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Mean-variance investment and reinsurance optimization with stochastic interest rate and volatility

LIHUA BIAN†, YANG SHEN ‡*, WENJUN ZHANG § and BIN ZOU ¶

†School of Mathematics and Statistics, Xuzhou University of Technology, Xuzhou 221018, People's Republic of China
‡School of Risk and Actuarial Studies, University of New South Wales, Sydney, NSW 2052, Australia
§Department of Mathematical Sciences, Auckland University of Technology, Auckland, New Zealand
¶Department of Mathematics, University of Connecticut, Storrs, USA

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This paper studies a continuous-time investment and reinsurance problem for an insurer. The financial market consists of one money market asset, one stock, and two zero-coupon bonds. Interest rates follow a generalized Cox-Ingersoll-Ross model, and the stock price is given by the Heston stochastic volatility model, with the stochastic interest rate and volatility processes being correlated. The insurer purchases proportional reinsurance to mitigate its risk and invests the surplus in the financial market, with the purpose to optimize mean-variance preferences in the precommitment sense. By the theory of backward stochastic differential equations and linear-quadratic control, the insurer's efficient strategy and efficient frontier are both obtained in closed form. An empirical analysis using market data validates the proposed financial model, and a numerical study offers valuable insights into the impact of model inputs on the insurer's efficient frontier.

Keywords: Mean-variance; Investment and reinsurance; Correlated stochastic interest rate and volatility; Backward stochastic differential equations

1. Introduction

Insurers play a critical role in managing risk and providing financial security for individuals and businesses, allowing them to transfer potential losses in exchange for a premium. While the foundation of insurance is the 'Law of Large Numbers', which involves pooling risks from a large group of insureds, insurers take further actions to manage the assumed risks and to enhance their profit stability. The standard practices include purchasing reinsurance to cede part of their assumed risks to reinsurers and investing received premiums in financial markets. To explore the joint implementation of these standard practices, this paper studies an optimal investment and reinsurance problem for a representative insurer equipped with mean-variance (MV) preferences.

We begin with a brief background on reinsurance in the real world. Reinsurance is a form of secondary-level insurance provided to insurers by reinsurers, which is highly customizable and can differ in coverage types, contract forms, and terms (see Albrecher *et al.* 2017, Chapters 1 and 2). The global reinsurance market has demonstrated strong growth in recent years, and the dedicated capital totaled USD $766 billion in the first half of 2024, an increase of 5.4% from the previous year (see Gallagher Re 2024). Next, moving to the investment side, Wong (2024) notes that 'the U.S. insurance industry reported $8.5 trillion in total cash and invested assets at year-end 2023, an increase of 4.4% versus year-end 2022', and among all asset classes, the top two invested classes are bonds and common stocks. According to the National Association of Insurance Commissioners (Wong 2024), as of year-end 2023, U.S. insurers held more than 60% of their total invested assets in bonds. Given those astronomical numbers in reinsurance and investment by insurers, and the fact that insurers are foundational to economic and social stability, an in-depth study into the operations of insurers on their reinsurance and investment strategies is well justified.

This paper responds to the above call for research by formulating and solving an optimal investment and reinsurance

*Corresponding author. Email: y.shen@unsw.edu.au, skyshen87@gmail.com

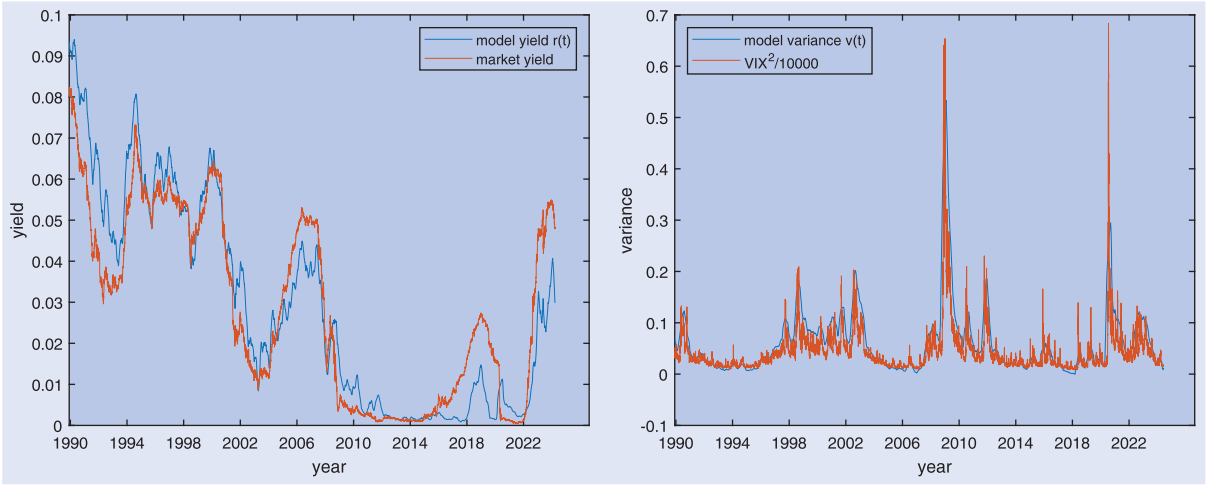

Figure 1. Comparison between the model estimated and market values of $r(t)$ (left) and $v(t)$ (right).

problem in a continuous-time framework. Inspired by the empirical statistics on the top asset classes invested by insurers in the U.S. (see Wong 2024), we consider a stylized financial market consisting of three risky assets, including two bonds and one common stock, along with one risk-free money market asset (used for short-term borrowing and leading). Our model features both stochastic interest rate and stochastic (stock) volatility, and further allows the two processes to be correlated. We explain the motivation for such a financial model as follows. On the one hand, interest rates have exhibited cyclical patterns historically, and movements have been particularly significant in recent years. As an example, the federal funds rate in the U.S. dropped to 0.05% in April 2020 but bounced to 5.33% in July 2024; see the left panel of figure 1 for a dynamic movement of the interest rates in the U.S. from 1990 to 2023. Consequently, stochastic interest rate models are self-justified, and two earlier developments are the Vasicek model (Vasicek 1977) and the Cox-Ingersoll-Ross (CIR) model (Cox *et al.* 1985).† On the other hand, despite the profound impact on both academia and industry, the celebrated Black-Scholes model assumes a *constant* volatility for the stock price, which is in sharp contrast to empirical observations (see the right panel of figure 1 for the volatility movement of S&P 500 from 1990 to 2023). This generates a strong interest of developing stochastic volatility (SV) models that offer better fitting to market data. One highly regarded SV model is the Heston model (Heston 1993), which essentially adopts a CIR-type process for the stock volatility.‡ Furthermore, interest rate $r(\cdot)$ and stock volatility $\sqrt{v(\cdot)}$ are not only stochastic but also correlated (see, e.g. Trolle and Schwartz 2009, for empirical evidence). In Section 5.1, we also test this hypothesis by considering models with and without the correlation between $r(\cdot)$ and $\sqrt{v(\cdot)}$, and investigate how market data fit under each model. The empirical results in table 1 offer compelling support for the (positive) correlation between $r(\cdot)$ and $\sqrt{v(\cdot)}$. To our awareness, the first continuous-time financial model with correlated stochastic interest rate and stochastic volatility is proposed by Scott (1997) to study option pricing. The financial model considered in this paper draws inspiration from Scott (1997) and extends it (excluding jumps) to allow for more freedom on several key model parameters.§ On the insurance risk side, we model the insurer's aggregate claims by the so-called diffusion approximation model, which has enjoyed great success in the study of continuous-time optimal reinsurance

Table 1. Model estimation and goodness-of-fit.

Model setup	Full model with correlation	Partial model without correlation
Parameter	Estimated value (Standard error)	Estimated value (Standard error)
α	0.20 (0.05)	0.00 (–)
β	0.32 (0.06)	0.36 (0.08)
γ	0.54 (0.10)	0.57 (0.14)
ρ	− 0.34 (0.16)	− 0.39 (0.19)
κ_1	1.18 (0.62)	1.21 (0.54)
κ_2	0.66 (0.35)	0.70 (0.32)
$\overline{m}_1$	0.23 (0.08)	0.25 (0.10)
$\overline{m}_2$	0.14 (0.06)	0.17 (0.07)
σ_1	0.18 (0.08)	0.20 (0.08)
σ_2	0.14 (0.10)	0.18 (0.08)
b_0	0.05 (0.02)	0.05 (0.02)
b_1	− 0.03 (0.01)	− 0.03 (0.01)
b_2	0.05 (0.03)	0.05 (0.02)
Measure	Goodness-of-fit	Goodness-of-fit
AIC	28 647	29 120
BIC	29 433	30 143
RMSE	0.07	0.11
MAPE	3.15%	4.25%

† The Vasicek model is widely considered to be the first stochastic interest rate model, and it uses a mean-reversion Ornstein–Uhlenbeck process to describe interest rates. But interest rates may take negative values under the Vasicek model. The CIR model overcomes this drawback by using a mean-reversion square-root process. We refer the reader to Brigo and Mercurio (2001)[Part II] for a comprehensive overview on popular stochastic interest rate models.

‡ The book edited by Shephard (2005) contains several important papers on SV models, and the Introduction chapter provides an excellent overview of the development of SV models and their applications in finance.

§ Scott (1997) sets $r(t) = m_1(t) + m_2(t)$, i.e. $\alpha = \beta = 1$ in (5) under our notation, but we allow general $\alpha \geq 0$ and $\beta > 0$. The calibration results in table 1 show that $\hat{\alpha} = 0.20 \neq \hat{\beta} = 0.32$, supporting more flexible relation between α and β.

problems.¶ The insurer purchases proportional (quota-share) reinsurance to mitigate its risk exposure, and reinsurance contracts are priced by the expected-value premium principle (see Schmidli 2001, Shen and Zeng 2015, for the same setup).

With the combined financial and insurance model described above, we proceed to introduce the main investment and reinsurance problem faced by the insurer of MV preferences. We consider a continuous-time model over a finite time horizon, and the insurer's decision includes an investment strategy $\boldsymbol{\pi}(\cdot)$ and a reinsurance strategy $q(\cdot)$, in which $\boldsymbol{\pi}(t) \in \mathbb{R}^3$ denote the dollar amounts invested in the stock and two bonds, respectively, and $q(t) \geq 0$ denotes the risk proportion retained by the insurer, at time t. The insurer seeks an 'optimal' strategy $\boldsymbol{u}^*(\cdot) := (\boldsymbol{\pi}^*(\cdot)^\top, q^*(\cdot))^\top$ to maximize its MV preferences of the terminal wealth. However, the concept of 'optimality' under MV preferences requires caution and clarification, since dynamic MV optimization problems are time-inconsistent, due to the failure of Bellman optimality principle (see Bjork and Murgoci 2010). To address the time-inconsistency in decision-making, Strotz (1955) proposes *two* possible solutions—'the strategy of precommitment' and 'the strategy of consistent planning.' While time-consistent strategies ensure optimal decision-making at every point in time, they tend to be overly conservative (Bjork and Murgoci 2010), as they are designed to avoid future regret rather than globally maximizing objective functions. Empirical studies indicate that institutional investors, including insurers, dynamically adjust their bond and equity allocations based on their wealth levels (Greenwood and Vayanos 2014), a behavior that is not well captured by time-consistent models. In contrast, the precommitment approach allows insurers to commit to an optimal strategy at the outset, leading to superior investment performance while effectively managing risk at the same time. Recognizing these advantages, this paper adopts the precommitment framework to derive an insurer's optimal investment and reinsurance strategy in a market with correlated stochastic interest rate and volatility processes. We refer the reader to Li *et al.* (2012) for the time-consistent solutions to MV investment and reinsurance optimization problems.

In this paper, the insurer solves its MV investment and reinsurance problem at time 0 and precommits to this solution $\boldsymbol{u}^*(\cdot)$, called the *efficient strategy* or *optimal precommitment strategy*, over the entire planning period $[0, T]$. By Lagrangian duality, we solve the insurer's original MV problem, formulated in (P1), in two steps. In the first step, we study a family of quadratic loss minimization (QLM) problems, parameterized by the Lagrangian multiplier $\lambda \in \mathbb{R}$ (see (P2–u)); denote the solution of the QLM problem under a given λ by $\boldsymbol{u}^*_\lambda(\cdot)$. In the second step, we solve for the optimal Lagrangian multiplier λ^* (see (P2–λ)), which immediately yields the efficient strategy $\boldsymbol{u}^*(\cdot) = \boldsymbol{u}^*_\lambda(\cdot)|_{\lambda=\lambda^*}$ (upon verifying its admissibility). By employing the theory of backward stochastic differential equations (BSDEs) and linear-quadratic (LQ) control, we obtain the insurer's efficient (investment and reinsurance) strategy $\boldsymbol{u}^*(\cdot)$ in closed form (see (33) and (34)). In addition, we derive the corresponding efficient frontier explicitly and show that it is in a 'complete square' form (see (37)). The main results rely on the analysis of three BSDEs: two of them are the so-called (backward) stochastic Riccati equations (SREs), given by (18) and (19), and the third one is a linear BSDE in (20). These BSDEs help us implement the 'completing the square' technique, which in turn leads to a candidate solution to the QLM problem in (45) and (46). Note that the drivers of the two SREs in (18) and (19) are *not* globally Lipschitz continuous and involve *random* parameters that are *unbounded*, which significantly complicates the solvability analysis.

Schmidli (2002) is likely the first paper that studies a joint optimal investment and (proportional) reinsurance problem for an insurer.† In his work, the insurer invests its surplus in a risky asset whose price follows the Black-Scholes model and purchases proportional reinsurance to mitigate its risk exposure given by the Cramér-Lundberg (CL) model; the goal is to seek an optimal investment and reinsurance strategy that minimizes the insurer's ruin probability. Since then, there has been an explosive development on this topic, exploring different asset price and risk models, incorporating additional constraints or controls, and optimizing under different criteria.‡ Since both the interest rate $r(\cdot)$ and stock volatility $\sqrt{v(\cdot)}$ are stochastic in our model, and they impact both the analysis and results, we limit our attention to those papers with stochastic $r(\cdot)$ or $\sqrt{v(\cdot)}$ in their models. In what follows, we review several key references which consider MV investment-reinsurance optimization problems with stochastic $r(\cdot)$ or $\sqrt{v(\cdot)}$, and discuss the main differences of the current paper with those papers.

Li *et al.* (2012) consider a market with one bank account (money market asset) earning a constant interest rate and one risky asset whose price follows the Heston SV model, and the goal is to obtain the time-consistent equilibrium strategies under MV preferences in the sense of Bjork and Murgoci (2010). Wang *et al.* (2019) assume that both interest rate and stock volatility are stochastic and apply the BSDE approach under the time-consistent framework; but they require one of the two processes to be deterministic when solving related BSDEs to derive equilibrium MV strategies in closed form. Yan and Wong (2020) look for open-loop equilibrium strategies when stock volatility is modeled by a general diffusion process, but interest rates are deterministic. Note that Li *et al.* (2012), Wang *et al.* (2019), and Yan and Wong (2020) all seek time-consistent equilibrium strategies, which often lead to an investment strategy that is *independent* of wealth.§ However, this independence result contradicts

¶ See, e.g. Browne (1995), Højgaard and Taksar (1998), Schmidli (2001), Liang and Yuen (2016), Jin *et al.* (2024), and Yao and Zhu (2024).

† Browne (1995) is among the earliest to study optimal investment, *without* reinsurance, for an insurer. On the other hand, optimal dynamic proportional reinsurance, *without* investment, for an insurer is first solved by Højgaard and Taksar (1998); see Ceci and Colaneri (2025), Jin *et al.* (2024), and Ma *et al.* (2023) for recent contributions.

‡ See, e.g. Promislow and Young (2005), Bai and Guo (2008), Zeng and Li (2011), Shen and Zou (2021), and Wang *et al.* (2024). Note that all these paper assume constant interest rates and constant stock volatility.

§ One possible exception is when the insurer's risk aversion is state-dependent, and the corresponding equilibrium investment strategy

with empirical evidence that wealthier institutional investors allocate more funds to fixed income securities than their less wealthy counterparts (see Greenwood and Vayanos 2014); also, recall that insurers in the U.S. hold more than 60% of their investment portfolio in bonds (see Wong 2024). This empirical finding offers support to the precommitment solution framework adopted by this paper, which is also used in Shen and Zeng (2015) and Wang *et al.* (2023). Regarding the financial model, Shen and Zeng (2015) apply an affine square-root process to model the market price of risk, which includes the CEV model and Heston SV model as special cases, but interest rates are assumed to be deterministic and uniformly bounded. Under a general 4/2 SV model but still with a deterministic interest rate, Wang *et al.* (2023) derive semi-closed-form efficient strategy and efficient frontier via the PDE approach. Similar to this paper, the insurer in Shen and Zeng (2015) and Wang *et al.* (2023) seeks an efficient strategy under the precommitment framework. In summary, the aforementioned works focus on either stochastic volatility models or simplified uncorrelated (but unrealistic) models.

We summarize the main contributions of this paper as follows. First, by using measure change techniques and modifying the definition of admissible strategies, we extend the BSDE approach that works mainly for the case with bounded model parameters (Lim and Zhou 2002) to the unbounded case. This extension is indispensable because the stochastic interest rate and stochastic volatility models are unbounded in this paper. Second, we are the first to derive a closed-form solution for the efficient strategy and efficient frontier to the precommitment MV investment and reinsurance optimization problems under correlated stochastic interest rate and stock volatility models. To the best of our knowledge, even among the broad literature on general portfolio optimization problems *without* reinsurance (see, e.g. Noh and Kim 2011), our paper is still the first to obtain the closed-form optimal investment strategy when stochastic interest rate and volatility models are correlated. Third, we develop a tailor-made procedure to calibrate model parameters to real stock and bond prices data, leading to more trustful economic insights from subsequent numerical analysis on the efficient frontier. Whereas, most existing works (see the aforementioned literature and references therein) on investment and reinsurance optimization do not calibrate their models to market data, limiting the practical relevance of their numerical analysis. Fourth, based on the calibrated model, we obtain several new economic insights in the presence of correlated stochastic interest rate and stock volatility. (1) Positive correlation between stochastic interest rate and stock volatility processes enhances investment performance. This is because when interest rates and stock volatility move in the same direction, returns are more predictable, which in turn reduces investment risk and yields higher expected terminal wealth. (2) The mean-reversion speed of different stochastic factors may have opposite impact on the efficient frontier. More specifically, a higher mean-reversion speed of the stochastic interest rate improves investment stability, shifting the efficient frontier upward. In contrast, increasing the mean-reversion speed of the stock volatility reduces potential returns, leading to a worsened efficient frontier. (3) Higher volatility in interest rate and stock volatility models improves the efficient frontier. Increasing the volatility of the stochastic interest rate and stock volatility results in a higher expected return but also an elevated risk. However, the effect of increased expected return dominates that of higher risk, leading to an improved efficient frontier.

The rest of the paper is organized as follows. Section 2 formulates the insurer's MV investment and reinsurance problem. Section 3 presents several preliminary results and related BSDEs. Section 4 solves the problem and obtains the efficient strategy (optimal precommitment strategy) and the efficient frontier. Section 5 first calibrates our model to real market data and then conducts a numerical study focusing on the impact of various model parameters on the efficient frontier. Section 6 concludes the study. Technical proofs are placed in Appendix 1, while a summary of the calibration procedure is presented in Appendix 2.

depends on wealth. However, for this case, no closed-form solution is available even under the Heston SV model (see Yan and Wong 2020) or geometric Brownian motion (GBM) stock price model.

2. Problem formulation

2.1. Notations

We fix a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} := \{\mathcal{F}_t\}_{t\in[0,\overline{T}]}, \mathbb{P})$ over a finite time horizon $[0, \overline{T}]$, in which $\mathbb{P}$ is the real-world probability measure. Here, $\overline{T} \in (0, \infty)$ is chosen to be sufficiently large so that all processes can be defined within the time horizon $[0, \overline{T}]$. In particular, we will consider three different terminal times: the terminal time of decision-making T and the maturity times of two zero-coupon bonds T_1 and T_2 which satisfy $0 < T \leq T_1 < T_2 \leq \overline{T} < \infty$. The filtration $\mathbb{F}$ is the natural filtration generated by four mutually independent, standard, one-dimensional Brownian motions $W_i(\cdot) := \{W_i(t)\}_{t\in[0,\overline{T}]}$, $i = 0, 1, 2$, and $Z(\cdot) := \{Z(t)\}_{t\in[0,\overline{T}]}$; $W_i(\cdot)$'s and $Z(\cdot)$ capture the randomness in the financial and insurance markets, respectively. Let $\boldsymbol{W}(t) := (W_0(t), W_1(t), W_2(t))^\top$, with $\top$ denoting the transpose operator, and write $\boldsymbol{W}(\cdot) := \{\boldsymbol{W}(t)\}_{t\in[0,\overline{T}]}$. In the sequel, $\mathbb{E}[\cdot]$ and $\mathrm{Var}[\cdot]$ denote expectation and variance taken under $\mathbb{P}$, respectively, and $\mathbb{E}_t[\cdot] := \mathbb{E}[\cdot \,|\, \mathcal{F}_t]$ is the conditional expectation given the information contained by $\mathcal{F}_t$. All (in)equalities involving random subjects are understood in the $\mathbb{P}$ almost surely (a.s.) sense.

