent work and the role of the lecturer is needed for learning to mathematically model (Kaiser, 2020). Little is known about the intricacies and nuances of this balance. The lecturer's coaching style can have an effect on how dependent a student remains on the lecturer for guidance. Students who perceive themselves weak in mathematics and/or are not fluent in the language medium being used may potentially struggle more. The results indicate the struggle experienced for these students can potentially be navigated through working collaboratively in student groups. How necessary is the independent component of the student experience for learning to mathematically model? Is independence more important than the coaching role the lecturer plays? What things contribute to the balance between students' independent work and the role of the lecturer?

Frameworks are needed that promote independence and reduce dependency, while at the same time balancing struggle with moving forward, frameworks acknowledge the need for independent grappling with the problem but also provide direction so students do not get 'bogged' down in the problem and give up.

The question is how can instructors set up an environment within which opportunities to be mathematically creative can occur, while also giving students some direction so the creativity can flow? This needs further investigation.

9.6 Teaching implications from findings[16]

From this discussion and review of the results, teaching implications have been identified for developing students' ability to connect mathematics with contexts, work within a collaborative space, take ownership and responsibility for their work, and work undirected. Teaching implications have also been identified for the lecturer's role in developing students mathematical modelling competencies. Teaching implications for providing guidance through facilitating a modelling experience, preparing students through role modelling of mathematical modelling behaviours and model development, and providing support frameworks for students to utilise during their modelling

[16] This section includes re-presentation of material published in Spooner (2022).

experience have been identified. The teaching implications are captured in Table 10 below:

Table 10. Teaching implications

Connecting context and mathematics
• Include using authentic or relatable contexts and relating these to mathematics during lectures, both in mathematical modelling lectures and in general mathematics courses. • Include using authentic contexts and relating these to mathematics earlier in mathematics education both at secondary and primary school. • Allow time for students to collaboratively explore connections between mathematics and the real world. • Start with using simple contextual situations when first starting to relate contexts and mathematics.
Building a collaborative working space
• Students carry out modelling within student groups. • Include tutorial sessions/workshops focussed on developing behaviours for brainstorming and sharing ideas within groups. • Encourage individual thinking time within the collaborative space, including within student groups, for students to individually think about approaches and then share these within their groups. • Students to share their workings for models developed with the wider cohort. • Students are working on the same open-ended problem and/or situations using the same base model, students to share their work as final formal presentations. • Students are working on the same open-ended problem and/or situations using the same base model, students to share their work at differing stages of the modelling process.
Developing student responsibility and ownership
• Use teaching and learning activities that involve students making their own decisions. • Use teaching and learning activities that involve students choosing from a range of solution pathways to explore. • Allow students to choose from a selection of modelling situations which situation to work on. • Provide opportunities for students to work undirected. • Provide time when lecturer is not available for advice or guidance. • Provide time for independent thinking time within groups and within the wider collaborative working space. • Instructors provide support that encourages successful independent work without minimising/removing struggle. The lecturer needs to consciously monitor their involvement to provide balance between encouraging independence and minimising unproductive struggle. • Provide relevant support that students can utilise on their own.

• Provide guidance in the form of questions that places responsibility for answers on the student. • Lecturer to use students' own work to give advice for steps forward. • Support given needs to help students that need it while promoting students' own discovery of solutions and processes.
Allowing free scope
• Encourage exploration through incorporating exploratory activities in teaching activities. • Provide opportunities for students to work undirected. • Lecturer to be unavailable for advice and guidance for different sections of workings. • Lecturer to be available 'after the moment'.
Guiding through facilitation
• Professional development in facilitation techniques and methods for instructors that promote independence for students. • Guidance non-prescriptive to allow for creativity. • Lecturer to draft/have available a set of pre-determined non-prescriptive facilitatory questions aligned with the modelling process to use as prompts for students to reflect on as they move through the modelling process. • Provide guidance in the form of questions and prompts that places responsibility for answers on the student. • If lecturer present, guidance given through facilitator type questions and/or prompts. • If the lecturer is not present, the use of a set of facilitatory questions aligned to the modelling process should be used. • Lecturer to use students' own work to give facilitatory advice for steps forward. • Lecturer to be aware of being respectful about students' work while providing advice. • Lecturer to use students' own work to respectfully clear up misconceptions while not limiting students' learning process.
Guiding through role modelling
• Lecturer to role model different models that could be produced for a situation. • Lecturer to share different situations using the same foundation model. • Lecturer to role model developing a model for a problem they have never previously attempted for students. • Lecturer to role model behaviours of uncertainty and unfamiliarity. • Lecturer to provide tools to deal with uncertainty including exploration, heuristic strategies, producing imperfect models for the situation. • Use a generic modelling cycle and explicitly reference the modelling cycle throughout lectures. • Provide commentary of the modelling process alongside the models during lectures.

Guiding through preparation
• Preparation in modelling behaviours before students participate in the modelling experience. • Preparation in how to effectively work within a collaborative space.
Characteristics of a support framework
• Facilitatory in nature. • Connected to modelling cycle. • When using a case study approach, engagement with a clear modelling framework to support students' modelling process development.

9.6.1 Connecting context and mathematics

The results show it is important to include authentic and/or relatable contexts and establish connections between the context and mathematics during lectures. This also implies that contextualising mathematics within lectures is beneficial for developing relating mathematics and contexts for students. Mathematising the situation is an essential part of the mathematical modelling process, hence it makes sense to develop students' ability to relate mathematics to contexts. Literature has identified developing students' ability to mathematise contexts and contextualise mathematics is an important and essential consideration for teaching and learning mathematical modelling (Alsina, 2007; Stillman, 2002; Villa-Ochoa & Berrío, 2015; Wake, 2016). According to Legé (2005a) students need to be able to recognise features of a contextual situation that can be mathematised in order to successfully model. The results of this current study imply the earlier contextualising mathematics starts to occur in a student's education the better prepared a student will be for mathematical modelling. Starting contextualising mathematics early in a student's education may help minimise some of the difficulty identified by Soon et al. (2011) that is associated with modelling for the first time.

Research shows hands-on experiences using contexts helps students connect mathematical concepts to reality (Armstrong & Bajpai, 1998; Quimson, 2021) and using relatable contexts makes it easier for students (Carducci, 1996). In conjunction with moving between contexts and mathematics in lectures, and vice versa, lecturers also need to allow students to 'have a go' at moving backwards and forwards between familiar contexts and possible mathematical connections themselves. Verschaffel, Greer and De Corte (2000) recommend lecturers are conscious to use authentic contexts wherever possible, while Quiroz et al. (2015) say students using authentic contexts in mathematical modelling started to see themselves as useful mathematical citizens.

As already discussed and supported by Ainley et al. (2006), Blomhøj (2019), Boaler (1993b), Dindyal and Kaur (2010), Dreyfus (1991), Freudenthal (1973), Julie (2002), and Winkel (2016), applying mathematics in context has the potential to deepen and broaden students understanding and retention of mathematics. Beswick (2011), Boaler (1993b) and Brown (2017) suggest that using context alone is not enough to develop mathematical understanding. While Lesh, Post and Behr (1987) provide evidence of using visual representations (images and diagrams) to aid translating between contexts and mathematics can deepen students' understanding of mathematics. Brown (2017) considers genuine collaboration and 'interthinking' while using context with mathematics is needed. There is a further teaching implication that group work, discussing ways mathematics relates to the task, and visual representations should be utilised while mathematising context and contextualising mathematics. Further research into the nuances of using contexts for teaching mathematics is still needed. It is clear that the collaborative nature of mathematical modelling involving group work, discussion and free exploration provides opportunities for students to work with contextual situations to deepen mathematical understanding and hence provides further opportunities for research in this area.

While literature shows that context can be beneficial to developing mathematical understanding, there are also some issues to be aware of. Wake (2016) states “it is at these key moments of connecting mathematics and a context outside of mathematics that students and others (such as workers) have difficulties” (Wake, 2016, p. 173). Research has shown students’ mathematical knowledge, backgrounds, previous experience and beliefs can all influence students’ experience with contexts and mathematics (Beswick, 2011). This suggests that collaborative group work potentially would help to minimise individual difficulties by providing a collective pool of resources for students to work with. With contextualising mathematics being potentially difficult (Beswick, 2011; Wake, 2016) and considering the range of different knowledge students bring to the activity while working collaboratively, it would make sense to start with relatively simple and straightforward situations for students to mathematise before attempting something as messy and complicated as modelling. It is also important to avoid the use of trivial contextual problems (Verschaffel, Greer & De Corte, 2000) as this can result in students later ignoring contexts as they develop a belief that contexts are not important.

Research in tertiary education is starting to provide evidence that students learn better in context (Alsina, 2007; Sijmkens, 2022). Alsina (2007) suggests the contexts involved with mathematical modelling have the potential to generate interest and motivation and thus enhance learning opportunities. Sijmkens (2022) carried out a study using contexts for learning differential equations. The study found that using contextual problems had a significant effect on students’ conceptual understanding and procedural knowledge of differential equations for the experimental group of first-year engineering students. Further testing revealed that using contexts removed difficulties students were known to experience with learning differential equations. Sijmkens et al. (2022) concluded “that the procedural knowledge of the students in the experimental group also improved significantly [was] against our expectations. Arslan (2010) concluded that an improved conceptual knowledge of DEs also led to equal knowledge of procedures, but not vice versa" (Sijmkens et al., 2022, p. 12). Even though more research is needed there is an implicit implication that this improvement in both procedural and conceptual knowledge may be the result of students using contexts to relate mathematics and contexts. As this study is only in mathematics and even though it is key to understand how differential equations connect with modelling, the results may support mathematics students starting off simply, as opposed to using messy open-ended modelling situations, when beginning to work with connecting contexts and mathematics.

To enable lecturers and teachers to have suitable contexts to use with students it is recommended that a database of contexts is developed for mathematics and mathematical modelling educators.

9.6.2 Building a collaborative working space

The results show it is important for modelling to be taught as a collaborative activity. This involves working within student groups, providing opportunities for groups to share their workings and activity within the wider cohort, including with the lecturer. Preparing students for effectively working within a group environment is an important teaching implication. Providing training for effectively working within a group can involve how to: effectively brainstorm; manage individual 'thinking' time with sharing ideas and workings with others; oversee time management; and manage conflict.

Modelling as an activity commonly takes place in group/team environments (Bender, 2000; Dym, 2004; Spooner, 2017; Tam, 2011; Treilibs et al., 1980; Toews, 2012). Toews (2012) promotes group work for developing mathematical modelling skills as fundamental for future work in industry and science and is reflective of industry modelling behaviours (Spooner, 2013). Interesting to note, within this study, no lecturer allowed students to self-select groups. The results showed that while it is commonplace for students to model in groups, it was the collaborative nature of the experience that allowed student progress to be made. The results show that for a productive student experience, modelling should be undertaken in groups as opposed to individually. The collaborative environment enables active learning to take place, as seen through students exchanging and discussing ideas on how to proceed and create their model. Hernandez-Martinez and Vos (2018) found students found it useful modelling in groups, relating this to their improvement in mathematical understanding and future work skills. As already mentioned, the results of Brown (2017) suggest group work is needed to develop connections between context and mathematics, an essential component of mathematical modelling. This study shows participating in the practice of collaborative mathematical modelling helps students' development of group work skills. The literature is clearly supportive of group work for the development of mathematical modelling competency and mathematical understanding. It therefore makes sense that modelling is undertaken in groups and students are prepared beforehand in how to effectively participate in a mathematical modelling group.

The results showed students found it beneficial for groups to share their work within the wider class cohort. Sharing work enabled students to see the workings of other groups. This is supported by Hernandez-Martinez and Vos (2018) who found students found it useful to explain to others through presentations. The results of this current study also show students found it useful to see others presentations as this contributed to their understanding of mathematical modelling and knowledge of possible approaches to use while developing their model.

