Overview of V.A. Plotnikov 's research on averaging of differential inclusions ☆

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Abstract

In this review we will first look in detail at V.A. Plotnikov's results on the substantiation of full and partial schemes of averaging for differential inclusions in the standard form on final and infinite interval. Then we will consider the algorithms where there is no average, but there is a possibility to find its estimation from below and from above. Such approach is also used when the detection of an average is approximate. This situation is especially typical at consideration of differential inclusions with fast and slow variables. In the last part we will give the results concerning the substantiation of the full and partial averaging method for impulsive differential inclusions on final and infinite intervals.

Keywords: differential inclusion differential inclusions with impulses averaging method

Introduction

Many important problems of analytical dynamics are described by the nonlinear mathematical models that as a rule are presented by the nonlinear differential or integro - differential equations. The absence of exact universal research methods for nonlinear systems has caused the development of numerous approximate analytic and numerically-analytic methods that can be realized in effective computer algorithms.

All these methods are constructed by an iterative principle, i.e. either consecutive approximations or chains of consecutive transformations of phase variables or functional series with members decreasing on size, etc. are used. It means that first somehow the initial approximation is chosen then the additives of various order are found using the iterations to approach the true solution. This rule is especially effective at research of the mathematical models described by regular on small parameters nonlinear equations. Also there exist various methods of the initial approximation choice: solving of some linear problem (the linearization method) or solving of some nonlinear but essentially more "simple" system (often the averaging method).

Recently, the averaging methods combined with the asymptotic representations (in Poincare sense) began to be applied as the basic constructive tool for solving the complicated problems of analytical dynamics described by the differential equations. It became possible due to the works of N.N. Bogolyubov, Yu.A. Mitropolskij, A.M. Samojlenko, V.M. Volosov, E.A. Grebennikov, M.A. Krasnoselskiy, S.G. Krein, A.N. Filatov, etc. The application of the averaging method in optimal control problems contains in the works of N.N. Moiseev, V.N. Lebedev, F.L. Chernousko, L.D. Akulenko, V.A. Plotnikov, etc.

The development of the theory of differential inclusions began from the works of T. Wazewski and A.F. Filippov in which the basic results on existence and properties of the solutions of the differential inclusions have been received. The differential inclusions are valuable not only as the generalization of the theory of the differential equations, but also for their numerous applications to the research of optimal control problems, the game theory and economics. The possibility of the application of the averaging method in the theory of differential inclusions was considered by V.A. Plotnikov.

Victor Aleksandrovich Plotnikov was born on January 5, 1938 in Leningrad (nowadays St. Petersburg). During the World war II he was the inhabitant of blockade Leningrad. Then in 1944 the family moved to Odessa. In 1960 V.A. Plotnikov graduated from Odessa State University named after I.I. Mechnikov, where afterwards worked in positions of the assistant, associate professor, department chief and the dean up to his death on September 4, 2006. In 1969 V.A. Plotnikov defended the kandidat thesis "Research of a class of optimal control problems for systems with two degrees of freedom" in Odessa State University and in 1980 defended the doctoral thesis "Asymptotical methods in optimal control problems" in Leningrad State University. V.A. Plotnikov's scientific works cover a wide range of complex and actual problems in the theory of differential equations and optimal control that concern a new direction of these theories - the differential equations with multivalued and discontinuous right-hand side, the quasidifferential equations in the metric spaces. V.A. Plotnikov developed the algorithms of asymptotic solving for quite a wide class of differential inclusions and proved deep theorems by N.N. Bogolyubov and A.N. Tikhonov on a substantiation of the asymptotic methods for the differential equations with the multivalued and discon-

☆Overview of V.A. Plotnikov 's research

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tinuous right-hand side and the quasidifferential equations, developed algorithms of numerically asymptotical solving of the control problems, proved the theorems of existence and uniqueness of solutions of the quasidifferential equations in locally compact and full metric spaces. The achievements in this direction initiated the mathematical researches of asymptotical methods in the theory of the differential inclusions in Russia, Belarus, Bulgaria, Poland, France, the USA, etc. V.A. Plotnikov published over 250 scientific nworks, including 6 monographes [1, 2, 3, 4, 5, 6].

In this review we will first look in detail at V.A. Plotnikov's results on the substantiation of full and partial schemes of averaging for differential inclusions in the standard form on final and infinite interval. Then we will consider the algorithms where there is no average, but there is a possibility to find its estimation from below and from above. Such approach is also used when the detection of an average is approximate. This situation is especially typical at consideration of differential inclusions with fast and slow variables. In the last part we will give the results concerning the substantiation of the full and partial averaging method for impulsive differential inclusions on final and infinite intervals.

1. The averaging of differential inclusions

For differential inclusions the theorem which is the analogue of the first N.N. Bogolyubov's theorem has been proved by V.A. Plotnikov in [3, 7, 8]. It became a push for the further development of the given method for this type of the equations.

1.1. The full averaging scheme

1.1.1. The averaging on the finite interval

Consider the differential inclusion

$$\dot{x} \in \varepsilon X(t,x), \quad x(0) = x_0, \tag{1}$$

where $t \in \mathbb{R}^+$ is time, $x \in \mathbb{R}^n$ is a phase vector, $\varepsilon > 0$ is a small parameter, $X : \mathbb{R}^+ \times \mathbb{R}^n \to comp(\mathbb{R}^n)$ is a multivalued mapping, $comp(\mathbb{R}^n)$ ($conv(\mathbb{R}^n)$) is the set of all nonempty compact (and convex) subsets of $\mathbb{R}^n$ with Hausdorff metric:

$$h(A,B) = \min\{r \geq 0 \;:\; A \subset B + S_r(0), B \subset A + S_r(0)\},$$

$S_r(a)$ is the ball in $\mathbb{R}^n$ with radius $r \geq 0$ and center in the point $a \in \mathbb{R}^n$.

Let us associate with the inclusion (1) the following averaged differential inclusion

$$\dot{\xi} \in \varepsilon \overline{X}(\xi), \quad \xi(0) = x_0, \tag{2}$$

where

$$\overline{X}(x) = \lim_{T\to\infty} \frac{1}{T} \int_0^T X(t,x)\,dt. \tag{3}$$

Here the integral of the multivalued mapping is understood in Aumann sense [9] and the convergence - in sense of the Hausdorff metric.

Theorem 1. *[3, 7]. Let in the domain $Q = \{t \geq 0, x \in D \subset \mathbb{R}^n\}$ the following hold:*

1) *the mapping $X(t,x)$ is continuous, uniformly bounded with constant M, satisfies the Lipschitz condition in x with constant λ;*
2) *uniformly with respect to x in the domain D the limit (3) exists;*
3) *for any $x_0 \in D' \subset D$ and $t \geq 0$ the solutions of the inclusion (2) together with a ρ−neighborhood belong to the domain D.*

Then for any $\eta \in (0,\rho]$ and $L > 0$ there exists $\varepsilon^0(\eta, L) > 0$ such that for all $\varepsilon \in (0,\varepsilon^0]$ and $t \in [0, L\varepsilon^{-1}]$ the following statements fulfill:

1) *for any solution $\xi(t)$ of the inclusion (2) there exists a solution $x(t)$ of the inclusion (1) such that*

$$\|x(t) - \xi(t)\| \leq \eta; \tag{4}$$

2) *for any solution $x(t)$ of the inclusion (1) there exists a solution $\xi(t)$ of the inclusion (2) such that the inequality (4) holds.*

Thereby,

$$h(\overline{R}(t), clR(t)) \leq \eta, \tag{5}$$

where $\overline{R}(t)$ is the section of the family of the solutions of the averaged inclusion, $clR(t)$ is the closure of the section of the family of the solutions of the initial inclusion.

Proof. Using the conditions 1), 2) and the properties of Aumann's integral we obtain that the set $\overline{X}(x)$ is convex and compact. Besides

$$\overline{X}(x) = \lim_{T\to\infty} \frac{1}{T} \int_0^T coX(t,x)\,dt,$$

so the differential inclusion (2) is also averaged to the differential inclusion

$$\dot{\overline{x}} \in \varepsilon coX(t,\overline{x}), \quad \overline{x}(0) = x_0. \tag{6}$$

From the conditions 1), 2) follows that the multivalued mapping $\bar{X}(x)$ is uniformly bounded with constant M and satisfies the Lipschitz condition with constant λ. Really in view of the condition 2) of the theorem for any $\delta > 0$ it is possible to find $T(\delta) > 0$ such that for all $T > T(\delta)$ the estimate is fair:

$$h\left(\overline{X}(x), \frac{1}{T}\int_0^T X(t,x)dt\right) < \delta.$$

Then choosing $T > T(\delta)$ we obtain

$$\left|\overline{X}(x)\right| = h\left(\overline{X}(x), \{0\}\right) \leq$$

$$\leq h\left(\overline{X}(x), \frac{1}{T}\int_0^T X(t,x)dt\right) + h\left(\frac{1}{T}\int_0^T X(t,x)dt, \{0\}\right) <$$

$$< \delta + \frac{1}{T}\int_0^T h\left(X(t,x),\{0\}\right)dt \le \delta + M,$$

$$h(\overline{X}(x'),\overline{X}(x'')) \le h\left(\overline{X}(x'),\frac{1}{T}\int_0^T X(t,x')dt\right) +$$

$$+h\left(\frac{1}{T}\int_0^T X(t,x')dt,\frac{1}{T}\int_0^T X(t,x'')dt\right)+$$

$$+h\left(\frac{1}{T}\int_0^T X(t,x'')dt,\overline{X}(x'')\right) <$$

$$< 2\delta + \frac{1}{T}\int_0^T h\left(X(t,x'),X(t,x'')\right)dt \le$$

$$\le 2\delta + \frac{1}{T}\int_0^T \lambda \left\|x'-x''\right\| dt \le 2\delta + \lambda\left\|x'-x''\right\|.$$

As the value δ is chosen arbitrarily, in a limit we will receive:

$$\left|\overline{X}(x)\right| \le M, \quad h(\overline{X}(x'),\overline{X}(x'')) \le \lambda\left\|x'-x''\right\|.$$

The solutions of the inclusions (1), (2), (6) exist and are continuable on an interval $[0, L\varepsilon^{-1}]$. According to [10] the family of solutions $H_1(x_0)$ of the inclusion (1) is everywhere dense in the compact set $H(x_0)$ of the family of solutions of the inclusion (6).

Hence, it is enough to prove the theorem for the inclusions with the convex right-hand side.

The families of the solutions of the inclusions (2) and (6), and also their sections $\overline{R}(t)$ and $clR(t)$ accordingly, are compact sets [11].