For every real number $x \in \mathbb{R}$, denote $x^+ := \max\{x, 0\}$ and $x^- := \max\{-x, 0\}$. Moreover, we use E to denote a generic Euclidean space, equipped with the standard Euclidean norm $|\cdot|$. Let $p \geq 1$ and $q \geq 1$ be two generic real numbers, we define the following spaces that will be used throughout this paper:

- $\mathcal{L}^p_{\mathcal{F}_T,\mathbb{P}}(\Omega; E)$ is the space of all E-valued, $\mathcal{F}_T$-measurable random variables ξ satisfying $\|\xi\|_{\mathcal{L}^p_{\mathcal{F}_T,\mathbb{P}}(\Omega;E)} := \{\mathbb{E}[|\xi|^p]\}^{\frac{1}{p}} < \infty$;
- $\mathcal{L}^p_{\mathbb{F},\mathbb{P}}(0, T; E)$ is the space of all E-valued, $\mathbb{F}$-adapted processes $f(\cdot)$ satisfying $\|f(\cdot)\|_{\mathcal{L}^p_{\mathbb{F},\mathbb{P}}(0,T;E)} := \{\mathbb{E}[(\int_0^T |f(t)|^2 \,\mathrm{d}t)^{\frac{p}{2}}]\}^{\frac{1}{p}} < \infty$;

- $\mathcal{S}^p_{\mathbb{F},\mathbb{P}}(0,T;E)$ is the space of all E-valued, $\mathbb{F}$-adapted, càdlàg processes $f(\cdot)$ satisfying $\|f(\cdot)\|_{\mathcal{S}^p_{\mathbb{F},\mathbb{P}}(0,T;E)} := \{\mathbb{E}[\sup_{t\in[0,T]}|f(t)|^p]\}^{\frac{1}{p}} < \infty$;
- $\mathcal{L}^{2,loc}_{\mathbb{F},\mathbb{P}}(0,T;E)$ is the space of all E-valued, $\mathbb{F}$-adapted processes $f(\cdot)$ satisfying $\int_0^T |f(t)|^2\,\mathrm{d}t < \infty$.

Let $\mathbb{Q}$ be a probability measure equivalent to $\mathbb{P}$. All the above spaces can be defined similarly with $\mathbb{P}$ replaced by $\mathbb{Q}$, e.g. $\mathcal{L}^p_{\mathcal{F}_T,\mathbb{Q}}(\Omega;E)$.

2.2. Model

Financial market. We consider an 'ideal' financial market that consists of four tradable assets.

The first asset is a money market asset that can be used for instantaneous borrowing and lending at the same rate. Its price process $S_0(\cdot) := \{S_0(t)\}_{t\in[0,\overline{T}]}$ follows the dynamics

$$\mathrm{d}S_0(t) = r(t)S_0(t)\,\mathrm{d}t, \quad S_0(0) = 1, \tag{1}$$

in which $r(t)$ is the instantaneous interest rate at time t. The interest rate process $r(\cdot) := \{r(t)\}_{t\in[0,\overline{T}]}$ is *stochastic* and will be specified shortly by (3)–(5).

The second asset is a risky asset, which we call '*stock*' hereafter. The stock price process $S(\cdot) := \{S(t)\}_{t\in[0,\overline{T}]}$ is governed by the following stochastic differential equation (SDE):

$$\begin{aligned}\mathrm{d}S(t) &= S(t)\left(r(t) + \theta_0(t)\sqrt{v(t)}\right)\mathrm{d}t\\ &\quad + S(t)\sqrt{v(t)}\,\mathrm{d}W_0(t), \quad S(0) > 0,\end{aligned} \tag{2}$$

in which $\sqrt{v(t)}$ is the instantaneous volatility, and $r(t) + \theta_0(t)\sqrt{v(t)}$ is the return rate of the stock at time t. In (2), the variance process $v(\cdot) := \{v(t)\}_{t\in[0,\overline{T}]}$ is also *stochastic*, as specified later by (3)–(5), while $\theta_0(\cdot) := \{\theta_0(t)\}_{t\in[0,\overline{T}]}$ denotes the stock's market price of risk (corresponding to $W_0(\cdot)$) which is also stochastic and will be specified by (6).

The third and fourth assets are two zero-coupon bonds, each paying one unit fiat currency (e.g. U.S. dollar) at the respective maturity time T_i, for $i = 1, 2$. Recall that $T \le T_1 < T_2$ is assumed. Let $P(\cdot,T_i) := \{P(t,T_i)\}_{t\in[0,T_i]}$ denote the bond price process, with $P(T_i,T_i) = 1$, for each $i = 1, 2$. In Appendix A.1, we provide closed-form expressions and dynamics for the bond price processes (refer to Lemma 5).

To model the stochastic interest rate process $r(\cdot)$ in (1) and the stock variance process $v(\cdot)$ in (2), we introduce two mean-reverting square-root processes $m_i(\cdot) := \{m_i(t)\}_{t\in[0,\overline{T}]}$, $i = 1, 2$, by

$$\begin{aligned}\mathrm{d}m_1(t) &= \kappa_1(\overline{m}_1 - m_1(t))\,\mathrm{d}t\\ &\quad + \sigma_1\sqrt{m_1(t)}\left[\rho\,\mathrm{d}W_0(t) + \sqrt{1-\rho^2}\,\mathrm{d}W_1(t)\right],\\ &\quad m_1(0) > 0,\end{aligned} \tag{3}$$

$$\begin{aligned}\mathrm{d}m_2(t) &= \kappa_2(\overline{m}_2 - m_2(t))\,\mathrm{d}t + \sigma_2\sqrt{m_2(t)}\,\mathrm{d}W_2(t),\\ &\quad m_2(0) > 0,\end{aligned} \tag{4}$$

in which $\kappa_i > 0$ is the mean-reverting speed, $\overline{m}_i > 0$ denotes the long-term level, and $\sigma_i > 0$ is the volatility coefficient, of the process $m_i(\cdot)$ for each $i = 1, 2$. In addition, $\rho \in [-1, 1]$ captures the correlation between the stock price $S(\cdot)$ and $m_1(\cdot)$, or equivalently the stock volatility. Note that $m_1(\cdot)$ and $m_2(\cdot)$ are mutually independent, as $W_i(\cdot)$, $i = 0, 1, 2$, are three independent Brownian motions introduced in Subsection 2.1. We model the interest rate process $r(\cdot)$ and the stock variance process $v(\cdot)$ by

$$\begin{aligned}r(t) = \alpha m_1(t) + \beta m_2(t) \quad &\text{and} \quad v(t) = \gamma^2 m_1(t),\\ &\text{for all } t \in [0,\overline{T}],\end{aligned} \tag{5}$$

in which $\alpha \ge 0$, $\beta > 0$, and $\gamma > 0$ are constants.

Remark 1 The stock price $S(\cdot)$ specified by (2) and (5) follows the celebrated Heston stochastic volatility model (see Heston 1993). The stochastic interest rate process $r(\cdot)$ and the stock variance process $v(\cdot)$ given by (3)–(5) generalize those introduced in Scott (1997) because α and β there are restricted to $\alpha = \beta = 1$, but we allow $\alpha \ge 0$ and $\beta > 0$ in (5). Furthermore, our model for $r(\cdot)$ reduces to the Cox-Ingersoll-Ross (CIR) model (see Cox *et al.* 1985) when $\alpha = 0$; in this special case, $r(\cdot)$ is independent of the stock variance process $v(\cdot)$. However, when $\alpha > 0$, the three processes $S(\cdot)$, $r(\cdot)$, and $v(\cdot)$ are correlated with each other. To see how these three processes are correlated, using (2)–(5), we obtain the results regarding the quadratic covariations of these processes by

$$\begin{aligned}\langle S(t), v(t)\rangle &= \frac{\rho\sigma_1}{\gamma}\int_0^t S(s)v(s)\,\mathrm{d}s,\\ \langle v(t), r(t)\rangle &= \alpha\sigma_1^2\int_0^t v(s)\,\mathrm{d}s,\\ \langle S(t), r(t)\rangle &= \frac{\alpha\rho\sigma_1}{\gamma}\int_0^t S(s)v(s)\,\mathrm{d}s.\end{aligned}$$

We impose the following assumption on the financial market throughout the paper.

Assumption 2 We assume that (1) $2\kappa_i\overline{m}_i > \sigma_i^2$ for $i = 1, 2$ in (3)–(4); (2) the market price of risk $\boldsymbol{\theta}(\cdot) := (\theta_0(\cdot), \theta_1(\cdot), \theta_2(\cdot))^\top$, corresponding to $(W_0(\cdot), W_1(\cdot), W_2(\cdot))^\top$, is given by

$$\begin{aligned}\boldsymbol{\theta}(t) &= (\theta_0(t), \theta_1(t), \theta_2(t))^\top\\ &= \left(b_0\sqrt{m_1(t)}, b_1\sqrt{m_1(t)}, b_2\sqrt{m_2(t)}\right)^\top, \quad t \in [0,\overline{T}],\end{aligned} \tag{6}$$

in which b_i's are constants; (3) there exists a sufficiently large $p \ge \frac{1}{2}$ such that $\mathbb{E}[e^{p\int_0^{\overline{T}}|\boldsymbol{\theta}(s)|^2\mathrm{d}s}] < \infty$.

Remark 3 Condition (1) in Assumption 2 is the so-called Feller condition, and it ensures that the mean-reverting processes $m_1(\cdot)$ and $m_2(\cdot)$ in (3)–(4) are strictly positive almost surely for all $t \in [0,\overline{T}]$. Condition (2) on the market price of risk is common in the literature when square-root factor models are considered; see, e.g. Shen and Zeng (2015) and Wang *et al.* (2019). Condition (3) is imposed so that the change of measure techniques are applicable to the aforementioned financial model which has *unbounded* parameters.

Refer to Shen and Zeng (2015) and Yan and Wong (2020) for similar conditions.

For the given market price of risk $\boldsymbol{\theta}(\cdot)$ in (6), the corresponding risk-neutral probability measure $\widetilde{\mathbb{P}}$ is given by

$$\left.\frac{\mathrm{d}\widetilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t} = \exp\left\{-\frac{1}{2}\int_0^t |\boldsymbol{\theta}(s)|^2\,\mathrm{d}s - \int_0^t \boldsymbol{\theta}(s)^\top \mathrm{d}\boldsymbol{W}(s)\right\}. \quad (7)$$

The process $\widetilde{\boldsymbol{W}}(\cdot) := \{\widetilde{\boldsymbol{W}}(t)\}_{t\in[0,\overline{T}]}$, defined by

$$\widetilde{\boldsymbol{W}}(t) := \boldsymbol{W}(t) + \int_0^t \boldsymbol{\theta}(s)\,\mathrm{d}s, \quad t \in [0,\overline{T}], \quad (8)$$

is a standard three-dimensional Brownian motion under $\widetilde{\mathbb{P}}$. In what follows, we also use the one-dimensional notation $\widetilde{W}_i(\cdot) := \{\widetilde{W}_i(t)\}_{t\in[0,\overline{T}]}$, $i = 0, 1, 2$; note that $\widetilde{\boldsymbol{W}}(\cdot) = (\widetilde{W}_0(\cdot), \widetilde{W}_1(\cdot), \widetilde{W}_2(\cdot))^\top$.

Using (8), we rewrite the dynamics of $m_1(\cdot)$ in (3) and $m_2(\cdot)$ in (4) under $\widetilde{\mathbb{P}}$ by

$$\begin{aligned}\mathrm{d}m_1(t) &= \widetilde{\kappa}_1(\widetilde{m}_1 - m_1(t))\,\mathrm{d}t + \sigma_1\sqrt{m_1(t)}\left[\rho\,\mathrm{d}\widetilde{W}_0(t)\right.\\ &\quad\left. + \sqrt{1-\rho^2}\,\mathrm{d}\widetilde{W}_1(t)\right], \quad m_1(0) > 0, \quad (9)\\ \mathrm{d}m_2(t) &= \widetilde{\kappa}_2(\widetilde{m}_2 - m_2(t))\,\mathrm{d}t + \sigma_2\sqrt{m_2(t)}\,\mathrm{d}\widetilde{W}_2(t),\\ &\quad m_2(0) > 0, \quad (10)\end{aligned}$$

in which the new model parameters are defined by

$$\widetilde{\kappa}_1 := \kappa_1 + b_0\sigma_1\rho + b_1\sigma_1\sqrt{1-\rho^2}, \quad \widetilde{\kappa}_2 := \kappa_2 + b_2\sigma_2,$$
$$\widetilde{m}_1 := \frac{\kappa_1\overline{m}_1}{\widetilde{\kappa}_1}, \quad \widetilde{m}_2 := \frac{\kappa_2\overline{m}_2}{\widetilde{\kappa}_2}.$$

Since $\widetilde{\mathbb{P}}$ is equivalent to $\mathbb{P}$, the two factor processes $m_1(\cdot)$ and $m_2(\cdot)$ remain strictly positive under $\widetilde{\mathbb{P}}$. Alternatively, such a result can be derived by noticing that the Feller condition, $2\widetilde{\kappa}_i\widetilde{m}_i = 2\kappa_i\overline{m}_i > \sigma_i^2$, for $i = 1, 2$, holds for the dynamics (9)–(10) under $\widetilde{\mathbb{P}}$.

With the risk-neutral probability measure $\widetilde{\mathbb{P}}$ identified by (7), we apply the standard risk-neutral pricing approach to obtain the price processes $P(\cdot, T_i)$ of the two zero-coupon bonds by (see Lemma 5)

$$\begin{aligned}P(t,T_i) &= \exp\{-N_1(t,T_i)\,m_1(t) - N_2(t,T_i)\,m_2(t)\\ &\quad - N_3(t,T_i)\}, \quad t \in [0,T_i],\ i = 1,2, \quad (11)\end{aligned}$$

in which $N_1(t,T_i)$, $N_2(t,T_i)$, $N_3(t,T_i)$ are defined by (A2), (A3), and (A4), respectively, with s replaced by T_i.

Remark 4 As seen from the above analysis, there exists a unique risk-neutral probability measure $\widetilde{\mathbb{P}}$, given by (7). Recall that the randomness of the financial market is driven by the 3-dimensional Brownian motion $\boldsymbol{W}$. As such, to ensure the uniqueness of $\widetilde{\mathbb{P}}$, we assume that, in addition to the risky stock, there are *two* zero-coupon bonds available for trading.

Insurance market. We next introduce an insurance risk model and consider a representative insurer who sells insurance policies to customers and resorts to reinsurance for risk management. Following Browne (1995), we apply a diffusion process $\mathcal{C}(\cdot) := \{\mathcal{C}(t)\}_{t\in[0,\overline{T}]}$ to model the insurer's aggregate claims by

$$\mathrm{d}\mathcal{C}(t) = a_Z\,\mathrm{d}t - \sigma_Z\,\mathrm{d}Z(t),$$

in which $a_Z > 0$ and $\sigma_Z > 0$ denote the mean and volatility of the aggregate claims, respectively. One drawback of the diffusion risk model is that the aggregate claims $\mathcal{C}(t)$ may become negative. To alleviate this issue, we impose a technical condition that $a_Z/\sigma_Z \gg 3$, under which the probability of negative aggregate claims is very small (assuming $\mathcal{C}(0) > 0$). As introduced in Subsection 2.1, $Z(\cdot) = \{Z(t)\}_{t\in[0,\overline{T}]}$ is a standard one-dimensional Brownian motion, independent of the three-dimensional Brownian motion $\boldsymbol{W}(\cdot) = (W_0(\cdot), W_1(\cdot), W_2(\cdot))^\top$ that drives the financial market. Suppose that the insurer applies the expected-value principle to calculate premiums (see Schmidli 2001, Shen and Zeng 2015); as such, the premium rate c is given by $c = (1+\eta)a_Z$, in which $\eta > 0$ is the insurer's relative safety loading.

Suppose that in the insurance market, *proportional* reinsurance policies are available to the insurer for managing the risk exposure to the aggregate claims (see Højgaard and Taksar 1998, Schmidli 2001, Zhang *et al.* 2024). Let $q(\cdot) = \{q(t)\}_{t\in[0,T]}$ denote a proportional reinsurance strategy,† in which $q(t)$ is the proportion retained by the insurer at time t. We assume that the reinsurance premium is also calculated by the expected-value principle, but with a higher loading η_r, i.e. $\eta_r \geq \eta > 0$. Given a reinsurance strategy $q(\cdot)$, the insurer's wealth process $\mathcal{R}^q(\cdot) := \{\mathcal{R}^q(t)\}_{t\in[0,T]}$, *without* investment, follows an arithmetic Brownian motion:

$$\begin{aligned}\mathrm{d}\mathcal{R}^q(t) &= \underbrace{c\,\mathrm{d}t - \mathrm{d}\mathcal{C}(t)}_{\text{original surplus}} + \underbrace{(1-q(t))\,\mathrm{d}\mathcal{C}(t)}_{\text{reinsurance coverage}}\\ &\quad - \underbrace{(1-q(t))\,(1+\eta_r)\,\mathbb{E}[\mathrm{d}\mathcal{C}(t)]}_{\text{reinsurance premium}}\\ &= (\eta_r a_Z\,q(t) - (\eta_r - \eta)a_Z)\,\mathrm{d}t + \sigma_Z q(t)\,\mathrm{d}Z(t). \quad (12)\end{aligned}$$

Remark 5 Given that Z and $\boldsymbol{W}$ are independent, our model assumes that the insurance market (the insurer's claims process $\mathcal{C}$) is independent of the financial market. This assumption is reasonable because claims from most traditional insurance lines, such as auto, property and casualty (P&C), health, and life insurance, are not impacted by the financial market. This explains why actuarial ratemaking models often do not use financial data as covariates; see, for instance, Part IV in Frees (2009). We further note that such an independence assumption is frequently imposed in the study of optimal investment and (re)insurance problems; see Schmidli (2002), Bai and Guo (2008), and Li *et al.* (2012).

2.3. *The insurer's problem*

We assume that the insurer invests its surplus in the financial market, without frictions and taxes, and purchases proportional reinsurance in the insurance market. Let $\pi_S(t)$, $\pi_{P_1}(t)$,

† Note in the rest of the paper, we concentrate on the decision-making time horizon $[0,T]$ for all stochastic processes although some of those processes (e.g. $P(\cdot,T_i)$) need to be defined over a longer horizon in the previous subsections.

and $\pi_{P_2}(t)$ denote, respectively, the *amounts* invested in the stock and the two zero-coupon bonds at time t (with the remaining invested in the money market asset by the self-financing rule); denoting $\boldsymbol{\pi}(t) := (\pi_S(t), \pi_{P_1}(t), \pi_{P_2}(t))^\top$, we call $\boldsymbol{\pi}(\cdot) := \{\boldsymbol{\pi}(t)\}_{t\in[0,T]}$ an investment strategy. Following the notation in Subsection 2.2, we call $q(\cdot) := \{q(t)\}_{t\in[0,T]}$ a reinsurance strategy. When $0 \le q(t) \le 1$, the insurer retains $100q(t)\%$ of its claims and cedes the remaining proportion to the reinsurer; when $q(t) > 1$, the insurer acts as a reinsurer and assumes additional risks, equivalent to $100(q(t) - 1)\%$ of its own claims, from other insurers in the market. Together, we call $\boldsymbol{u}(\cdot) := (\pi(\cdot)^\top, q(\cdot))^\top$ the insurer's investment-reinsurance strategy or simply a strategy.

Given a strategy $\boldsymbol{u}(\cdot)$, let $X(\cdot) := \{X(t)\}_{t\in[0,T]}$ denote the insurer's wealth process. For the market model introduced in Subsection 2.2, we obtain the dynamics of $X(\cdot)$ by†

$$\begin{aligned} \mathrm{d}X(t) = &\left[r(t)X(t) + \boldsymbol{\pi}(t)^\top \boldsymbol{b}(t) + \eta_r a_Z q(t) - (\eta_r - \eta)a_Z\right]\mathrm{d}t \\ &+ \boldsymbol{\pi}(t)^\top \boldsymbol{\sigma}(t)\,\mathrm{d}\boldsymbol{W}(t) + \sigma_Z q(t)\,\mathrm{d}Z(t), \end{aligned} \tag{13}$$

in which $\boldsymbol{b}(\cdot) := \{\boldsymbol{b}(t)\}_{t\in[0,T]}$ is the risk premium process, with

$$\boldsymbol{b}(t) := \begin{pmatrix} b_0\gamma m_1(t) \\ -\left(b_0\rho + b_1\sqrt{1-\rho^2}\right)\sigma_1 N_1(t,T_1)m_1(t) - b_2\sigma_2 N_2(t,T_1)m_2(t) \\ -\left(b_0\rho + b_1\sqrt{1-\rho^2}\right)\sigma_1 N_1(t,T_2)m_1(t) - b_2\sigma_2 N_2(t,T_2)m_2(t) \end{pmatrix}, \tag{14}$$

and $\boldsymbol{\sigma}(\cdot) := \{\boldsymbol{\sigma}(t)\}_{t\in[0,T]}$ is the volatility process of the three risky assets, with

$$\boldsymbol{\sigma}(t) := \begin{pmatrix} \gamma\sqrt{m_1(t)} & 0 & 0 \\ -\rho\sigma_1 N_1(t,T_1)\sqrt{m_1(t)} & -\sqrt{1-\rho^2}\sigma_1 N_1(t,T_1)\sqrt{m_1(t)} & -\sigma_2 N_2(t,T_1)\sqrt{m_2(t)} \\ -\rho\sigma_1 N_1(t,T_2)\sqrt{m_1(t)} & -\sqrt{1-\rho^2}\sigma_1 N_1(t,T_2)\sqrt{m_1(t)} & -\sigma_2 N_2(t,T_2)\sqrt{m_2(t)} \end{pmatrix}. \tag{15}$$

By the definition of $\boldsymbol{\theta}(\cdot)$ in (6), $\boldsymbol{b}(t) = \boldsymbol{\sigma}(t)\boldsymbol{\theta}(t)$ for all $t \in [0,T]$. The functions $N_1(t,T_i)$ and $N_2(t,T_i)$ in (14) and (15) are from the bond prices $P(t,T_i)$ in (11), for $i = 1, 2$, and formally defined by (A2) and (A3), respectively.