Students working within groups and presenting their work makes students responsible for the work being done and thus promotes equitable interactions within the group

(Boaler, 2008). Using activities that all students are unfamiliar with also helped to promote equitable relations and encouraged students to remain open-minded to others ideas.

Little research addresses how to minimise and overcome student difficulties when a student group is working independently of the instructor. While further research is still needed, the results of this study imply that by allowing students to utilise their group members as resources can help students to navigate the challenges of modelling independently of the lecturer. This in turn allows students to discover their own process and enables learning. In order to make the most of this, lecturers should focus time on preparing students for effective group work for modelling.

Part of effective group work is knowing how to manage the group processes effectively. Student use and knowledge of metacognitive strategies is likely to contribute to effective group work. Metacognitive strategies such as planning, monitoring and self-regulation appear to be positively related to students success (Stillman & Galbraith, 1998; Vorhölter, 2019; Vorhölter & Kaiser, 2016). According to Krug & Schukajlow (2020) prompting students to develop multiple solutions improves planning, monitoring and self-regulation. Anticipatory metacognition (Galbraith et al., 2017), where "the emphasis shifts towards reflection that points forward to actions not yet taken, that is, recognizing (noticing) possibilities of what 'might be'" (p. 84)" (Schukajlow, Kasier & Stillman 2021 p. 6), is also considered an important metacognitive strategy. The importance of metacognitive strategies strongly suggests the need for lecturers to prepare students to manage group processes effectively and develop metacognitive skills useful for group processes.

As discussed in relating contexts and mathematics, discussing things within groups is necessary in order for students to utilise the collaborative working space. In addition, Sofroniou and Poutos (2016) noticed that, through discussion, students realised the potential and importance of different proposed solutions and the benefits of group work as opposed to individual work for modelling.

Discussion cycles are one way of promoting discussion within groups and are recommended for use both at secondary and tertiary level. Discussion cycles have been used successfully in secondary classes, as demonstrated by the work of Legé (2005b) and Spooner (2013). Legé (2005b) called the approach "modelling discussion" "in which groups (or the entire class) are engaged in a conversation about what they know, what they might need to know and how they might proceed forward from where they currently are." (Legé, 2005b, p. 31-32). Facilitator questions are used to stimulate conversation and engage the students in the modelling activity. A similar approach was taken by Spooner (2013) in her work with secondary students, while Soon et al. (2011) discuss using discussion cycles as a way to build students "confidence to tackle complicated problems, though it can be argued that the cognitive demand of students may be lowered to too great a degree" (Soon et al., 2011, p. 1028). Soon et al. (2011) remind us of being conscious of not removing too much of the struggle for students as they work through a modelling experience.

Discussion within groups is beneficial for learning to mathematically model and learning mathematics. Collaboratively learning within a group environment that utilises discussion produces an increase in student performance (Tarmizi & Bayata, 2012) and encourages greater understanding of the mathematics (Koçak et al., 2009).

9.6.3 Developing student responsibility and ownership

The results show that in order for students to take responsibility and ownership of their work, time needs to be provided for students to work independently. Yoon et al. (2011) are supportive of independent student work, believing that for students to have a productive learning experience students need to take responsibility for their own learning and not to expect the lecturer to provide all of the direction. The results of this study show students working in groups independent of the lecturer is important for enabling students to take responsibility and ownership of their work. Students individually having some independent thinking time away from their student groups may also be important. The way lecturers interact with students through support materials and verbal guidance can either promote student ownership and responsibility or promote dependence. To promote student ownership and responsibility, lecturer-student interactions should promote student decision making, students making choices and student exploration. Initial lecturer-student interactions should also prepare students for working independently. Support that students can utilise on their own is key to students moving into taking responsibility and ownership of their work. As seen from the results, it is important that verbal guidance acknowledges the student and leaves them feeling valued. Lecturers respectfully using students own work as teaching points for the wider cohort is one way of achieving this.

Students working in groups is important for providing students opportunity to take collective responsibility and ownership for their work. This is similar to Boaler (2008) who pointed out group work has the potential to make students responsible for the work being done and thus promotes equitable interactions within the group. The results of this study showed students working within a group can feel a responsibility towards their group and hence put in more effort. Soforniou and Poutos (2016) found that students did not feel strongly that group work allowed students to not pull their weight. Soforniou & Poutos (2016), along with the results of this current study, challenge the idea that group work enables 'free riders'. The results of this study also showed that approaching the modelling activity over the period of a day minimised time-management issues some student groups can have over long-term group projects. This suggests that to promote responsible group work, group work should be carried out in short-term projects as opposed to long-term projects, particularly while students are learning how to be responsible towards other group members.

As already seen, students in this study found it beneficial to work with others. While students know it is beneficial to work with others, Yoon et al. (2011) found they do not usually like to work with others unless directed by the lecturer. Yoon et al.'s (2011) study showed even though students viewed working collaboratively positively, they did not want to impose themselves on others and thus avoided spontaneously working with others. From the results of this study we have already established that modelling should be done in groups, with no student participants mentioning resisting group work. This suggests that students undertaking modelling is one way to bring about change in student working culture for mathematics, making group work a norm and not something that is outside of the norms of mathematical behaviour.

To promote student ownership and responsibility, lecturer-student interactions should promote student decision making, students making choices and student exploration. All of these activities involve the student working independently of the lecturer and place the lecturer in a facilitatory role. Pollak (2007), Burkhardt (2006) and Legé (2005b) propose students actively create models, which involves students taking ownership and responsibility of the modelling process. In conjunction with creating models Pollak (2007), Burkhardt (2006) and Legé (2005a) also promote 'getting to know models'. This implies students need preparation to be able to create models and thus enable student responsibility for the modelling process. As to whether students should 'get to know models' first or 'create models', Pollak (2007) was unsure which approach should occur first. While Legé's (2005a) work shows that it does not matter if you get students 'getting to know models' first or get students to 'create models' first, finding both approaches viable for modelling. Legé (2005a) thought the key to both approaches was that the teacher acted as a guide as opposed to an expert. This is consistent with the results of this present study which show the instructor's role should be one of facilitator. Instead of asking should students 'create models' or 'get to know models' first, maybe the question should be how do we as instructors balance between the two approaches for maximum student learning?

If students working independently, taking responsibility and ownership of the modelling process is our goal, how do we make the independent component of modelling achievable for students? While there is a lot of research into student difficulties, none of the research addresses difficulties students have while working independent of the lecturer. This is because most of the research has focussed on intervention strategies that assume the teacher or lecturer is present. This aside, lessons can be learnt from how successful intervention strategies can be modified and included in guides for independent student use or as part of students' preparation prior to modelling. Some successful teacher interventions that have the potential to be used independently by students include students talking out loud their current workings and thinking (Kaiser & Stender, 2013), and giving strategic prompts including "imagine the real situation clearly, make a sketch of what is still missing, does this result fit to the real situation" (Blum, 2011, p. 23). Blum (2011) also suggests strategic prompts can be part of a written prompt sheet for students to use to guide them through the process.

While Kaiser and Stender (2013) ask students to talk out loud their workings as a way to access students' thoughts, getting students to talk out loud also gives students a chance to collect their thoughts which must, in turn, help students move forward. Students presenting their work to an audience involves students collecting their thoughts, making decisions and can result in students moving forward. Moving forward can also be achieved by presenting work in a written report. Deadlines associated with presenting work may stop students getting stuck down ‘academic rabbit holes’ and forces movement forward. Though the results showed a lot of students said they needed more time, it also appears important to have good time deadlines associated with formalising work, either as a presentation or written report, for the modelling activities. The need for students to present work on its own will not guarantee students moving forward with the modelling. Clear guidelines of what is expected in the presentation and report are also needed to help students move forward.

Literature has identified a lot of strategies, that while in secondary school modelling experiences these strategies have been given as ‘interventions’ by the teacher, these ‘interventions’ could be given by the lecturer to the student as part of a ‘toolkit’ in preparation for the modelling experience. If the students are prepared, then it makes it easier for them to be responsible for their own experience. However, as Wedelin and Adwai (2014) say “some students seem to get along quite well by solving the problems on their own” (Wedelin & Adwai, 2014, p. 54).

Students who received feedback from their lecturer during the modelling process, had mixed feelings towards receiving the feedback. Some students viewed the feedback favourably as a way of verifying they were on the right track. At other times, due to the personal investment in student solutions, students felt criticised by the feedback, particularly when their contributions were not acknowledged by the lecturer. Feedback is part of the facilitator role with Carducci (1996) and Lyon and Magana (2020) emphasising its importance. Findings from Lyon and Magana (2020) found that as modelling activities are occurring, feedback by a lecturer to student needs to be both positive and needed. Kaur (2009) revealed general mathematics students viewed feedback on “student work/group presentations to individuals or the whole class” as a characteristic of a good teaching (Kaur, 2009, p. 345). Lyon and Magana (2020) found that positive feedback on modelling activities helped students clear up misconceptions, however, negative feedback discouraged students from persevering with solutions. The results of this current study extend on this, showing that it is important to acknowledge students' work and the value in what they have done before offering any constructive criticism. Just as the student felt valued when the lecturer used their work positively as teaching points, the lecturer also needs to make sure the student feels valued when offering any advice contradictory to the student's current workings, particularly when this advice is given publicly. Valuing students' work has the potential to promote student responsibility and ownership by keeping students engaged with their own process. Further research is needed into this area to gain a better understanding of the dynamics that can occur between lecturer - student feedback interactions.

Due to the personal investment involved in producing workings, there is also the potential that students will have better retention of any feedback received on their own work, both positive and negative. The same may also apply to retention from any learning happening from their own personal work. There appears to be very little further literature in this area, hence another area for investigation.

9.6.4 Allowing free scope

The results show in order for students to begin to take ownership of the modelling process, periods of exploration where the lecturer is not the sole authority of information is needed. Having periods where the lecturer in not available, allows students to explore, go down blind alleys and question what they are doing (Solomon, 2007). This exploration and discussion allows students to share responsibility of knowledge with the lecturer (Yoon et al., 2011) and their peers. An outcome of this is students taking responsibility for negotiating their knowledge development (Solomon, 2007). By moving from a position of the lecturer as the expert to one where students take ownership of their process is part of cognitive apprenticeship (Collins & Kapur, 2006).

9.6.5 Guiding through facilitation

The results imply that guidance from instructors should be of a facilitatory nature and aligned with the modelling process. Guidance given should place responsibility on the student to find the answers and be non-prescriptive to allow student creativity. Wherever possible, students' own work should be used by the lecturer to provide steps forward. Facilitatory support resources can be developed that contain prompts and questions aligned with the modelling process. The prompts and questions should promote independent student work involving exploration and discovery of solutions and the modelling process, while placing the responsibility of working through the modelling process on the student. Scaffolding is the main form of facilitatory advice identified in literature for modelling (Kaiser, 2020).

9.6.6 Guiding through role modelling

It appears the lecturer's role as a role model for modelling is important. It is important for students to know what to expect. This includes knowing that different models can be produced for the same situation, characteristics of situations that indicate a foundation model is suitable for use, and that the same foundation model can be modified for similar situations. It is important for lecturers to role model the behaviours of uncertainty and unfamiliarity to students. One way lecturers can do this is to attempt modelling a situation they have not tried before during lecturers. Part of this experience is to provide students with tools to deal with uncertainty. These tools include exploration, heuristic strategies and the idea that there is no such thing as a perfect model. These tools should be demonstrated for students to start to comprehend and have a way to reference the tools by when working independently.

Alongside students developing an idea of what the modelling experience entails and thereby building a concept of mathematical modelling, it is important students have explicit exposure to the mathematical modelling process via a generic modelling cycle. This is consistent with the findings of Caron and Bélair (2007) for tertiary students. The results of this current study imply that it would be useful to have a running commentary of the modelling cycle alongside the exposure to models during lecturers.