Let us prove the first statement of the theorem and hence the validity of the inclusion

$$\overline{R}(t) \subset S_\eta(clR(t)). \tag{7}$$

Divide the interval $[0, L\varepsilon^{-1}]$ on the partial intervals with the points $t_i = \frac{Li}{m\varepsilon}$, $i = \overline{0,m}$, $m \in \mathbb{N}$. Let $\xi(t)$ be a solution of the inclusion (2). Then there exists a measurable selector $v(t) \in \overline{X}(\xi(t))$ such that

$$\xi(t) = \xi(t_i) + \varepsilon\int_{t_i}^t v(\tau)d\tau, \ \ t\in[t_i,t_{i+1}], \ \ \xi(0)=x_0. \tag{8}$$

Consider the function

$$\xi^1(t) = \xi^1(t_i) + \varepsilon v_i(t-t_i), \ \ t\in[t_i,t_{i+1}], \ \ \xi^1(0)=x_0, \tag{9}$$

where vector v_i satisfies the condition

$$\left\|\frac{L}{m\varepsilon}v_i - \int_{t_i}^{t_{i+1}} v(t)dt\right\| = \min_{v\in\overline{X}(\xi^1(t_i))}\left\|\frac{L}{m\varepsilon}v - \int_{t_i}^{t_{i+1}} v(t)dt\right\|. \tag{10}$$

The vector v_i exists and is unique in view of the compactness and convexity of the set $\overline{X}(\xi^1(t_i))$ and the strong convexity of the function being minimized.

Set $\delta_i = \|\xi(t_i) - \xi^1(t_i)\|$. As

$$\|\,\xi(t)-\xi(t_i)\,\| = \varepsilon\left\|\int_{t_i}^t v(\tau)d\tau\right\| \le \varepsilon M(t-t_i) \le \frac{ML}{m}, \tag{11}$$

then

$$\left\|\,\xi(t)-\xi^1(t_i)\,\right\| \le \|\,\xi(t)-\xi(t_i)\,\| + \left\|\,\xi(t_i)-\xi^1(t_i)\,\right\| \le$$

$$\le \delta_i + \varepsilon M(t-t_i),$$

$$h(\overline{X}(\xi(t)),\ \overline{X}(\xi^1(t_i))) \le \lambda[\delta_i + \varepsilon M(t-t_i)], \ t\in[t_i,t_{i+1}]. \tag{12}$$

From (10) and (12) follows that

$$\left\|\int_{t_i}^{t_{i+1}}[v(t)-v_i]dt\right\| \le h\left(\int_{t_i}^{t_{i+1}}\overline{X}(\xi(t))dt,\ \int_{t_i}^{t_{i+1}}\overline{X}(\xi^1(t_i))dt\right) \le$$

$$\le \int_{t_i}^{t_{i+1}} h(\overline{X}(\xi(t)),\ \overline{X}(\xi^1(t_i)))dt \le$$

$$\le \lambda\left[\delta_i(t_{i+1}-t_i) + \varepsilon M\frac{(t_{i+1}-t_i)^2}{2}\right] =$$

$$= \lambda\left[\delta_i\frac{L}{\varepsilon m} + \frac{L^2M}{2\varepsilon m^2}\right]. \tag{13}$$

Then according to (8), (9) and (13) we get

$$\delta_{i+1} = \|\xi(t_{i+1}) - \xi^1(t_{i+1})\| \le$$

$$\le \|\xi(t_i)-\xi^1(t_i)\| + \varepsilon\left\|\int_{t_i}^{t_{i+1}}[v(t)-v_i]dt\right\| \le$$

$$\le \delta_i + \varepsilon\lambda\left[\delta_i\frac{L}{\varepsilon m} + \frac{L^2M}{2\varepsilon m^2}\right] =$$

$$= \frac{\lambda ML^2}{2m^2} + \left(1+\frac{\lambda L}{m}\right)\delta_i \le \frac{\lambda ML^2}{2m^2} +$$

$$+\left(1+\frac{\lambda L}{m}\right)\left(\frac{\lambda ML^2}{2m^2} + \left(1+\frac{\lambda L}{m}\right)\delta_{i-1}\right) \le \ldots \le$$

$$\le \left(1+\frac{\lambda L}{m}\right)^{i+1}\delta_0 + \frac{\lambda ML^2}{2m^2}\sum_{k=0}^{i}\left(1+\frac{\lambda L}{m}\right)^k \le$$

$$\le \frac{ML}{2m}\left[\left(1+\frac{\lambda L}{m}\right)^{i+1} - 1\right] \le \frac{ML}{2m}(e^{\lambda L}-1), \tag{14}$$

$$i = \overline{0, m-1}.$$

As

$$\|\,\xi^1(t)-\xi^1(t_i)\,\| = \varepsilon\,\|\,v_i\,\|\,(t-t_i) \le \frac{ML}{m}, \tag{15}$$

then from (11) and (14) follows that

$$\| \xi(t) - \xi^1(t) \| \le \| \xi(t) - \xi(t_i) \| + \| \xi(t_i) - \xi^1(t_i) \| +$$

$$+ \| \xi^1(t_i) - \xi^1(t) \| \le \frac{ML}{2m}(e^{\lambda L} + 3). \quad (16)$$

From the condition 2) of the theorem follows that for any $\eta_1 > 0$ and fixed m the inequality holds

$$h\left(\frac{\varepsilon m}{L} \int_{t_i}^{t_{i+1}} X(t, \xi^1(t_i))dt,\ \overline{X}(\xi^1(t_i)) \right) \le \eta_1. \quad (17)$$

Hence, there exists such measurable selector $v^i(t) \in X(t, \xi^1(t_i)),\ i = \overline{0, m-1}$ that

$$\left\| \frac{\varepsilon m}{L} \int_{t_i}^{t_{i+1}} [v^i(t) - v_i]dt \right\| \le \eta_1. \quad (18)$$

Consider the family of functions

$$x^1(t) = x^1(t_i) + \varepsilon \int_{t_i}^{t} v^i(\tau)\, d\tau,\ t \in [t_i, t_{i+1}],\ x^1(0) = x_0. \quad (19)$$

From (18),(19) and (9) follows that

$$\| x^1(t_i) - \xi^1(t_i) \| \le \|x^1(t_{i-1}) - \xi^1(t_{i-1})\| +$$

$$+\varepsilon \left\| \int_{t_{i-1}}^{t_i} [v^{i-1}(t) - v_{i-1}]dt \right\| \le$$

$$\le \|x^1(t_{i-1}) - \xi^1(t_{i-1})\| + \frac{L\eta_1}{m} \le \ldots \le L\eta_1,\ i = \overline{1, m}. \quad (20)$$

As

$$\| x^1(t) - x^1(t_i) \| = \varepsilon \left\| \int_{t_i}^{t} v^i(\tau)d\tau \right\| \le \frac{ML}{m},\ t \in [t_i, t_{i+1}]$$

then from (15) and (20) we have

$$\| x^1(t) - \xi^1(t) \| \le \frac{2ML}{m} + L\eta_1 \quad (21)$$

and

$$h(X(t, x^1(t)),\ X(t, \xi^1(t_i))) \le h(X(t, x^1(t)),\ X(t, x^1(t_i))) +$$

$$+h(X(t, x^1(t_i)),\ X(t, \xi^1(t_i))) \le \lambda \frac{ML}{m} + \lambda L\eta_1. \quad (22)$$

Taking into consideration the choice of the function $v^i(t)$ and (22) we have

$$\rho(\dot{x}^1(t),\ \varepsilon X(t, x^1(t))) \le \varepsilon\lambda L \left(\frac{M}{m} + \eta_1 \right). \quad (23)$$

According to [12] there exists such a solution $x(t)$ of the inclusion (1) that the A.F. Filippov's theorem

$$\| x(t) - x^1(t) \| \le \varepsilon\lambda L \left(\frac{M}{m} + \eta_1 \right) \int_0^t e^{\varepsilon\lambda(t-\tau)}d\tau \le$$

$$\le L \left(\frac{M}{m} + \eta_1 \right) (e^{\lambda L} - 1). \quad (24)$$

From the estimates (16), (21) and (24) follows that

$$\|\xi(t) - x(t)\| \le$$

$$\le \|\xi(t) - \xi^1(t)\| + \|x(t) - x^1(t)\| + \|\xi^1(t) - x^1(t)\| \le$$

$$\le \frac{ML}{2m}(e^{\lambda L} + 3) + \frac{2ML}{m} + L\eta_1 + L\left(\frac{M}{m} + \eta_1 \right)(e^{\lambda L} - 1) =$$

$$= \frac{ML}{2m}(3e^{\lambda L} + 5) + Le^{\lambda L}\eta_1. \quad (25)$$

Choosing

$$m > \frac{ML}{\eta}(3e^{\lambda L} + 5), \quad \eta_1 < \frac{\eta}{2Le^{\lambda L}}$$

from (25) we get the first statement of the theorem.

The proof of the second part of the theorem is similar to the proof of the first one.

Remark 1. If the condition 3) doesn't hold it can be replaced by the following condition:

3') for any $x_0 \in D' \subset D$ the solutions of the inclusion (2) together with a $\rho-$ neighborhood belong to the domain D for $\tau \in [0, L^*]$, where $\tau = \varepsilon t$.

Then for any $\eta \in (0, \rho]$ and $L \in (0, L^*]$ there exists such $\varepsilon^0(\eta, L) > 0$ that for all $\varepsilon \in (0, \varepsilon^0]$ and $t \in [0, L\varepsilon^{-1}]$ the statements 1) and 2) of the theorem 1 fulfill.

In case when there is no uniform convergence in (3), V.A. Plotnikov proved the following theorem, which is the generalization of A.N. Filatov's result [13] on a case of differential inclusions:

Theorem 2. *[3, 7]. Let in the domain Q the following hold:*

1) *the mapping $X(t, x)$ is continuous, locally satisfies the Lipschitz condition in x;*
2) *in every point $x \in D$ the limit (3) exists;*
3) *for any $x_0 \in D' \subset D$ and $t \ge 0$ the solutions of the inclusion (2) together with a $\rho-$ neighborhood belong to the domain D.*

Then for any $\eta \in (0, \rho]$ and $L > 0$ there exists such $\varepsilon^0(\eta, L, x_0) > 0$ that for all $\varepsilon \in (0, \varepsilon^0]$ and $t \in [0, L\varepsilon^{-1}]$ the statements 1) and 2) of the theorem 1 fulfill.

Proof. Consider the set $D(L, x_0) = S_\rho(\overline{R}(t))$.

The set $D(L, x_0) \subset D$ is compact. Hence the limit (3) exists uniformly with respect to $x \in D(L, x_0)$. As at the proof of the theorem 1 it is enough to consider the domain $Q(L, x_0) = \{t \ge 0,\ x \in D(L, x_0)\}$ the statements of the theorem 2 follow from the justice of the theorem 1 for the domain $Q(L, x_0)$.

Remark 2. The estimates received in the theorem 2 qualitatively differ from the corresponding estimates of the theorem 1. The external coincidence of the statements of theorems 1 and 2 leads sometimes to their wrong understanding. Really, the theorem 1 affirms that the inequality (4) holds uniformly for all family of trajectories $x(t)$ and $\xi(t)$ with coincident initial conditions, i.e. the existence of $\varepsilon(\eta, L)$ is affirmed. The estimate received in the theorem 2 is fair only for solutions $x(t)$ and $\xi(t)$ beginning in the fixed initial point x_0, i.e. the existence of $\varepsilon(x_0, \eta, L)$ is affirmed.