Definition 1 *A strategy $\boldsymbol{u}(\cdot) = (\boldsymbol{\pi}(\cdot)^\top, q(\cdot))^\top$ is said to be admissible if (1) $\boldsymbol{u}(\cdot)$ is $\mathbb{F}$-adapted; (2) $\boldsymbol{\pi}(\cdot) = (\pi_S(\cdot), \pi_{P_1}(\cdot), \pi_{P_2}(\cdot))^\top$ is $\mathbb{R}^3$-valued, and $q(\cdot)$ is non-negative-valued; (3) $\boldsymbol{\sigma}(\cdot)^\top\boldsymbol{\pi}(\cdot) \in \mathcal{L}^{2,loc}_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R}^3)$ and $q(\cdot) \in \mathcal{L}^{2,loc}_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R})$; (4) the process $h_+(\cdot)(X(\cdot) - H(\cdot))^2$ is uniformly integrable, in which $h_+(\cdot)$ and $H(\cdot)$ are defined by (18) and (20), respectively. The set of all admissible strategies is denoted by $\mathcal{A}$.*

Suppose that the insurer is endowed with MV preferences and, in recognition of time inconsistency, chooses to *pre-commit* to its 'best' strategy at the initial time. This leads to the following MV investment and reinsurance optimization problem.

† Note that the insurer's wealth process $X(\cdot)$ obviously depends on its strategy $\boldsymbol{u}(\cdot)$, and if such a dependence relation needs to be emphasized, we use the notation $X^{\boldsymbol{u}}(\cdot)$.

Problem 1 *The insurer's MV investment and reinsurance optimization problem is a constrained stochastic control problem, parameterized by $d \in \mathbb{R}$, defined as follows:*

$$\begin{cases} \min_{\boldsymbol{u}(\cdot)\in\mathcal{A}} J(x_0, r_0, v_0; \boldsymbol{u}(\cdot)) \\ \quad := \min_{\boldsymbol{u}(\cdot)\in\mathcal{A}} \mathbb{E}\left[(X(T) - d)^2\right], \\ \text{subject to } \mathbb{E}[X(T)] = d,\ X(0) = x_0,\ r(0) = r_0, \\ \quad \text{and } v(0) = v_0. \end{cases} \tag{P1}$$

A solution $\boldsymbol{u}^*(\cdot)$ to Problem (P1) *is called an efficient strategy. Denote the insurer's wealth process associated with strategy $\boldsymbol{u}^*(\cdot)$ by $X^*(\cdot)$.*

In the above formulation, the insurer seeks an efficient strategy to Problem (P1), which is a precommitment solution (see Zhou and Li 2000). We refer to Shen and Zou (2021), Yuan *et al.* (2023), Zhang *et al.* (2024), and the references therein for detailed discussions on both the precommitment and time-consistent (equilibrium) strategies to MV investment-reinsurance optimization problems.

3. Preliminary results and backward stochastic differential equations

3.1. Subproblems

In this section, we obtain several preliminary results for Problem (P1) and break it into two subproblems. Note that there is an expectation constraint $\mathbb{E}[X(T)] = d$ in Problem (P1), and it is not immediately clear that the problem itself is feasible for any given $d \in \mathbb{R}$. The next lemma offers a positive answer.

Lemma 1 *Problem* (P1) *is always feasible in the sense that, for every $d \in \mathbb{R}$, there exists at least one admissible strategy $\boldsymbol{u}(\cdot) \in \mathcal{A}$ such that $\mathbb{E}[X(T)] = d$.*

Proof See Appendix A.2. ■

Remark 6 Although Problem (P1) is feasible for every $d \in \mathbb{R}$, thanks to Lemma 1, not all values of d yield a non-trivial solution to Problem (P1). We define a threshold on the expectation by

$$d_{\min} := \frac{x_0}{\widetilde{\mathbb{E}}\left[e^{-\int_0^T r(s)\,\mathrm{d}s}\right]} - (\eta_r - \eta)a_Z \frac{\widetilde{\mathbb{E}}\left[\int_0^T e^{-\int_0^t r(s)\,\mathrm{d}s}\,\mathrm{d}t\right]}{\widetilde{\mathbb{E}}\left[e^{-\int_0^T r(s)\,\mathrm{d}s}\right]}. \tag{16}$$

By Lemma 1, there exists a strategy, denoted by $\boldsymbol{u}_{\min}(\cdot)$, such that $\mathbb{E}[X^{\boldsymbol{u}_{\min}}(T)] = d_{\min}$. It will become clear in Section 4 that the variance of $X^{\boldsymbol{u}_{\min}}(T)$ attains the minimum value of zero, explaining the subscript in $d_{\min}$ and $\boldsymbol{u}_{\min}(\cdot)$. Because of this result, we impose the standing assumption $d \ge d_{\min}$ when we solve Problem (P1). Indeed, for all $d < d_{\min}$, feasible strategies are *inferior* to $\boldsymbol{u}_{\min}(\cdot)$ because $\boldsymbol{u}_{\min}(\cdot)$ leads to a higher mean but a smaller variance; however, $\boldsymbol{u}_{\min}(\cdot)$ is *not* feasible to Problem (P1) for all $d < d_{\min}$. As such, solving Problem (P1) for $d < d_{\min}$ has no economic meaning.

As pointed out above, $\mathrm{Var}[X^{\boldsymbol{u}_{\min}}(T)] = 0$, and immediately, one may think that the associated strategy should be $\boldsymbol{u}_0(\cdot) \equiv$

$(0,0,0,0)^\top \in \mathcal{A}$. However, this is *not* correct, i.e. $\boldsymbol{u}_{\min}(\cdot) \neq \boldsymbol{u}_0(\cdot)$. To see this, note that under strategy $\boldsymbol{u}_0(\cdot)$, the insurer invests only in the money market account and cedes all of its insurance risk to the reinsurer, but because the interest rate $r(\cdot)$ is stochastic, $X^{\boldsymbol{u}_0}(T)$ is a non-trivial $\mathcal{F}_T$-measurable random variable, and its expectation d_0 can be obtained by

$$\begin{aligned} d_0 &= \mathbb{E}\left[X^{\boldsymbol{u}_0}(T)\right] \\ &= \mathbb{E}\left[x_0\, e^{\int_0^T r(s)\,\mathrm{d}s} - (\eta_r - \eta)a_Z \int_0^T e^{\int_t^T r(s)\,\mathrm{d}s}\,\mathrm{d}t\right]. \end{aligned} \tag{17}$$

By comparing (16) with (17), we note that $d_0 \neq d_{\min}$ in general, and the two are equal when $r(\cdot)$ is deterministic, under which $X^{\boldsymbol{u}_0}(\cdot)$ becomes deterministic as well.

A standard approach of solving constrained optimization problems is the classical Lagrangian approach, and we follow the one in Shen and Zou (2021) to solve Problem (P1) and outline the key steps as follows. To start, note an elementary result, $\mathbb{E}[X^2(T)] - d^2 - 2\lambda(\mathbb{E}[X(T)] - d) = \mathbb{E}[(X(T) - \lambda)^2] - (\lambda - d)^2$, in which $\lambda \in \mathbb{R}$ is a free parameter, called the Lagrangian multiplier. By the Lagrangian duality theorem, Problem (P1) is equivalent to the following min-max problem:

$$\begin{aligned} &\max_{\lambda\in\mathbb{R}} \min_{\boldsymbol{u}(\cdot)\in\mathcal{A}} J_1(x_0, r_0, v_0; \boldsymbol{u}(\cdot), \lambda) \\ &\quad := \max_{\lambda\in\mathbb{R}} \min_{\boldsymbol{u}(\cdot)\in\mathcal{A}} \left\{\mathbb{E}\left[(X(T) - \lambda)^2\right] - (\lambda - d)^2\right\}. \end{aligned} \tag{P2}$$

Careful readers may notice that the constraints on the initial values ($X(0) = x_0$, $r(0) = r_0$, and $v(0) = v_0$ in Problem (P1)) are *not* explicitly stated in the above formulation, which is done for notational simplicity and, in the meantime, does not cause confusion because the arguments in J_1 clearly indicate the initial values. Such a practice is also followed below.

We break Problem (P2) into two subproblems and solve them sequentially. Inspired by the inner problem of Problem (P2), we first study the following quadratic loss minimization (QLM) problem:

$$\min_{\boldsymbol{u}(\cdot)\in\mathcal{A}} J_2(x_0, r_0, v_0; \boldsymbol{u}(\cdot)) := \min_{\boldsymbol{u}(\cdot)\in\mathcal{A}} \mathbb{E}\left[(X(T) - \lambda)^2\right], \tag{P2–u}$$

which is parameterized by λ. Let $\boldsymbol{u}^*_\lambda(\cdot)$ denote an optimal solution to Problem (P2–u), in which the subscript λ is the same parameter from the problem. The second subproblem is the outer problem of Problem (P2), and we formulate it as

$$\begin{aligned} &\max_{\lambda\in\mathbb{R}} J_1(x_0, r_0, v_0; \boldsymbol{u}^*_\lambda(\cdot), \lambda) \\ &\quad := \max_{\lambda\in\mathbb{R}} \left\{\mathbb{E}\left[\left(X^{\boldsymbol{u}^*_\lambda}(T) - \lambda\right)^2\right] - (\lambda - d)^2\right\}. \end{aligned} \tag{P2–λ}$$

Let λ^* denote an optimal solution to Problem (P2–λ); then, $\boldsymbol{u}^*(\cdot) = \boldsymbol{u}^*_{\lambda^*}(\cdot)$ is an *efficient strategy*, to Problem (P1). Furthermore, by varying d over $[d_{\min}, \infty)$, we obtain the *efficient frontier* by the coordinates $(\mathrm{Var}(X^*(T)), d)$.

3.2. Related BSDEs

The model introduced in Subsection 2.2 involves *random* parameters, and consequently, the risk premium $\boldsymbol{b}(\cdot)$ in (14) and the volatility $\boldsymbol{\sigma}(\cdot)$ in (15) are stochastic processes. As such, our main problem, Problem (P1) or Problem (P2), cannot be solved by the standard HJB equation approach. Instead, we apply the theory of BSDEs and LQ control to find the efficient strategy $\boldsymbol{u}^*(\cdot)$ (see Lim and Zhou 2002, Shen and Zeng 2015, Sun *et al.* 2020). Below, we introduce several related BSDEs that will play a key role in finding the efficient strategy in the next section.

We first introduce two nonlinear BSDEs:

$$\begin{aligned} \mathrm{d}h_+(t) = &\left[\left(-2r(t) + |\boldsymbol{\theta}(t)|^2\right)h_+(t) + 2\boldsymbol{\theta}(t)^\top \boldsymbol{g}_+(t)\right. \\ &\left. + \frac{|\boldsymbol{g}_+(t)|^2}{h_+(t)}\right]\mathrm{d}t + \boldsymbol{g}_+(t)^\top \mathrm{d}\boldsymbol{W}(t), \end{aligned} \tag{18}$$

$$\begin{aligned} \mathrm{d}h_-(t) = &\left[\left(-2r(t) + |\boldsymbol{\theta}(t)|^2 + \frac{\eta_r^2 a_Z^2}{\sigma_Z^2}\right)h_-(t)\right. \\ &\left. + 2\boldsymbol{\theta}(t)^\top \boldsymbol{g}_-(t) + \frac{|\boldsymbol{g}_-(t)|^2}{h_-(t)}\right]\mathrm{d}t + \boldsymbol{g}_-(t)^\top \mathrm{d}\boldsymbol{W}(t), \end{aligned} \tag{19}$$

with terminal conditions $h_+(T) = h_-(T) = 1$. Both (18) and (19) belong to the family of backward stochastic Riccati equations (SREs). The third BSDE we consider is a linear BSDE, given by

$$\begin{aligned} \mathrm{d}H(t) = &\left[r(t)H(t) + \boldsymbol{\theta}(t)^\top \boldsymbol{G}(t) - (\eta_r - \eta)a_Z\right]\mathrm{d}t \\ &+ \boldsymbol{G}(t)^\top \mathrm{d}\boldsymbol{W}(t), \quad H(T) = \lambda. \end{aligned} \tag{20}$$

In the above BSDEs, for all $t \in [0, T]$, $r(t)$ is the interest rate defined by (5), and $\boldsymbol{\theta}(t)$ is the market price of risk given by (6); η, η_r, a_Z, and σ_Z are positive constants in the insurance model (see (12)), and $\lambda \in \mathbb{R}$ in (20) is the same Lagrangian multiplier from Problem (P2–u).

The next two lemmas provide existence and uniqueness results of the BSDEs in (18)–(20).

Lemma 2 *The BSDEs in* (18) *and* (19) *admit unique solutions* $(h_+(\cdot), \boldsymbol{g}_+(\cdot))$ *and* $(h_-(\cdot), \boldsymbol{g}_-(\cdot))$, *respectively, and they are given by*

$$h_+(t) = \exp\left\{A(t)m_1(t) + B(t)m_2(t) + C(t)\right\}, \tag{21}$$

$$\begin{aligned} \boldsymbol{g}_+(t) = h_+(t)\Big(&\rho\sigma_1\sqrt{m_1(t)}A(t),\ \sigma_1\sqrt{1-\rho^2}\sqrt{m_1(t)}A(t), \\ &\sigma_2\sqrt{m_2(t)}B(t)\Big)^\top, \end{aligned} \tag{22}$$

$$\begin{aligned} h_-(t) = \exp\Big\{&A(t)m_1(t) + B(t)m_2(t) \\ &+ C(t) - \frac{\eta_r^2 a_Z^2}{\sigma_Z^2}(T-t)\Big\}, \end{aligned} \tag{23}$$

$$\begin{aligned} \boldsymbol{g}_-(t) = h_-(t)\Big(&\rho\sigma_1\sqrt{m_1(t)}A(t),\ \sigma_1\sqrt{1-\rho^2}\sqrt{m_1(t)}A(t), \\ &\sigma_2\sqrt{m_2(t)}B(t)\Big)^\top, \end{aligned} \tag{24}$$

in which the functions A, B, and C are defined by (A10), (A11), *and* (A12), *respectively.*

Proof See Appendix A.3. ■

To study the solvability of the linear BSDE in (20), we introduce *synthetic* zero-coupon bonds that pay one unit of currency at time $s > 0$; denote their price process by $P(\cdot, s) := \{P(t,s)\}_{t\in[0,s]}$. By definition, we have

$$P(t,s) = \widetilde{\mathbb{E}}_t\left[e^{-\int_t^s r(u)\mathrm{d}u}\right] \quad \text{and} \quad P(s,s) = 1, \tag{25}$$

in which $\widetilde{\mathbb{E}}$ is taken under the risk-neutral probability measure $\widetilde{\mathbb{P}}$ defined by (7). Lemma 5 obtains a closed-form expression for the price $P(t,s)$ (see (A1)). Using the notation $P(t,s)$ in (25), we present the solution to (20) in the next lemma.

LEMMA 3 *The BSDE in* (20) *admits a unique solution* $(H(\cdot), \boldsymbol{G}(\cdot))$*, with* $G(\cdot) := (G_0(\cdot), G_1(\cdot), G_2(\cdot))^\top$*, and it is given by*

$$H(t) = \lambda P(t,T) + (\eta_r - \eta)a_Z \int_t^T P(t,s)\,\mathrm{d}s, \tag{26}$$

$$G_0(t) = -\sigma_1\rho\sqrt{m_1(t)}\left[\lambda N_1(t,T)P(t,T) + (\eta_r - \eta)a_Z\int_t^T N_1(t,s)P(t,s)\,\mathrm{d}s\right], \tag{27}$$

$$G_1(t) = -\sigma_1\sqrt{1-\rho^2}\sqrt{m_1(t)}\left[\lambda N_1(t,T)P(t,T) + (\eta_r - \eta)a_Z\int_t^T N_1(t,s)P(t,s)\,\mathrm{d}s\right], \tag{28}$$

$$G_2(t) = -\sigma_2\sqrt{m_2(t)}\left[\lambda N_2(t,T)P(t,T) + (\eta_r - \eta)a_Z\int_t^T N_2(t,s)P(t,s)\,\mathrm{d}s\right], \tag{29}$$

in which $P(t,s)$ *is defined by* (25) *for all* $s > 0$ *and all* $t \in [0,s]$ *and explicitly solved in* (A1)*, and* $N_1(t,s)$ *and* $N_2(t,s)$ *are defined by* (A2) *and* (A3)*, respectively.*

Proof See Appendix A.4. ■

REMARK 7 When model coefficients are bounded, Lim and Zhou (2002) and Lim (2004) show that the SREs corresponding to their MV portfolio optimization problems have unique solutions in desired spaces. In our paper, the unbounded stochastic interest rate and variance processes, $r(\cdot)$ and $v(\cdot)$, make it more challenging to establish similar existence and uniqueness results of the two SREs (18) and (19). We overcome this challenge by using measure change techniques. While the current paper involves two SREs, other works which also rely on the BSDE approach to tackle MV optimization problems (see, e.g. Shen and Zeng 2015, Sun *et al.* 2020) need only one. The two SREs (18) and (19) are required due to the application of the Itô-Tanaka formula (refer to the proof of Theorem 4). Such complexity is caused by the more intricate structure in the correlated stochastic factors $r(\cdot)$ and $v(\cdot)$ and the non-negativity of the reinsurance strategy in our paper.

With $P(t,s)$ from (25), define two constants, $\overline{\Gamma}$ and $\widehat{\Gamma}$, by

$$\overline{\Gamma} := \widetilde{\mathbb{E}}\left[e^{-\int_0^T r(t)\,\mathrm{d}t}\right] = P(0,T) \quad \text{and}$$
$$\widehat{\Gamma} := \widetilde{\mathbb{E}}\left[\int_0^T e^{-\int_0^s r(t)\,\mathrm{d}t}\,\mathrm{d}s\right] = \int_0^T P(0,s)\,\mathrm{d}s. \tag{30}$$

Then, we obtain the initial value of $H(\cdot)$ in (26) by

$$H(0) = \lambda\overline{\Gamma} + (\eta_r - \eta)a_Z\widehat{\Gamma}, \tag{31}$$

and rewrite $d_{\min}$ in (16) by

$$d_{\min} = \frac{1}{\overline{\Gamma}}\left(x_0 - (\eta_r - \eta)a_Z\widehat{\Gamma}\right). \tag{32}$$

4. Main results

In this section, we present the main results of this paper in Theorem 4, a closed-form solution to the insurer's MV investment and reinsurance optimization problem (P1). The proof of Theorem 4 relies on the completing the square technique from LQ theory.

THEOREM 4 *The efficient strategy* $\boldsymbol{u}^*(\cdot) = (\boldsymbol{\pi}^*(\cdot)^\top, q^*(\cdot))^\top$ *to the insurer's MV investment and reinsurance problem* (P1) *is given by*

$$\boldsymbol{\pi}^*(t) = -\boldsymbol{\Sigma}(t)^{-1}\boldsymbol{\sigma}(t)\boldsymbol{\Theta}(t)\left(X^*(t) - H^*(t)\right) + \boldsymbol{\Sigma}(t)^{-1}\boldsymbol{\sigma}(t)\boldsymbol{G}^*(t), \tag{33}$$

$$q^*(t) = -\frac{\eta_r a_Z}{\sigma_Z^2}\left(X^*(t) - H^*(t)\right), \tag{34}$$

in which η_r*,* a_Z*, and* σ_Z *are positive constants from the insurance model (see* (12)*),* $\boldsymbol{\Sigma}(t) := \boldsymbol{\sigma}(t)\boldsymbol{\sigma}(t)^\top$ *is the covariance matrix, and the adjusted market price of risk* $\boldsymbol{\Theta}(\cdot)$ *is defined by*

$$\boldsymbol{\Theta}(t) := \boldsymbol{\theta}(t) + \frac{\boldsymbol{g}_+(t)}{h_+(t)} = \boldsymbol{\theta}(t) + \frac{\boldsymbol{g}_-(t)}{h_-(t)} = \Big(\left(b_0 + \rho\sigma_1 A(t)\right)\sqrt{m_1(t)}, \left(b_1 + \sigma_1\sqrt{1-\rho^2}A(t)\right)\sqrt{m_1(t)}, \left(b_2 + \sigma_2 B(t)\right)\sqrt{m_2(t)}\Big)^\top. \tag{35}$$

In (33) *and* (34)*,* $X^*(\cdot)$ *is the insurer's wealth process satisfying (13) under the efficient strategy* $\boldsymbol{u}^*(\cdot)$*,* $(H^*(\cdot), \boldsymbol{G}^*(\cdot))$ *is the unique solution to the linear BSDE* (20)*, as given by* (26)–(29)*, with the free parameter* λ *replaced by the optimal Lagrangian multiplier* λ^* *that is obtained by*

$$\lambda^* = \frac{h_-(0)\overline{\Gamma}\left[x_0 - (\eta_r - \eta)a_Z\widehat{\Gamma}\right] - d}{h_-(0)\overline{\Gamma}^2 - 1}, \tag{36}$$

in which $h_-(0)$ *is from* (23) *by setting* t *to 0, and* $\overline{\Gamma}$ *and* $\widehat{\Gamma}$ *are defined in* (30)*.*

Furthermore, the efficient frontier of Problem (P1) *is given by*

$$\mathrm{Var}\left(X^*(T)\right) = \frac{h_-(0)\overline{\Gamma}^2}{1-h_-(0)\overline{\Gamma}^2}\,(d-d_{\min})^2\,, \qquad (37)$$

in which $d_{\min}$ *is defined in* (16)*, and* $d = \mathbb{E}[X^*(T)] \geq d_{\min}$.