9.6.7 Guiding through preparation

As already discussed, preparing students beforehand for the modelling experience is an important step. Whether students are working with open-ended, ill-defined problems, modifying already formed models or modelling a piece of popular culture, preparation beforehand is important. When modelling is undertaken in a group, the results suggest that preparation for working within a group would be beneficial to students. Experience with relating mathematics to contexts is also useful preparation for students. As already discussed, ideally, relating mathematics to contexts should start before students encounter modelling. Research suggests word problems as one way students can have experiences relating mathematics and context before they experience modelling (Quimson, 2021).

9.6.8 Characteristics of a support framework

The results of this study imply that a support framework for mathematical modelling is facilitatory in nature and based on a modelling cycle. A support framework would contain probing questions to help guide students' decision making. The next stage of this study is to identify the facilitatory prompts used, and to suggest prompts from the data, to form a set of probing questions aligned with the modelling that could be used by lecturers for guiding modelling. These prompts would include both process prompts and metacognitive prompts.

9.7 Summary[17]

The results of this study discuss how the experience of mathematical modelling pushed students outside of their normal experience of mathematics and presents the mathematical modelling experience as one that required students to be willing to consider new ways of thinking about, behaving, and approaching a problem. This practice is different to that which most students of mathematics have previously experienced (Hernandez-Martinez, 2020; Kotze, 2020; Solomon, 2007; Yoon et al., 2011). Results show the student experience involved being flexible and open to new ways of doing things including being actively involved, relating mathematics to context, working in groups, and being open to considering and adopting other people's ideas. This was due to the nature of the modelling process being indirect and complex.

Challenges involved with this change of culture include students being unsure of what they are doing. Results show that while students are being given responsibility for discovering their own process, they still expressed a need for reassurance. Unsure if it is right, no model answer, unrealistic results, unrealistic first time around and querying learning correct information come under this theme. Solutions for these challenges include finding ways to provide reassurance for students. These include role modelling the messy behaviour of modelling for students and forming a supportive framework aligned with the modelling process.

Role modelling the messy behaviour of modelling for students, followed by coaching students in these behaviours is in line with cognitive apprenticeship (Collins & Kapur, 2014). Learning to model needs to be viewed as a cognitive apprenticeship, where lecturers first model the behaviours of modelling such that students can observe teachers performing the art of modelling while making explicit what they are thinking

[17] This section includes re-presentation of material published in Spooner (2021) and Spooner (2022).

and doing, and why. Role modelling is then followed by students having active experiences of modelling where they are coached, scaffolded through the process, while being encouraged to explore as they are articulating and reflecting on what they are doing.

9.8 Limitations

Within this research there are some limitations that need to be acknowledged. In particular: the diversity of perspectives is only representative of the participants; Interpretive Description only illuminates experiences – it does not develop theory; the final choice of methodology occurred after data collection; and the limited number of modelling courses offered at New Zealand universities.

The diversity of perspectives is limited to those students that chose to participate and so can only be representative of the participants. The aim of the study was to capture a breadth and depth of student experiences with mathematical modelling courses. The diversity of participants is limited to the students who choose to participate from the researcher-selected mathematical modelling courses. The final participants were those that self-selected and came forward to participate, meaning the participants we had were the participants we had. Within those that chose to participate, there were the perspectives of first-year students and third-year students, students with ranging mathematics abilities, and a few students with exposure at high school to mathematical modelling. These student backgrounds had potential influence on individual student experience, and the experience of the students they were working with. What was common between all student participants was this was the first tertiary mathematical modelling experience for all students.

Interpretive Description can only illuminate experiences, not theorise. Insight into a phenomenon is obtained from the participants' and researcher's subjective and experiential knowledge of the subject under investigation. It is not possible to attain a full or objective knowledge (Hunt, 2009), only insights into participants' experiences. As such, the results of an Interpretive Description study can highlight areas where further research can be helpful while offering insight into experience from a practical perspective.

A further limitation of this study was the decision to use Interpretive Description after data collection occurred. While it meant that rich data on the student experience had occurred, data collection was not influenced by researcher insights gained from later analysis. Data collection was primarily focussed on the student experience.

Another limitation was the limited number of stage one courses in modelling that teach modelling as a process within New Zealand. A stage three course was also included in the study. As discussed, common across the three courses was that this was the first experience of mathematical modelling for all students at tertiary level.

As seen in the findings chapters, some students discuss drawing on previous experiences with physics, however, students' previous experience with physics was not recorded as a demographic. On reflection, recording students' past study would have been useful.

In spite of the limitations of this study, the study has shed light on experiences of students learning to mathematically model in tertiary modelling courses. A study focussed on the student experience had not been explored using Interpretive Description methodology before. The findings add to the body of knowledge by providing insight into the student experience and how these new understandings can potentially be incorporated in lecturer teaching practice. The findings have a number of implications for teaching practice.

9.9 Further work

Further research into the nuances of using context for teaching mathematics and making connections between mathematics and contexts is still needed. It is clear that the collaborative nature of mathematical modelling involving group work, discussion and free exploration provides opportunities for students to work with contextual situations to deepen mathematical understanding, hence providing further opportunities for research in this area.

Students with previous experience with physics, an applied mathematics subject, made relating mathematics to contexts easier for these students. Further research is needed into the nuances of a student's previous experience with physics and how this helps prepare them for relating mathematics and contexts. Further research is also need into the nuances of how previous physics experience can influence making connections between mathematics and contexts within a modelling situation.

With the nature of modelling being complex and uncertain, further work is needed into how the lecturer can expose the modelling behaviours of uncertainty and unfamiliarity as positive behaviours rather than negative behaviours. Extra research is needed into how to provide additional strategies for dealing with uncertainty and unfamiliarity while creating a culture where students start to become comfortable with these behaviours and view them as positive attributes as opposed to negative ones. In order to create an environment where mathematical creativity can flourish, space for not having all the answers is needed. Further work is needed into how instructors can set up a learning environment within which opportunities to be mathematically creative can occur while also giving students some direction without removing all uncertainty for the student. Additional work is needed into formulating facilitatory prompts the lecturer can use to help students maintain direction with the modelling without fully removing uncertainty.

It is out of the uncertainty that direction is found by students. Further investigation is needed into how we support this process, as opposed to the process becoming too overwhelming that students give up.

The boundaries between promoting independence vs dependence, together with teacher and lecturer guidance being complex, means further research is needed into the circumstances and nuances for when guidance can help promote student independence as opposed to dependence.

While this study provides insights into how a lecturer or teacher might guide independent student work, more research is needed into the nuances of when and how guidance can be given.

With the need to manage struggle as productive struggle within the student experience, as opposed to struggle that results in students becoming despondent, it would be useful to carry out further research into the role of struggle in the student modelling experience. How much struggle is too much struggle? What circumstances produce productive struggle?

Chapter 10. Conclusion

10.1 Introduction

Compared to research into teaching at secondary school, more research into tertiary teaching, and in particular mathematics teaching and mathematical modelling teaching at the tertiary level, is needed. The overarching aim of this study was to investigate tertiary students' experiences of learning mathematical modelling and how this might impact mathematical modelling teaching practices. The study addressed the complex experiential question "what are the student experiences for mathematical modelling?" The study looked at tertiary student experiences with mathematical modelling activities used within three different mathematical modelling courses. While the study focusses on student experiences, the study uses Interpretive Description methodology (Thorne, 2008) to gain insight from students' experiences to better inform teaching practices. This is the first time an Interpretive Description methodology study has taken place within mathematical modelling education research.

From the literature, it is known there should be a balance between instructor guidance and student autonomy for mathematical modelling development (Blum, 2011, 2015; Kaiser, 2020). However, little is known about the teaching considerations for providing guidance while maintaining student independence. This study focusses on the student experience, in conjunction with the lecturer's role within that experience, to gain insight into key lecturer behaviours that keep students moving forward while promoting student independence. This study revealed key considerations for teaching.

To enable independent student work, lecturers and teachers can: provide students with material to refer to and resources to utilise; use facilitatory questions and prompts aligned with the modelling process; give non-prescriptive guidance; and create environments for students to share their thoughts and workings. Resources for students to independently use include: a student group to work with; heuristic strategies such as ‘keep it simple’, draw a figure, explain and discuss your thinking with someone else, reflect on what you are doing; tools such as dimensional analysis and mathematical techniques; and the modelling cycle to use as a scaffold. A key finding for providing reassurance to students while they are modelling is for the lecturer to try role modelling developing a model to an unfamiliar problem for the students and see if this has a positive impact on the student experience.

Overall, the findings show that students have an experience that pushes their boundaries. Students experienced modelling as a change of culture. Students were not used to relating mathematics to context and needing to work independently, or be creative, in mathematics. To allow students to mathematically model, a balance between student independent work and guidance is needed. The lecturer plays an important role in providing guidance that promotes and/or maintains independent creative work.

The conclusion is structured using the subthemes used to present the findings and discussion: *sources of pushing boundaries, moving forward on own, being moved forward* and *going forward together*. Within the subthemes, I summarise the key findings of this study, the wider significance of these findings including their implications for general mathematics education, and future research possibilities.

10.2 Key findings, implications, and further research

10.2.1 Sources of boundaries

The mathematical modelling experience was a change of culture for students, involving new behaviours and ways of thinking for students. One of the key new behaviours/ways of thinking for students is relating mathematics and context.

The findings showed relating mathematics to context was a change of culture for students. This finding has implications for teaching practice as relating mathematics to context is an essential component of modelling (Carducci,1996; de Almeida, 2018; Stillman, 2002). Students who had previous experience with applied mathematics subjects, in particular physics, coped better with relating context to mathematics than those who had no previous exposure. This finding draws attention to the importance of providing students opportunities for relating mathematics to context. Hands-on experience relating mathematics to contexts is needed to prepare students for mathematical modelling (Armstrong & Bajpai, 1998; Quimson, 2021). In conjunction with Blomhøj (2019), the findings show relating mathematics to context can help develop students mathematical understanding. Contrary to the literature (Alsina, 2007; Galbraith, 1995; Kaiser-Messmer, 1993; Palm, 2007), the findings suggest the context does not necessarily need to be authentic but it needs to be relatable for students.

The benefit of relating mathematics to contexts for mathematics education is that it can make mathematics relatable for students. Making mathematics relatable to students gives students something to anchor the mathematics to and thus opens up the possibilities of challenging and/or deepening students' previous understandings of the mathematics (Blomhøj, 2019). As relating mathematics to contexts was a change of culture for students, a recommendation from this study is to regularly apply mathematics to contexts in mathematics and modelling courses to normalise using mathematics with contexts wherever possible.

The findings showed previous experience with physics can help students see connections between mathematics and contexts. Studies have been undertaken that focus on students linking physics and mathematics (Chaachoua & Saglam, 2006; Domínguez, Garza & Zavala, 2015; Nguyen & Rebello, 2011; Quimson, 2021; Rowland & Jovanoski, 2004; Rowland, 2006). However, none of the studies investigate how experience with physics influences using mathematics in context. Further research is needed into how previous physics experience contributes to successful mathematical modelling.

The findings revealed that students who had a fully independent modelling experience found embracing relating mathematics to contexts easier than students who had access to the lecturer. This finding draws attention to the importance of providing students opportunities to independently explore connections between mathematics and the real world. Independent exploration encourages students to take responsibility for looking for connections and thus start to develop this new way of thinking. However, students still struggle with relating mathematics to context. Specific activities and guidance focussed solely on connecting mathematics and contexts would be beneficial for all students. Further research is needed into the key characteristics of activities and guidance to help students who continue to struggle to make connections between mathematics while they are modelling. It is suggested that experience helps students navigate the challenges associated with connecting mathematics and reality. It is recommended that applications of mathematics to contexts are included in students' mathematics education to prepare them for future mathematical modelling study.

Understanding the context is needed for successful modelling (Carducci, 1996; de Almeida, 2018; Stillman, 2002). It was not surprising the findings showed students needed to understand the context in order to make progress. This adds further evidence that it is important for lecturers and teachers to provide some form of experience and/or guidance for connecting contexts and mathematics. This is particularly important for students participating in partial-immersion modelling experiences.