Example 1. *Consider the differential inclusion*

$$\dot{x} \in \{\varepsilon a x \sin t,\ a \in [1, 2]\},\ x(0) = x_0. \tag{26}$$

The averaged system will be $\dot{\xi} = 0,\ \xi(0) = x_0$. *Therefore*

$$|x(t) - \xi(t)| = \left| x_0 e^{\varepsilon a \int\limits_0^t \sin s ds} - x_0 \right| =$$

$$= |x_0| \left(e^{\varepsilon a (1-\cos t)} - 1 \right). \tag{27}$$

It is easy to check that for the system (26) the conditions of the theorem 1 do not fulfill and the conditions of the theorem 2 fulfill. Really the right-hand side is not uniformly bounded and

$$h\left(\frac{1}{T} \int\limits_0^T X(t, x) dt, \overline{X}(x) \right) \le$$

$$\le \frac{2|x|}{T} \int\limits_0^T \sin t dt = \frac{2|x|}{T}(1 - \cos T) \tag{28}$$

does not exceed $\frac{4|x|}{T}$ *and converges to* 0 *when* $T \to \infty$, *but the value* $T(\delta)$ *depends on* x, *though* $T(\delta)$ *converges to infinity when* $x \to \infty$. *So the condition of the uniform convergence in (28) is not fair.*

From (27) follows that there exists $\varepsilon_0(\eta, L, x_0) > 0$ *such that for all* $\varepsilon \in (0, \varepsilon_0]$ *and* $t \in (0, L\varepsilon^{-1}]$ *the estimate* $|x(t) - \xi(t)| < \varepsilon$ *is fair. For example one can take* $\varepsilon_0 = \frac{1}{2} \ln\left(1 + \frac{\eta}{|x_0|}\right)$. *But for fixed* η *and L the function* $\varepsilon_0(\eta, L, x_0) \to 0$ *when* $|x_0| \to \infty$, *so there is no uniform estimate (27) with respect to* $x_0 \in \mathbb{R}$.

If the mapping $X(t, x)$ is periodic in t, one can receive the more exact estimate.

Theorem 3. *Let in the domain Q the conditions 1), 3) of the theorem 1 fulfill and besides the mapping* $X(t, x)$ *is* 2π*−periodic in* t.

Then for any $L > 0$ *there exist* $\varepsilon^0(L) > 0$ *and* $C(L) > 0$ *such that for all* $\varepsilon \in (0, \varepsilon^0]$ *and* $t \in [0, L\varepsilon^{-1}]$ *the following statements fulfill:*

1) *for any solution* $\xi(t)$ *of the inclusion (2) there exists a solution* $x(t)$ *of the inclusion (1) such that*

$$\| x(t) - \xi(t) \| \le C\varepsilon; \tag{29}$$

2) *for any solution* $x(t)$ *of the inclusion (1) there exists a solution* $\xi(t)$ *of the inclusion (2) such that the inequality (29) holds.*

Proof. If the multivalued mapping is 2π−periodic in t and uniformly bounded then

$$\overline{X}(x) = \frac{1}{2\pi} \int\limits_0^{2\pi} X(t, x) dt$$

is the uniform average for $X(t, x)$.

Really for $2k\pi \le T < 2(k+1)\pi$ we have

$$h\left(\frac{1}{T} \int\limits_0^T X(t, x) dt, \overline{X}(x) \right) =$$

$$= h\left(\frac{1}{T} \sum_{i=0}^{k-1} \int\limits_{2\pi i}^{2\pi(i+1)} X(t, x) dt + \frac{1}{T} \int\limits_{2k\pi}^T X(t, x) dt, \right.$$

$$\left. \frac{1}{T} \sum_{i=0}^{k-1} \int\limits_{2\pi i} 2\pi(i+1) \overline{X}(x) dt + \frac{1}{T} \int\limits_{2k\pi}^T \overline{X}(x) dt \right) \le$$

$$\le \frac{1}{T} \sum_{i=0}^{k-1} h\left(\int\limits_{2\pi i}^{2\pi(i+1)} X(t, x) dt, \int\limits_{2\pi i}^{2\pi(i+1)} \overline{X}(x) dt \right) +$$

$$+ \frac{1}{T}(T - 2k\pi)\left(|X(t, x)| + |\overline{X}(x)|\right) \le \frac{4M\pi}{T}.$$

Hence for $T > \frac{4M\pi}{\delta}$ we have $h\left(\frac{1}{T} \int\limits_0^T X(t, x) dt, \overline{X}(x) \right) < \delta$ for all $x \in D$.

Let us prove the first statement of the theorem. Divide the interval $[0, L\varepsilon^{-1}]$ on the partial intervals with the points $t_i = 2\pi i,\ i = 0, 1 \ldots$ Let $x(t)$ be a solution of the inclusion (1). Then there exists a measurable selector $v(t)$ of the multivalued mapping $X(t, x(t))$ such that

$$x(t) = x(t_i) + \varepsilon \int\limits_{t_i}^t v(\tau) d\tau,\ t \in [t_i, t_{i+1}],\ x(0) = x_0. \tag{30}$$

Consider the mapping

$$x^1(t) = x^1(t_i) + \varepsilon \int\limits_{t_i}^t v^1(\tau) d\tau,\ t \in [t_i, t_{i+1}],\ x^1(0) = x_0. \tag{31}$$

where $v^1(t)$ is the measurable selector of the multivalued mapping $X(t, x^1(t_i))$ such that

$$\|v(t) - v^1(t)\| = \min_{v^1 \in X(t, x^1(t_i))} \|v(t) - v^1\|. \tag{32}$$

Denote by $\delta_i = \|x(t_i) - x^1(t_i)\|$, then we have

$$\|v(t) - v^1(t)\| \le h(X(t, x(t)), X(t, x^1(t_i))) \le$$

$$\leq \lambda\|x(t)-x^1(t_i)\| \leq \lambda\left[\ \|x(t)-x(t_i)\|+\|x(t_i)-x^1(t_i)\|\right] \leq$$

$$\leq \lambda\left[\varepsilon\int\limits_{t_i}^{t}\|v(\tau)\|d\tau+\delta_i\right] \leq \lambda[\delta_i+\varepsilon M(t-t_i)].$$

Therefore from (30), (31) and (32) follows

$$\delta_{i+1}=\|x(t_{i+1})-x^1(t_{i+1})\|=$$

$$=\left\|x(t_i)+\varepsilon\int\limits_{t_i}^{t_{i+1}}v(\tau)d\tau-x^1(t_i)-\varepsilon\int\limits_{t_i}^{t_{i+1}}v^1(\tau)d\tau\right\| \leq$$

$$\leq \delta_i+\varepsilon\int\limits_{t_i}^{t_{i+1}}\|v(\tau)-v^1(\tau)\|d\tau \leq$$

$$\leq \delta_i+\varepsilon\lambda\int\limits_{t_i}^{t_{i+1}}[\delta_i+\varepsilon M(\tau-t_i)]d\tau=$$

$$=\delta_i(1+2\pi\varepsilon\lambda)+2\pi^2\varepsilon^2\lambda M.$$

Hence, as $2\pi(i+1)\leq L\varepsilon^{-1}$, we get

$$\delta_{i+1} \leq (1+2\pi\varepsilon\lambda)\delta_i+2\pi^2\varepsilon^2\lambda M \ \leq$$

$$\leq (1+2\varepsilon\lambda\pi)\ ((1+2\pi\varepsilon\lambda)\delta_{i-1}+2\pi^2\varepsilon^2\lambda M\ )+2\varepsilon^2\lambda M\pi^2 \leq$$

$$\leq \cdots \leq (1+2\pi\varepsilon\lambda)^{i+1}\delta_0+2\pi^2\varepsilon^2\lambda M\sum_{k=0}^{i}(1+2\pi\varepsilon\lambda)^k=$$

$$=2\pi^2\varepsilon^2\lambda M\sum_{k=0}^{i}(1+2\pi\varepsilon\lambda)^k=$$

$$=\varepsilon M\pi\left((1+2\varepsilon\lambda\pi)^{i+1}-1\right) \leq \varepsilon M\pi(e^{\lambda L}-1),$$

i.e.

$$\delta_i \leq M\pi(e^{\lambda L}-1)\varepsilon, \quad i=0,1,\ldots. \tag{33}$$

Taking into account that for $t\in[t_i,t_{i+1}]$ the following inequalities hold

$$\begin{aligned}&\|x(t)-x(t_i)\| \leq \varepsilon\left\|\int\limits_{t_i}^{t}v(\tau)d\tau\right\| \leq\\ &\leq \varepsilon M(t-t_i) \leq 2\pi M\varepsilon,\\ &\|x^1(t)-x^1(t_i)\| \leq \varepsilon\left\|\int\limits_{t_i}^{t}v^1(\tau)d\tau\right\| \leq\\ &\leq \varepsilon M(t-t_i) \leq 2\pi M\varepsilon,\end{aligned} \tag{34}$$

using (33) we obtain

$$\|x(t)-x^1(t)\|=\pi M(e^{\lambda L}+3)\varepsilon. \tag{35}$$

Calculate the value of the mapping $x^1(t)$ in the points t_{i+1}:

$$x^1(t_{i+1})=x^1(t_i)+\varepsilon\int\limits_{t_i}^{t_{i+1}}v^1(t)dt=x^1(t_i)+2\varepsilon v_i\pi, \tag{36}$$

where $v_i\in\frac{1}{2\pi}\int\limits_{t_i}^{t_{i+1}}X(t,x^1(t_i))dt=\frac{1}{2\pi}\int\limits_{0}^{2\pi}X(t,x^1(t_i))dt=\overline{X}(x^1(t_i))$.