Several remarks on the above theorem are due as follows. First, we immediately observe that the efficient frontier obtained by (37) is in a 'complete square' form, which resembles the result in MV investment problems (see Lim and Zhou 2002, Eq.(50)) or MV investment and reinsurance problems (see Shen and Zeng 2015, Eq.(4.49)). However, $d_{\min}$ in (16) is different from d_0 in (17), the expected return of the zero strategy u_0; recall that u_0 only invests in the money market asset and cedes all risk to the reinsurer. If $r(\cdot)$ is deterministic, we have $d_{\min} = d_0$, as in Shen and Zeng (2015) (see Eq.(4.14) therein); additionally, if reinsurance control is ignored, we further obtain $d_{\min} = d_0 = x_0\, e^{\int_0^T r(t)\,\mathrm{d}t}$, as in Zhou and Li (2000). Second, because our model features unbounded stochastic interest rate and stochastic volatility (with possible correlation between them), and an additional risk process from the insurance market, Problem (P1) in this paper is considerably challenging than the classical MV portfolio optimization problem studied in Zhou and Li (2000) and Lim and Zhou (2002) and more recent MV works in an insurance context, such as Zhang and Chen (2016) and Sun *et al.* (2020); see Remark 7 for a detailed discussion on related BSDEs. Note that due to the presence of correlated stochastic interest rate and volatility, and insurance risk, the efficient investment strategy $\boldsymbol{\pi}^*(\cdot)$ includes an additional hedging term, $\boldsymbol{\Sigma}(t)^{-1}\boldsymbol{\sigma}(t)\boldsymbol{G}^*(t)$ in (33). This hedging term involves trading the risky stock and two bonds, and it serves the purpose of hedging correlated stochastic interest rate and volatility risks. This differs from the literature which utilizes less risky assets to combat one type of stochastic factor risk (see, e.g. Shen and Zeng 2015, Zhang and Chen 2016, Sun *et al.* 2020). In addition, the 'distorted' risk-adjusted discounting factor in (37) takes a more complicated form given by $h_-(0)\overline{\Gamma}^2$, with $h_-(\cdot)$ solving the BSDE (19) and $\overline{\Gamma} = \widetilde{\mathbb{E}}[e^{-\int_0^T r(t)\mathrm{d}t}]$, i.e. the expectation of the discount factor under the risk-neutral measure; in comparison, such a factor in Shen and Zeng (2015) reads as $h_-(0)e^{-2\int_0^T r(t)\mathrm{d}t}$ in the notation of the current paper. Recall from the definition of admissible strategies (see Definition 1), an admissible reinsurance strategy $q(\cdot)$ must be non-negative. Therefore, given the efficient reinsurance strategy $q^*(\cdot)$ in (34), we show that $X^*(\cdot) \leq H^*(\cdot)$, i.e. the associated efficient (optimal) wealth process $X^*(\cdot)$ is bounded above by $H^*(\cdot)$, in which $H^*(\cdot)$ solves the linear BSDE (20) with parameter λ^* and is explicitly obtained in (26).

Proof of Theorem 4. We prove the desired results in Theorem 4 in three steps.

Step 1. Solve the QLM problem in (P2–u).

We fix a Lagrangian multiplier $\lambda \in \mathbb{R}$ and often suppress the dependence of processes on λ in this step if there is no risk of confusion. For every admissible strategy $\boldsymbol{u}(\cdot) \in \mathcal{A}$, we define a new process $Y(\cdot) = \{Y(t)\}_{t\in[0,T]}$ by

$$Y(t) := X(t) - H(t), \quad t \in [0,T],$$

in which $X(\cdot)$ is the insurer's wealth process satisfying (13) under $\boldsymbol{u}(\cdot)$, and $H(\cdot)$ satisfies the linear BSDE in (20) and is obtained by (26). Using (13) and (20), we obtain the dynamics of $Y(\cdot)$ by

$$\begin{aligned}\mathrm{d}Y(t) = &\big[r(t)Y(t) + \boldsymbol{\pi}(t)^\top\boldsymbol{\sigma}(t)\boldsymbol{\theta}(t)\\ &+ \eta_r a_Z q(t) - \boldsymbol{\theta}(t)^\top\boldsymbol{G}(t)\big]\,\mathrm{d}t\\ &+ \left(\boldsymbol{\pi}(t)^\top\boldsymbol{\sigma}(t) - \boldsymbol{G}(t)^\top\right)\mathrm{d}\boldsymbol{W}(t) + \sigma_Z q(t)\,\mathrm{d}Z(t),\end{aligned}$$

in which $Y(0) = x_0 - H(0)$. Note that $x_0 = X(0) \in \mathbb{R}$ is given, and $H(0)$ is obtained by (31). Using $Y(T)$, we can rewrite the objective of Problem (P2–u) as $\min_{\boldsymbol{u}(\cdot)\in\mathcal{A}} \mathbb{E}[Y^2(T)]$.

Recall $Y^{\pm}(\cdot) = \max\{\pm Y(\cdot), 0\}$. By the Itô-Tanaka formula (see Equation (6.11), p.205 in Karatzas and Shreve 1991), we get

$$\begin{aligned}\mathrm{d}Y^+(t) &= \mathbb{1}_{\{Y(t)>0\}}\,\mathrm{d}Y(t) + \frac{1}{2}\,\mathrm{d}L(t) \quad \text{and}\\ \mathrm{d}Y^-(t) &= -\mathbb{1}_{\{Y(t)\leq 0\}}\,\mathrm{d}Y(t) + \frac{1}{2}\,\mathrm{d}L(t),\end{aligned}$$

in which $\mathbb{1}$ is the indicator function, and $L(t)$ denotes the local time spent by $Y(\cdot)$ at the level of 0 up to time t (see Definition 6.3, Chapter 3 in Karatzas and Shreve 1991). Applying Itô's formula to $(Y^+(\cdot))^2$ and noting $Y^+(t)\mathrm{d}L(t) = 0$, we obtain

$$\begin{aligned}\mathrm{d}\left(Y^+(t)\right)^2 = &\Big\{2Y^+(t)\big[r(t)Y(t) + \boldsymbol{\pi}(t)^\top\boldsymbol{\sigma}(t)\boldsymbol{\theta}(t)\\ &+ \eta_r a_Z q(t) - \boldsymbol{\theta}(t)^\top\boldsymbol{G}(t)\big]\\ &+\mathbb{1}_{\{Y(t)>0\}}\left[\left|\boldsymbol{\sigma}(t)^\top\boldsymbol{\pi}(t) - \boldsymbol{G}(t)\right|^2 + \sigma_Z^2 q^2(t)\right]\Big\}\,\mathrm{d}t\\ &+ 2Y^+(t)\left[\left(\boldsymbol{\pi}(t)^\top\boldsymbol{\sigma}(t) - \boldsymbol{G}(t)^\top\right)\mathrm{d}\boldsymbol{W}(t)\right.\\ &\left.+ \sigma_Z q(t)\,\mathrm{d}Z(t)\right], \qquad (38)\end{aligned}$$

in which we have used the elementary identity $Y^+(t)\,\mathbb{1}_{\{Y(t)>0\}} = Y^+(t)$. By using the above result and (18), and applying the stochastic product rule, we get

$$\begin{aligned}&\mathrm{d}\left[h_+(t)(Y^+(t))^2\right]\\ &= h_+(t)\,\mathbb{1}_{\{Y(t)>0\}}\,\mathfrak{f}_+(t,\boldsymbol{\pi}(t),q(t))\,\mathrm{d}t\\ &\quad + 2\sigma_Z h_+(t)Y^+(t)q(t)\,\mathrm{d}Z(t)\\ &\quad + \left[2h_+(t)Y^+(t)\,\left(\boldsymbol{\pi}(t)^\top\boldsymbol{\sigma}(t) - \boldsymbol{G}(t)^\top\right)\right.\\ &\quad \left.+ (Y^+(t))^2\boldsymbol{g}_+(t)^\top\right]\mathrm{d}\boldsymbol{W}(t), \qquad (39)\end{aligned}$$

in which $\mathfrak{f}_+(\cdot,\cdot,\cdot) : [0,T]\times\mathbb{R}^3\times\mathbb{R}_+ \mapsto \mathbb{R}_+$ is defined by

$$\begin{aligned}\mathfrak{f}_+(t,\boldsymbol{\pi},q) := &\left|\boldsymbol{\sigma}(t)^\top\boldsymbol{\pi} - \boldsymbol{G}(t) + Y^+(t)\left(\boldsymbol{\theta}(t) + \frac{\boldsymbol{g}_+(t)}{h_+(t)}\right)\right|^2\\ &+ \left[\sigma_Z^2 q^2 + 2\eta_r a_Z Y^+(t)q\right]. \qquad (40)\end{aligned}$$

Repeating the same derivation for $Y^-(\cdot)$ and $h_-(\cdot)$ yields

$$\begin{aligned}
&\mathrm{d}\left[h_-(t)(Y^-(t))^2\right] \\
&\quad = h_-(t)\,\mathbb{1}_{\{Y(t)\le 0\}}\,\mathfrak{f}_-(t,\boldsymbol{\pi}(t),q(t))\,\mathrm{d}t \\
&\qquad - 2\sigma_Z h_-(t)Y^-(t)q(t)\,\mathrm{d}Z(t) \\
&\qquad + \left[-2h_-(t)Y^-(t)\left(\boldsymbol{\pi}(t)^\top\boldsymbol{\sigma}(t) - \boldsymbol{G}(t)^\top\right)\right. \\
&\qquad \left. + (Y^-(t))^2\boldsymbol{g}_-(t)^\top\right]\mathrm{d}\boldsymbol{W}(t),
\end{aligned} \tag{41}$$

in which $\mathfrak{f}_-(\cdot,\cdot,\cdot):[0,T]\times\mathbb{R}^3\times\mathbb{R}_+\mapsto\mathbb{R}_+$ is defined by

$$\begin{aligned}
\mathfrak{f}_-(t,\boldsymbol{\pi},q) &:= \left|\boldsymbol{\sigma}(t)^\top\boldsymbol{\pi} - \boldsymbol{G}(t) - Y^-(t)\left(\boldsymbol{\theta}(t) + \frac{\boldsymbol{g}_-(t)}{h_-(t)}\right)\right|^2 \\
&\quad + \left(\sigma_Z\, q - \frac{\eta_r a_Z}{\sigma_Z}\,Y^-(t)\right)^2.
\end{aligned} \tag{42}$$

To use standard localization techniques, we define $\{\tau_n\}_{n=1,2,\ldots,}$ to be a localization sequence of stopping times such that all Itô integrals in (39) and (41) are $\mathbb{P}$-local martingales. Then, integrating the two equations (39) and (41) from 0 to $T\wedge\tau_n$ and summing them up gives

$$\begin{aligned}
&\mathbb{E}\left[h_+(T\wedge\tau_n)\left(Y^+(T\wedge\tau_n)\right)^2\right] \\
&\quad + \mathbb{E}\left[h_-(T\wedge\tau_n)\left(Y^-(T\wedge\tau_n)\right)^2\right] \\
&= \mathbb{E}\left[\int_0^{T\wedge\tau_n} h_+(t)\,\mathbb{1}_{\{Y(t)>0\}}\,\mathfrak{f}_+(t,\boldsymbol{\pi}(t),q(t))\,\mathrm{d}t\right] \\
&\quad + \mathbb{E}\left[\int_0^{T\wedge\tau_n} h_-(t)\,\mathbb{1}_{\{Y(t)\le 0\}}\,\mathfrak{f}_-(t,\boldsymbol{\pi}(t),q(t))\,\mathrm{d}t\right] \\
&\quad + h_+(0)\left(Y^+(0)\right)^2 + h_-(0)\left(Y^-(0)\right)^2,
\end{aligned} \tag{43}$$

in which we have used the integrability conditions in Definition 1 to ensure that all the Itô integrals stopped by $\{\tau_n\}_{n=1,2,\ldots,}$ in (39) and (41) are martingales and thus have zero expectation.

On the one hand, because

$$\begin{aligned}
&h_+(t)\left(Y^+(t)\right)^2 \le h_+(t)\,(Y(t))^2 \quad \text{and} \\
&h_-(t)\left(Y^-(t)\right)^2 \le h_+(t)\left(Y^-(t)\right)^2 \le h_+(t)\,(Y(t))^2,
\end{aligned}$$

the uniform integrability condition (4) in Definition 1 implies that for all $\boldsymbol{u}(\cdot)\in\mathcal{A}$, both $h_+(\cdot)(Y^+(\cdot))^2$ and $h_-(\cdot)(Y^-(\cdot))^2$ are uniformly integrable. On the other hand, from the definitions of $\mathfrak{f}_+$ and $\mathfrak{f}_-$ in (40) and (42) and the positivity of $h_+(\cdot)$ and $h_-(\cdot)$, we easily see that both integrands in (43) are nonnegative. Thereby, using the dominated convergence theorem and monotone convergence theorem to the left-hand side and right-hand side of (43), respectively, we obtain

$$\begin{aligned}
&\mathbb{E}\left[(X(T)-\lambda)^2\right] \\
&\quad = \mathbb{E}[Y^2(T)] = \mathbb{E}\left[h_+(T)\left(Y^+(T)\right)^2\right] \\
&\qquad + \mathbb{E}\left[h_-(T)\left(Y^-(T)\right)^2\right] \\
&\quad = \mathbb{E}\left[\int_0^T h_+(t)\,\mathbb{1}_{\{Y(t)>0\}}\,\mathfrak{f}_+(t,\boldsymbol{\pi}(t),q(t))\,\mathrm{d}t\right] \\
&\qquad + \mathbb{E}\left[\int_0^T h_-(t)\,\mathbb{1}_{\{Y(t)\le 0\}}\,\mathfrak{f}_-(t,\boldsymbol{\pi}(t),q(t))\,\mathrm{d}t\right] \\
&\qquad + h_+(0)\left(Y^+(0)\right)^2 + h_-(0)\left(Y^-(0)\right)^2.
\end{aligned} \tag{44}$$

Consequently, minimizing $\mathbb{E}[(X(T)-\lambda)^2]$ leads to the following (candidate) solution $\boldsymbol{u}^*_\lambda(\cdot) = (\boldsymbol{\pi}^*_\lambda(\cdot)^\top, q^*_\lambda(\cdot))^\top$ to Problem (P2–*u*):

$$\begin{aligned}
\boldsymbol{\pi}^*_\lambda(t) &= \left\{-\boldsymbol{\Sigma}(t)^{-1}\boldsymbol{\sigma}(t)\left(\boldsymbol{\theta}(t) + \frac{\boldsymbol{g}_+(t)}{h_+(t)}\right)Y^+(t)\right. \\
&\qquad \left. + \boldsymbol{\Sigma}(t)^{-1}\boldsymbol{\sigma}(t)\boldsymbol{G}(t)\right\}\mathbb{1}_{\{Y(t)>0\}} \\
&\qquad + \left\{\boldsymbol{\Sigma}(t)^{-1}\boldsymbol{\sigma}(t)\left(\boldsymbol{\theta}(t) + \frac{\boldsymbol{g}_-(t)}{h_-(t)}\right)Y^-(t)\right. \\
&\qquad \left. + \boldsymbol{\Sigma}(t)^{-1}\boldsymbol{\sigma}(t)\boldsymbol{G}(t)\right\}\mathbb{1}_{\{Y(t)\le 0\}} \\
&= -\boldsymbol{\Sigma}(t)^{-1}\boldsymbol{\sigma}(t)\left(\boldsymbol{\theta}(t) + \frac{\boldsymbol{g}_+(t)}{h_+(t)}\right)Y(t) \\
&\qquad + \boldsymbol{\Sigma}(t)^{-1}\boldsymbol{\sigma}(t)\boldsymbol{G}(t),
\end{aligned} \tag{45}$$

$$\begin{aligned}
\text{and}\quad q^*_\lambda(t) &= 0\cdot\mathbb{1}_{\{Y(t)>0\}} + \frac{\eta_r a_Z}{\sigma_Z^2}Y^-(t) \\
&\quad \cdot\mathbb{1}_{\{Y(t)\le 0\}} = \frac{\eta_r a_Z}{\sigma_Z^2}\,Y^-(t).
\end{aligned} \tag{46}$$

To derive (45), we have used the results that $Y^+(t)\mathbb{1}_{\{Y(t)>0\}} - Y^-(t)\mathbb{1}_{\{Y(t)\le 0\}} = Y(t)$ and $\frac{g_+(t)}{h_+(t)} = \frac{g_-(t)}{h_-(t)}$ (see (21)–(24) in Lemma 2 for the latter identity). Note that the subscript λ in $\boldsymbol{\pi}^*_\lambda(\cdot)$ and $q^*_\lambda(\cdot)$ indicates their dependence on the fixed Lagrangian multiplier λ, which appears in the objective of Problem (P2–*u*).

Step 2. Solve the outer problem in (P2–λ) to obtain the optimal Lagrangian multiplier λ^*, which in turn yields the (candidate) efficient strategy, $\boldsymbol{u}^*_{\lambda^*}(\cdot)$, to Problem (P1).

Now following the strategy $\boldsymbol{u}^*_\lambda(\cdot) = (\boldsymbol{\pi}^*_\lambda(\cdot)^\top, q^*_\lambda(\cdot))^\top$ found in (45)–(46), we obtain the value function of Problem (P2–*u*) from (44) by

$$\begin{aligned}
V_2(x_0,r_0,v_0;\lambda) &:= \min_{\boldsymbol{u}(\cdot)\in\mathcal{A}}\mathbb{E}\left[(X(T)-\lambda)^2\right] \\
&= h_+(0)\left(Y^+(0)\right)^2 + h_-(0)\left(Y^-(0)\right)^2.
\end{aligned}$$

Immediately, the objective of Problem (P2–λ) becomes

$$\begin{aligned}
&J_1(x_0,r_0,v_0;\boldsymbol{u}^*_\lambda(\cdot),\lambda) \\
&\quad = V_2(x_0,r_0,v_0;\lambda) - (\lambda-d)^2 \\
&\quad = h_+(0)\left[\left(x_0 - \lambda\overline{\Gamma} - (\eta_r-\eta)a_Z\widehat{\Gamma}\right)^+\right]^2 \\
&\qquad + h_-(0)\left[\left(x_0 - \lambda\overline{\Gamma} - (\eta_r-\eta)a_Z\widehat{\Gamma}\right)^-\right]^2 - (\lambda-d)^2,
\end{aligned} \tag{47}$$

in which the last equality is due to $Y(0) = x_0 - H(0)$ and (31).

From the first-order condition of Problem (P2–λ),

$$0 = \frac{\partial J_1}{\partial\lambda}$$

$$= \begin{cases} 2h_+(0)\overline{\Gamma}\left[\lambda\overline{\Gamma} + (\eta_r - \eta)a_Z\widehat{\Gamma} - x_0\right] & \\ \quad -2(\lambda - d), & \text{if } x_0 > H(0), \\ 2h_-(0)\overline{\Gamma}\left[\lambda\overline{\Gamma} + (\eta_r - \eta)a_Z\widehat{\Gamma} - x_0\right] & \\ \quad -2(\lambda - d), & \text{if } x_0 \le H(0), \end{cases}$$

we obtain a candidate solution λ^* by

$$\lambda^* = \begin{cases} \dfrac{h_+(0)\overline{\Gamma}\left[x_0 - (\eta_r - \eta)a_Z\widehat{\Gamma}\right] - d}{h_+(0)\overline{\Gamma}^2 - 1}, & x_0 > H(0), \\ \dfrac{h_-(0)\overline{\Gamma}\left[x_0 - (\eta_r - \eta)a_Z\widehat{\Gamma}\right] - d}{h_-(0)\overline{\Gamma}^2 - 1}, & x_0 \le H(0). \end{cases} \tag{48}$$

Our next agenda is to show that the first-order condition is also sufficient. To that end, we compute the second-order derivative of J_1 (with respect to λ) by

$$\frac{\partial^2 J_1}{\partial \lambda^2} = 2\left(h_+(0)\overline{\Gamma}^2 - 1\right)\mathbb{1}_{\{x_0 > H(0)\}} + 2\left(h_-(0)\overline{\Gamma}^2 - 1\right)\mathbb{1}_{\{x_0 \le H(0)\}}.$$

By Itô's formula and (A5), we have

$$\mathrm{d}P^2(t,T) = \left[2r(t)P^2(t,T) + P(t,T)\boldsymbol{\theta}(t)^\top \boldsymbol{Q}(t,T) + |\boldsymbol{Q}(t,T)|^2\right]\mathrm{d}t + 2P(t,T)\boldsymbol{Q}(t,T)^\top \mathrm{d}\boldsymbol{W}(t),$$

and

$$\begin{aligned} &\mathrm{d}\{h_+(t)P^2(t,T)\} \\ &= h_+(t)\left|P(t,T)\boldsymbol{\theta}(t) + P(t,T)\frac{\boldsymbol{g}_+(t)}{h_+(t)} + \boldsymbol{Q}(t,T)\right|^2 \mathrm{d}t \\ &\quad + \left[2h_+(t)P(t,T)\boldsymbol{Q}(t,T)^\top + P^2(t,T)\boldsymbol{g}_+(t)^\top\right]\mathrm{d}\boldsymbol{W}(t), \end{aligned}$$

which, along with (30), implies

$$\begin{aligned} h_-(0)\overline{\Gamma}^2 < h_+(0)\overline{\Gamma}^2 &= h_+(0)P^2(0,T) \\ &= h_+(T)P^2(T,T) - \mathbb{E}\left[\int_0^T h_+(t)\left|P(t,T)\boldsymbol{\theta}(t) + P(t,T)\frac{\boldsymbol{g}_+(t)}{h_+(t)} + \boldsymbol{Q}(t,T)\right|^2 \mathrm{d}t\right] < 1. \end{aligned}$$

With the above inequalities, we obtain $\frac{\partial^2 J_1}{\partial \lambda^2} < 0$. Hence, λ^* in (48) is indeed the optimal solution to Problem (P2–λ).