10.2.2 Moving forward on own

Research shows that modelling is best learnt by 'doing' (Blum, 2011, 2015; Boaler, 2001; Blomhøj & Kjeldsen, 2006; Caron & Bélair, 2007; Legé, 2007; Toews, 2012;

Wake, 2011). The findings of this study suggest that learning to model is more than just ‘doing’ modelling. Learning to model involves having an independent experience creating a, or part of a, model where students take responsibility for the student group process. Blum (2015), Blomhøj and Kjeldsen (2006), Carducci (1996), Caron and Bélair (2007), Kaiser (2020), and Stillman (2011) all acknowledge that some independent modelling experience is needed to develop mathematical modelling competencies. The findings showed students were unaccustomed to working independently of the lecturer. There is an implication that lecturers and teachers need to provide opportunities for students to work independently of themselves. In addition to this, lecturers and teachers need to train students how to work independently of themselves as instructors.

The findings showed students taking responsibility for applying independent thought processes to their workings was a change of culture for students. Students initially went looking for resources to show them what to do, resisting taking responsibility for their own process. While students are used to independently following step-by-step instructions for mathematics (Solomon, 2007), they are not used to applying independent thought to these instructions. Resistance to independent thought by students is most likely due to students having little previous experience within their mathematics education to work independently solving open problems (Kaiser & Stender, 2013; Soon et al., 2011). The inclusion of exploratory problem solving within general mathematics education would be beneficial for preparing students for future mathematical modelling.

There is a gap in the literature examining what happens when the student starts to become stuck and lost and there is no lecturer readily available to provide guidance. This study contributes to the general literature by showing students initially look for something to show them what to do, revealing an initial dependence on the instructor and/or resources and revealing that though they may be used to doing mathematics, they are not used to applying an independent thought process in a mathematical context.

The findings showed that, given the right support, it is possible for students to model independently of the lecturer. In addition to this, students can successfully model without the lecturer being present during the modelling activity. For independent modelling behaviour to occur successfully, lecturers need to prepare students. The findings showed through preparing students, giving advice to students, and providing resources for students to refer to, lecturers impact the independent component of the student experience. This is irrespective of whether the lecturer is present or not during the modelling activity. The findings showed that advice given, and resources provided, should be of a facilitatory nature to preserve student independence. More research is needed into preparing students for independent student modelling experiences to expand on these findings.

Students working independently of the lecturer, taking responsibility for their modelling process, involves struggle. Struggle is a known part of the student modelling experience (Chaachoua & Saglam, 2006; Crouch & Haines, 2004; Klymchuk & Zverkova, 2001; Klymchuk et al., 2010; Palm, 2007; Soon et al., 2011; Stillman, 2004). Producing experiences involving productive struggle is needed for students. The findings of this study showed lecturers and teachers need to be aware of finding the balance between providing enough support that struggle is productive while avoiding students struggling detrimentally. With research focussing on teacher interventions, further research is needed to explore how much struggle is good for developing modelling skills. The students in this study perceived group work made them more resilient to unproductive struggle than if they were working individually. Further investigation is also needed into additional ways to support productive struggle for mathematical modelling students.

The findings showed students learnt how to navigate the struggle associated with being unsure what to do by developing coping behaviours to enable them to move forward independently of the lecturer. These coping behaviours included: taking action, including backing themselves; being open-minded to others' inputs; and finding ways to feel reassured they were progressing in a productive way. Students developed coping behaviours, used tools provided by the lecturer, and each other to provide reassurance and direction. The results of this current study also suggest it would be beneficial for a lecturer to role model working through the development of a model for a problem the lecturer was unfamiliar with. This would also allow students to witness the behaviours of mathematical modelling, including uncertainty and how to deal with uncertainty. Role modelling the behaviours of mathematical modelling would be in addition to teaching and frequently alluding to the mathematical modelling cycle as recommended by Caron and Bélair (2007). Modelling involves going down dead ends and not having immediate solutions (Drake, 2012). If lecturers role model these behaviours, students will feel reassured that being uncertain is part of the modelling process and view uncertainty as an opportunity as opposed to a negative behaviour. Normalising uncertainty and mistakes as okay, along with providing strategies to deal with these, is needed within mathematics education.

The study showed that modelling guidance can be solely provided by the lecturer before a modelling activity as opposed to during a modelling activity. By providing and setting up the necessary support structures and guidance before the modelling activity, student dependence on the lecturer during the activity is minimised. The lecturer being unavailable during the modelling activity would further encourage student independence. Struggle within the students' experience would not be minimised and is a necessary component of the modelling experience. Further research is needed into how lecturers can ensure students experience struggle that is productive, as opposed to discouraging, while independently modelling.

10.2.3 Being moved forward

Students working independently of their lecturer and teacher is known to be beneficial for developing mathematical modelling competency (Burkhardt, 2006; Blum, 2011, 2015; Carducci, 1996; Durandt et al., 2021; Legé, 2005b; Legé, 2007). The findings of this study showed that lecturers play a role in promoting student independence. Common to all lecturer coaching styles was promoting students working independent of perceived authorities as a key student behaviour. While working independently is important, it is known that students struggle when modelling independently (Soon et al,. 2011; Stillman et al., 2010). The findings of this study concur with Blum (2011, 2015), Durandt et al. (2021) and Kasier (2020) that both independent work and guidance is needed while students are learning to mathematically model.

The findings of this study extend on the current literature of independent student modelling experiences by identifying key factors that contribute to a student group being able to have an independent modelling experience when the lecturer was not present. These factors include: having the modelling cycle explicitly available and frequently alluded to by the lecturer during lectures; students sharing ideas through discussion and debate; students being open-minded; and using an activity structure that allows for student decision making. Key factors identified that appear to contribute to enabling a dependent relationship on the lecturer include: not clearly having a scaffold for the modelling process readily available for students; the lecturer being too readily available, allowing students to ask the lecturer for help when stuck as opposed to entering productive struggle to find their own way forward; and the activity structure not allowing for student decision making.

Characteristics of modelling guidance provided by the lecturer include: providing relevant support students can utilise on their own, leading to personal direction being found by students; and allowing students to be actively thinking and creating and discovering their own modelling process. Guidance involves preparing students beforehand, providing resources, and role modelling behaviours. An outcome from this study is to suggest lecturers provide a commentary on the modelling process alongside the models they share in lectures. This will help students to see the connections between a model and its development. Further research into this would be beneficial.

10.2.4 Going forward together

The findings showed that lecturer and student experiences are interwoven and contribute to the collective experience. The lecturer played a key role in moving student groups forward. Collaboration within students' groups was also key to moving student groups forward. It is known that students need to be actively involved with the modelling process (Bassanezi, 1994; Wake, 2011). What is unknown is the division of labour between the student and the lecturer for mathematical modelling (Wake, 2016). The responsibility of the development of the model should be the student's, with the lecturer playing a facilitatory role. From analysing the independent component of the student modelling experience, it was seen that the lecturer contributes to the student experience, whether they are present or absent for the modelling activity. The results showed there is a danger that if students have not been given enough space for independent exploration, direction from the lecturer could limit independent student thought. The teaching implication is for the lecturer to have periods of time, or all the time, being unavailable to the student. The key is to prepare students beforehand. Further research is needed into how to best prepare students for independent mathematical modelling experiences.

To navigate different support required by students while modelling (Wedelin & Adwai, 2014), the findings of this study suggest lecturers: are conscious of unconfident mathematics and/or language students who may struggle more than others; set up student groups for modelling activities that contain a range of mathematical and language abilities; and provide training for students in time management when mathematical modelling.

Working in groups is a key component for productive mathematical modelling (Bender, 2000; Dym, 2004; Spooner, 2017, 2022; Tam, 2011; Treilibs et al., 1980) including developing mathematical modelling competencies (Caron, 2019; Maaß, 2006; Spooner, 2022; Toews, 2012). The findings support being able to work in a group as a key competency of the process of mathematical modelling (Toews, 2012) and key for developing modelling abilities. Students pooled resources, resulting in students progressing further than they would have individually. Working in groups provided a way for students to negotiate challenges, suggesting working within a group makes students more resilient than working individually.

The findings suggest students sharing their workings with the wider cohort is beneficial. The benefits of students sharing their work include: becoming aware that there is no one way of approaching the model; developing conceptual meaning for modelling; sharing techniques for approaches; and engaging in cognitive conflict to critique and compare approaches. These findings extend Hernandez-Martinez and Vos' (2018) conclusions that students find it useful to be able to work in a group and share their working with others. Hernandez-Martinez and Vos (2018) suggest students explaining work to others improves mathematical understanding. The findings support this and further suggest students presenting their work enables students to collect their thoughts through explaining their work to someone else.

The findings showed students like to see peer workings of the problem they are working on. For some students, cognitive conflict can be produced from seeing others workings using different approaches. This conflict may take time to resolve. Overall, students found it beneficial to observe how others approached and developed their models either for the same or similar situations. This finding draws attention to the importance of lecturers creating opportunities for students to witness peer workings of model development. Further research is needed into ways to support students who may not easily resolve cognitive conflict produced from witnessing different student approaches.

10.2.5 Independent work vs guidance

The findings showed lecturers need to be careful that the guidance given promotes independence as opposed to dependency. Durandt et al. (2021) claims directive teaching approaches are not favourable for developing modelling skills and competencies. In conjunction with Durandt et al. (2021), this finding draws attention to the need for the lecturer to provide non-prescriptive guidance that facilitates students to self-discover, be creative, engage in productive struggle, and take ownership of the modelling process.

Learning experiences that support students independently modelling involve lecturers playing a ‘behind the scenes’ role and include providing generic modelling cycles and self-guiding resources to scaffold the experience, encouraging students to reflect on their process while modelling, and, as the findings of this study further suggest, asking students to share and/or present their work to the wider cohort.

In more general terms, this study showed key lecturer behaviours that help students to take responsibility for their own modelling process include: providing students with material to refer to; using facilitatory questions/prompts aligned with the modelling process; and creating environments for students to share their thoughts and workings.

Lecturer support needs to allow for students to experience struggle, explore, be creative and become self-reliant. Experiencing these key student behaviours allows students to move from being uncertain to finding direction and thus take responsibility and ownership for their own modelling process.

Uncertainty is a key modelling behaviour that should not be minimised. Students were unaccustomed to uncertainty being an acceptable behaviour while working with mathematics. This finding draws attention to the need for lecturers to expose students to modelling behaviours such as uncertainty and unfamiliarity, and to provide additional strategies for dealing with these in lectures and tutorials. To prepare students for the modelling experience it is recommended that lecturers role model working through at least one problem unfamiliar to the lecturer, making sure they are clearly making explicit their process to students. This would involve a change of culture not only for students but also for lecturers, that is, moving from a culture of the lecturer presenting pre-meditated solutions to one of role modelling strategies for dealing with uncertainty. While the struggle will not be the same for the lecturer as it would be for a student it is important for students to witness these behaviours. A change of culture, where struggle and uncertainty is seen as a productive attribute, needs to be developed within mathematics and mathematical modelling education.

10.2.6 Teaching implications

The outcomes of this study have teaching implications for mathematics and mathematical modelling lecturers and teachers. The following recommendations are things that lecturers and teachers could try with their students.

To help students develop the ability to see connections between context and mathematics, this study suggests lecturers and teachers incorporate connecting mathematics to a context as a normal part of their teaching practice. Findings show the need to use contexts students can relate to. The context does not necessarily need to be authentic. Allowing students exploratory time to explore connections between mathematics and contexts could also be beneficial. To enable lecturers and teachers to have suitable contexts to use with students, it is recommended that an easily accessible database of contexts be made available for mathematics and mathematical modelling educators.