Consider the mapping

$$\xi^1(t)=\xi^1(t_i)+\varepsilon v_i(t-t_i),\ t\in[t_i,t_{i+1}],\ \xi^1(0)=x_0. \tag{37}$$

It is obviously that $x^1(t_i)=\xi^1(t_i), \quad i=0,1,\ldots.$

From (34), (37) we have

$$\|x^1(t)-\xi^1(t)\| \leq 4\pi M\varepsilon. \tag{38}$$

As for $t\in[t_i,t_{i+1}], \quad i=0,1,\ldots$

$$\|\xi^1(t)-\xi^1(t_i)\| \leq 2\pi M\varepsilon,$$

$$h(\overline{X}(\xi^1(t_i)),\overline{X}(\xi^1(t))) \leq 2\lambda\pi M\varepsilon,$$

then

$$\rho\left(\dot{\xi}^1(t),\varepsilon\overline{X}(\xi^1(t))\right) \leq$$

$$\leq h\left(\varepsilon\overline{X}(\xi^1(t_i)),\varepsilon\overline{X}(\xi^1(t))\right) \leq 2\lambda\pi M\varepsilon^2. \tag{39}$$

According to [12] from the inequality (39) follows that there exists such a solution $\xi(t)$ of the inclusion (2), that

$$\|\xi(t)-\xi^1(t)\| \leq$$

$$\leq 2\varepsilon^2\pi\lambda M\int\limits_{0}^{t}e^{\lambda\varepsilon(t-\tau)}d\tau \leq 2\varepsilon\pi M(e^{\lambda L}-1). \tag{40}$$

From (35), (38) and (40) follows that

$$\|x(t)-\xi(t)\| \leq$$

$$\leq \|x(t)-x^1(t)\|+\|x^1(t)-\xi^1(t)\|+\|\xi^1(t)-\xi(t)\| \leq$$

$$\leq \pi M\varepsilon\left(e^{\lambda L}+3\right)+4\pi M\varepsilon+2\varepsilon\pi M\left(e^{\lambda L}-1\right)=$$

$$=\pi M\varepsilon\left(3e^{\lambda L}+5\right).$$

Denote by $C_1=\pi M\left(3e^{\lambda L}+5\right)$, then

$$D(x(t),\xi(t)) \leq C_1\varepsilon. \tag{41}$$

The first part of the theorem is proved.

Taking any solution $\xi(t)$ of the inclusion (2) and making the calculations similar to the previous, it is possible to find a solution $x(t)$ of the inclusion (1) such that inequality similarly to (41) with some constant C_2 is fair. Choosing $C=\max(C_1,C_2)$ we will receive the justice of all statements of the theorem.

1.1.2. The averaging on the infinite interval

For generalization of the theorem 1 on an infinite interval V.A. Plotnikov has extended the concept of stability of solutions of the differential equations on a case of differential inclusions [4, 14]. In addition the concept of R-solution of the differential inclusion introduced in [15, 16] was used.

Definition 1. [15, 16]. The absolutely continuous multivalued maping $R : \mathbb{R} \to comp(\mathbb{R}^n)$, $R(0) = X_0$, is called the $R-$solution of the differential inclusion

$$\dot{x} \in X(t, x), \;\; x(0) = x_0 \in X_0 \in comp(\mathbb{R}^n) \tag{42}$$

if for almost every t

$$\lim_{\Delta\downarrow 0} \frac{1}{\Delta} h\left(R(t+\Delta), \bigcup_{x\in R(t)} \left\{ x + \int_t^{t+\Delta} X(s, x) ds \right\}\right) = 0. \tag{43}$$

Definition 2. [5] $R-$ solution $R(t), t \in [t_0, +\infty)$ of the differential inclusion

$$\dot{x} \in coX(t, x) \tag{44}$$

is called stable if for any $\varepsilon > 0$ there exists such $\delta(\varepsilon) > 0$ that all $R-$ solutions $\bar{R}(t)$ of the inclusion (44), satisfying the initial condition

$$h\left(R(t_0), \overline{R}(t_0)\right) < \delta \tag{45}$$

are defined for all $t > t_0$ and $h\left(R(t), \overline{R}(t)\right) < \varepsilon$.

Definition 3. [5] *The $R-$ solution $R(t), t \in [t_0, +\infty)$ of the differential inclusion (44) is called asymptotically stable if it is stable and for any $R-$ solution $\bar{R}(t)$ of the inclusion (44), satisfying the initial condition (45)*

$$\lim_{t\to\infty} h\left(R(t), \bar{R}(t)\right) = 0.$$

Theorem 4. *[4, 14]. Let in the domain Q the following hold:*

1) *the mapping $X(t, x)$ is continuous, uniformly bounded, satisfies the Lipschitz condition in x;*
2) *uniformly with respect to t and x in the domain Q the limit*

$$\overline{X}(x) = \lim_{T\to\infty} \frac{1}{T} \int_t^{t+T} X(t, x) dt \tag{46}$$

exists;

3) *for any $x_0 \in D' \subset D$ and $t \geq 0$ the solutions of the inclusion (2) together with a $\rho-$ neighborhood belong to the domain* D;
4) *the $R-$solution of the differential inclusion (2) is asymptotically stable.*

Then for any $\eta \in (0, \rho]$ there exists $\varepsilon^0(\eta) > 0$ such that for all $\varepsilon \in (0, \varepsilon^0]$ and $t \geq 0$ the following statements fulfill:

1) *for any solution $\xi(t)$ of the inclusion (2) there exists a solution $x(t)$ of the inclusion (1) such that the inequality (4) fulfills;*
2) *for any solution $x(t)$ of the inclusion (1) there exists a solution $\xi(t)$ of the inclusion (2) such that the inequality (4) holds.*

Thereby,

$$h(R(t), \overline{R}(t)) \leq \eta, \tag{47}$$

where $R(t), \overline{R}(t)$ are the $R-$ solutions of the differential inclusions (1) and (2) accordingly, $R(0) = \overline{R}(0) \subset D'$.

The proof of the theorem is carried on similarly to the proof of the Banfy's theorem [17] with changing references to the first N.N. Bogolyubov's theorem with references to the theorem 1.

Example 2. *Consider the following differential inclusion*

$$\dot{x} \in \varepsilon(-x + [-1, 1] + \cos t), \;\; x(0) \in [2, 3],$$

where $x \in D = [-6, 6]$.

The averaged inclusion is

$$\dot{\xi} \in \varepsilon(-\xi + [-1, 1]), \;\; \xi(0) \in [2, 3].$$

The $R-$solution of the averaged inclusion

$$\overline{R}(t) = [3e^{-\varepsilon t} - 1,\, 2e^{-\varepsilon t} + 1]$$

is asymptotically stable. The fulfillment of all other conditions of the theorem 4 is checked evidently.

The $R-$solution of the initial inclusion is

$$R(t) = \left[\left(3 - \frac{\varepsilon^2}{1+\varepsilon^2}\right)e^{-\varepsilon t} + \frac{\varepsilon^2}{1+\varepsilon^2}\cos t + \frac{\varepsilon}{1+\varepsilon^2}\sin t - 1; \right.$$
$$\left.\left(2 - \frac{\varepsilon^2}{1+\varepsilon^2}\right)e^{-\varepsilon t} + \frac{\varepsilon^2}{1+\varepsilon^2}\cos t + \frac{\varepsilon}{1+\varepsilon^2}\sin t + 1\right].$$

Therefore

$$h(R(t), \overline{R}(t)) \leq \frac{\varepsilon^2}{1+\varepsilon^2}e^{-\varepsilon t} + \frac{\varepsilon^2}{1+\varepsilon^2} + \frac{\varepsilon}{1+\varepsilon^2} < 3\,\varepsilon.$$

Thus when $\varepsilon^0 = \frac{\eta}{3}$ the conclusion of the theorem 4 holds.

In V.A. Plotnikov's works the possibility of averaging of the differential inclusions on the infinite interval using the stability of separate trajectories was also considered.

Definition 4. [18]. The solution $\psi(t), \; t \in [t_0, +\infty)$ of the differential inclusion (42) is called stable if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for all $\tilde{x}_0 : \|\tilde{x}_0 - \psi(t_0)\| < \delta$ any solution $\tilde{x}(t)$ with the initial condition $\tilde{x}(t_0) = \tilde{x}_0$ exists for all $t \in [t_0, +\infty)$ and satisfies the inequality

$$\|\tilde{x}(t) - \psi(t)\| < \varepsilon.$$

Definition 5. [18]. The solution $\psi(t), \; t \in [t_0, +\infty)$ of the differential inclusion (42) is called weakly stable if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for all $\tilde{x}_0 : \|\tilde{x}_0 - \psi(t_0)\| < \delta$ some solution $\tilde{x}(t)$ with the initial condition $\tilde{x}(t_0) = \tilde{x}_0$ exists for all $t \in [t_0, +\infty)$ and satisfies the inequality

$$\|\tilde{x}(t) - \psi(t)\| < \varepsilon.$$

Definition 6. [18]. The solution $\psi(t),\ t \in [t_0, +\infty)$ of the differential inclusion (42) is called asymptotically stable if it is stable and

$$\lim_{t\to\infty} \|\tilde{x}(t) - \psi(t)\| = 0.$$

Definition 7. [18]. The solution $\psi(t),\ t \in [t_0, +\infty)$ weakly asymptotically stable if it is weakly stable and

$$\lim_{t\to\infty} \|\tilde{x}(t) - \psi(t)\| = 0.$$

Theorem 5. *[19]. Let in the domain Q the conditions 1) – 3) of the theorem 4 hold and besides*

4) *the solution $\xi(t)$ of the inclusion (2) is weakly asymptotically stable.*

Then for any $\eta \in (0, \rho]$ there exists $\varepsilon^0 > 0$ such that for all $\varepsilon \in (0, \varepsilon^0]$ there exists a solution $x(t)$ of the inclusion (1) such that the inequality (4) holds for $t \geq 0$.

Proof. Let $\xi(t)$ be a weakly asymptotically stable solution of the inclusion (2). It means that for any $\bar{t} > 0$ and η there exist $\rho < \eta$ and the solution $\xi^1(t)$ of the inclusion (2) such that if at the moment $\bar{t}$ we have

$$\|\xi(\bar{t}) - \xi^1(\bar{t})\| \leq \rho,$$

then for any $t > \bar{t}$ the inequality holds

$$\|\xi^1(t) - \bar{\xi}(t)\| < \frac{\eta}{2}.$$

For $\xi(t)$ and $\xi^1(t)$ it is possible to find the constant L such that for any $t > \bar{t} + L\varepsilon^{-1}$ the inequality holds

$$\|\xi(t) - \xi^1(t)\| \leq \frac{\rho}{2}.$$

From the theorem 1 follows that for the given ρ and L it is possible to choose $\varepsilon^0(\rho, L) > 0$ such that there exists a solution $x(t)$ of the differential inclusion (1) such that for any $\varepsilon \in (0, \varepsilon^0]$ and $t \in [0, L\varepsilon^{-1}]$ the inequality is true

$$\| \xi(t) - x(t) \| < \rho < \eta. \tag{48}$$

Let us prove that the inequality (4) fulfills on an infinite interval. We will assume that the theorem statement is incorrect and on the infinity an inequality (4) is not true, i.e. there exists a moment of time $t^* > L\varepsilon^{-1}$ such that

$$\min_{x(t)\in R(t^*)} \|\xi(t^*) - x(t^*)\| = \eta$$

and for any $t < t^*$ we have

$$\min_{x(t)\in R(t)} \|\xi(t) - x(t)\| < \eta,$$

where $R(t)$ is the section of the set of solutions of the differential inclusion (1) and t^* is the first moment of time in which the inequality (4) fails.