Recall from Lemma 3 that $(H(\cdot), \boldsymbol{G}(\cdot))$ is the unique solution to the linear BSDE in (20), for a given (Lagrangian multiplier) $\lambda \in \mathbb{R}$, which determines the terminal condition by $H(T) = \lambda$. As such, when we derive the optimal Lagrangian multiplier λ^* in (48), we compare x_0 with $H(0)$ which is part of the solution to (20) with $\lambda = \lambda^*$. To avoid potential confusion, let $(H^*(\cdot), \boldsymbol{G}^*(\cdot))$ denote the unique solution to (20) under the optimal Lagrangian multiplier λ^*; the two conditions in (48) now become $x_0 > H^*(0)$ and $x_0 \le H^*(0)$, respectively. It appears that λ^* in (48) is intertwined with $H^*(\cdot)$, and our next goal is to disentangle them by showing that only $x_0 \le H^*(0)$ is possible. For that purpose, assume to the contrary that $x_0 > H^*(0)$ holds; this assumption, together with (48), implies

$$\begin{aligned} \lambda^* - d &= \frac{h_+(0)\overline{\Gamma}\left[x_0 - (\eta_r - \eta)a_Z\widehat{\Gamma} - d\overline{\Gamma}\right]}{h_+(0)\overline{\Gamma}^2 - 1} \\ &= \frac{h_+(0)\overline{\Gamma}^2\left[d_{\min} - d\right]}{h_+(0)\overline{\Gamma}^2 - 1} \ge 0, \end{aligned}$$

in which the last inequality is due to $d_{\min} \le d$ and $0 < h_+(0)\overline{\Gamma}^2 < 1$. By using $\lambda^* \ge d \ge d_{\min}$, $H^*(0)$ from (31) with $\lambda = \lambda^*$, and $d_{\min}$ in (32), we obtain

$$H^*(0) = \lambda^*\overline{\Gamma} + (\eta_r - \eta)a_Z\widehat{\Gamma} \ge d\,\overline{\Gamma} + (\eta_r - \eta)a_Z\widehat{\Gamma} \ge x_0,$$

which is a contradiction to the assumption of $x_0 > H^*(0)$. Therefore, $x_0 \le H^*(0)$ must hold true, and thus the solution λ^* is given by the second case in (48) or equivalently by (36). Finally, as explained in Subsection 3.1, by substituting λ^* in (36) for λ in (45)–(46), we obtain the efficient strategy $\boldsymbol{u}^*(\cdot) = \boldsymbol{u}^*_{\lambda^*}(\cdot)$ to Problem (P1) given in (33)–(34), pending the verification that $\boldsymbol{u}^*(\cdot)$ is an admissible strategy.

Now we are ready to derive the efficient frontier. Since $x_0 \le H^*(0)$, we have $(x_0 - H^*(0))^+ = 0$. Using this result, along with (32), (36), and (47), the objective value of Problem (P1) under $\boldsymbol{u}^*(\cdot)$ is given by

$$\begin{aligned} &J(x_0, r_0, v_0; \boldsymbol{u}^*(\cdot)) \\ &= h_-(0)\left[\left(x_0 - \lambda^*\overline{\Gamma} - (\eta_r - \eta)a_Z\widehat{\Gamma}\right)^-\right]^2 - (\lambda^* - d)^2 \\ &= h_-(0)\overline{\Gamma}^2(\lambda^* - d_{\min})^2 - (\lambda^* - d)^2 \\ &= \frac{h_-(0)\overline{\Gamma}^2}{(h_-(0)\overline{\Gamma}^2 - 1)^2}(d - d_{\min})^2 \\ &\quad - \frac{(h_-(0)\overline{\Gamma}^2)^2}{(h_-(0)\overline{\Gamma}^2 - 1)^2}(d - d_{\min})^2 \\ &= \frac{h_-(0)\overline{\Gamma}^2}{1 - h_-(0)\overline{\Gamma}^2}(d - d_{\min})^2, \end{aligned} \tag{49}$$

which yields the efficient frontier in (37).

Step 3. Verify $\boldsymbol{u}^*(\cdot) = (\boldsymbol{\pi}^*(\cdot)^\top, q^*(\cdot))^\top$, given by (33)–(34), is admissible.

Recall there are four conditions in Definition 1 of admissible strategies; the objective in this step is to show that all the conditions hold for $\boldsymbol{u}^*(\cdot)$. First, $\boldsymbol{u}^*(\cdot)$ given by (33)–(34) is clearly adapted to the underlying filtration $\mathbb{F}$, and Condition (1) is satisfied.

Condition (2). Since $\boldsymbol{\pi}^*(t) \in \mathbb{R}^3$ is trivially satisfied, it remains to verify that $q^*(t) \ge 0$ or equivalently $X^*(t) \le H^*(t)$, for all $t \in [0,T]$. To that end, introduce $Y^*(\cdot) := X^*(\cdot) - H^*(\cdot)$. Similar to (38), we obtain

$$\begin{aligned} &\mathrm{d}\left((Y^*)^+(t)\right)^2 \\ &= \left((Y^*)^+(t)\right)^2\left\{2\left[r(t) - \left(\boldsymbol{\theta}(t) + \frac{\boldsymbol{g}_+(t)}{h_+(t)}\right)^\top \boldsymbol{\theta}(t)\right]\right. \end{aligned}$$

$$+\left|\boldsymbol{\theta}(t)+\frac{\boldsymbol{g}_+(t)}{h_+(t)}\right|^2\Bigg\}\,\mathrm{d}t$$
$$-2\left((Y^*)^+(t)\right)^2\left(\boldsymbol{\theta}(t)+\frac{\boldsymbol{g}_+(t)}{h_+(t)}\right)^\top\mathrm{d}\boldsymbol{W}(t),$$
$$\left((Y^*)^+(0)\right)^2=0.$$

Note that the zero initial value is due to $x_0 \le H^*(0)$. Since $((Y^*)^+(\cdot))^2$ has a geometric structure with zero initial value, $((Y^*)^+(t))^2 \equiv 0$ for all t, which naturally implies $X^*(t) \le H^*(t)$ for all $t \in [0,T]$.

Condition (4). We next verify the uniform integrability condition in Definition 1. Recall that

$$\frac{\mathrm{d}\left[h_-(t)((Y^*)^-(t))^2\right]}{\left[h_-(t)((Y^*)^-(t))^2\right]}=2\frac{\eta_r a_Z}{\sigma_Z}\,\mathrm{d}Z(t)+\left[2\boldsymbol{\theta}(t)+\frac{\boldsymbol{g}_-(t)}{h_-(t)}\right]^\top\mathrm{d}\boldsymbol{W}(t),$$

whose solution is given by the following Doléans-Dade exponential

$$h_-(t)\left((Y^*)^-(t)\right)^2=K\exp\left\{-\frac{1}{2}\int_0^t\left[4\left(\frac{\eta_r a_Z}{\sigma_Z}\right)^2+\left|2\boldsymbol{\theta}(s)+\frac{\boldsymbol{g}_-(s)}{h_-(s)}\right|^2\right]\mathrm{d}s+2\frac{\eta_r a_Z}{\sigma_Z}Z(s)+\int_0^t\left[2\boldsymbol{\theta}(s)+\frac{\boldsymbol{g}_-(s)}{h_-(s)}\right]^\top\mathrm{d}\boldsymbol{W}(s)\right\},\tag{50}$$

where $K := h_-(0)((Y^*)^-(0))^2$.

In what follows, to show that $h_-(t)((Y^*)^-(t))^2$ is a uniformly integrable exponential martingale, we verify that K is finite and Novikov's condition holds, that is,

$$\mathbb{E}\left[\exp\left\{\frac{1}{2}\int_0^T\left[4\left(\frac{\eta_r a_Z}{\sigma_Z}\right)^2+\left|2\boldsymbol{\theta}(s)+\frac{\boldsymbol{g}_-(s)}{h_-(s)}\right|^2\right]\mathrm{d}s\right\}\right]<\infty.$$

On the one hand, by the derivation in equation (49) and the proof of Lemma 2, we obtain

$$\begin{aligned}K&\le h_-(0)\left(Y^*(0)\right)^2\\&=\frac{h_-(0)\overline{\Gamma}^2}{(h_-(0)\overline{\Gamma}^2-1)^2}(d-d_{\min})^2\\&=\frac{h_-(0)}{(h_-(0)\overline{\Gamma}^2-1)^2}\left[d\overline{\Gamma}-(x_0-(\eta_r-\eta)a_Z\widehat{\Gamma})\right]^2\\&=\frac{\exp\{-\eta_r^2a_Z^2T/\sigma_Z^2\}\cdot\overline{h}_+(0)}{\left[\exp\{-\eta_r^2a_Z^2T/\sigma_Z^2\}\overline{\Gamma}^2-\overline{h}_+(0)\right]^2}\\&\quad\times\left[d\overline{\Gamma}-(x_0-(\eta_r-\eta)a_Z\widehat{\Gamma})\right]^2,\end{aligned}\tag{51}$$

where $\overline{h}_+(0)$ is the first component of the time-0 solution to the auxiliary BSDE (A17).

It can be shown that the denominator in (51) is bounded above zero. Indeed,

$$\begin{aligned}&\left[\exp\{-\eta_r^2a_Z^2T/\sigma_Z^2\}\overline{\Gamma}^2-\overline{h}_+(0)\right]^2\\&=\left\{\exp\{-\eta_r^2a_Z^2T/\sigma_Z^2\}\cdot\left(\widetilde{\mathbb{E}}[e^{-\int_0^T r(t)\,\mathrm{d}t}]\right)^2-\overline{\mathbb{E}}[e^{\int_0^T(-2r(s)+|\boldsymbol{\theta}(s)|^2)\mathrm{d}s}]\right\}^2\\&=\Big\{\exp\{-\eta_r^2a_Z^2T/\sigma_Z^2\}\cdot\left(\mathbb{E}\left[e^{\int_0^T(-r(s)-\frac{1}{2}|\boldsymbol{\theta}(s)|^2)\mathrm{d}s-\int_0^T\boldsymbol{\theta}(s)^\top\mathrm{d}\boldsymbol{W}(s)}\right]\right)^2-\mathbb{E}\left[e^{\int_0^T(-2r(s)-|\boldsymbol{\theta}(s)|^2)\mathrm{d}s-\int_0^T2\boldsymbol{\theta}(s)^\top\mathrm{d}\boldsymbol{W}(s)}\right]\Big\}^2\\&\ge\Bigg\{\left(\mathbb{E}\left[e^{\int_0^T(-r(s)-\frac{1}{2}|\boldsymbol{\theta}(s)|^2)\mathrm{d}s-\int_0^T\boldsymbol{\theta}(s)^\top\mathrm{d}\boldsymbol{W}(s)}\right]\right)^2-\exp\{-\eta_r^2a_Z^2T/\sigma_Z^2\}\cdot\left(\mathbb{E}\left[e^{\int_0^T(-r(s)-\frac{1}{2}|\boldsymbol{\theta}(s)|^2)\mathrm{d}s-\int_0^T\boldsymbol{\theta}(s)^\top\mathrm{d}\boldsymbol{W}(s)}\right]\right)^2\Bigg\}^2\\&=\overline{\Gamma}^4\left\{1-\exp\{-\eta_r^2a_Z^2T/\sigma_Z^2\}\right\}^2\ge\epsilon>0.\end{aligned}\tag{52}$$

In the fourth and fifth lines of (52), we have used the Cauchy-Schwarz inequality and the fact that $\exp\{-\eta_r^2a_Z^2T/\sigma_Z^2\}<1$; in the last line of (52), we have applied the result in Lemma 5 to $\overline{\Gamma}=P(0,T)$, in particular, the boundedness of $N_i(0,T)$, $i=1,2,3$. Evidently, all the remaining terms in (51) are finite. Hence, we can conclude that $K<\infty$.

On the other hand, recalling (24), we obtain

$$\mathbb{E}\left[\exp\left\{\frac{1}{2}\int_0^T\left|2\boldsymbol{\theta}(s)+\frac{\boldsymbol{g}_-(s)}{h_-(s)}\right|^2\mathrm{d}s\right\}\right]\le\mathbb{E}\left[\exp\left\{\left[2+\max\left(\frac{\sigma_1^2\sup_{s\in[0,T]}A^2(s)}{b_0^2+b_1^2},\frac{\sigma_2^2\sup_{s\in[0,T]}B^2(s)}{b_2^2}\right)\right]\int_0^T|\boldsymbol{\theta}(s)|^2\mathrm{d}s\right\}\right],$$

in which functions A and B are defined by (A10) and (A11), respectively. If we can show that $\sup_{s\in[0,T]}A^2(s)<\infty$ and $\sup_{s\in[0,T]}B^2(s)<\infty$, then Novikov's condition holds by setting

$$p=2+\max\left\{\frac{\sigma_1^2\sup_{s\in[0,T]}A^2(s)}{b_0^2+b_1^2},\frac{\sigma_2^2\sup_{s\in[0,T]}B^2(s)}{b_2^2}\right\}$$

in Condition (3) of Assumption 2.

Suppose that there exist an $s_0\in[0,T]$ such that $A(s_0)=\infty$. It then follows from Lemma 2 and its proof that

$$\begin{aligned}h_+(s_0)&=\exp\{A(s_0)m_1(s_0)+B(s_0)m_2(s_0)+C(s_0)\}\\&=\frac{1}{\overline{\mathbb{E}}_{s_0}\left[e^{\int_{s_0}^T(-2r(s)+|\boldsymbol{\theta}(s)|^2)\mathrm{d}s}\right]}=\infty,\end{aligned}$$

in which $\overline{\mathbb{E}}$ denotes taking expectation under $\overline{\mathbb{P}}$ (defined by (A18)), and this result implies

$$\overline{\mathbb{E}}_{s_0}\left[e^{\int_{s_0}^{T}(-2r(s)+|\boldsymbol{\theta}(s)|^2)\mathrm{d}s}\right]=0. \tag{53}$$

On the other hand, by Jensen's inequality, we have

$$\begin{aligned}&\overline{\mathbb{E}}_{s_0}\left[e^{\int_{s_0}^{T}(-2r(s)+|\boldsymbol{\theta}(s)|^2)\mathrm{d}s}\right]\\&\quad\geq\overline{\mathbb{E}}_{s_0}\left[e^{\int_{s_0}^{T}-2r(s)\mathrm{d}s}\right]\geq\exp\left\{\overline{\mathbb{E}}_{s_0}\left[-2\int_{s_0}^{T}r(s)\mathrm{d}s\right]\right\},\end{aligned}$$

in which the exponent can be computed by

$$\begin{aligned}&\overline{\mathbb{E}}_{s_0}\left[-2\int_{s_0}^{T}r(s)\mathrm{d}s\right]\\&\quad=\overline{\mathbb{E}}_{s_0}\left[-2\int_{s_0}^{T}(\alpha m_1(s)+\beta m_2(s))\mathrm{d}s\right]\\&\quad=-2\alpha\int_{s_0}^{T}\overline{\mathbb{E}}_{s_0}[m_1(s)]\mathrm{d}s-2\beta\int_{s_0}^{T}\overline{\mathbb{E}}_{s_0}[m_2(s)]\mathrm{d}s\\&\quad=-2\alpha\int_{s_0}^{T}\left[m_1(s_0)e^{-\kappa_A(T-s_0)}\right.\\&\qquad\left.+\frac{\kappa_1\overline{m}_1}{\kappa_A}\left(1-e^{-\kappa_A(T-s_0)}\right)\right]\mathrm{d}s\\&\qquad-2\beta\int_{s_0}^{T}\left[m_2(s_0)e^{-\kappa_B(T-s_0)}\right.\\&\qquad\left.+\frac{\kappa_2\overline{m}_2}{\kappa_B}\left(1-e^{-\kappa_B(T-s_0)}\right)\right]\mathrm{d}s.\end{aligned}$$

Evidently, the above expression is greater than $-\infty$ with a positive probability. Here, the calculation of $\overline{\mathbb{E}}_{s_0}[m_1(s)]$ is standard and can be found in Theorem 6.3.3.1 of Jeanblanc *et al.* (2009). Thus, $\overline{\mathbb{E}}_{s_0}[e^{\int_{s_0}^{T}(-2r(s)+|\boldsymbol{\theta}(s)|^2)\mathrm{d}s}]$ is strictly greater than 0 with a positive probability, which contradicts with (53). Therefore, it must be true that $\sup_{s\in[0,T]}A^2(s)<\infty$. By a similar argument, $\sup_{s\in[0,T]}B^2(s)<\infty$ follows as well.† ■

Using the facts that $h_+(t)\leq h_-(t)\exp(\frac{\eta_r^2a_Z^2}{\sigma_Z^2}T)$ and $Y^*(t)=(Y^*)^-(t)$, for all $t\in[0,T]$, we immediately obtain that the process $h_+(\cdot)(Y^*(\cdot))^2=h_+(\cdot)(X(\cdot)-H(\cdot))^2$ is also uniformly integrable.

Condition (3). We last verify the integrability assumptions in Condition (3) of Definition 1. In what follows, we only show

$$\begin{aligned}&h_-(t)Y^-(t)\left(\boldsymbol{\sigma}(t)^\top\boldsymbol{\pi}^*(t)-\boldsymbol{G}(t)\right)\\&\quad=-\boldsymbol{\Theta}(t)h_-(t)\left(Y^-(t)\right)^2\in\mathcal{L}^{2,loc}_{\mathbb{F},\mathbb{P}}(0,T;\mathbb{R}^3),\end{aligned}$$

as the same argument can be applied to show all other Itô integrals are local martingales. Recall that $\boldsymbol{\Theta}(\cdot)$ is the adjusted market price of risk, and it is defined by (35).

† From (A10) and (A11), we can show that for certain set of model parameters, $\sup_{s\in[0,T]}A^2(s)<\infty$ and/or $\sup_{s\in[0,T]}B^2(s)<\infty$ are not necessarily true. However, one should bear in mind that the above derivation is conducted under Assumption 2 (particularly, Condition (3)), which excludes those unfavorable parameter values.

First, by using Assumption 2, the Taylor expansion, and Markov's inequality, we obtain

$$\int_0^T|\boldsymbol{\Theta}(t)|^4\mathrm{d}t<e^{p\int_0^T|\boldsymbol{\Theta}(t)|^2\mathrm{d}t}<\infty.$$

Moreover, it follows from (50) that

$$\begin{aligned}&\left\{h_-(t)\left(Y^-(t)\right)^2\right\}^4\\&\quad=\left\{h_-(0)\left((Y^*)^-(0)\right)^2\right\}^4\mathcal{E}(t)\exp\left\{\int_0^t\left[24\left(\frac{\eta_ra_Z}{\sigma_Z}\right)^2\right.\right.\\&\qquad\left.\left.+6\left|2\boldsymbol{\theta}(t)+\frac{\boldsymbol{g}_-(t)}{h_-(t)}\right|^2\right]\mathrm{d}s\right\},\end{aligned}$$

in which

$$\begin{aligned}\mathcal{E}(t):=\exp\left\{-\frac{1}{2}\int_0^t\left[64\left(\frac{\eta_ra_Z}{\sigma_Z}\right)^2\right.\right.&\\\left.+16\left|2\boldsymbol{\theta}(t)+\frac{\boldsymbol{g}_-(t)}{h_-(t)}\right|^2\right]\mathrm{d}s+8\frac{\eta_ra_Z}{\sigma_Z}Z(t)&\\\left.+4\int_0^t\left[2\boldsymbol{\theta}(s)+\frac{\boldsymbol{g}_-(s)^\top}{h_-(s)}\right]\mathrm{d}\boldsymbol{W}(s)\right\}.&\end{aligned}$$

As in the previous step, we can show $\mathcal{E}(\cdot)$ is a uniformly integrable martingale, thereby satisfying

$$\mathbb{E}\left[\int_0^T\mathcal{E}(t)\mathrm{d}t\right]\leq T\sup_{t\in[0,T]}\mathbb{E}\left[\mathcal{E}(t)\right]<\infty.$$

On the other hand, by setting an appropriate p in Condition (3) of Assumption 2, we have

$$\begin{aligned}&\mathbb{E}\left[\exp\left\{\int_0^T\left[24\left(\frac{\eta_ra_Z}{\sigma_Z}\right)^2+6\left|2\boldsymbol{\theta}(s)+\frac{\boldsymbol{g}_-(s)}{h_-(s)}\right|^2\right]\mathrm{d}s\right\}\right]\\&\quad<\infty.\end{aligned}$$

Therefore, by Markov's inequality, we get

$$\begin{aligned}&\int_0^T\left\{h_-(t)\left(Y^-(t)\right)^2\right\}^4\mathrm{d}t\\&\quad\leq K^4\sup_{t\in[0,T]}\exp\left\{\int_0^t\left[24\left(\frac{\eta_ra_Z}{\sigma_Z}\right)^2\right.\right.\\&\qquad\left.\left.+6\left|2\boldsymbol{\theta}(s)+\frac{\boldsymbol{g}_-(s)}{h_-(s)}\right|^2\right]\mathrm{d}s\right\}\times\int_0^T\mathcal{E}(s)\mathrm{d}s\\&\quad=K^4\exp\left\{\int_0^T\left[24\left(\frac{\eta_ra_Z}{\sigma_Z}\right)^2+6\left|2\boldsymbol{\theta}(s)+\frac{\boldsymbol{g}_-(s)}{h_-(s)}\right|^2\right]\mathrm{d}s\right\}\\&\qquad\times\int_0^T\mathcal{E}(s)\mathrm{d}s<\infty,\end{aligned}$$

where $K=h_-(0)((Y^*)^-(0))^2$ is a finite constant as shown earlier. Then, by the Cauchy–Schwarz inequality, we obtain

$$\int_0^T\left|\boldsymbol{\Theta}(t)h_-(t)\left(Y^-(t)\right)^2\right|^2\mathrm{d}t$$

$$\leq \sqrt{\int_0^T |\boldsymbol{\Theta}(t)|^4 \mathrm{d}t \int_0^T \left\{h_-(t)\,(Y^-(t))^2\right\}^4 \mathrm{d}t} < \infty,$$

$\mathbb{P}-$a.s..