The findings show lecturers should provide opportunities for students to carry out modelling within groups. Lecturers can place students in groups, either consciously or through an unbiased, randomised selection process. Avoiding students self-selecting groups may increase collaboration within the wider cohort and allows lecturers to intentionally place unconfident mathematics and/or language students in groups with complementary skill sets if deemed necessary. To further enhance the collaborative space, lecturers and teachers can provide opportunities for students to share their workings with the wider cohort. Formal presentations of student work at key stages of the modelling process, followed by class discussion, is one way of doing this. Lecturers and teachers respectfully using students' presented work to teach the wider cohort new mathematical skills and techniques is something to consider.

To develop student responsibility and ownership of the modelling process, lecturers and teachers can experiment with using modelling activities that allow for students to make their own decisions. This could involve allowing students to: choose which modelling situation to work on; choose from a range of possible solution pathways to explore for a situation; be responsible for all decisions associated with the modelling process. Allowing students free scope for exploring potential outcomes without the pressure of a right or wrong answer is also important. Providing time when the teacher or lecturer is not available for guidance is an important part of encouraging students to make their own decisions and learn to draw on their group members as valuable resources.

Students need to be prepared and guided through the modelling process. Preparing students to work effectively within a group, as well as exposure to modelling behaviours would be beneficial for students. Lecturers should aim to teach modelling techniques and processes together (Spooner, 2020). Frequently alluding to, and aligning teaching to, the modelling process throughout the course is suggested. It is proposed during lectures, for lecturers to role model the modelling behaviours of unfamiliarity and uncertainty by working through problems unfamiliar to the lecturer (Spooner, 2021). Promoting exploration, using heuristic strategies, and knowing it is okay to produce an imperfect solution are strategies for students to deal with uncertainty. Modelling-process guidance given by the lecturer or teacher during a modelling activity should be non-prescriptive and of a facilitatory nature. A set of facilitatory questions aligned with the modelling process is one way of providing a support framework for modelling.

10.2.7 Interpretive Description as a methodology

The review of methodologies used in mathematical modelling education and education shows that while there are some studies that use established methodologies (Brown, 2013; 2015a; 2105b; Hernandez-Martinez & Vos, 2018; Ng et al., 2015; Rosa, Orey, & de Sousa Mesquita, 2021), the methodologies for the majority of research consists of the use of a variety of methods for data collection and analysis (for example: Blum & Leiß, 2007; Caron & Bélair, 2007; Czocher, 2018). When reviewing research described as qualitative, these were limited. Other than studies using established methodologies, most papers citing qualitative methods used deductive analysis looking for pre-determined aspects (for example: Czocher, 2018; Lu & Kaiser, 2021). While some studies take their findings and use them to suggest recommendations for modelling courses (Caron & Bélair, 2007) or discussion papers on different teaching approaches (Bassanezi, 1994; Li, 2013), the suggestions are suggestions and generally lack research backing them. What is missing from the literature is a methodological qualitative study that aims to understand the nuances of student experiences and use these to inform future teaching practice. This study addresses this gap by using Interpretive Description (Thorne, 2008) to move beyond mere description and begin to address the 'so what' of that which is occurring for participants, and what this means for practitioners within the discipline field (Thorne, 2008). This study contributes to the field of knowledge by providing potential teaching practices for mathematical modelling based on robust data analysis of students' experiences with mathematical modelling.

10.2.8 Summary of key contributions and findings

This thesis contributes to both knowledge and to practice. This thesis expands on the limited existing knowledge on the student experience while students learn mathematical modelling, and on the role of the lecturer in this context, within tertiary education. It also suggests practical guidance for lecturers who teach mathematical modelling. The thesis indicates that modelling requires a change of culture for both students and lecturers, and it investigates the dynamics between students' independent work and the role of the lecturer.

Productive struggle is a key part of learning to mathematically model. The thesis provides recommendations that enable tertiary lecturers to produce and then deal with productive struggle being experienced by students who are engaging with mathematical modelling for the first time. Enabling students to engage in productive struggle involves careful preparation of students beforehand and the use of particular types of non-directive teaching.

Another contribution of this thesis is in its first application of interpretative description methodology to mathematical modelling and mathematics education research, resulting not only in better understanding but also the practical guidance for lecturers.

A summary of the key findings of this thesis include:

- Modelling is a change of culture for mathematics students and lecturers.
- A balance between lecturer guidance and independent student work is important for students to move through a modelling experience.
- Independent student work is needed to develop modelling competencies.
- A role of the lecturer is to enable students to develop skills to independently model and support students to participate in independent modelling.
- Students working in groups, independent of the lecturer, develop coping behaviours, use tools provided by the lecturer and use each other as resources.
- A fully independent student modelling experience is possible with careful preparation beforehand and access to self-guiding resources to scaffold the experience.
- Struggle is an important part of both the modelling process and learning to model. However, a balance between struggle and achievement is needed for the student.
- Providing hints and minimising struggle can hinder internalisation of the modelling process for students and thereby hinder learning to model.
- It is the independent total immersion experience that helps students embrace applying themselves to modelling, grapple with and discover the process, and make it more likely to internalise the process.

- The use of Interpretive Description as a methodological contribution to mathematics education as a method that is robust for looking at student experience and their implications to teaching practice.
- The use of Interpretive Description as a methodological contribution as a way of better understanding student modelling experiences to inform lecturer teaching practice.

Key practice applications include lecturers:

- Frequently alluding to the modelling cycle while they are teaching including aligning the modelling process to models and modelling examples while they are teaching.
- Regularly applying mathematics to contexts in mathematics and modelling courses.
- Role modelling, modelling an unfamiliar situation they have not worked on before, without a pre-rehearsal.
- Providing opportunities for students to take responsibility and ownership of their work including making their own decisions.
- Understanding that modelling is a creative process.
- Acknowledging that productive struggle is necessary for learning mathematical modelling and therefore create opportunities for students to experience and feel supported while engaging in productive struggle.
- Developing a change of culture where struggle and being creative in mathematics is viewed by students as a positive attribute.
- Providing student support through preparing students beforehand, role modelling behaviours characteristic of modellers, resources and non-directive guidance.

- Providing guidance and scaffolding that enables students to self-discover, be creative, engage in productive struggle and take ownership of the modelling process.
- Structuring modelling activities for students to collaboratively model in student groups.
- Providing opportunities for students to share their modelling work and modelling process, and to witness other students modelling work and modelling process.

10.2.9 Final words [18]

To conclude, lecturers need to be conscious that they are connecting contexts and mathematics, building collaborative working spaces, developing student responsibility and ownership for their work, and allowing students free scope for mathematical exploration in their mathematical and mathematical modelling teaching practices. Guidance for modelling activities should occur through preparing students beforehand, 'role modelling' modelling behaviours and be facilitatory in nature. A support framework for modelling should be aligned to the modelling cycle.

Mathematical modelling should involve students actively engaged in their learning process, resulting in learning by doing. Activities should involve students making their own decisions and deciding on their own solution paths while also providing guidance on the modelling process. For a productive student experience, modelling should be undertaken in groups as opposed to individually, opportunities to make connections between reality and mathematics are provided, students are encouraged to be open-

[18] This section includes re-presentation of material published in Spooner (2022).

minded and think differently about mathematics and its potential, and involves some productive struggle combined with lecturer or teacher guidance.

Mathematical modelling offers a practice that provides new and unique situations where exploration is part of the process, and places students on an even playing field. Mathematical modelling utilises assisted discovery as an instructional method. An implication from this study is that the change of culture mathematical modelling provides will help mathematics education by making some of the practices that are commonly associated with effective mathematics learning more commonplace. This has the potential to influence students to feel good about mathematics and provide alternative approaches and opportunities for teachers to rethink the way they will present mathematics.

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Appendix A

- Student participant information sheet
- Lecturer participant information sheet
- Head of Department participant information sheet

Appendix Ai). Student participant information sheet

Participant Information Sheet (Student)

TE WĀNANGA ARONUI O TĀMAKI MAKAU RAU

Date Information Sheet Produced: 20/09/2016

Project title: Identifying effective teaching practices to overcome student difficulties in the formulating mathematical models.

Supervisor: Sergiy Klymchuk; Associate Professor; School of Engineering, Computer and Mathematical Sciences.
Researcher: Kerri Spooner; PhD student; School of Engineering, Computer and Mathematical Sciences.

Researcher Introduction
Greetings, my name is Kerri Spooner. I am a lecturer in the School of Engineering, Computer and Mathematical Sciences and a PhD student in Mathematics Education at Auckland University.

What is the purpose of the research project?
My research project is about identifying effective teaching practices for mathematical modelling experiences for tertiary students. Mathematical modelling is an important way of understanding the world and is a useful tool for finding solutions to everyday problems. Your participation in the research project will help to develop guidelines for effective teaching practices for mathematical modelling for secondary and tertiary students. This research will contribute towards a PhD qualification.

Why are you being invited to participate in the research project?
I would like to invite you to participate in this research as you are a student taking a mathematical modelling course or a course that has a mathematical modelling component.

How do I agree to participate in this research?
Participation in this study is entirely voluntary. Whether or not you choose to participate will neither advantage nor disadvantage you. You are able to withdraw from the study at any time. You may choose to no longer participate, and any information collected will be destroyed. However, you maybe video and audio recorded with other students. If this is the case, it will be impossible to remove any information that has already been recorded in a group setting once the project has started. Also, once the findings have been produced, removal of your data may not be possible. However, your identity will remain confidential. A written copy of results of research will be given to all participants who request it.

The project
If you choose to participate, you will be participating in a fifteen-to-thirty-minute audio recorded interview, allowing me to video record and use your written record of your the mathematical modelling activity in your course and keeping a reflective log of what you are doing while you are participating in the mathematical modelling activity to be used for analysis. The assessment activity and interview will be recorded so I can refer back to it at a later date. The video recorder and audio recorder can be switched off at any time during the assessment and interview if you so wish. Analysing of your assessments, reflective log and interviews will occur after you have finished your course.

The video tapes assessments, written assessments, reflective logs and interviews will be used to collect evidence of student strategies for formulating mathematical models and help me to understand how students have experienced the teaching of mathematical modelling in your course.

Participation in the mathematical modelling activity is a normal component of your course. The difference to the activity will be that I, the researcher, will video and observe the activity and you will be asked to keep a voluntary reflective log of your experience. Whether or not you choose to participate, you will have the opportunity to keep a voluntary reflective log of the activity if you so choose to do so. It is expected that up to one hour outside of normal class time will be involved in participating in a post activity interview.

Data Storage,
All collected data (i.e., video recordings, audio recordings and written work) will be securely stored by my supervisor for a period of six years.

Confidentiality and Future Use
The findings of this research will be used for academic publications and presentations. If the information you provide is reported/published/presented, this will be done in a way that does not identify you as its source. No video recordings, or parts of, will be used in any presentations or publications.

What opportunity do I have to consider this invitation?
Please let me know within one week if you would like to participate or decline this invitation.

Please keep this Information Sheet and a copy of the Consent Form for your future reference. If you have any further questions you may contact:

Researcher: Kerri Spooner
kspooner@aut.ac.nz
Research Supervisor: Sergiy Klymchuk
sergiy.klymchuk@aut.ac.nz
Head of Mathematical Sciences Department: Jiling Cao
jiling.cao@aut.ac.nz

Any concerns regarding the nature of this project should be notified in the first instance to the Project Supervisor, Sergiy Klymchuk, sergiy.klymchuk@aut.ac.nz 921 9999 ext 8431

Concerns regarding the conduct of the research should be notified to the Executive Secretary of AUTEC, Kate O'Connor, ethics@aut.ac.nz, 921 9999 ext 6038.

Approved by the Auckland University of Technology Ethics Committee on 16th January 2017 AUTEC Reference number 16/387
Note: The Participant should retain a copy of this form.

Appendix Aii). Lecturer participant information sheet

PARTICIPANT INFORMATION SHEET

(University Lecturer)

Date Information Sheet Produced: 20/09/2016

Project title: Identifying effective teaching practices to overcome student difficulties in the formulating mathematical models.