Then from the inequality (48) and our assumption follows that there is a moment t_ρ when the following equality holds

$$\min_{x(t_\rho)\in R(t_\rho)} \|\xi(t_\rho) - x(t_\rho)\| = \rho. \tag{49}$$

Let us consider that t_ρ is the maximum point in which the equality (49) fulfills. Then for any $t > t_\rho$ we have

$$\min_{x(t)\in R(t)} \|\xi(t) - x(t)\| > \rho. \tag{50}$$

As it is possible to take t_ρ as the moment $\bar{t}$ there exists a solution $\xi^0(t)$ of the averaged inclusion such that

$$\rho = \|x(t_\rho) - \xi(t_\rho)\| = \|\xi^0(t_\rho) - \xi(t_\rho)\|,$$

$$\|\xi(t) - \xi^0(t)\| \leq \frac{\eta}{2} \quad \text{for } t > t_\rho$$

and

$$\|\xi(t) - \xi^0(t)\| \leq \frac{\rho}{2} \quad \text{for any } t > t_\rho + L\varepsilon^{-1}.$$

Now it is possible to find $\varepsilon_1\left(\frac{\rho}{2}, L\right) \in (0, \varepsilon^0]$ such that for any $t_\rho \leq t < t_\rho + L\varepsilon^{-1}$ and $\varepsilon \in (0, \varepsilon_1]$ there exists a solution $x(t)$ of the differential inclusion (1) that satisfies the following inequality

$$\|\xi(t) - x(t)\| \leq \frac{\rho}{2}.$$

From the other side if $t \in \left[t_\rho, t_\rho + L\varepsilon^{-1}\right]$ then

$$\|x(t) - \xi(t)\| \leq \|x(t) - \xi^0(t)\| + \|\xi^0(t) - \xi(t)\| \leq$$

$$\leq \frac{\rho}{2} + \frac{\eta}{2} < \eta.$$

Thus, we receive that $t_\rho + L\varepsilon^{-1} < t^*$. But for $\tilde{t} = t_\rho + L\varepsilon_1^{-1} > t_\rho$ it is possible to write down the following estimate

$$\|x(\tilde{t}) - \xi(\tilde{t})\| \leq \|x(\tilde{t}) - \xi^0(\tilde{t})\| + \|\xi(\tilde{t}) - \xi^0(\tilde{t})\| <$$

$$< \frac{\rho}{2} + \frac{\rho}{2} = \rho.$$

The received estimate contradicts the inequality (50). Hence, our assumption is incorrect.

Remark 3. The conclusion of the theorem 5 concerns not to all solutions of differential inclusion (2), but only to the solution $\xi(t)$. Therefore the differential inclusion (2) can have non - continuable solutions for $t \geq 0$ and the solutions which are not weakly asymptotically stable.

Thus, from this theorem the closeness of the $R-$solutions of the initial and the averaged inclusions does not follow.

Example 3. *Consider the following differential inclusion*

$$\dot{x} \in \varepsilon([-2, 2]x + e^{-t}), \quad x(0) \in [1, 2]. \tag{51}$$

The averaged inclusion is

$$\dot{\xi} \in \varepsilon[-2, 2]\xi, \quad \xi(0) \in [1, 2].$$

The $R-$solution of the averaged inclusion $\overline{R}(t) = [e^{-2\varepsilon t}, 2e^{2\varepsilon t}]$ is not asymptotically stable as

$$h(\overline{R}_1(t), \overline{R}\,(t)) = h(\overline{R}_1(0), \overline{R}\,(0))\, e^{2\varepsilon t},$$

where $\overline{R}_1(t)$ is the $R-$solution of the averaged differential inclusion with the initial set $\overline{R}_1(0)$.

Thus for the solution $\xi(t) = 2e^{2\varepsilon t}$ of the averaged inclusion the closest solution of the initial inclusion is

$$x(t) = \left(2 - \frac{\varepsilon}{1+2\varepsilon}\right)e^{2\varepsilon t} + \frac{\varepsilon}{1+2\varepsilon}e^{-t}$$

and

$$\lim_{T\to\infty} \|\xi(t) - x(t)\| = \lim_{T\to\infty} \frac{\varepsilon}{1+2\varepsilon}(e^{2\varepsilon t} - e^{-t}) = \infty.$$

At the same time, for example, the solution $\xi_1(t) = 1.5e^{\varepsilon t}$ is weakly asymptotically stable and it is directly checked that this solution is also the solution of the inclusion (51).

Remark 4. In the theorem 5 it is possible to replace the condition 4) with the following:

4') the solution $\xi(t)$ of the inclusion (2) is asymptotically stable.

Then the conclusion of the theorem will be the following: *for any $\eta \in (0, \rho]$ there exist $\varepsilon^0 > 0$ and $\sigma > 0$ such that for all $\varepsilon \in (0, \varepsilon^0]$ for all solutions $x(t)$ of the inclusion (1) with the initial conditions $\left\|x(t_0) - x^0\right\| \le \sigma$ the inequality (4) holds for all $t \ge t_0$.*

When the condition 4') holds the closeness of the $R-$solutions of the initial and the averaged inclusions follows from the theorem.

Example 4. *Consider the following differential inclusion*

$$\dot{x} \in \varepsilon(\,[-2,-1]\,x + 2\cos t\,), \quad x(0) = x_0. \tag{52}$$

For any solution $\xi(t)$ of the averaged inclusion

$$\dot{\xi} \in \varepsilon\,[-2,-1]\,\xi, \quad \xi(0) = x_0$$

the following estimate fulfills:

$$x_0 e^{-2\varepsilon t} \le \xi(t) \le x_0 e^{-\varepsilon t}. \tag{53}$$

According to (53) the solution $\xi(t) \equiv 0$ of the averaged inclusion is asymptotically stable.

For the solutions of the initial inclusion (52) the following inequality holds:

$$\left(x_0 - \frac{4\varepsilon^2}{1+4\varepsilon^2}\right)e^{-2\varepsilon t} + \frac{4\varepsilon^2}{1+4\varepsilon^2}\cos t + \frac{2\varepsilon}{1+4\varepsilon^2}\sin t \le x(t) \le$$

$$\le \left(x_0 - \frac{2\varepsilon^2}{1+\varepsilon^2}\right)e^{-\varepsilon t} + \frac{2\varepsilon^2}{1+\varepsilon^2}\cos t + \frac{2\varepsilon}{1+\varepsilon^2}\sin t.$$

Thus for all solutions $x(t)$ of the initial inclusion we have $\|\, x(t) - \xi(t) \,\| \le \eta$ *for* $\varepsilon^0 \le \dfrac{\eta}{5}$.

1.1.3. Differential inclusions with semicontinuous right-hand sides

In V.A. Plotnikov's works the often meeting in the applications case, when the right-hand side is not continuous but only upper semicontinuous in a phase variable was considered.

Theorem 6. *[5]. Let in the domain Q the following hold:*

1) *the mapping $X : Q \to conv(\mathbb{R}^n)$ is measurable in t, upper semicontinuous in x, uniformly bounded by a summable function $M(t)$ such that for all $t_2 > t_1 \ge 0$ the inequality holds*
$$\int_{t_1}^{t_2} M(t)\,dt \le M_0(t_2 - t_1);$$
2) *the mapping $\overline{X}(x)$ satisfies the Lipschitz condition;*
3) *for any $x_0 \in D' \subset D$ and $t \ge 0$ the solutions of the inclusion (2) together with a $\rho-$ neighborhood belong to the domain D.*

Then for any $\eta \in (0, \rho]$ and $L > 0$ there exists $\varepsilon^0(\eta, L) > 0$ such that for all $\varepsilon \in (0, \varepsilon^0]$ and $t \in [0, L\varepsilon^{-1}]$ for any solution $x(t)$ of the inclusion (1) there exists a solution $\xi(t)$ of the inclusion (2) such that

$$\|\xi(t) - x(t)\| \le \eta. \tag{54}$$

Remark 5. The theorem affirms that $R(t,\varepsilon) \subset R(t,0) + S_\eta(0)$, where $R(t,\varepsilon)$ is the R-solution of the inclusion (1) corresponding to the parameter ε. The validity of the inclusion $R(t,0) \subset R(t,\varepsilon) + S_\eta(0)$ is not affirmed, i.e. only the upper semicontinuity in ε of the multivalued mapping $R(t,\varepsilon)$ at the point $\varepsilon = 0$ is proved.

1.1.4. The approximation of the solution bunches in case when the average does not exist

In [5, 19] V.A. Plotnikov considered the case when the limit (3) does not exist but there exist multivalued mappings $X^-, X^+ : D \to conv(\mathbb{R}^n)$ such that

$$\lim_{T\to\infty} \beta\left(X^-(x), \frac{1}{T}\int_0^T X(t,x)dt\right) = 0, \tag{55}$$

$$\lim_{T\to\infty} \beta\left(\frac{1}{T}\int_0^T X(t,x)dt, X^+(x)\right) = 0, \tag{56}$$

where $\beta(\cdot,\cdot)$ is the semideviation of the sets in the sense of Hausdorff:

$$\beta(A,B) = \sup_{a\in A}\inf_{b\in B}\|a-b\|.$$

Along with the differential inclusion (1) we will consider the following differential inclusions:

$$\dot{x}^- \in \varepsilon X^-(x^-), \quad x^-(0) = x_0, \tag{57}$$

$$\dot{x}^+ \in \varepsilon X^+(x^+), \quad x^+(0) = x_0. \tag{58}$$

Theorem 7. *[19]. Let in the domain Q the following hold:*

1) *the mapping $X(t,x)$ is uniformly bounded with constant M, measurable in t, satisfies the Lipschitz condition in x with constant λ;*
2) *the mapping $X^-(x)$ is uniformly bounded with constant M, satisfies the Lipschitz condition in x with constant λ;*
3) *uniformly with respect to x in the domain D the limit (55) exists;*
4) *for any $x_0 \in D' \subset D$ and $t \geq 0$ the solutions of the inclusion (57) together with a $\rho-$ neighborhood belong to the domain D.*

Then for any $\eta \in (0,\rho]$ and $L > 0$ there exists $\varepsilon^0(\eta, L) > 0$ such that for all $\varepsilon \in (0, \varepsilon^0]$ and $t \in [0, L\varepsilon^{-1}]$ for any solution $x^-(t)$ of the inclusion (57) there exists a solution $x(t)$ of the inclusion (1) such that

$$\| x^-(t) - x(t) \| \leq \eta. \tag{59}$$

Proof. Divide the interval $[0, L\varepsilon^{-1}]$ on the partial intervals with the points $t_i = \frac{Li}{m\varepsilon}$, $i = \overline{0,m}$, $m \in \mathbb{N}$. Let $x^-(t)$ be a solution of the inclusion (57). Then there exists a measurable selector $\overline{u}(t) \in X^-(\xi(t))$ such that

$$x^-(t) = x^-(t_i) + \varepsilon \int_{t_i}^{t} \overline{u}(\tau)d\tau,\ t \in [t_i, t_{i+1}],\ x^-(0) = x_0\ . \tag{60}$$

Consider the following function

$$\xi^1(t) = \xi^1(t_i) + \varepsilon u_i\,(t - t_i),\ t \in [t_i, t_{i+1}],\ \xi^1(0) = x_0, \tag{61}$$

where

$$\left\| \frac{L}{m\varepsilon} u_i - \int_{t_i}^{t_{i+1}} \overline{u}(t)dt \right\| = \min_{u \in X^-(\xi^1(t_i))} \left\| \frac{L}{m\varepsilon} u - \int_{t_i}^{t_{i+1}} \overline{u}(t)dt \right\|. \tag{62}$$

As in (62) the function being minimized is strongly convex and the set $X^-(x^-(t_i))$ is compact and convex then there exists the unique vector u_i.