The proof is completed.

5. Numerical analysis

In this section, we first estimate the model parameters using bond price and S&P500 data and then conduct a numerical analysis on the efficient frontier.

5.1. *Model estimation*

This subsection estimates the financial model, described by (1)–(5), and conducts an empirical study to confirm the correlation between stochastic interest rates and stochastic volatility. Recall from Subsection 2.2 that there are three risky assets traded in the financial market, including one risky stock and two zero-coupon bonds. For the calibration purpose, we select the S&P 500 to be the risky stock and the US treasury bonds to be the zero-coupon bonds. Our analysis is based on their daily price data from January 2, 1990 to December 29, 2023.† Note that both the stochastic interest rate $r(\cdot)$ and the stochastic variance of the risky stock $v(\cdot)$ are driven by the two square-root processes $m_1(\cdot)$ and $m_2(\cdot)$ in (3)–(4), both of which are *latent* processes and cannot be directly observed. In the estimation procedure, we apply the unscented Kalman filter (UKF) to handle these latent processes and use the maximum likelihood estimation (MLE) method to calibrate the model. Please see Appendix 2 for a detailed description of the estimation procedure.

We consider two models in the estimation. The first model is a full model in which the correlation between the interest rate $r(\cdot)$ and the stock volatility $\sqrt{v(\cdot)}$ is estimated from real data. The second model is a partial model in which the correlation between $r(\cdot)$ and $\sqrt{v(\cdot)}$ is assumed to be zero, or equivalently, α is set to be zero (see (5)). For convenience, we call these two models 'full model with correlation' and 'partial model without correlation' in what follows. The upper panel of table 1 reports the estimated values of all parameters in each model and their corresponding standard errors, while the lower panel of table 1 compares the goodness-of-fit measures between the two models. We consider four goodness-of-fit measures: the Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), root-mean-square error (RMSE), and mean absolute percentage error (MAPE); note that, under each of the four measures, a model with a smaller value indicates a better fitting to data. We observe from table 1 that the full model with correlation consistently outperforms the partial model without correlation under all the four goodness-of-fit measures considered. This result offers strong support to incorporating the correlation between the interest rates and the stock volatility, and is consistent with earlier empirical findings in, for instance, Trolle and Schwartz (2009). In Assumption 2, we impose the Feller condition, $2\kappa_i\overline{m}_i > \sigma_i^2$ for $i = 1, 2$, which are satisfied by the estimated parameters in table 1. Indeed, we easily verify that $2\kappa_1\overline{m}_1 = 0.5428 > \sigma_1^2 = 0.0324$ and $2\kappa_2\overline{m}_2 = 0.1848 > \sigma_2^2 = 0.0196$.

With the above conclusion, we further examine the performance of the full model. To that end, we plot the model-estimated interest rates against the daily market yields (derived from the US treasury bonds) in the left panel of figure 1, and the model-estimated daily S&P500 variance against the market implied values, defined as $\frac{\text{VIX}^2}{10000}$ (VIX represents the Chicago Board Options Exchange (CBOE) volatility index),‡ in the right panel of figure 1. Both plots show that the full model can capture the overall movements of the stochastic interest rate process $r(\cdot)$ and the variance process $v(\cdot)$; in particular, the fitting to daily stock (S&P500) variance is remarkably well.

5.2. *Numerical analysis on the efficient frontier*

In this subsection, we investigate how various model parameters impact the efficient frontier, obtained by (37) in Theorem 4. In the subsequent analysis, we consider the full (financial) model with correlation and adopt the parameter values estimated in table 1 as the baseline. Regarding the insurance model (see Subsection 2.2), we follow Chen and Shen (2018) and set the baseline parameter values by $a_z = 10$, $\sigma_z = 3$, $\eta = 0.35$, and $\eta_r = 0.45$. In addition, we set the remaining parameters by $T = 3$ (investment horizon), $T_1 = 5$ (maturity of the first bond), $T_2 = 10$ (maturity of the second bond), $x_0 = 1$ (initial wealth), $r_0 = 0.05$ (initial interest rate), and $v_0 = 0.15^2$ (initial variance).

We first study the effect of ρ on the efficient frontier; note from (3) that ρ captures the correlation between the stock price $S(\cdot)$ and the process $m_1(\cdot)$. From table 1, the estimated value of ρ is -0.34, and we consider two alternative values, $\rho = -0.14$ and $\rho = -0.54$. We plot the efficient frontier under these three values of ρ in figure 2. We observe that as ρ decreases, the efficient frontier moves downward (i.e. for all fixed variance level, the expected mean of the optimal wealth $X^*(T)$ decreases). This is because a higher negative correlation ρ leads to more 'volatile' investment returns and thus a worsened efficient frontier.

Next, we study the effects of α and β on the efficient frontier; recall from (5) that α and β determine the dependency of $r(t)$ on $m_1(t)$ and $m_2(t)$, respectively. We plot the efficient frontier under three different levels of α and β in figure 3. A higher α increases the correlation between $S(\cdot)$ and $r(\cdot)$, which in turn stabilizes investment returns and shifts the efficient frontier upward. This is confirmed by the left panel of figure 3. In contrast, increasing β does not affect the correlations between $S(\cdot)$, $v(\cdot)$ and $r(\cdot)$, but it amplifies the variation of $r(\cdot)$, leading to a more pronounced investment risk (in the zero-coupon bonds) and a downward shift of the efficient frontier, as shown by the right panel of figure 3.

† The S&P500 data is downloaded from Yahoo!Finance, while the US treasury bonds data is obtained from https://www.treasury.gov/resource-center/data-chart-center/interest-rates/pages/TextView.aspx?data=yieldAll.

‡ The VIX data is available from https://fred.stlouisfed.org/series/VIXCLS.

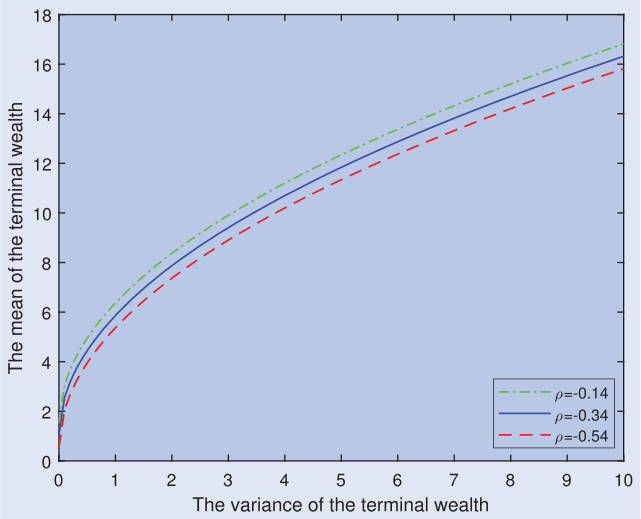

Figure 2. Effects of ρ on the efficient frontier.

We proceed to analyze how the mean-reversion speed parameters κ_1 and κ_2 affect the efficient frontier and plot it under different κ_1 or κ_2 in figure 4. When κ_1 (resp., κ_2) increases, $m_1(\cdot)$ (resp., $m_2(\cdot)$) mean reverts to its long-term level faster and thus has a reduced variation. Because $m_2(\cdot)$ only affects the stochastic interest rate $r(\cdot)$ but not the stock volatility $\sqrt{v(\cdot)}$ (see (5)), the increase of κ_2 stabilizes the returns on the two zero-coupon bonds and consequently improves the efficient frontier; this is exactly what the right panel of figure 4 demonstrates. However, the impact of κ_1 is more subtle because $m_1(\cdot)$ enters both $r(\cdot)$ and $\sqrt{v(\cdot)}$, and $\sqrt{v(\cdot)}$ affects both the expected return and volatility of the stock in the same direction. The left panel of figure 4 shows that the increase of κ_1 leads to a worsened efficient frontier, which implies that the decrease effect from the stock volatility outweighs the increase effect from the stock return, on the efficient frontier.

Lastly, we study the impact of two volatility parameters, σ_1 and σ_2, on the efficient frontier and plot the graphs in figure 5; σ_i is the volatility of the square-root process $m_i(\cdot)$, $i = 1, 2$. Increasing σ_1 raises both the stochastic interest rate and stock volatility, leading to a higher expected return but also an elevated risk. Unlike the finding obtained previously for the left panel of figure 4, the stock return now has a more significant impact than the stock volatility on the efficient frontier (at least for the estimated model). As such, we observe that the efficient frontier improves as σ_1 increases, which is clearly

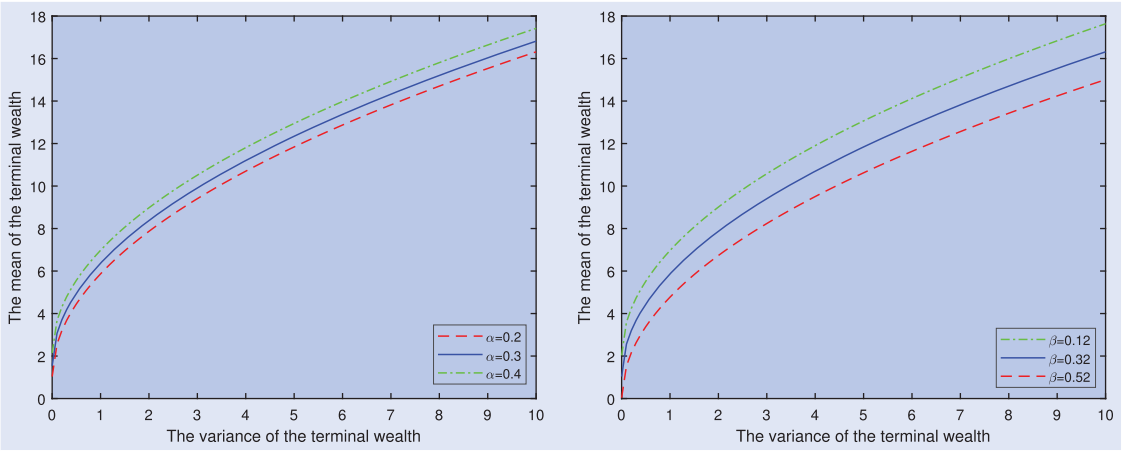

Figure 3. Effects of α (left) and β (right) on the efficient frontier.

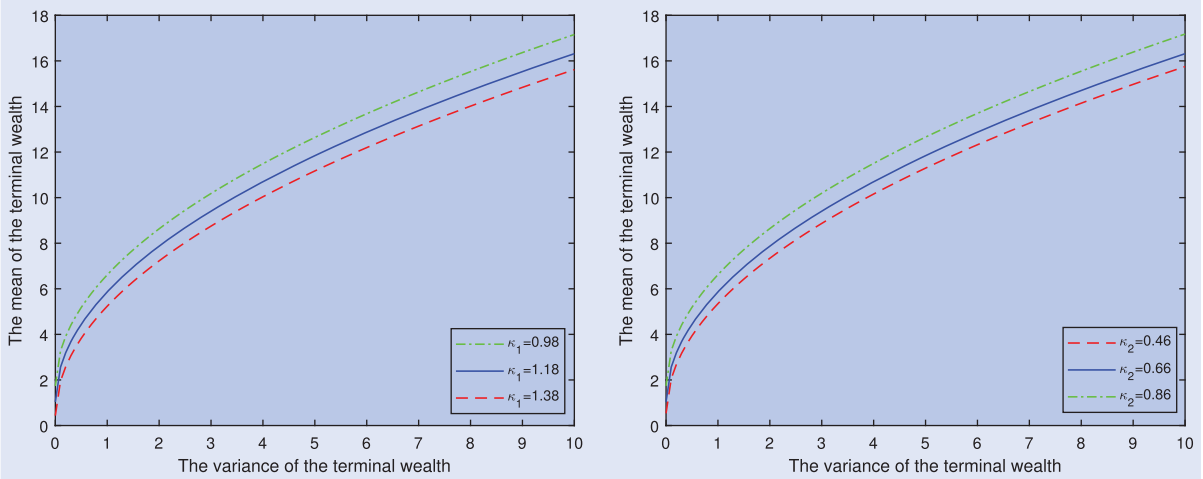

Figure 4. Effects of κ_1 (left) and κ_2 (right) on the efficient frontier.

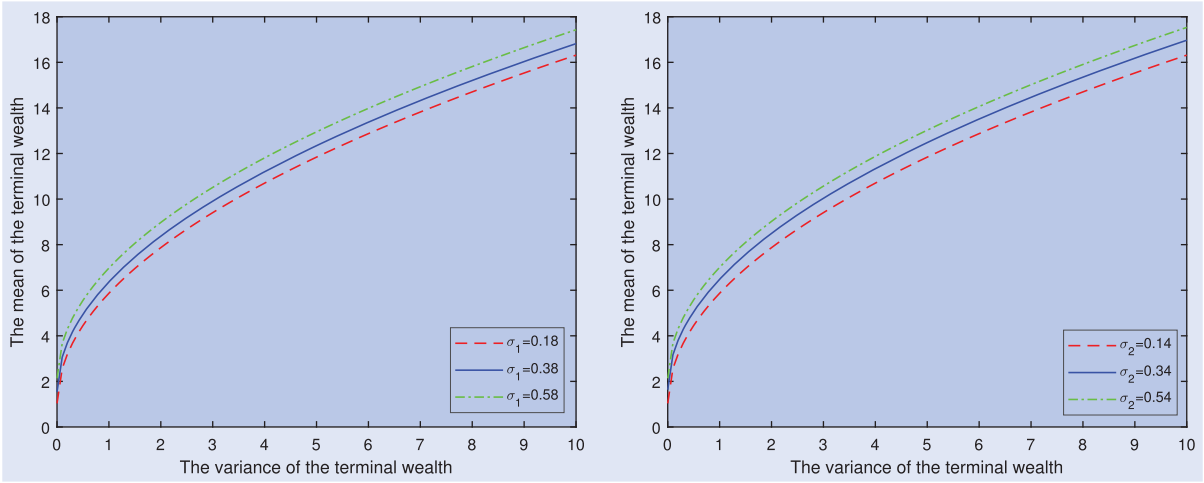

Figure 5. Effects of σ_1 (left) and σ_2 (right) on the efficient frontier.

shown in the left panel of figure 5. A similar argument applies to σ_2 and is consistent with the result in the right panel of figure 5.

6. Conclusion

This study presents a novel BSDE approach to optimal investment and reinsurance for insurers with mean-variance preferences, incorporating both stochastic interest rates and stock volatility. By employing a generalized Cox-Ingersoll-Ross process for interest rates and a Heston model for stock volatility, we capture realistic market dynamics and their correlation. Utilizing the backward stochastic differential equation (BSDE) and linear-quadratic (LQ) control framework, we derive closed-form solutions for the insurer's efficient strategy and efficient frontier, setting our work apart from prior studies that rely on numerical solutions or asymptotic analysis when stochastic interest rate and volatility are correlated. A key contribution of this paper is extending the BSDE approach to handle unbounded stochastic factors, overcoming limitations in existing literature that primarily considers bounded parameter cases.

Future research directions include investigating time-consistent equilibrium strategies under similar model settings, extending our framework to multi-factor stochastic processes, and analyzing the impact of model uncertainty on insurers' decision-making. Such extensions could provide deeper insights into risk management and optimal portfolio strategies in dynamic financial environments.

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ORCID

Yang Shen http://orcid.org/0000-0002-0559-2231
Wenjun Zhang http://orcid.org/0000-0001-7992-8204
Bin Zou http://orcid.org/0000-0002-7390-7979

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Appendices

Appendix 1. Technical proofs

A.1. Bond price

Recall from Subsection 3.2 that we have introduced *synthetic* zero-coupon bonds that pay one unit of currency at time $s > 0$. Denoting their price by $P(\cdot, s) := \{P(t,s)\}_{t\in[0,s]}$, we have $P(s,s) = 1$ and $P(t,s) = \widetilde{\mathbb{E}}_t[e^{-\int_t^s r(u)\mathrm{d}u}]$ for all $t \in [0,s]$ (see equation (25)). The lemma below provides an explicit characterization of the bond price $P(t,s)$.

LEMMA 5 *Let Assumption 2 hold. The zero-coupon bond price $P(t,s)$ in (25) is given by*

$$P(t,s) = \exp\{-N_1(t,s)\, m_1(t) - N_2(t,s)\, m_2(t) - N_3(t,s)\}, \quad t \in [0,s], \tag{A1}$$

in which the functions $N_i(\cdot,\cdot)$, $i = 1, 2, 3$, are defined by

$$N_1(t,s) := \frac{2\alpha(e^{\widetilde{\Delta}_1(s-t)} - 1)}{(\widetilde{\Delta}_1 + \widetilde{\kappa}_1)(e^{\widetilde{\Delta}_1(s-t)} - 1) + 2\widetilde{\Delta}_1}, \tag{A2}$$

$$N_2(t,s) := \frac{2\beta(e^{\widetilde{\Delta}_2(s-t)} - 1)}{(\widetilde{\Delta}_2 + \widetilde{\kappa}_2)(e^{\widetilde{\Delta}_2(s-t)} - 1) + 2\widetilde{\Delta}_2}, \tag{A3}$$

$$\text{and } N_3(t,s) := -\sum_{i=1}^{2} \frac{2\widetilde{\kappa}_i \widetilde{m}_i}{\sigma_i^2} \log\left(\frac{2\widetilde{\Delta}_i\, e^{(\widetilde{\Delta}_i + \widetilde{\kappa}_i)(s-t)/2}}{(\widetilde{\Delta}_i + \widetilde{\kappa}_i)(e^{\widetilde{\Delta}_i(s-t)} - 1) + 2\widetilde{\Delta}_i}\right), \tag{A4}$$

with $\widetilde{\Delta}_1 := \sqrt{\widetilde{\kappa}_1^2 + 2\alpha\sigma_1^2}$ and $\widetilde{\Delta}_2 := \sqrt{\widetilde{\kappa}_2^2 + 2\beta\sigma_2^2}$.