Supervisor: Sergiy Klymchuk; Associate Professor; School of Engineering, Computer and Mathematical Sciences, Auckland University of Technology.
Researcher: Kerri Spooner; PhD student; School of Engineering, Computer and Mathematical Sciences, Auckland University of Technology.

Researcher Introduction

Greetings, my name is Kerri Spooner. I am a lecturer in the School of Engineering, Computer and Mathematical Sciences and a Phd student in Mathematics Education at Auckland University of Technology.

The project

I am conducting research on identifying effective teaching practices for mathematical modelling. The goal of the research is to be able identify practices that enable students to successful formulate a mathematical model. One of the purposes of the research is to investigate teaching styles and their potential influence on student approaches to modelling. Knowledge gained from the research will contribute to information about procedures and strategies that are effective for teaching mathematical modelling. This research is part of my PhD Thesis.

Invitation to Participate

I would like to invite you to participate in this research as you are currently teaching a mathematical modelling course or a course that has a mathematical modelling component. Your participation is voluntary. Whether or not you choose to participate will neither advantage nor disadvantage you.

Project Procedures

If you agree to take part in this research, you will need to:

- Allowing us to make a 2-minute presentation announcement in class after you have finished lecturing in which we invite students to volunteer to participate.
- Allowing student participants from your class to be observed, video recorded, and audio recorded as they work on one of the mathematical modelling activities within your course and participate in a fifteen-to-thirty-minute interview outside of class time. The mathematical modelling activity observations will be used for evidence of student strategies for formulating mathematical models. If you are willing, I would appreciate your guarantee that a student's decision to participate will not affect the student's school learning, standing or assessment.
- I will be asking your students to keep a voluntary reflective log on the mathematical modelling activity used for this research. The reflective log will be used for evidence of student strategies for formulating mathematical models. Student work on the reflective log will take place while they are participating on the mathematical modelling activity.
- Participating in pre lecture and post lecture audio taped interviews, of approximately sixty minutes each, to establish your teaching methods for formulation of a mathematical model and to identify student participation activities used in your course for formulating a mathematical modelling. The interviews will be audio-taped so it can be transcribed and referred back to at a later date. The audio recorder can be switched off at any time during interview if you so wish.
- Allowing your lecturers to be observed and video recorded for collection of data on teaching methods for formulation of a mathematical model. The observation will be video-taped so it can be transcribed and referred back to at a later date. The video recorder can be switched off at any time during the lecture if you so wish.
- Allowing your course notes to be examined for strategies for formulating a mathematical model.

It is expected that you will give two hours outside of normal teaching time. The only difference to your normal teaching time will be that I, the researcher, will be present in class for lecturer observations. The amount of time for lecturer observations will depend on the length of teaching time allocated for teaching formulating mathematical modelling.

Data Storage, Retention, Destruction, and Future Use

The video-recordings and audio-recordings will be transcribed by a third party, who will sign a confidentiality agreement. A pseudonym will be used instead of your real name to label the audio-recordings and video-recordings, as well as in the transcripts and any publications, ensuring that your work will not be traceable back to you. The data (i.e., video recordings, audio recordings, and the students' work on the task) will be securely stored on university password-protected computers, accessible only to the researchers, and will be securely destroyed after six years.

Right to Withdraw from Participation

Participation in this study is entirely voluntary. In addition, at any stage of the research process you may choose to no longer participate and any information collected will be destroyed. If you choose to withdraw from the study, then you will be offered the choice between having any data that is identifiable as belonging to you removed or allowing it to continue to be used. However, once the findings have been produced, removal of your data may not be possible. A written copy of results of research will be given to all participants who request it.

Confidentiality

The preservation of confidentiality is paramount. Only research personnel will have sole access to data collected. All participants' identities will remain confidential. When I am writing up the findings I will do my best to exclude any information that may identify you, however as you know we are a small population someone may be able to guess who is involved. Un-identified transcripts of data collection recordings will be kept indefinitely for research purposes and publication. The findings of this research will be used for academic publications and presentations. If the information you provide is reported/published/presented, this will be done in a way that it will not identify you as its source. No video recordings, or parts of, will be used in any presentations or publications.

Timeframe for response

Please let me know within one week if you would like to participate or decline this invitation.

CONTACT DETAILS and APPROVAL

Thank you for your help.

Please feel free to contact us now or at any time if you have any questions about the project or procedures.

Researcher: Kerri Spooner
kspooner@aut.ac.nz
Research Supervisor: Sergiy Klymchuk
sergiy.klymchuk@aut.ac.nz
Head of Mathematical Sciences Department: Jiling Cao
jiling.cao@aut.ac.nz

Any concerns regarding the nature of this project should be notified in the first instance to the Project Supervisor, Sergiy Klymchuk, sergiy.klymchuk@aut.ac.nz 921 9999 ext 8431

Concerns regarding the conduct of the research should be notified to the Executive Secretary of AUTEC, Kate O'Connor, *ethics@aut.ac.nz* , 921 9999 ext 6038.

Approved by the Auckland University of Technology Ethics Committee on 16th January 2017 AUTEC Reference number 16/387

Note: The Participant should retain a copy of this form.

Appendix Aiii). Head of Department participant information sheet

INFORMATION SHEET

(HOD Mathematics)

Date Information Sheet Produced: 20/09/2016

Project title
Identifying effective teaching practices to overcome student difficulties in the formulating mathematical models

Supervisor
Sergiy Klymchuk; Associate Professor; School of Engineering, Computer and Mathematical Sciences, Auckland University of Technology.
Researcher
Kerri Spooner; Phd student; School of Engineering, Computer and Mathematical Sciences, Auckland University of Technology.

Researcher Introduction
My name is Kerri Spooner. I am a lecturer in the School of Engineering, Computer and Mathematical Sciences and a Phd student in Mathematics Education at Auckland University of Technology.

The project
I am conducting research on identifying effective teaching practices for mathematical modelling. The goal of the research is to be able identify practices that enable students to successful formulate a mathematical model. Knowledge gained from the research will contribute to information about procedures and strategies that are effective for teaching mathematical modelling. This research is part of my PhD Thesis.

Request for Access
I would like to invite students who are taking mathematical modelling courses, or a course with a mathematical component in it, to participate in this research. Students' participation is voluntary and they may decline this invitation to participate without penalty. I would appreciate your guarantee that a student's decision to participate will not affect the student's university learning, standing or assessment.

We will also invite a lecturer, responsible for teaching the above course, from your university to participate in the project. The lecturer's participation would involve:

- Allowing us to make a 2-minute presentation announcement in their class after they have finished lecturing in which we invite students to volunteer to participate as described above
- Participating in pre lecture and post lecture audio taped interviews, of approximately sixty minutes each.
- Allowing their lecturers to be observed and video recorded for collection of data on teaching methods for formulation of a mathematical model.
- Allowing their course notes to be examined for evidence of teaching methods for formulations of a mathematical model.

Project Procedures

Students who consent to participate in this research will be observed and video recorded while they work on a mathematical modelling activity that's is part of their course. The written assessment work of the student will also be analysed. They will also participate in a fifteen-to-thirty-minute interview outside of class time regarding the task. The interview will be audio recorded and the assessment observation will be video recorded. The work they do on the assessment and the follow up interview will be analysed by myself and an independent researcher. We will create unidentifiable reports of those tasks, using pseudonyms to replace the participants' real names when the recordings are transcribed. I would also appreciate your guarantee that your students' performance on these tasks will not affect their grade or their standing in the class. The expected time commitment from students will be between one and eight hours. Time involvement is dependent on length of time the student's lecturer allocates for the mathematical modelling activity.

Lecturers who consent to participate in this research will be asked to participate in pre and post lecture interviews and be observed during lecturers on 'formulation of a mathematical model'. The interview will be audio recorded and the lecture observation will be video recorded. They can ask for the recordings to be turned off at any time without having to give a reason.

Data Storage, Retention, Destruction, and Future Use

Pseudonyms will be used instead of the participants' real names to label the audio-recordings and video-recordings, as well as in the transcripts and any publications, ensuring that the participants' data will not be traceable back to them. The data (i.e., video recordings, audio recordings, and the students' written work on the task) will be securely stored on university password-protected computers, accessible only to the researchers, and will be securely destroyed after six years.

Right to Withdraw from Participation

Participation in this study is entirely voluntary. Even if students and lecturers agree to participate in this project, they may withdraw at any stage. Because students may be video and audio recorded together with two other students (while working on the assessment activities and during the interview) it is not possible to turn off the recording during the activities or during the interview. However, if a student does not wish to be recorded he/she may choose to leave the room. Also, because a student may be video and audio recorded together with two other students, it will be impossible to remove any of a single student's data that he/she has already given if he/she withdraws after the project has started. Students who do not wish to withdraw will be requested to maintain the confidentiality of the withdrawing student's data, and will be requested not to reveal his/her identity. Moreover, if a student chooses to withdraw from the project after having started working on the project tasks (i.e., assessment activities and interview), he/she will be requested to maintain the confidentiality of any information shared by the student participant(s) who do not wish to withdraw from the study.

Information collected from lecturers who withdraw will be destroyed. However, once the findings have been produced, removal of your data may not be possible.

Confidentiality

The preservation of confidentiality is paramount. Only research personnel will have sole access to data collected. All participants' identities will remain confidential. The un-identified transcripts of these tapes will be kept indefinitely for research purposes and publication. The findings of this research will be used for academic publications and presentations. If the information students and lecturers provide is reported/ published/ presented, this will be done in a way that does not identify the programme, the lecturer, or any students you as its source.

We will do our best to protect a student's anonymity by taking names of volunteering student participants when the lecturer is not present, and where possible conducting the activity sessions in a location that is inaccessible to the lecturer/teacher for the duration of the events.

CONTACT DETAILS and APPROVAL
Thank you for your help.

Please feel free to contact us now or at any time if you have any questions about the project or procedures.

Researcher: Kerri Spooner
kspooner@aut.ac.nz
Research Supervisor: Sergiy Klymchuk
sergiy.klymchuk@aut.ac.nz
Head of Mathematical Sciences Department: Jiling Cao
jiling.cao@aut.ac.nz

Any concerns regarding the nature of this project should be notified in the first instance to the Project Supervisor, Sergiy Klymchuk, sergiy.klymchuk@aut.ac.nz 921 9999 ext 8431

Concerns regarding the conduct of the research should be notified to the Executive Secretary of AUTEC, Kate O'Connor, *ethics@aut.ac.nz* , 921 9999 ext 6038.

Approved by the Auckland University of Technology Ethics Committee on 16th January 2017 AUTEC Reference number 16/387
Note: The Participant should retain a copy of this form.

Appendix B

- Student consent sheet
- Lecturer consent sheet
- Head of Department permission to access

Appendix B i). Student consent form

CONSENT FORM

(University student)

THIS FORM WILL BE HELD FOR A PERIOD OF 6 YEARS

Project title: Identifying effective teaching practices to overcome student difficulties in the formulating mathematical models.

Supervisor: Sergiy Klymchuk; Associate Professor; School of Engineering, Computer and Mathematical Sciences; Auckland University of Technology. Contact email address of supervisor: sklymchuk@aut.ac.nz

Researcher: Kerri Spooner; PhD student; School of Engineering, Computer and Mathematical Sciences; Auckland University of Technology. Contact email address of researcher: kspooner@aut.ac.nz

- I have read and understood the information provided about this research project in the Participant Information Sheet dated 20/09/2016.
- I have had an opportunity to ask questions and to have them answered.
- I agree to take part in this research.
- I understand that taking part in this study is voluntary (my choice) and that I may withdraw from the study at any time without being disadvantaged in any way.
- I understand that I will be observed, video recorded, and audio recorded as I work on a mathematical modelling activity within my course. The modelling activity maybe an individual activity or maybe a group activity. I understand that notes will be taken during the modelling activity and the recordings of the activity will be transcribed.
- I understand I will also be asked to participate in a fifteen-to-thirty-minute post activity audio recorded interview with the researcher. I understand that notes will be taken during the interviews and that the audio-taped recording will be transcribed.
- I agree to be video-recorded and audio-recorded, and that a pseudonym will be used instead of my real name to label the transcripts of the audio-recordings and video-recordings.
- I understand that not video recordings, or parts of will be used in any presentations or publications.
- I understand my written work on the mathematical modelling activity, including my reflective log will be used to collect data.
- I understand that a third party who has signed a confidentiality agreement may transcribe and analyse the recordings and my written work.