Let $\delta_i = \| x^-(t_i) - \xi^1(t_i) \|$, then for $t \in [t_i, t_{i+1}]$ we have

$$\| x^-(t) - \xi^1(t_i) \| \leq \| x^-(t) - \xi(t_i) \| + \| x^-(t_i) - \xi^1(t_i) \| \leq$$

$$\leq \delta_i + \varepsilon M(t - t_i); \tag{63}$$

$$h\left(X^-(x^-(t)), X^-(\xi^1(t_i))\right) \leq \lambda[\delta_i + \varepsilon M(t - t_i)]. \tag{64}$$

From (62),(64) and the properties of the support function [20] follow that

$$\left\| \int_{t_i}^{t_{i+1}} \overline{u}(t)dt - \int_{t_i}^{t_{i+1}} u_i dt \right\| \leq$$

$$\leq h\left(\int_{t_i}^{t_{i+1}} X^-(x^-(t))dt, \int_{t_i}^{t_{i+1}} X^-(\xi^1(t_i))dt \right) =$$

$$= \max_{\|\psi\|=1} \left| C\left(\int_{t_i}^{t_{i+1}} X^-(x^-(t))dt, \psi \right) - C\left(\int_{t_i}^{t_{i+1}} X^-(\xi^1(t_i))dt, \psi \right) \right| =$$

$$= \max_{\|\psi\|=1} \left| \int_{t_i}^{t_{i+1}} \left[C(X^-(x^-(t)), \psi) - C(X^-(\xi^1(t_i)), \psi) \right] dt \right| \leq$$

$$\leq \int_{t_i}^{t_{i+1}} \max_{\|\psi\|=1} \left| C(X^-(x^-(t)), \psi) - C(X^-(\xi^1(t_i)), \psi) \right| dt =$$

$$= \int_{t_i}^{t_{i+1}} h(X^-(x^-(t)), X^-(\xi^1(t_i)))dt \leq$$

$$\leq \lambda \left[\delta_i(t_{i+1} - t_i) + \frac{\varepsilon M(t_{i+1} - t_i)^2}{2} \right] =$$

$$= \lambda \left[\delta_i \frac{L}{\varepsilon m} + \frac{L^2 M}{2\varepsilon m^2} \right]. \tag{65}$$

Taking into account (60),(61) and (65) we get the following estimate:

$$\delta_{i+1} \leq \delta_i + \varepsilon\lambda \left[\delta_i \frac{L}{\varepsilon m} + \frac{ML^2}{2\varepsilon m^2} \right] = \frac{\lambda ML^2}{2m^2} + \left(1 + \frac{\lambda L}{m}\right)\delta_i \leq$$

$$\leq \frac{ML}{2m} \left[\left(1 + \frac{\lambda L}{m}\right)^{i+1} - 1 \right] \leq \frac{ML}{2m}\left(e^{\lambda L} - 1\right). \tag{66}$$

As

$$\|x^-(t) - x^-(t_i)\| = \varepsilon \left\| \int_{t_i}^{t} \overline{u}(\tau)d\tau \right\| \leq \frac{ML}{m},$$

$$\| \xi^1(t) - \xi^1(t_i) \| \leq \frac{ML}{m},$$

so then using (66) we obtain

$$\| x^-(t) - \xi^1(t) \| \leq \frac{ML}{m} + \frac{ML}{m} + \frac{ML}{2m}(e^{\lambda L} - 1) =$$

$$= \frac{ML}{2m}(e^{\lambda L} + 3). \tag{67}$$

From the condition 2) of the theorem follows that for any $\eta_1 > 0$ there exists $\varepsilon^0(L, \eta_1) > 0$ such that for all $\varepsilon \leq \varepsilon^0$ the inclusion holds

$$X^-(\xi^1(t_i)) \subset \frac{\varepsilon m}{L} \int_{t_i}^{t_{i+1}} X(\tau, \xi^1(t_i))d\tau + S_{\eta_1}(0). \tag{68}$$

So there exists a measurable function $u^1(t) \in X(t, \xi^1(t_i))$, $t \in [t_i, t_{i+1}]$ such that

$$\left\| \frac{\varepsilon m}{L} \int_{t_i}^{t_{i+1}} [u^1(t) - u_i]dt \right\| \leq \eta_1.$$

Consider the function

$$x^1(t) = x^1(t_i) + \varepsilon \int_{t_i}^{t} u^1(\tau)d\tau,\ t \in [t_i, t_{i+1}],\ x^1(0) = x_0. \quad (69)$$

Then from (68),(69) follows that

$$\| x^1(t_i) - \xi^1(t_i) \| \le L\eta_1.$$

As

$$\| x^1(t) - x^1(t_i) \| \le \frac{ML}{m},$$

we obtain the following inequalities:

$$\| x^1(t) - \xi^1(t) \| \le \frac{2ML}{m} + L\eta_1, \quad (70)$$

$$h\left(X(t, x^1(t)), X(t, \xi^1(t_i))\right) \le \frac{\lambda ML}{m} + \lambda L\eta_1 =$$

$$= \lambda L\left(\frac{M}{m} + \eta_1\right). \quad (71)$$

From the inequality (71) and the way of choosing the function $u^1(t)$ we get

$$\rho\left(\dot{x}^1(t), \varepsilon X(t, x^1(t)\right) \le \varepsilon\lambda L\left(\frac{M}{m} + \eta_1\right).$$

According to [21] there exists such a solution $x(t)$ of the inclusion (1) that

$$\| x(t) - x^1(t) \| \le \varepsilon\lambda L\left(\frac{M}{m} + \eta_1\right)\int_0^t e^{\varepsilon\lambda(t-\tau)}d\tau \le$$

$$\le L\left(\frac{M}{m} + \eta_1\right)\left(e^{\lambda L} - 1\right). \quad (72)$$

From (67), (70), (72) follows that

$$\| x^-(t) - x(t) \| \le (3e^{\lambda L} + 5)\frac{ML}{2m} + L\eta_1 e^{\lambda L}.$$

Choosing $m \ge (3e^{\lambda L} + 5)\frac{ML}{\eta}$ and $\eta_1 \le \frac{\eta}{2Le^{\lambda L}}$, we get

$$\| x^-(t) - x(t) \| \le \eta$$

and the theorem is proved.

Theorem 8. *[19]. Let in the domain Q the following hold:*

1) *the mapping $X(t, x)$ is uniformly bounded, measurable in t, satisfies the Lipschitz condition in x;*
2) *the mapping $X^+(x)$ is uniformly bounded, satisfies the Lipschitz condition in x;*
3) *uniformly with respect to x in the domain D the limit (56) exists;*
4) *for any $x_0 \in D' \subset D$ and $t \ge 0$ the solutions of the inclusion (58) together with a $\rho-$ neighborhood belong to the domain D.*

Then for any $\eta \in (0, \rho]$ and $L > 0$ there exists $\varepsilon^0(\eta, L) > 0$ such that for all $\varepsilon \in (0, \varepsilon^0]$ and $t \in [0, L\varepsilon^{-1}]$ for any solution $x(t)$ of the inclusion (1) there exists a solution $x^+(t)$ of the inclusion (58) such that

$$\| x^+(t) - x(t) \| \le \eta. \quad (73)$$

The proof of the theorem is carried on similarly to the proof of the theorem 7.

Remark 6. If $R(t)$, $R^-(t)$, $R^+(t)$ are the sections of the families of the solutions of the inclusions (1), (57) and (58) accordingly then

$$R^-(t) \subset R(t) + S_\eta(0), \quad R(t) \subset R^+(t) + S_\eta(0). \quad (74)$$

Remark 7. In the capacity of the mappings $X^-(x)$ and $X^+(x)$ one can use the superior and inferior limit of the sequence of sets [22]:

$$\overline{X}^-(x) = \underline{\lim}_{T\to\infty} \frac{1}{T}\int_0^T X(t, x)dt, \qquad \overline{X}^+(x) = \overline{\lim}_{T\to\infty} \frac{1}{T}\int_0^T X(t, x)dt.$$

The sets $\overline{X}^-(x)$ and $\overline{X}^+(x)$ are the maximum and the minimum with respect to the inclusion among the sets $X^-(x)$ and $X^+(x)$, that is for any $X^-(x)$ and $X^+(x)$ the inclusions hold

$$X^-(x) \subset \overline{X}^-(x), \qquad \overline{X}^+(x) \subset X^+(x).$$

Remark 8. If the limit (3) exists then $\overline{X}^-(x) = \overline{X}^+(x) = \overline{X}(x)$ and from theorems 7,8 the theorem 1 follows.

Remark 9. If the limit (3) exists, its calculation is usually carried out by means of support function. Thus it is often impossible to calculate the exact value of the support function and the sets $X^-(x)$ and $X^+(x)$ appear as the result of the approximate calculation of the set $\overline{X}(x)$.

Example 5. *Consider the linear differential inclusion*

$$\dot{x} \in \varepsilon\left[\begin{pmatrix} 2\cos^2 t & 0 \\ 0 & 2\sin^2 t \end{pmatrix} x + S_{r(t)}(0)\right],\ x(0) = x_0, \quad (75)$$

where $r(t) = 2 + e^{-t} + 0.5\sqrt{2}\sin(\ln(t + 1))$.