In addition, the dynamics of $P(\cdot,s)$ satisfies

$$\mathrm{d}P(t,s) = \Big[r(t)P(t,s) + \boldsymbol{\theta}(t)^\top \boldsymbol{Q}(t,s)\Big]\mathrm{d}t + \boldsymbol{Q}(t,s)^\top \mathrm{d}\boldsymbol{W}(t), \tag{A5}$$

for all $t \in [0,s]$, in which

$$\boldsymbol{Q}(t,s) := P(t,s)\Big(-\sigma_1\rho N_1(t,s)\sqrt{m_1(t)}, -\sigma_1\sqrt{1-\rho^2}N_1(t,s)\sqrt{m_1(t)}, -\sigma_2 N_2(t,s)\sqrt{m_2(t)}\Big)^\top.$$

Proof Please see Proposition 6.3.4.1 in Jeanblanc *et al.* (2009) for a proof. ∎

A.2. Proof to Lemma 1

Proof Consider the following linear BSDE:

$$\mathrm{d}\psi(t) = -r(t)\psi(t)\,\mathrm{d}t + \boldsymbol{\zeta}(t)^\top \mathrm{d}\boldsymbol{W}(t), \quad \psi(T) = 1. \tag{A6}$$

Although (A6) does not satisfy the Lipschitz condition due to the unboundedness of $r(\cdot)$, by the product rule we can construct a related BSDE as follows:

$$\mathrm{d}\widetilde{\psi}(t) = \widetilde{\boldsymbol{\zeta}}(t)^\top \mathrm{d}\boldsymbol{W}(t), \quad \widetilde{\psi}(T) = e^{\int_0^T r(t)dt}, \tag{A7}$$

in which $(\psi, \boldsymbol{\zeta})$ and $(\widetilde{\psi}, \widetilde{\boldsymbol{\zeta}})$ are related to each other through $(\widetilde{\psi}(t), \widetilde{\boldsymbol{\zeta}}(t)) = e^{\int_0^T r(t)dt}(\psi(t), \boldsymbol{\zeta}(t))$, for every $t \in [0,T]$. Note that now the driver of (A7) is zero (thereby Lipschitz continuous), and the terminal value is square integrable thanks to Condition (3) in Assumption 2. Therefore, it follows from Theorem 2.1 in El Karoui *et al.* (1997) that (A7) admits a unique solution and so does (A6). In fact, by applying the martingale representation theorem to (A7) and using the relation between $(\psi, \boldsymbol{\zeta})$ and $(\widetilde{\psi}, \widetilde{\boldsymbol{\zeta}})$, we obtain

$$\begin{aligned}
\psi(t) &= e^{-\int_0^t r(s)\mathrm{d}s}\widetilde{\psi}(t) = \mathbb{E}_t\left[e^{\int_t^T r(s)\mathrm{d}s}\right] \\
&= \mathbb{E}_t\left[e^{\int_t^T (\alpha m_1(s) + \beta m_2(s))\mathrm{d}s}\right] \\
&= \exp\left(\Psi_1(t)m_1(t) + \Psi_2(t)m_2(t) + \Psi_3(t)\right), \\
\boldsymbol{\zeta}(t) &= \psi(t)\Big(\rho\sigma_1\Psi_1(t)\sqrt{m_1(t)}, \sigma_1\sqrt{1-\rho^2}\Psi_1(t)\sqrt{m_1(t)}, \\
&\qquad \sigma_2\Psi_2(t)\sqrt{m_2(t)}\Big)^\top,
\end{aligned}$$

in which Ψ_i, $i = 1, 2, 3$, satisfy the following ordinary differential equations (ODEs)

$$\Psi_1'(t) - \kappa_1\Psi_1(t) + \frac{1}{2}\sigma_1^2\Psi_1^2(t) + \alpha = 0, \quad \Psi_1(T) = 0,$$

$$\Psi_2'(t) - \kappa_2\Psi_2(t) + \frac{1}{2}\sigma_2^2\Psi_2^2(t) + \beta = 0, \quad \Psi_2(T) = 0,$$

$$\Psi_3'(t) + \kappa_1\overline{m}_1\Psi_1(t) + \kappa_2\overline{m}_2\Psi_2(t) = 0, \quad \Psi_3(T) = 0.$$

We further solve the above three ODEs and obtain the closed-form solutions by ($\tau := T - t$)

$$\begin{aligned}
\Psi_1(t) &= \frac{\alpha\sinh(\delta_1\tau)}{\delta_1\cosh(\delta_1\tau) + \frac{\kappa_1}{2}\sinh(\delta_1\tau)}, \\
\Psi_2(t) &= \frac{\beta\sinh(\delta_2\tau)}{\delta_2\cosh(\delta_2\tau) + \frac{\kappa_2}{2}\sinh(\delta_2\tau)}, \\
\Psi_3(t) &= \Psi_{31}(t) + \Psi_{32}(t), \\
\Psi_{31}(t) &= -\frac{2\kappa_1\overline{m}_1}{\sigma_1^2}\log\left|\cosh(\delta_1\tau) + \frac{\kappa_1}{2\delta_1}\sinh(\delta_1\tau)\right| + \frac{\kappa_1^2\overline{m}_1\tau}{\sigma_1^2}, \\
\Psi_{32}(t) &= -\frac{2\kappa_2\overline{m}_2}{\sigma_2^2}\log\left|\cosh(\delta_2\tau) + \frac{\kappa_2}{2\delta_2}\sinh(\delta_2\tau)\right| + \frac{\kappa_2^2\overline{m}_2\tau}{\sigma_2^2},
\end{aligned}$$

where parameters δ_1 and δ_2 are defined by

$$\delta_1 := \frac{1}{2}\sqrt{|\Delta_1|}, \quad \delta_2 := \frac{1}{2}\sqrt{|\Delta_2|}, \quad \text{with} \quad \Delta_1 := \kappa_1^2 + 2\sigma_1^2\alpha, \quad \Delta_2 := \kappa_2^2 + 2\sigma_2^2\beta.$$

Following Lim (2004), we can show that Problem (P1) is feasible if and only if

$$\mathbb{E}\left[\int_0^T |\boldsymbol{b}(t)\psi(t) + \boldsymbol{\sigma}(t)\boldsymbol{\zeta}(t)|^2\mathrm{d}t\right] > 0, \tag{A8}$$

in which $\boldsymbol{b}(t)$ and $\boldsymbol{\sigma}(t)$ are defined by (14) and (15), respectively. Using the above results on $\psi(t)$ and $\boldsymbol{\zeta}(t)$, we obtain

$$\begin{aligned}
&|\boldsymbol{b}(t)\psi(t) + \boldsymbol{\sigma}(t)\boldsymbol{\zeta}(t)|^2 \\
&= \psi^2(t)\left|\boldsymbol{\sigma}(t)\left[\boldsymbol{\theta}(t) + \Big(\rho\sigma_1\Psi_1(t)\sqrt{m_1(t)}, \right.\right. \\
&\qquad \left.\left.\sigma_1\sqrt{1-\rho^2}\Psi_1(t)\sqrt{m_1(t)}, \sigma_2\Psi_2(t)\sqrt{m_2(t)}\Big)^\top\right]\right|^2 \\
&\geq \psi^2(t)\gamma^2(b_0 + \rho\sigma_1\Psi_1(t))^2 m_1^2(t) \geq 0.
\end{aligned} \tag{A9}$$

Note that (A8) holds true unless $|\boldsymbol{b}(t)\psi(t) + \boldsymbol{\sigma}(t)\boldsymbol{\zeta}(t)|^2 = 0$ almost surely, for all $t \in [0,T]$. It should be noted that (i) $\psi^2(t) > 1 > 0$ and $m_1(t) > 0$ thanks to the Feller condition (see Remark 3); (ii) the function $\Psi_1(t)$ is time-varying, thereby $b_0 + \rho\sigma_1\Psi_1(t)$ could not be zero for all $t \in [0,T]$. It then can be observed from (A9) that $\mathbb{P}(|\boldsymbol{b}(t)\psi(t) + \boldsymbol{\sigma}(t)\boldsymbol{\zeta}(t)|^2 = 0, \forall t \in [0,T]) < 1$. Therefore, (A8) must be true, implying Problem (P1) is feasible. ∎

A.3. Proof to Lemma 2

Proof We only show the existence and uniqueness results for the BSDE (18), because the same argument also applies to the BSDE (19).

Denoting $\tau := T - t$, we define the following functions for later use:

$$A(t) := \begin{cases} \dfrac{(2\alpha - b_0^2 - b_1^2)\sin(\delta_A\tau)}{\delta_A\cos(\delta_A\tau) + \frac{\kappa_A}{2}\sin(\delta_A\tau)}, & \Delta_A < 0, \\ \dfrac{(2\alpha - b_0^2 - b_1^2)\tau}{\delta_A + \frac{\kappa_A}{2}\tau}, & \Delta_A = 0, \\ \dfrac{(2\alpha - b_0^2 - b_1^2)\sinh(\delta_A\tau)}{\delta_A\cosh(\delta_A\tau) + \frac{\kappa_A}{2}\sinh(\delta_A\tau)}, & \Delta_A > 0, \end{cases} \tag{A10}$$

$$B(t) := \begin{cases} \dfrac{\left(2\beta - b_2^2\right)\sin(\delta_B \tau)}{\delta_B \cos(\delta_B \tau) + \frac{\kappa_B}{2}\sin(\delta_B \tau)}, & \Delta_B < 0, \\ \dfrac{\left(2\beta - b_2^2\right)\tau}{\delta_B + \frac{\kappa_B}{2}\tau}, & \Delta_B = 0, \\ \dfrac{\left(2\beta - b_2^2\right)\sinh(\delta_B \tau)}{\delta_B \cosh(\delta_B \tau) + \frac{\kappa_B}{2}\sinh(\delta_B \tau)}, & \Delta_B > 0, \end{cases} \tag{A11}$$

$$C(t) := C_1(t) + C_2(t), \tag{A12}$$

$$C_1(t) := \begin{cases} \dfrac{2\kappa_1 \overline{m}_1}{\sigma_1^2} \log\left|\cos(\delta_A \tau) + \dfrac{\kappa_A}{2\delta_A}\sin(\delta_A \tau)\right| \\ \quad - \dfrac{\kappa_1 \overline{m}_1 \kappa_A \tau}{\sigma_1^2}, & \Delta_A < 0, \\ \dfrac{2\kappa_1 \overline{m}_1}{\sigma_1^2} \log\left|1 + \dfrac{\kappa_A}{2\delta_A}\tau\right| - \dfrac{\kappa_1 \overline{m}_1 \kappa_A \tau}{\sigma_1^2}, & \Delta_A = 0, \\ \dfrac{2\kappa_1 \overline{m}_1}{\sigma_1^2} \log\left|\cosh(\delta_A \tau) + \dfrac{\kappa_A}{2\delta_A}\sinh(\delta_A \tau)\right| \\ \quad - \dfrac{\kappa_1 \overline{m}_1 \kappa_A \tau}{\sigma_1^2}, & \Delta_A > 0, \end{cases} \tag{A13}$$

$$C_2(t) := \begin{cases} \dfrac{2\kappa_2 \overline{m}_2}{\sigma_2^2} \log\left|\cos(\delta_B \tau) + \dfrac{\kappa_B}{2\delta_B}\sin(\delta_B \tau)\right| \\ \quad - \dfrac{\kappa_2 \overline{m}_2 \kappa_B \tau}{\sigma_2^2}, & \Delta_B < 0, \\ \dfrac{2\kappa_2 \overline{m}_2}{\sigma_2^2} \log\left|1 + \dfrac{\kappa_B}{2\delta_B}\tau\right| - \dfrac{\kappa_2 \overline{m}_2 \kappa_B \tau}{\sigma_2^2}, & \Delta_B = 0, \\ \dfrac{2\kappa_2 \overline{m}_2}{\sigma_2^2} \log\left|\cosh(\delta_B \tau) + \dfrac{\kappa_B}{2\delta_B}\sinh(\delta_B \tau)\right| \\ \quad - \dfrac{\kappa_2 \overline{m}_2 \kappa_B \tau}{\sigma_2^2}, & \Delta_B > 0, \end{cases} \tag{A14}$$

with parameters defined by

$$\begin{aligned} &\kappa_A := \kappa_1 + 2b_0\rho\sigma_1 + 2b_1\sqrt{1-\rho^2}\sigma_1, \\ &\Delta_A := \kappa_A^2 + 2\sigma_1^2(2\alpha - b_0^2 - b_1^2), \quad \delta_A := \frac{1}{2}\sqrt{|\Delta_A|}, \\ &\kappa_B := \kappa_2 + 2b_2\sigma_2, \quad \Delta_B := \kappa_B^2 + 2\sigma_2^2(2\beta - b_2^2), \\ &\delta_B := \frac{1}{2}\sqrt{|\Delta_B|}. \end{aligned} \tag{A15, A16}$$

Let us begin by studying an auxiliary BSDE:

$$\mathrm{d}\overline{h}_+(t) = \overline{\boldsymbol{g}}_+(t)\,\mathrm{d}\overline{\boldsymbol{W}}(t), \quad \overline{h}_+(T) = e^{\int_0^T(-2r(s)+|\boldsymbol{\theta}(s)|^2)\mathrm{d}s}. \tag{A17}$$

In (A17), $\overline{\boldsymbol{W}}(\cdot) := \{\overline{\boldsymbol{W}}(t)\}_{t\in[0,T]}$, in which $\overline{\boldsymbol{W}}(t) := \boldsymbol{W}(t) + 2\int_0^t \boldsymbol{\theta}(s)\,\mathrm{d}s$, is a standard Brownian motion under $\overline{\mathbb{P}}$ that is equivalent to the real-world measure $\mathbb{P}$ defined via

$$\left.\frac{\mathrm{d}\overline{\mathbb{P}}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t} = \exp\left\{-2\int_0^t |\boldsymbol{\theta}(s)|^2\mathrm{d}s - 2\int_0^t \boldsymbol{\theta}(s)^\top \mathrm{d}W(s)\right\}. \tag{A18}$$

Let $\overline{\mathbb{E}}[\cdot]$ denote the expectation operator under $\overline{\mathbb{P}}$. The terminal value $\overline{h}_+(T)$ is square integrable under $\overline{\mathbb{P}}$ because

$$\begin{aligned} \overline{\mathbb{E}}\left[|\overline{h}_+(T)|^2\right] &= \overline{\mathbb{E}}\left[e^{\int_0^T(-4r(s)+2|\boldsymbol{\theta}(s)|^2)\mathrm{d}s}\right] \\ &= \mathbb{E}\left[e^{-4\int_0^T r(s)ds - 2\int_0^T \boldsymbol{\theta}(s)^\top \mathrm{d}W(s)}\right] \\ &\le \mathbb{E}\left[e^{-2\int_0^T \boldsymbol{\theta}(s)^\top \mathrm{d}W(s)}\right] \le \left\{\mathbb{E}\left[e^{8\int_0^T |\boldsymbol{\theta}(s)|^2\mathrm{d}s}\right]\right\}^{\frac{1}{2}} < \infty, \end{aligned}$$

in which the last inequality is due to Condition (3) in Assumption 2, and the second last inequality follows from Theorem 43 in Chapter III of Protter (2005). As the driver of the auxiliary BSDE in (A17) is zero, it then follows from Theorem 2.1 in El Karoui *et al.* (1997) that (A17) admits a unique solution $(\overline{h}_+(\cdot), \overline{\boldsymbol{g}}_+(\cdot)) \in \mathcal{S}^2_{\mathbb{F},\overline{\mathbb{P}}}(0,T;\mathbb{R}) \times \mathcal{L}^2_{\mathbb{F},\overline{\mathbb{P}}}(0,T;\mathbb{R}^3)$. Furthermore, the standard comparison theorem implies $\overline{h}_+(t) > 0$ for all $t \in [0,T]$. An application of Itô's formula shows that $(h_+(\cdot), \boldsymbol{g}_+(\cdot))$, with

$$\begin{aligned} h_+(t) &= \frac{\exp\left\{\int_0^t(-2r(s) + |\boldsymbol{\theta}(s)|^2)\mathrm{d}s\right\}}{\overline{h}_+(t)} \quad \text{and} \\ \boldsymbol{g}_+(t) &= -\frac{\overline{\boldsymbol{g}}_+(t)\exp\left\{\int_0^t(-2r(s) + |\boldsymbol{\theta}(s)|^2)\mathrm{d}s\right\}}{\overline{h}_+^2(t)}, \end{aligned} \tag{A19}$$

solves the BSDE in (18). The above one-to-one relation between $(\overline{h}_+(\cdot), \overline{\boldsymbol{g}}_+(\cdot))$ and $(h_+(\cdot), \boldsymbol{g}_+(\cdot))$ implies that the pair defined in (A19) is the unique solution to the BSDE in (18); otherwise, the auxiliary BSDE in (A17) would also have more than one solution, contradicting the uniqueness result for (A17).

Having established the existence and uniqueness of a solution to the BSDE in (18), we proceed to show that such a unique solution is indeed given by (21)–(22). To that end, we guess an *ansatz* of $h_+(\cdot)$ in the following form:

$$h_+(t) = \exp\{A(t)m_1(t) + B(t)m_2(t) + C(t)\}, \quad t \in [0,T],$$

in which the deterministic functions A, B, and C are yet to be identified. By applying Itô's formula to the above form of $h_+(\cdot)$, we obtain

$$\begin{aligned} \mathrm{d}h_+(t) = h_+(t)\Bigg\{&\left[A'(t) - \kappa_1 A(t) + \frac{1}{2}\sigma_1^2 A^2(t)\right] m_1(t) \\ &+ \left[B'(t) - \kappa_2 B(t) + \frac{1}{2}\sigma_2^2 B^2(t)\right] m_2(t) \\ &+ \left[C'(t) + \kappa_1\overline{m}_1 A(t) + \kappa_2\overline{m}_2 B(t)\right]\Bigg\}\,\mathrm{d}t \\ &+ h_+(t)\left\{A(t)\sigma_1\sqrt{m_1(t)}\left[\rho\,\mathrm{d}W_0(t) + \sqrt{1-\rho^2}\,\mathrm{d}W_1(t)\right]\right. \\ &\left. + B(t)\sigma_2\sqrt{m_2(t)}\,\mathrm{d}W_2(t)\right\}, \end{aligned} \tag{A20}$$

in which $A'(t)$, $B'(t)$ and $C'(t)$ denote the corresponding derivatives with respect to t. Comparing the diffusion terms of (18) and (A20) immediately leads to the explicit expression of $\boldsymbol{g}_+(\cdot)$ in (22). Thus, the pair $(h_+(\cdot), \boldsymbol{g}_+(\cdot))$, given by (21)–(22), constitutes the unique solution to the BSDE (18), once we are able to determine the functions A, B and C, which is the remaining task of this proof.

To identify the functions A, B and C, we substitute $h_+(\cdot)$ by (21) and $\boldsymbol{g}_+(\cdot)$ by (22) into the BSDE in (18) and obtain the drift term (i.e. $\mathrm{d}t$ term) in the dynamics of $h_+(t)$ by (recall $r(t) = \alpha m_1(t) + \beta m_2(t)$ from (5))

$$\begin{aligned} h_+(t)\Bigg\{&\left[-2\alpha + b_0^2 + b_1^2 + \left(2b_0\rho\sigma_1 + 2b_1\sigma_1\sqrt{1-\rho^2}\right)A(t)\right. \\ &\left. + \sigma_1^2 A^2(t)\right] m_1(t) \\ &+ \left[-2\beta + b_2^2 + 2b_2\sigma_2 B(t) + \sigma_2^2 B^2(t)\right] m_2(t)\Bigg\}. \end{aligned}$$

Matching the coefficients of $m_1(t)$ and $m_2(t)$ in the above expression with those in the drift term of (A20) yields three ODEs:

$$A'(t) - \kappa_A A(t) - \frac{1}{2}\sigma_1^2 A^2(t) + 2\alpha - b_0^2 - b_1^2 = 0, \quad A(T) = 0, \tag{A21}$$

$$B'(t) - \kappa_B B(t) - \frac{1}{2}\sigma_2^2 B^2(t) + 2\beta - b_2^2 = 0, \quad B(T) = 0, \tag{A22}$$

$$C'(t) + \kappa_1\overline{m}_1 A(t) + \kappa_2\overline{m}_2 B(t) = 0, \quad C(T) = 0, \tag{A23}$$

in which $\kappa_A = \kappa_1 + 2b_0\rho\sigma_1 + 2b_1\sigma_1\sqrt{1-\rho^2}$ and $\kappa_B = \kappa_2 + 2b_2\sigma_2$ are defined in (A15) and (A16). Notice that both (A21) and (A22) are Riccati equations, while (A23) is a linear ODE, whose solution depends on those of (A21) and (A22).

To solve (A21), we guess an *ansatz* of the solution in the following parametric form:

$$A(t) = \frac{R_2(T-t)}{R_1(T-t)} = \frac{R_2(\tau)}{R_1(\tau)}, \quad \text{with } \tau := T-t, \tag{A24}$$

in which R_1 and R_2 are two functions that satisfy $R_1(0) = 1$ and $R_2(0) = 0$. Substituting (A24) into (A21) gives

$$\frac{1}{R_1(\tau)}\left\{\left[-R_2'(\tau) - \kappa_A R_2(\tau) + (2\alpha - b_0^2 - b_1^2)R_1(\tau)\right] + A(t)\left[R_1'(\tau) - \frac{1}{2}\sigma_1^2 R_2(\tau)\right]\right\} = 0.$$

Setting the coefficients of the zero-th and the first order terms of A in the above equation to zero, we get

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\begin{pmatrix} R_1(\tau) \\ R_2(\tau) \end{pmatrix} = \boldsymbol{M}\cdot\begin{pmatrix} R_1(\tau) \\ R_2(\tau) \end{pmatrix}, \quad \text{with } \boldsymbol{M} := \begin{pmatrix} 0 & \frac{1}{2}\sigma_1^2 \\ 2\alpha - b_0^2 - b_1^2 & -\kappa_A \end{pmatrix}.$$

The vector solution to the above ODE system is obtained by

$$\begin{pmatrix} R_1(\tau) \\ R_2(\tau) \end{pmatrix} = \exp(\tau\boldsymbol{M})\cdot\begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{A25}$$

Straightforward but tedious calculation shows that the exponential matrix $\exp(\tau\boldsymbol{M})$ has the following explicit expression

$$\exp(\tau\boldsymbol{M}) = \begin{cases} e^{-\frac{1}{2}\kappa_A\tau}\begin{pmatrix} \cos(\delta_A\tau) + \frac{\kappa_A}{2\delta_A}\sin(\delta_A\tau) & \frac{\sigma_1^2}{2\delta_A}\sin(\delta_A\tau) \\ \frac{2\alpha - b_0^2 - b_1^2}{\delta_A}\sin(\delta_A\tau) & \cos(\delta_A\tau) - \frac{\kappa_A}{2\delta_A}\sin(\delta_A\tau) \end{pmatrix}, & \Delta_A < 0, \\ e^{-\frac{1}{2}\kappa_A\tau}\begin{pmatrix} 1 + \frac{\kappa_A}{2\delta_A}\tau & \frac{\sigma_1^2}{2\delta_A}\tau \\ \frac{2\alpha - b_0^2 - b_1^2}{\delta_A}\tau & 1 - \frac{\kappa_A}{2\delta_A}\tau \end{pmatrix}, & \Delta_A = 0, \\ e^{-\frac{1}{2}\kappa_A\tau}\begin{pmatrix} \cosh(\delta_A\tau) + \frac{\kappa_A}{2\delta_A}\sinh(\delta_A\tau) & \frac{\sigma_1^2}{2\delta_A}\sinh(\delta_A\tau) \\ \frac{2\alpha - b_0^2 - b_1^2}{\delta_A}\sinh(\delta_A\tau) & \cosh(\delta_A\tau) - \frac{\kappa_A}{2\delta_A}\sinh(\delta_A\tau) \end{pmatrix}, & \Delta_A > 0, \end{cases} \tag{A26}$$

in which $\Delta_A = \kappa_A^2 + 2\sigma_1^2(2\alpha - b_0^2\gamma^2 - b_1^2)$ and $\delta_A = \frac{1}{2}\sqrt{|\Delta_A|}$ are defined in (A15). Combining (A24), (A25) and (A26) leads to the expression of the function A in (A10). Following the same analysis to solve the ODE in (A22) of the function B leads to the solution in (A11).