- I understand that if I withdraw from the study then, while it may not be possible to destroy all records of the focus group discussion of which I was part, I will be offered the choice between having any data that is identifiable as belonging to me removed or allowing it to continue to be used. However, once the findings have been produced, removal of my data may not be possible.

- I understand that identity of my fellow participants and our discussions within the activity group and group interview is confidential to the group, and I agree to keep this information confidential.

- I wish to receive a summary of findings, which can be emailed to me at this email address: ______________________. ☐

Participant 's signature :
....

Participant 's name : ....

Participant 's Contact Details (if appropriate) :
....
....
....
....

Date :

Approved by the Auckland University of Technology Ethics Committee on 16th January 2017 AUTEC Reference number 16/387
Note: The Participant should retain a copy of this form.

Appendix B ii). Lecturer consent form

CONSENT FORM

(University Lecturer)

THIS FORM WILL BE HELD FOR A PERIOD OF 6 YEARS

Project title: Identifying effective teaching practices to overcome student difficulties in the formulating mathematical models

Supervisor: Sergiy Klymchuk; Associate Professor; School of Engineering, Computer and Mathematical Sciences; Auckland University of Technology. Contact email address of supervisor: sklymchuk@aut.ac.nz

Researcher: Kerri Spooner; PhD student; School of Engineering, Computer and Mathematical Sciences; Auckland University of Technology. Contact email address of researcher: kspooner@aut.ac.nz

- I have read and understood the information provided about this research project in the Participant Information Sheet dated 20/09/2016.
- I have had an opportunity to ask questions and to have them answered.
- I agree to take part in this research.
- I understand that taking part in this study is voluntary (my choice) and that I may withdraw from the study at any time without being disadvantaged in any way.
- I understand that I agree to take part in this research, my participation would involve:
  1) Allowing us to make a 2-minute presentation announcement in class after you have finished lecturing in which we invite students to volunteer to participate.
  2) Allowing student participants from your class to be observed, video recorded, and audio recorded as they work on one of the mathematical modelling activities within your course and participate in a fifteen to thirty minute interview outside of class time.
  3) Participating in pre lecture and post lecture audio taped interviews, of approximately sixty minutes each, to establish your teaching methods for formulation of a mathematical model and to identify student participation activities used in your course for formulating a mathematical modelling.
  4) Allowing your lecturers to be observed and video recorded for collection of data on teaching methods for formulation of a mathematical model.
  5) Allowing your course notes to be examined for evidence of teaching methods for formulations of a mathematical model.
- I agree to be video-recorded and audio-recorded during a one-hour interview and lecture observations, and that a pseudonym will be used instead of my real name to label the audio-recordings and video-recordings. I can ask for the recordings to be turned off at any time during the interview or observation without having to give a reason.
- I understand that I will not have the opportunity to review the transcript of my interview, but if I have anything to add to my responses after the interview, I can communicate with the researchers directly.
- I understand that a third party who has signed a confidentiality agreement may transcribe and analyse the recordings.

- I understand that if I withdraw from the study then I will be offered the choice between having any data that is identifiable as belonging to me removed or allowing it to continue to be used. However, once the findings have been produced, removal of my data may not be possible.

- I wish to receive a summary of findings, which can be emailed to me at this email address: ________________________. ☐

Participant 's signature :
....

Participant 's name : ....

Participant 's Contact Details (if appropriate) :
....
....
....
....

Date :

Approved by the Auckland University of Technology Ethics Committee on 16th January 2017 AUTEC Reference number 16/387
Note: The Participant should retain a copy of this form.

Appendix B iii). Head of Department permission to access form

Permission to Access FORM

(HOD)

THIS FORM WILL BE HELD FOR A PERIOD OF 6 YEARS

Project title: Identifying effective teaching practices to overcome student difficulties in the formulating mathematical models.

Supervisor: Sergiy Klymchuk; Associate Professor; School of Engineering, Computer and Mathematical Sciences; Auckland University of Technology.
Contact email address of supervisor: sklymchuk@aut.ac.nz

Researcher: Kerri Spooner; PhD student; School of Engineering, Computer and Mathematical Sciences; Auckland University of Technology.
Contact email address of researcher: kspooner@aut.ac.nz

I have read the Participant Information Sheet and have understood the nature of the research. I have had the opportunity to ask questions and have them answered to my satisfaction.

- I agree to give permission for the researchers to approach lecturers in my department, inviting them to participate in the study.
- I agree to give permission for the researchers to approach students in the relevant course within my department, inviting them to participate in the study.

Participant 's signature : ....

Participant 's name : ....

Participant 's Contact Details (if appropriate) :

....
....
....
....

Date :

Approved by the Auckland University of Technology Ethics Committee on 16th January 2017 AUTEC Reference number 16/387
Note: The Participant should retain a copy of this form.

Appendix C

- Indicative student interview questions
- Indicative lecturer interview questions

Appendix C i). Indicative student interview questions

Post activity Indicative Student Interview Questions

Interview Protocol:

A set of structured open-ended questions will be used. Follow-up questions will also be asked based on participant responses to provide more detailed information and/or develop rapport.

Recording of interviews will be done by audiotape to allow for information to be analysed at a later date and by research assistant. Audio recording will be as taken on the day of the interview. No editing will be carried out of any audio recordings.

Because students may be audio recorded together with two other students in a group setting it is not possible to turn off the recording during the activities or during the interview. However, if a student does not wish to be recorded, he/she may choose to leave the room. Students who do not wish to withdraw will be requested to maintain the confidentiality of the withdrawing student's data and will be requested not to reveal his/her identity. Moreover, if a student chooses to withdraw from the project after having started working on the project tasks, he/she will be requested to maintain the confidentiality of any information shared by the student participant(s) who do not wish to withdraw from the study.

Interview Purpose:

Interviews will be used to establish the student experience of the mathematical modelling activities of their course and to establish their experience to the course in general.

Interviews will also be used to confirm findings from the assessment activity observations and where appropriate verify the lecturer's hypothetical learning trajectory (HLT).

To establish approach student or student group took in formulating the mathematical model.

To determine which parts of the HLT are evident in the students' work.

To help determine actual HLT.

To discover difficulties students had with the formulation stage of mathematical modelling.

To obtain in-depth information of the process of teaching and learning mathematical modelling.

Questions will be asked that are designed to uncover the approach was taken by the student and why.

<u>Lead-in questions</u>

Do you remember the modelling activity done for this paper?

Can you tell me about it?

For the activity here can you tell me how you and/or your group went about formulating the model?

Remember back to lectures, can you remember the lecture(s) on formulating a mathematical model?

What do you remember from it? (When your lecturer taught formulating the model what do you remember?

Did you use any of this information in the modelling activity?

How would you say your lecturer taught you to formulate a model?

Did your lecturer give you any steps scaffolding to help formulate the model?

If yes, what were these steps?

Did you use any of these things when you formulated the model?

Is there anything that stands out as useful from the lecturer when your group formulated the model?

Do you remember any specific strategies from your course notes for formulating the model?

Can you describe these strategies?

Did you use any of these strategies in the activity?

Tell me more.

Where any of these strategies useful?

Looking at the workings from your classroom activity here, lets look at the formulating stage of the model, can you tell me what was happening.

Using your workings, how did you start? Did you start by modelling what you had seen in lecturers?

For the next part of the working do you attempt this part by modelling what you had seen in lecturers?

Or did you attempt this part by exploring other approaches that were outside the scope of the lecturers?

For the next part of the working do you attempt this part by modelling what you had seen in lecturers?

Or did you attempt this part by exploring other approaches that were outside the scope of the lecturers?

What do you think are the key points for formulating a mathematical model?

Looking back, what difficulties did you experience during the modelling activity?

Is there anything extra the lecturer could have provided to minimise this difficulty?

What would you have done differently if you were to repeat the activity now?

Is this the first time you have taken this paper?

Which degree are you currently undertaking?

Appendix C ii). Indicative lecturer interview questions

Indicative Lecturer Interview Questions

Interview Protocol:

A set of structured open-ended questions will be used. Follow-up questions will also be asked based on participant responses to provide more detailed information and/or develop rapport.

Recording of interviews will be done by audiotape to allow for information to be analysed at a later date and by research assistant. Audio recording will be as taken on the day of the interview. No editing will be carried out of any audio recordings. The lecturer participant can request the audio recorder is turned off at any stage of the interview without giving a reason.

Should a lecturer participant withdraw their consent, any audio recordings already taken will be immediately destroyed.

Pre observation Interview:

Interview Purpose:

Interviews will be used to obtain in-depth information of the process of teaching and learning mathematical modelling and to establish Hypothetical Learning Trajectory (HLT) of lecturer.

In particular, to determine lecturer's learning goals, planned instructional activities and conjectured learning process for the students.

To identify, establish and confirm approach to teaching formulation stage of mathematical modelling and to determine"the lecturer's experience of teaching mathematical modelling".

*Question*s *will be asked that are designed to uncover why the approach was taken by the lecturer.*

Lead-in questions

How long teaching mathematical modelling for?

What is mathematical modelling to you? (Can you define mathematical modelling?)

What is the objective of the mathematical modelling course you teach on?

For formulating a mathematical model how many lectures do you spend on formulating a mathematical model?

When thinking about the formulation stage of mathematical modelling, what do you think are the key points for teaching formulating a mathematical model?

How do you teach the formulation stage of mathematical modelling? Can you describe it for me?

Why do you teach it this way?

Do you involve students in your classes when teaching formulating mathematical models?

Why or why not?

Is your lecture space one where students are doing things with you during class or sitting and listening/taking notes?

Do you think that students need to be active in the teaching learning process for mathematical modelling? (Do students need to be actively doing things in class/lecture or can they learn how to formulate models by direct transmission of information from the lecturer to the student?)

What do you think the difficulties students have with formulating the model?

When student difficulties do you keep in mind when teaching the formulation of the mathematical model?

How do these difficulties influence your teaching?

What do you think common student difficulties with mathematical modelling are?

When structuring the delivery/design of your course content what student difficulties do you keep in mind?

Can you give me an example of a delivery that addresses a student difficulty?

How do student difficulties influence the design and delivery of your course?

Thinking about an upcoming lecture on formulating a mathematical model what are your learning goals for the students?

Can you describe what you plan to take place during this lecture?

Do you have a specific way of delivery for teaching the formulation stage of creating a model?

Tell me about it.

What lecture activities will use to teach it?

In-lecture activity?

Worksheets?

Readings?

How will they use your delivery to be able to formulate a mathematical model?

What are your ideas/theory about how students learn to formulate a mathematical model?

How will you assess the formulation of the mathematical model?

Is there anything you find difficult to teach in regards to the formulation of a mathematical model?

Tell me about it.

Please expand.

Using what we have discussed, are you happy to work with me now and we write up a formal HLT for your upcoming lecture(s) on formulating a mathematical model?

A HLT has three main parts: learning goals, planned instructional activities and conjectured learning process for the students.

Can you define what your student learning goals are for the lecture(s)?

What is your planned instructional delivery for this lecture(s)?

What is your vision for the learning process the students will take part in from attending your lectures on this topic?

We can use it to review what happens later.

Post-observation Interview:

Interview purpose:

To obtain in-depth information about the process of teaching and learning mathematical modelling.

To confirm findings from the lecturer observations.

To review how HLT was delivered.

To discuss modifications of the HLT for future use.