It is obvious that the matrix

$$\overline{A} = \lim_{T\to\infty} \frac{1}{T}\int_0^T \begin{pmatrix} 2\cos^2 t & 0 \\ 0 & 2\sin^2 t \end{pmatrix} dt = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Let us average the multivalued mapping $U(t) = S_{r(t)}(0)$. As

$$\frac{1}{T}\int_0^T S_{r(t)}(0)dt =$$

$$= S_1(0)\left[\frac{2T - e^{-T}}{T} + \frac{(T + 1)\sqrt{2}}{4T}(\sin(\ln(T + 1)) - \right.$$
$$\left. - \cos(\ln(T + 1))) + \frac{\sqrt{2} + 4}{4T}\right],$$

then the average $\overline{U(t)}$ in this case does not exist, but obviously there exist such sets $U^- = S_{r_1}(0),\ 0 \le r_1 \le 1.5$ and $U^+ = S_{r_2}(0),\ r_2 \ge 2.5$ that the following hold

$$\lim_{T\to\infty} \beta\left(U^-, \frac{1}{T}\int_0^T U(t)dt \right) = 0, \tag{76}$$

$$\lim_{T\to\infty} \beta\left(\frac{1}{T}\int_0^T U(t)dt, U^+ \right) = 0. \tag{77}$$

Then the inclusions (57) and (58) assume the form:

$$\dot{x}^- \in \varepsilon\left[x^-(t) + S_{r_1}(0)\right], x^-(0) = x_0,$$

$$\dot{x}^+ \in \varepsilon\left[x^+(t) + S_{r_2}(0)\right], x^+(0) = x_0.$$

Let us find the $R-$solutions of these inclusions with the help of the Cauchy formula

$$R^-(t) = e^{\varepsilon t}x_0 + \varepsilon \int_0^t e^{\varepsilon(t-s)}S_{r_1}(0)ds = e^{\varepsilon t}x_0 + (e^{\varepsilon t}-1)S_{r_1}(0).$$

Similarly

$$R^+(t) = e^{\varepsilon t}x_0 + (e^{\varepsilon t}-1)S_{r_2}(0).$$

It is obvious that

$$\overline{U}^- = \underline{\lim_{T\to\infty}} \frac{1}{T}\int_0^T U(t)\,dt = S_{1.5}(0),$$

$$\overline{U}^+ = \overline{\lim_{T\to\infty}} \frac{1}{T}\int_0^T U(t)\,dt = S_{2.5}(0).$$

Then

$$R^-(t) \subset \overline{R}^-(t),\ \overline{R}^+(t) \subset R^+(t),$$

where

$$\overline{R}^-(t) = e^{\varepsilon t}x_0 + (e^{\varepsilon t}-1)S_{1.5}(0),\ \overline{R}^+(t) = e^{\varepsilon t}x_0 + (e^{\varepsilon t}-1)S_{2.5}(0).$$

For the initial inclusion (75) all the conditions of the theorems 7,8 hold. So for any $\eta \in (0,\rho]$ and $L > 0$ there exists $\varepsilon^0(\eta, L) > 0$ such that for all $\varepsilon \in (0, \varepsilon^0]$ and $t \in [0, L\varepsilon^{-1}]$ the inclusions (74) are true.

1.2. The partial averaging scheme

It is also possible to use the partial averaging of the differential inclusions, i.e. to average only some summands or factors. Such variant of the averaging method also leads to the simplification of the initial inclusion and happens to be useful when the average of some functions does not exist or their presence in the system does not complicate its research.

Along with the differential inclusion (1) we will consider the partially averaged differential inclusion

$$\dot{\xi} \in \varepsilon\overline{X}(t,\xi), \quad \xi(0) = x_0, \tag{78}$$

where

$$\lim_{T\to\infty} \frac{1}{T} h\left(\int_0^T X(t,x)\,dt, \int_0^T \overline{X}(t,x)\,dt \right) = 0. \tag{79}$$

Theorem 9. *[8]. Let in the domain Q the following hold:*

1) *the mappings $X(t,x), \overline{X}(t,x)$ are continuous, uniformly bounded with constant M, satisfy the Lipschitz condition in x with constant λ;*
2) *uniformly with respect to x in the domain D the limit (79) exists;*
3) *for any $x_0 \in D' \subset D$, $\varepsilon \in (0,\sigma]$ and $t \ge 0$ the solutions of the inclusion (78) together with a $\rho-$ neighborhood belong to the domain D.*

Then for any $\eta \in (0,\rho]$ and $L > 0$ there exists $\varepsilon^0(\eta, L) \in (0,\sigma]$ such that for all $\varepsilon \in (0,\varepsilon^0]$ and $t \in [0, L\varepsilon^{-1}]$ the following statements fulfill:

1) *for any solution $\xi(t)$ of the inclusion (78) there exists a solution $x(t)$ of the inclusion (1) such that*

$$\|x(t) - \xi(t)\| \le \eta; \tag{80}$$

2) *for any solution $x(t)$ of the inclusion (1) there exists a solution $\xi(t)$ of the inclusion (78) such that the inequality (80) holds.*

Thereby, $h(\overline{R^1(t)}, \overline{R^2(t)}) \le \eta$, where $\overline{R^1(t)}, \overline{R^2(t)}$ are the closures of the sections of the families of the solutions of the initial and the averaged inclusions.

Proof. Without loss of generality when proving the theorem we can suppose that the sets $X(t,x)$ and $\overline{X}(t,x)$ are convex.

Really if it is not true we will consider to the inclusions

$$\dot{x} \in \varepsilon coX(t,x),\ x(0) = x_0, \tag{81}$$

$$\dot{\xi} \in \varepsilon co\overline{X}(t,\xi),\ \xi(0) = x_0. \tag{82}$$

According to [10] the families of solutions of the inclusions (78), (79) are everywhere dense in the compact sets of the families of solutions of the inclusion (81),(82). Hence, it is enough to prove the theorem for the inclusions with the convex right-hand side.

Let us prove the second statement of the theorem and therefore the validity of the inclusion

$$\overline{R^1(t)} \subset S_\eta(\overline{R^2(t)}). \tag{83}$$

Divide the interval $[0, L\varepsilon^{-1}]$ on the partial intervals with the points $t_i = \frac{Li}{m\varepsilon}$, $i = \overline{0,m}$, $m \in \mathbb{N}$. Let $x(t)$ be a solution of the inclusion (78). Then there exists a measurable selector $v(t) \in X(t, x(t))$ such that

$$x(t) = x(t_i) + \varepsilon \int_{t_i}^t v(\tau)\,d\tau,\ \ t \in [t_i, t_{i+1}],\ x(0) = x_0. \tag{84}$$

Consider the function

$$y^1(t) = y^1(t_i) + \varepsilon \int\limits_{t_i}^{t} z^1(\tau)\, d\tau,\ \ t \in [t_i, t_{i+1}],\ y^1(0) = x_0, \quad (85)$$

where

$$\|v(t) - z^1(t)\| = \min_{z \in X(t, y^1(t_i))} \|v(t) - z\|. \quad (86)$$

The measurable function $z^1(t)$ exists [21] and is unique in view of the compactness and convexity of the set $X(t, y^1(t_i))$ and the strong convexity of the function being minimized.

From (84) - (86) we have

$$\|x(t) - y^1(t_i)\| \le \|x(t) - x(t_i)\| +$$

$$+\|x(t_i) - y^1(t_i)\| \le \delta_i + \varepsilon M(t - t_i), \quad (87)$$

$$\|v(t) - z^1(t)\| \le h(X(t, x(t)), X(t, y^1(t_i))) \le$$

$$\le \lambda(\delta_i + \varepsilon M(t - t_i)), \quad (88)$$

where $\delta_i = \|x(t_i) - y^1(t_i)\|,\ t \in [t_i, t_{i+1}],\ i = \overline{0, m-1}$.

From (84) - (88) follows the estimate

$$\delta_{i+1} \le \delta_i + \varepsilon \int\limits_{t_i}^{t_{i+1}} \|v(t) - z^1(t)\|\, dt \le$$

$$\le \delta_i + \varepsilon\lambda \left[\delta_i(t_{i+1} - t_i) + \varepsilon M \frac{(t_{i+1} - t_i)^2}{2} \right] = \frac{\lambda M L^2}{2m^2} + \left(1 + \frac{\lambda L}{m}\right)\delta_i.$$

Therefore

$$\delta_{i+1} \le \frac{ML}{2m} \left[\left(1 + \frac{\lambda L}{m}\right)^{i+1} - 1 \right] \le \frac{ML}{2m}(e^{\lambda L} - 1). \quad (89)$$

As for $t \in [t_i, t_{i+1}]$

$$\|x(t) - x(t_i)\| \le \frac{ML}{m}, \quad \|y^1(t) - y^1(t_i)\| \le \frac{ML}{m}, \quad (90)$$

then using (89) we obtain

$$\|x(t) - y^1(t)\| \le \|x(t) - x(t_i)\| + \|x(t_i) - y^1(t_i)\| +$$

$$+\|y^1(t_i) - y^1(t)\| \le \frac{ML}{2m}(e^{\lambda L} + 3). \quad (91)$$

Consider the function

$$y^2(t) = y^2(t_i) + \varepsilon \int\limits_{t_i}^{t} z^2(\tau) d\tau,\ t \in [t_i,\ t_{i+1}],\ \ y^2(0) = x_0, \quad (92)$$

where

$$\|z^1(t) - z^2(t)\| = \min_{z \in \overline{X}(t, y^1(t_i))} \|z^1(t) - z\|. \quad (93)$$

From the condition 2) of the theorem follows that for any $\eta_1 > 0$ there exists $\varepsilon^0(L, \eta_1) > 0$ such that for all $\varepsilon \le \varepsilon^0$ the inequality holds

$$\frac{\varepsilon m}{L} h \left(\int\limits_{t_i}^{t_{i+1}} X(t, y^1(t_i)) dt,\ \int\limits_{t_i}^{t_{i+1}} \overline{X}(t, y^1(t_i)) dt \right) \le \eta_1.$$

Hence

$$\int\limits_{t_i}^{t_{i+1}} \|z^1(t) - z^2(t)\| dt \le \frac{L\eta_1}{\varepsilon m}$$

and

$$\|y^1(t_{i+1}) - y^2(t_{i+1})\| \le$$

$$\le \varepsilon \int\limits_{t_i}^{t_{i+1}} \|z^1(t) - z^2(t)\| dt + \|y^1(t_i) - y^2(t_i)\| \le$$

$$\le \|y^1(t_i) - y^2(t_i)\| + \eta_1 \frac{L}{m} \le \ldots \le L\eta_1,\ \ i = \overline{0, m-1}. \quad (94)$$

As for $t \in [t_i,\ t_{i+1}]$

$$\|y^2(t) - y^2(t_i)\| \le \frac{ML}{m}, \quad (95)$$

then taking into account (90) and (94) we get

$$\|y^1(t) - y^2(t)\| \le L\eta_1 + \frac{2ML}{m}. \quad (96)$$

According to the condition 1) of the theorem and the inequalities (95), (96) we have

$$h\left(\overline{X}(t, y^2(t)), \overline{X}(t, y^1(t_i))\right) \le h\left(\overline{X}(t, y^2(t)), \overline{X}(t, y^2(t_i))\right) +$$

$$+h\left(\overline{X}(t, y^2(t_i)), \overline{X}(t, y^1(t_i))\right) \le \lambda\left(\frac{ML}{m} + L\eta_1\right)$$

and therefore using (93) we get

$$\rho(\dot{y}^2(t), \varepsilon\overline{X}(t, y^2(t))) \le \varepsilon\lambda L\left(\frac{M}{m} + \eta_1\right). \quad (97)$$

According to [12] from (97) follows the existence of such a solution $\xi(t)$ of the inclusion (78) that

$$\|y^2(t) - \xi(t)\| \le \varepsilon\lambda L\left(\frac{M}{m} + \eta_1\right) \int\limits_0^t e^{\varepsilon\lambda(t-\tau)} d\tau \le$$

$$\le L\left(\frac{M}{m} + \eta_1\right)(e^{\lambda L} - 1). \quad (98)$$

From the estimates (91), (96), (98) we get

$$\|x(t) - \xi(t)\| \le \frac{ML}{2m}(3e^{\lambda L} + 5) + Le^{\lambda L}\eta_1.$$

Choosing $m \ge \dfrac{ML(3e^{\lambda L} + 5)}{\eta}$ and $\eta_1 \le \dfrac{\eta}{2Le^{\lambda L}}$ we get the second statement of the theorem.