In the last step, we solve the linear ODE in (A23) to determine C. Integrating (A23) yields

$$C(t) = \underbrace{\kappa_1\overline{m}_1\int_t^T A(s)\,\mathrm{d}s}_{:=C_1(t)} + \underbrace{\kappa_2\overline{m}_2\int_t^T B(s)\,\mathrm{d}s}_{:=C_2(t)}.$$

For convenience, let us introduce a generic variable $\nu := T - s$, for any $s \in [t, T]$. Using (A24)–(A26) and $R_1(0) = 1$, we compute the integral in $C_1(t)$ by

$$\int_t^T A(s)\,\mathrm{d}s = \int_t^T \frac{R_2(T-s)}{R_1(T-s)}\,\mathrm{d}s = \int_0^{T-t}\frac{R_2(\nu)}{R_1(\nu)}\,\mathrm{d}\nu = \frac{2}{\sigma_1^2}\int_0^{T-t}\frac{\mathrm{d}R_1(\nu)/\mathrm{d}\nu}{R_1(\nu)}\,\mathrm{d}\nu = \frac{2}{\sigma_1^2}\log|R_1(\tau)|,$$

which, along with (A25) and (A26), leads to the expression of $C_1(t)$ in (A13). Similarly, we compute the integral in $C_2(t)$ and obtain (A14). As a result, $C(t)$ is confirmed as in (A12).

Finally, by comparing (18) and (19), we observe that

$$h_-(t) = h_+(t)\exp\left\{-\frac{\eta_r^2 a_Z^2}{\sigma_Z^2}(T-t)\right\} \quad \text{and}$$

$$\boldsymbol{g}_-(t) = \boldsymbol{g}_+(t)\exp\left\{-\frac{\eta_r^2 a_Z^2}{\sigma_Z^2}(T-t)\right\},$$

which shows the unique solution to the BSDE in (19) is given by (23)–(24). ■

A.4. Proof to Lemma 3

Proof Let us first consider an auxiliary BSDE, defined by

$$\begin{aligned} \mathrm{d}\widetilde{H}(t) &= e^{-\int_0^t r(s)\,\mathrm{d}s}(\eta_r - \eta)a_Z\,\mathrm{d}t + \widetilde{\boldsymbol{G}}(t)^\top\,\mathrm{d}\widetilde{\boldsymbol{W}}(t), \\ \widetilde{H}(T) &= \lambda e^{-\int_0^T r(s)\,\mathrm{d}s}, \end{aligned} \tag{A27}$$

which satisfies the Lipschitz continuous condition and thereby has a unique solution $(\widetilde{H}(\cdot), \widetilde{\boldsymbol{G}}(\cdot))$, according to Theorem 2.1 in El Karoui *et al.* (1997). In addition, we are able to obtain $\widetilde{H}(\cdot)$ explicitly as follows:

$$\widetilde{H}(t) = \lambda\,\widetilde{\mathbb{E}}_t\left[e^{-\int_0^T r(s)\,\mathrm{d}s}\right] + (\eta_r - \eta)a_Z\,\widetilde{\mathbb{E}}_t\left[\int_t^T e^{-\int_0^s r(\nu)\,\mathrm{d}\nu}\,\mathrm{d}s\right], \quad t \in [0,T]. \tag{A28}$$

By the product rule, we can show that $(H(\cdot), \boldsymbol{G}(\cdot))$, defined by

$$H(t) := e^{\int_0^t r(s)\,\mathrm{d}s}\widetilde{H}(t) \quad \text{and} \quad \boldsymbol{G}(t) := e^{\int_0^t r(s)\,\mathrm{d}s}\widetilde{\boldsymbol{G}}(t), \quad t \in [0,T], \tag{A29}$$

is a solution to the linear BSDE in (20). The uniqueness result then follows from the one-to-one relation between $(H(\cdot), \boldsymbol{G}(\cdot))$ and $(\widetilde{H}(\cdot), \widetilde{\boldsymbol{G}}(\cdot))$ in (A29), and the fact that the latter is a unique solution to the auxiliary BSDE in (A27).

Using (A28) and (A29), we easily observe

$$H(t) = \lambda\widetilde{\mathbb{E}}_t\left[e^{-\int_t^T r(s)\,\mathrm{d}s}\right] + (\eta_r - \eta)a_Z\widetilde{\mathbb{E}}_t\left[\int_t^T e^{-\int_t^s r(\nu)\,\mathrm{d}\nu}\,\mathrm{d}s\right],$$

which leads to the expression in (26) after recalling the definition of $P(t,s)$ in (25). Applying Itô's formula to $H(t)$ and using (A5), we obtain

$$\begin{aligned} \mathrm{d}H(t) = \text{drift term} &- \sigma_1\rho\sqrt{m_1(t)}\left[\lambda N_1(t,T)P(t,T) + (\eta_r - \eta)a_Z\int_t^T N_1(t,s)P(t,s)\,\mathrm{d}s\right]\mathrm{d}W_0(t) \\ &- \sigma_1\sqrt{1-\rho^2}\sqrt{m_1(t)}\left[\lambda N_1(t,T)P(t,T) + (\eta_r - \eta)a_Z\int_t^T N_1(t,s)P(t,s)\,\mathrm{d}s\right]\mathrm{d}W_1(t) \\ &- \sigma_2\sqrt{m_2(t)}\left[\lambda N_2(t,T)P(t,T) + (\eta_r - \eta)a_Z\int_t^T N_2(t,s)P(t,s)\,\mathrm{d}s\right]\mathrm{d}W_2(t). \end{aligned}$$

By matching the three diffusion terms above with those in (20), we conclude that the components $G_0(\cdot)$, $G_1(\cdot)$ and $G_2(\cdot)$ of $\boldsymbol{G}(\cdot)$ are given by (27), (28) and (29), respectively. ■

Appendix 2. Model calibration

In this section, we provide a step-by-step estimation procedure that we use to calibrate the financial model in Section 5.1.

We first derive the covariance between the log-stock price $\log S(\cdot)$ and the log-bond price $\log P(\cdot, T_i)$ and use the result later when we estimate the predicted state covariance. By Itô's formula, we obtain the dynamics of $\log S(\cdot)$ and $\log P(\cdot, T_i)$ by

$$\begin{aligned}
\mathrm{d}\log S(t) &= \left[r(t) + \theta_0(t)\sqrt{v(t)} - \frac{1}{2}v(t)\right]\mathrm{d}t + \sqrt{v(t)}\,\mathrm{d}W_0(t),\\
\mathrm{d}\log P(t,T_i) &= \Bigg[r(t) - \left(b_0\rho + b_1\sqrt{1-\rho^2}\right)\sigma_1 N_1(t,T_i)m_1(t)\\
&\quad - b_2\sigma_2 N_2(t,T_i)m_2(t)\\
&\quad + 0.5\sigma_1^2 N_1^2(t,T_i)m_1(t) + 0.5\sigma_2^2 N_2^2(t,T_i)m_2(t)\Bigg]\mathrm{d}t\\
&\quad - \sigma_1 N_1(t,T_i)\sqrt{m_1(t)}\left(\rho\,\mathrm{d}W_0(t)\right.\\
&\quad \left. + \sqrt{1-\rho^2}\,\mathrm{d}W_1(t)\right) - \sigma_2 N_2(t,T_i)\sqrt{m_2(t)}\,\mathrm{d}W_2(t).
\end{aligned}$$

Note that for $i = 1, 2$,

$$\mathbb{E}_t\left[m_i(s)\right] = \overline{m}_i + (m_i(t) - \overline{m}_i)\,e^{-\kappa_i(s-t)}, \quad s \geq t \geq 0.$$

Using the above results, we obtain the following covariances:

$$\begin{aligned}
\Sigma^S_{t+1\,|\,t} &:= \mathrm{Cov}_t\left(\log S(t+1), \log S(t+1)\right)\\
&= \mathbb{E}_t\left[\int_t^{t+1} v(s)\mathrm{d}s\right] = \gamma^2\mathbb{E}_t\left[\int_t^{t+1} m_1(s)\mathrm{d}s\right]\\
&= \gamma^2\left[\overline{m}_1 + (m_1(t) - \overline{m}_1)\frac{1-e^{-\kappa_1}}{\kappa_1}\right],\\
\Sigma^P_{t+1\,|\,t} &:= \mathrm{Cov}_t\left(\log P(t+1,T_i), \log P(t+1,T_j)\right)\\
&= \mathbb{E}_t\left[\int_t^{t+1}\left(\sigma_1^2 N_1(s,T_i)N_1(s,T_j)m_1(s)\right.\right.\\
&\quad \left.\left. + \sigma_2^2 N_2(s,T_i)N_2(s,T_j)m_2(s)\right)\mathrm{d}s\right]\\
&= \int_t^{t+1}\left(\overline{m}_1\sigma_1^2 N_1(s,T_i)N_1(s,T_j)\right.\\
&\quad \left. + \overline{m}_2\sigma_2^2 N_2(s,T_i)N_2(s,T_j)\right)\mathrm{d}s\\
&\quad + (m_1(t) - \overline{m}_1)\int_t^{t+1}\sigma_1^2 N_1(s,T_i)N_1(s,T_j)e^{-\kappa_1(s-t)}\mathrm{d}s\\
&\quad + (m_2(t) - \overline{m}_2)\int_t^{t+1}\sigma_2^2 N_2(s,T_i)N_2(s,T_j)e^{-\kappa_2(s-t)}\mathrm{d}s,
\end{aligned}$$

and

$$\begin{aligned}
\Sigma^{SP}_{t+1\,|\,t} &:= \mathrm{Cov}_t\left(\log S(t+1), \log P(t+1,T_i)\right)\\
&= -\gamma\rho\sigma_1\mathbb{E}_t\left[\int_t^{t+1} N_1(s,T_i)m_1(s)ds\right]\\
&= -\gamma\rho\sigma_1\left[\overline{m}_1\int_t^{t+1} N_1(s,T_i)\mathrm{d}s\right.\\
&\quad \left. + (m_1(t) - \overline{m}_1)\int_t^{t+1} N_1(s,T_i)e^{-\kappa_1(s-t)}\mathrm{d}s\right],
\end{aligned}$$

in which $i, j = 1, 2$.

Recall that functions $N_1(\cdot, T_i)$ and $N_2(\cdot, T_i)$ are defined by (A2) and (A3), respectively. The integrals involved in $\Sigma^P_{t+1\,|\,t}$ and $\Sigma^{SP}_{t+1\,|\,t}$ do not have explicit representations, and thereby a numerical method is needed to evaluate them. The only exception is $\int_t^{t+1} N_1(s,T_i)\mathrm{d}s$ in $\Sigma^{SP}_{t+1\,|\,t}$, which can be explicitly calculated by

$$\begin{aligned}
&\int_t^{t+1} N_1(s,T_i)\mathrm{d}s\\
&= \int_t^{t+1}\frac{2\alpha(e^{\Delta_1(T_i-s)} - 1)}{(\Delta_1 + \widetilde{\kappa}_1)(e^{\Delta_1(T_i-s)} - 1) + 2\Delta_1}\mathrm{d}s\\
&= \frac{2}{\sigma_1^2}\log\left(\frac{2\Delta_1\, e^{(\Delta_1+\widetilde{\kappa}_1)(T_i-s)/2}}{(\Delta_1 + \widetilde{\kappa}_1)(e^{\Delta_1(T_i-s)} - 1) + 2\Delta_1}\right)\Bigg|_{s=t}^{t+1}\\
&= \frac{2}{\sigma_1^2}\log\left(e^{-(\Delta_1+\widetilde{\kappa}_1)/2}\cdot\frac{(\Delta_1 + \widetilde{\kappa}_1)(e^{\Delta_1(T_i-t)} - 1) + 2\Delta_1}{(\Delta_1 + \widetilde{\kappa}_1)(e^{\Delta_1(T_i-t-1)} - 1) + 2\Delta_1}\right).
\end{aligned}$$

We now describe in full details how we estimate all the parameters of the financial model. The two square-root processes $m_1(\cdot)$ and $m_2(\cdot)$, defined in (3)–(4), are latent processes, and we use the unscented Kalman filter (UKF) to estimate the values of the latent variables to accelerate the speed of calibration and combine it with the maximum log-likelihood estimate method (MLE). We directly apply the MLE method to estimate the parameters involved in the observable processes (based on the data of S&P 500 and the US treasury bonds).

In order to search the maximum value of the total log-likelihood function, we employ the Nelder-Mead simplex algorithm which is a derivative-free method. We use this algorithm because it can be applied in the non-linear objective function which can be either continuous or discontinuous Wouk (see 1987). According to Ait-Sahalia and Kimmel (2007), the log-likelihood function at time $t+1$ conditional on time t is defined by

$$\begin{aligned}
l_{t+1\,|\,t}(\boldsymbol{\Theta}) = -\frac{1}{2}\Big\{&n_t\log(2\pi) + \log|\boldsymbol{\Sigma}^P_{t+1|t}| + (R_{t+1} - \hat{R}_{t+1})^2\\
&+ \left(\boldsymbol{P}_{t+1} - \hat{\boldsymbol{P}}_{t+1}\right)^\top\left(\boldsymbol{\Sigma}^P_{t+1\,|\,t}\right)^{-1}\left(\boldsymbol{P}_{t+1} - \hat{\boldsymbol{P}}_{t+1}\right)\Big\},
\end{aligned}$$

in which n_t denotes the total number of bond contracts during day t, R_{t+1} (resp. $\hat{R}_{t+1}$) is the actual (resp. estimated) stock return in day $t+1$, $\boldsymbol{P}_{t+1}$ (resp. $\hat{\boldsymbol{P}}_{t+1}$) is the vector of actual (resp. estimated) bond prices in day $t+1$, and $\Sigma^P_{t+1\,|\,t}$ refers to the covariance of bond prices conditional on time t. All the parameters in the model are stacked in $\boldsymbol{\Theta}$. The total log-likelihood function is the sum of all daily log-likelihood and reads as

$$\mathbb{L}(\boldsymbol{\Theta}) = \sum_{t=1}^{T} l_{t+1\,|\,t}(\boldsymbol{\Theta}),$$

in which T is the total sample period.

Denote the latent processes $m_1(\cdot)$ and $m_2(\cdot)$ in vector form $\mathbf{X}_t := (m_1(t), m_2(t))^\top$. Firstly, we discretize the latent processes by the Euler discretization method. Based on their dynamics in (3)–(4), we obtain the corresponding discrete version by $m_i(t+\Delta t) = (1 - \kappa_i\Delta t)m_i(t) + \kappa_i\overline{m}_i\,\Delta t + \sigma_i\sqrt{m_i(t)}w_i(t)$, $i = 1, 2$, in which $w_1(t)$ and $w_2(t)$ are independent increments of Brownian motions.

Secondly, the measurement errors between our estimated prices and market prices are considered in the procedure. Then, we let the measurement function be

$$\mathbf{Y}_t = \mathbf{f}(\mathbf{X_t}; R_t, \boldsymbol{\Theta}) + \boldsymbol{\epsilon}_t, \tag{A30}$$

in which the vector $\mathbf{Y}_t$ represents the market bond prices and the stock returns, the vector $\mathbf{f}$ denotes their estimated values, and $\boldsymbol{\epsilon}_t$ is the term for measurement errors. If there are a total of n_t bond contracts and one stock return at time t, then ϵ_t is an (n_t+1)-dimensional random vector, which follows an independent multivariate normal distribution $\mathcal{N}(\mathbf{0}, \Sigma^\epsilon_t)$, with the covariance matrix Σ^ϵ_t given by $\sigma^2_\epsilon \cdot \mathbf{I}_{n_t+1}$. In optimization, we employ a quasi-Newton algorithm, and the Hessian matrix is updated by Broyden-Fletcher-Goldfarb-Shanno

(BFGS) method. Specifically, we use the UKF method to estimate the latent state vector $\mathbf{X}_t$ with the non-linear measurement vector $\mathbf{f}$.

In the prediction step, we apply the linear forecast. Based on the information up to time $t-1$, we denote that the predicted state mean as $\hat{\mathbf{X}}_{t|t-1}$ and predicted state covariance as $\hat{\Sigma}^{\mathbf{X}}_{t|t-1}$; see Grewal and Andrews (2015) for a detailed discussion on linear forecasts.

In the update step, we employ scaled unscented transformation. As in Särkkä (2007), we generate $2n+1$ sigma points, in which n is the dimension of $\mathbf{X}_t$. These sigma points are computed as follows:

$$\begin{aligned}
\mathscr{S}_t^0 &= \hat{\mathbf{X}}_{t|t-1},\\
\mathscr{S}_t^i &= \hat{\mathbf{X}}_{t|t-1} + \left[\mathrm{Chol}\left(\left(n+\tilde{\lambda}\right)\hat{\Sigma}^{\mathbf{X}}_{t|t-1}\right)\right]_i, \quad i = 1,\dots,n,\\
\mathscr{S}_t^i &= \hat{\mathbf{X}}_{t|t-1} - \left[\mathrm{Chol}\left(\left(n+\tilde{\lambda}\right)\hat{\Sigma}^{\mathbf{X}}_{t|t-1}\right)\right]_{i-n},\\
&\quad i = n+1,\dots,2n.
\end{aligned} \tag{A31}$$

with the weights

$$\begin{aligned}
W_0^{(m)} &= \frac{\tilde{\lambda}}{n+\tilde{\lambda}}, \quad W_0^{(\Sigma)} = \frac{\tilde{\lambda}}{n+\tilde{\lambda}} + \left(1-\tilde{\alpha}^2+\tilde{\beta}\right),\\
W_i^{(m)} &= W_i^{(\Sigma)} = \frac{1}{2\left(n+\tilde{\lambda}\right)}, \quad i = 1,\dots,2n,
\end{aligned}$$

in which $\mathrm{Chol}(\cdot)$ denotes the Cholesky factorization, $[\cdot]_i$ denotes the ith column of a matrix, and $\tilde{\lambda} = \tilde{\alpha}^2(n+\tilde{\beta})$. For the normal distribution, the optimal value of $\tilde{\beta}$ is 2. We choose $\tilde{\alpha}$ to be a small positive number 10^{-3}.

Then, the predicted mean of the measurements (for measurement function see equation (A30)), variance of the measurements, and covariance of the latent variables and measurements can be computed by using the sigma points (A31) as

$$\begin{aligned}
\hat{\mathbf{Y}}_{t|t-1} &= \sum_{i=0}^{2n} W_i^{(m)} \mathbf{f}(\mathscr{S}_t^i; R_t, \Theta),\\
\hat{\Sigma}^{\mathbf{Y}}_{t|t-1} &= \sum_{i=0}^{2n} W_i^{(\Sigma)} \left(\mathbf{f}(\mathscr{S}_t^i; R_t, \Theta) - \hat{\mathbf{Y}}_{t|t-1}\right)\\
&\quad \times \left(\mathbf{f}(\mathscr{S}_t^i; R_t, \Theta) - \hat{\mathbf{Y}}_{t|t-1}\right)^{\top} + \Sigma_t^{\epsilon},\\
\hat{\Sigma}^{\mathbf{X},\mathbf{Y}}_{t|t-1} &= \sum_{i=0}^{2n} W_i^{(\Sigma)} \left(\mathscr{S}_t^i - \hat{\mathbf{X}}_{t|t-1}\right)\left(\mathbf{f}(\mathscr{S}_t^i; R_t, \Theta) - \hat{\mathbf{Y}}_{t|t-1}\right)^{\top}.
\end{aligned}$$

Finally, we calculate the Kalman filter gain by $K_t = \hat{\Sigma}^{\mathbf{X},\mathbf{Y}}_{t|t-1}(\hat{\Sigma}^{\mathbf{Y}}_{t|t-1})^{-1}$, the state mean by $\hat{\mathbf{X}}_t = \hat{\mathbf{X}}_{t|t-1} + K_t(\mathbf{Y}_t - \hat{\mathbf{Y}}_{t|t-1})$, and state covariance by $\hat{\Sigma}^{\mathbf{X}}_t = \hat{\Sigma}^{\mathbf{X}}_{t|t-1} - K_t\hat{\Sigma}^{\mathbf{Y}}_{t|t-1}K_t^{\top}$.