Lead-in questions

Tell me about your lecture(s) on formulating a mathematical model.

How well do you think your lecture(s) on formulating a mathematical model went?

Looking at your learning goals for the students, do you think these were met?

Why or why not?

How did your planned instructional activities go? Did they go to plan?

What modifications did you make?

What prompted these modifications?

What do you think the students' conjectured learning process was?

How do you think they were learning?

Were there any student difficulties that became apparent during the lecture?

How did you accommodate for these?

How well do you think the student modelling activities met your learning goals for the students?

What would you do differently next time?

Are you happy to work with me now and we review and update your HLT for future lectures on this topic?

Appendix D. Observation protocols

- Lecture observation protocols
- Modelling activity observation protocols

Lecture observations

Observations will be focussed on the lecturing process, to look for evidence of the practice of teaching mathematical modelling including how lecturers and students interact with each other. The venue of the observation is the lecture theatre where the class is taking place. The level of the class will be identified. The researcher will be present observing the class. Field notes will recorded during the observation on interactions between the lecturer and student and on the process for teaching mathematical modelling both for the student and the lecturer. A checklist based on each lecturer's hypothetical learning trajectory (HLT) will be used as a tool. Lectures/parts of lectures on formulating a mathematical model will video- and audio-taped so they can be referred back to at a later date. Video and audio recording will be as taken on the day of the interview. No editing will be carried out of any video and audio recordings. Video recording will capture the routine and sequence of lesson that may be missed or hard to uncover in an interview.

The lecturer participant can request the video and audio recorder is turned off at any stage of the observation without giving a reason. Should a lecturer participant withdraw their consent, any audio recordings already taken will be immediately destroyed.

To avoid video privacy issues for non-participants during the lecture, students who do not wish to be video-taped will be advised to sit away from the video. Students' audio responses will be transcribed as "a student said" to ensure confidentiality.

Activity observations

Student participants taking part in a mathematical modelling activity within the course will be observed. The activity observations will be video- and audio-recorded so they can be referred back to at a later date. This will allow for behaviours and interactions to be observed, which cannot be recorded in a written assessment or not easily uncovered during an interview. Observations will focus on how students are formulating a mathematical model, interactions between the student and student. A check list for behaviours for formulating a mathematical model will be used and a checklist of behaviours based on the lecturer's HLT will also be used.

To protect the privacy of non-participating students, an announcement will be made – "If you do not wish to be video recorded please sit over here".

Because students may be video- and audio recorded together with two other students in a group setting it is not possible to turn off the recording during the activities or during the interview. However, if a student does not wish to be recorded he/she may change groups to a non-participating group. Students who do not wish to withdraw will be requested to maintain the confidentiality of the withdrawing student's data, and will be requested not to reveal his/her identity. Moreover, if a student chooses to withdraw from the project after having started working on the project tasks, he/she will be requested to maintain the confidentiality of any information shared by the student participant(s) who do not wish to withdraw from the study.

Appendix E. Drew Interview summary

Drew is a repeating student, having done course during semester and now doing it a summer school. Doing the course twice helped ground the material and gave him the processing time needed for the content. He expressed that he had learnt from other students during the project day. He felt project added to the course and he could now participate in a modelling project if he had the opportunity.

The group's approach for the open project day was to spent the first half an hour getting their heads around the problem, then they spent the next half an hour making a plan of how they were going to approach things (which was an idea that came from Peter). Then people went off doing specific tasks, coming from plan. This took a while. The group then made a call to make some assumptions to get things moving. At first they struggled to find a base model to use as a foundation for the model.

Peter gave tips for managing the day – arrive 10 minutes early etc – which Drew talked about.

Had guidelines given during the course for the modelling process.

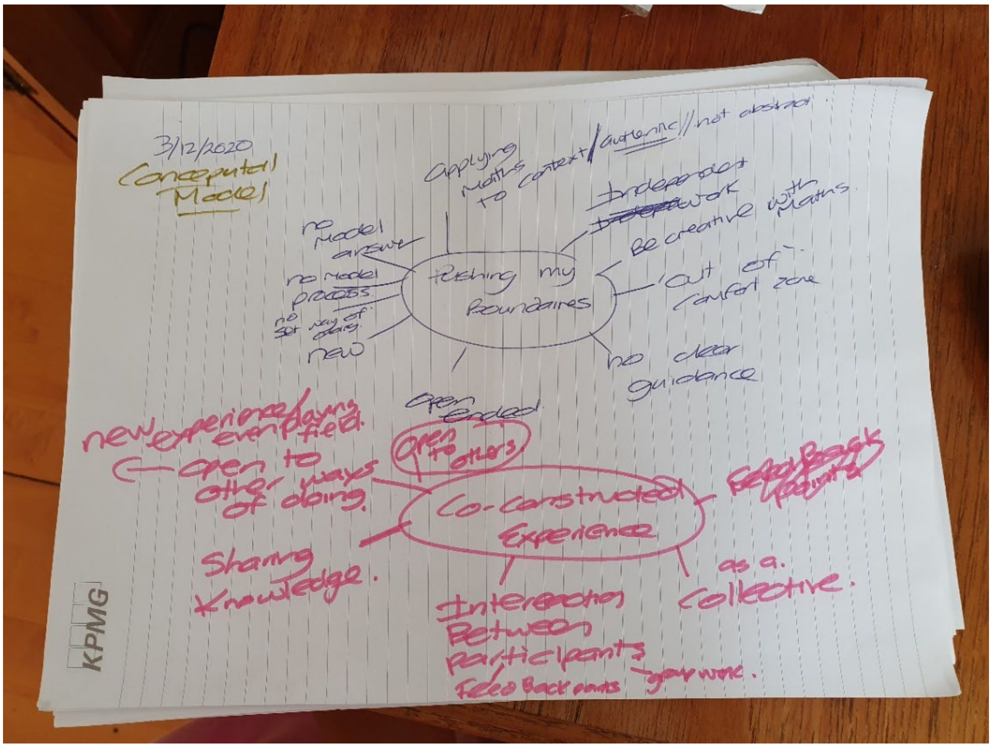
Drew said he learnt by doing in-course practice questions and learnt off others during the project day. Found that working in a group pushed each other's boundaries, something that doesn't happen as easily when working individually.

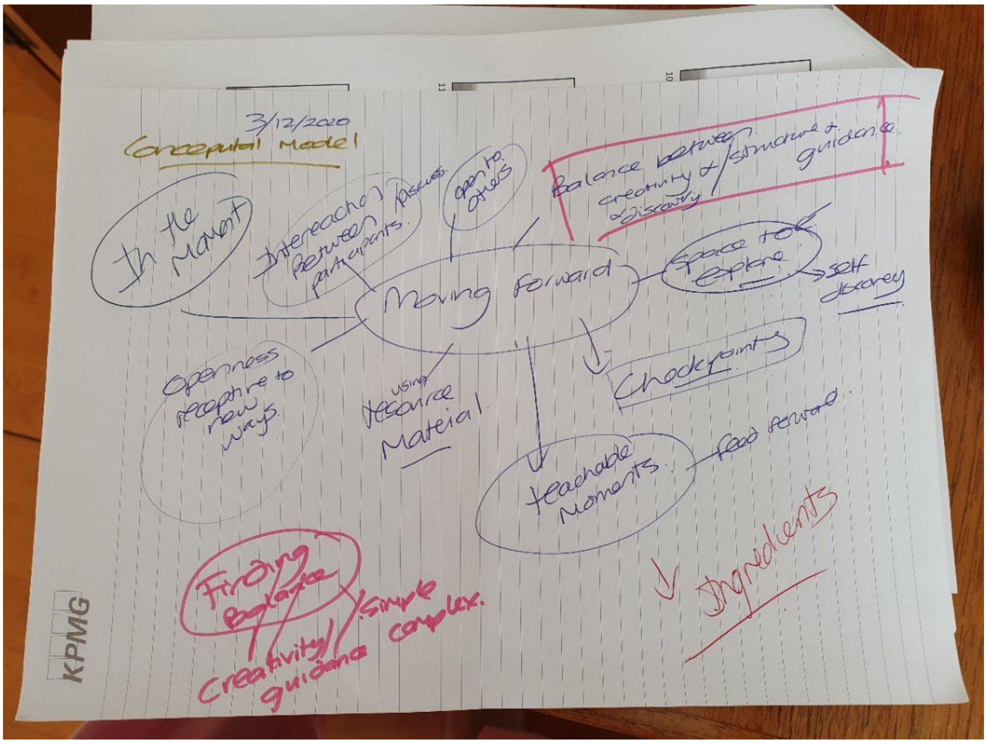
Felt he learnt modelling through in-course practice questions, studying for tests and doing project day.

Used diagrams to explain steps to solve the problem and made thinking clear while drawing diagram.

I had a restricted time for this interview. It felt that Drew's responses were surface and needed extra time to really get him to express his experience.

Appendix F. Mind maps to explore clusters and relationships





Appendix G. Initial candidate themes

Theme working title	Grappling with Uncertainty	Solid foundations	Taking responsibility	Finding ways forward
Working definition (Central organising concept)	Being unsure, not having a fixed direction, including not knowing what to do. Grappling with uncertainty refers to being unsure where is this going to end up, unsure how to proceed, unsure of direction taken.	Having something to lean into to support the experience. Solid foundations refers to support, ideas, or resourcing that students mentioned they utilised to help the group through the experience.	Students taking responsibility for the experience by making decisions, owning the process.	Things that move the experience out of 'being unstuck" and gives it direction, even if only temporarily.
Potential Codes	Wanting guidance Wanting reliable information Unclear how to start No model answer Questioning myself Lost with the maths Unsure where went wrong Unfamiliar territory Change direction of model Needing more information Not a straightforward process	External guidance Collaboration Student resourcefulness Lecturer support	Making decisions Freedom Student ownership of problem Actively involved.	Turning points Exploration Freedom Being flexible Change of direction of model Brainstorming Strategies Techniques for modelling Having to think

Theme working title	More than a maths course	Out of my comfort zone	Juggling act
Working definition (Central organising concept)	Students expressing a different experience than their normal mathematics course.	Students expressing their boundaries and capabilities being stretched.	Balancing different tensions.
Potential Codes	Using what learnt No right or wrong answer Freedom Exploration Take responsibility	Lots of new things Unfamiliar territory Change of culture Wanting guidance Unique experience	Balance Doing a jigsaw Not a straightforward process Simplicity vs accuracy

Appendix H. Course one marking scheme

Research

Group has consulted a good range of reliable resources (e.g. not just Wikipedia!)	/2
Group has found out the key relevant information	/2

Comments:

General Format

Report is well structured with clear headings and easy to follow layout	/2
Report is easily readable, using correct grammar and spelling	/2
Report makes good use of images/graphs/figures	/2

Comments:

Summary (not more than 1 page)

Clearly explains the problem in brief plus the approach taken and the solution obtained	/2

Comments:

Introduction (including identifying the problem)

Outlines the background to the problem	/2
Identifies the problem including	
• Clarifying	/2
• Considering Factors	/2
• Stating assumptions	/2
• Creating a precise problem statement	/2

Comments:

Main section (Modelling)

Details the mathematical modelling process undertaken, including	
• Formulating a model	/10
• Solving the model	/10
• Interpreting the solution	/2
• Comparing with reality	/2

Note that you may wish to go through more than one cycle of the modelling process

Comments:

Conclusion
Summary of the main findings of the report, including solution /2

Comments:

References
Materials used in the text are cited using a consistent style in and also listed in references /2

Note that failure to cite material you have used in your reports constitutes plagiarism and may result in a mark of zero for the entire project. You MUST cite material used. The above mark is for whether you have used an appropriate citation style and listed your cited material in the references section.

Comments:

Total /50

NOTE: you will also get a further 14 marks for participating in the group project tutorial session, and 6 marks for competing your personal log (four entries made during the courses of the project day), giving a possible 70 marks that can be earned.