The proof of the first part of the theorem is similar to the proof of the second one.

Remark 10. If one of the sets $X(t, x)$ or $\overline{X}(t, x)$ degenerates into a point then the corresponding inclusion becomes the differential equation which has the unique solution defined for $t \ge 0$. In this case the whole family of solutions of the second inclusion belongs to the η- neighborhood of the given solution.

Remark 11. If the convergence in (79) takes place in every point $x \in D$ then similarly to the theorem 2 one can prove the existence of such $\varepsilon^0(\eta, L, x_0) > 0$ that for all $\varepsilon \in (0, \varepsilon^0]$ the conclusions of the theorem 9 are true.

Remark 12. If the mappings $X(t, x)$ and $\overline{X}(t, x)$ are periodic in t then in the estimate (80) it is possible to replace η with $C\varepsilon$.

Remark 13. If $\overline{X}(t, x) \equiv \overline{X}(x)$ then the substantiation of the full averaging scheme (theorem 1) follows from the theorem 9.

Remark 14. Let the mapping $\overline{X}(t, x)$ be

$$\overline{X}(t, x) = \left\{ X_i(x) = \frac{1}{T} \int\limits_{i\omega}^{(i+1)\omega} X(t, x) dt, \quad i\omega < t \le (i+1)\omega, \ i = 0, 1, \ldots \right\}.$$

Dividing the interval $[0, L\varepsilon^{-1}]$ on partial intervals with the step ω, it is possible to show that the estimate (80) holds with $\eta = C\varepsilon$.

Remark 15. Similarly to the above the various schemes of averaging for integro - differential inclusions

$$\dot{x} \in \varepsilon X\left(t, x, \int\limits_0^t \varphi(t, s, x(s)) ds\right), \ x(0) = x_0,$$

where $X : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to comp(\mathbb{R}^n)$, $\varphi : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^m$ have been considered in [4, 23].

2. The averaging of impulsive differential inclusions

2.1. Differential inclusions with impulses in fixed moments of time

In this section we will discuss V.A.Plotnikov's results on the substantiation of the method of full and partial averaging on finite and infinite intervals for the differential inclusions which are exposed to impulse influence in the fixed moments of time.

2.1.1. The full averaging scheme

The averaging on the finite interval. Consider the differential inclusion with multivalued impulses

$$\dot{x} \in \varepsilon X(t, x), \ t \ne \tau_i, \ x(0) = x_0, \tag{99}$$

$$\Delta x|_{t=\tau_i} \in \varepsilon I_i(x).$$

If for any $x \in D$ there exists the limit

$$Y(x) = \lim_{T\to\infty} \left(\frac{1}{T} \int\limits_t^{t+T} X(t, x) dt + \frac{1}{T} \sum_{t \le \tau_i < t+T} I_i(x) \right), \tag{100}$$

then in the correspondence to the inclusion (99) we will set the following averaged inclusion

$$\dot{y} \in \varepsilon Y(y), \ y(0) = x_0. \tag{101}$$

Theorem 10. *[6]. Let in the domain Q the following hold:*

1) *the mappings $X : Q \to conv(\mathbb{R}^n)$, $I_i : D \to conv(\mathbb{R}^n)$ are continuous, uniformly bounded and satisfy the Lipschitz condition in x;*
2) *uniformly with respect to t and x in the domain Q the limit (100) exists and*

$$\frac{1}{T} i(t, t+T) \le d < \infty,$$

where $i(t, t+T)$ is the quantity of points of the sequence $\{\tau_i\}$ on the interval $(t, t+T]$;

3) *for any $x_0 \in D' \subset D$ and $t \ge 0$ the solutions of the inclusion (101) together with a $\rho-$ neighborhood belong to the domain D.*

Then for any $\eta \in (0, \rho]$ and $L > 0$ there exists $\varepsilon^0(\eta, L) > 0$ such that for all $\varepsilon \in (0, \varepsilon^0]$ and $t \in [0, L\varepsilon^{-1}]$ the following statements fulfill:

1) *for any solution $y(t)$ of the inclusion (101) there exists a solution $x(t)$ of the inclusion (99) such that*

$$\|x(t) - y(t)\| \le \eta; \tag{102}$$

2) *for any solution $x(t)$ of the inclusion (99) there exists a solution $y(t)$ of the inclusion (101) such that the inequality (102) holds.*

Proof. From the conditions 1), 2) follows that the multivalued mapping $Y : D \to conv(\mathbb{R}^n)$ is uniformly bounded with constant $M_1 = M(1 + d)$ and satisfies the Lipschitz condition with constant $\lambda_1 = \lambda(1 + d)$.

Let $y(t)$ be a solution of the inclusion (101). Divide the interval $[0, L\varepsilon^{-1}]$ on the partial intervals with the step $\gamma(\varepsilon)$ such that $\gamma(\varepsilon) \to \infty$ and $\varepsilon\gamma(\varepsilon) \to 0$ when $\varepsilon \to 0$. Then there exists a measurable selector $v(t)$ of the mapping $Y(y(t))$ such that

$$y(t) = y(t_j) + \varepsilon \int\limits_{t_j}^t v(s) ds, \ t \in [t_j, t_{j+1}], \ y(0) = x_0, \tag{103}$$

where $t_j = j\gamma(\varepsilon)$, $j = \overline{0, m}$, $m\gamma(\varepsilon) \le L\varepsilon^{-1} < (m+1)\gamma(\varepsilon)$.

Consider the function

$$y^1(t) = y^1(t_j) + \varepsilon v_j(t - t_j), \ t \in [t_j, t_{j+1}], \ y^1(0) = x_0, \tag{104}$$

where the vectors v_j satisfy the condition

$$\left\| \gamma(\varepsilon) v_j - \int\limits_{t_j}^{t_{j+1}} v(s) ds \right\| = \min_{v \in Y(y^1(t_j))} \left\| \gamma(\varepsilon) v - \int\limits_{t_j}^{t_{j+1}} v(s) ds \right\|. \tag{105}$$

The vector v_j exists and is unique in view of the compactness and convexity of the set $Y(y^1(t_j))$ and the strong convexity of the function being minimized.

Denote by $\delta_j = \|y(t_j) - y^1(t_j)\|$. For $t \in [t_j, t_{j+1}]$ using (103) and (104) we have

$$\|y(t) - y(t_j)\| \le M_1\varepsilon(t - t_j) \le M_1\varepsilon\gamma(\varepsilon),$$
$$\|y^1(t) - y^1(t_j)\| \le M_1\varepsilon(t - t_j) \le M_1\varepsilon\gamma(\varepsilon). \quad (106)$$

Therefore for $t \in [t_j, t_{j+1}]$ the following inequalities hold

$$\|y(t)-y^1(t_j)\| \le \|y(t_j)-y^1(t_j)\| + \|y(t)-y(t_j)\| \le \delta_j + \varepsilon M_1(t-t_j),$$

$$h(Y(y(t)), Y(y^1(t_j)) \le$$
$$\le \lambda_1\|y(t) - y^1(t_j)\| \le \lambda_1(\delta_j + \varepsilon M_1(t - t_j)). \quad (107)$$

From (105) and (107) follows that

$$\left\|\int\limits_{t_j}^{t_{j+1}} [v(s) - v_j]ds\right\| \le \int\limits_{t_j}^{t_{j+1}} h(Y(y(s)), Y(y^1(t_j)))ds \le$$
$$\le \lambda_1\left(\delta_j\gamma(\varepsilon) + \varepsilon M_1\frac{\gamma^2(\varepsilon)}{2}\right). \quad (108)$$

Considering (103) and (104) we obtain

$$\delta_{j+1} \le \delta_j + \varepsilon\lambda_1\left(\delta_j\gamma(\varepsilon) + \varepsilon M_1\frac{\gamma^2(\varepsilon)}{2}\right) =$$
$$= (1 + \lambda_1\varepsilon\gamma(\varepsilon))\delta_j + \lambda_1 M_1\frac{\varepsilon^2\gamma^2(\varepsilon)}{2}. \quad (109)$$

From the inequality (109) taking into account that $\delta_0 = 0$ we get

$$\delta_1 \le \lambda_1 M_1\frac{\varepsilon^2\gamma^2(\varepsilon)}{2},$$
$$\delta_2 \le (1 + \lambda_1\varepsilon\gamma(\varepsilon))\delta_1 + \lambda_1 M_1\frac{\varepsilon^2\gamma^2(\varepsilon)}{2} \le$$
$$\le \lambda_1 M_1\frac{\varepsilon^2\gamma^2(\varepsilon)}{2}((1 + \lambda_1\varepsilon\gamma(\varepsilon)) + 1)$$

and so on

$$\delta_{j+1} \le \lambda_1 M_1\frac{\varepsilon^2\gamma^2(\varepsilon)}{2}((1+\lambda_1\varepsilon\gamma(\varepsilon))^i + (1+\lambda_1\varepsilon\gamma(\varepsilon))^{i-1} + ... + 1) =$$
$$= \frac{M_1\varepsilon\gamma(\varepsilon)}{2}\left((1 + \lambda_1\varepsilon\gamma(\varepsilon))^{i+1} - 1\right) \le$$
$$\le \frac{M_1\varepsilon\gamma(\varepsilon)}{2}\left((1 + \lambda_1\varepsilon\gamma(\varepsilon))^{\frac{L}{\varepsilon\gamma(\varepsilon)}} - 1\right) \le$$
$$\le \frac{M_1\varepsilon\gamma(\varepsilon)}{2}(e^{\lambda_1 L} - 1). \quad (110)$$

So in view of the inequalities (106) the estimate is true:

$$\|y(t) - y^1(t)\| \le$$
$$\le \|y(t) - y(t_j)\| + \|y(t_j) - y^1(t_j)\| + \|y^1(t_j) - y^1(t)\| \le$$
$$\le 2M_1\varepsilon\gamma(\varepsilon) + \frac{M_1\varepsilon\gamma(\varepsilon)}{2}(e^{\lambda_1 L} - 1) \le \frac{M_1\varepsilon\gamma(\varepsilon)}{2}(e^{\lambda_1 L} + 3). \quad (111)$$

From the condition 2) of the theorem follows that for any $\eta_1 > 0$ exists $\varepsilon_1(\eta_1) > 0$ such that for $\varepsilon \le \varepsilon_1(\eta_1)$ the inequality holds

$$h\left(Y(y^1(t_j)), \frac{1}{\gamma(\varepsilon)}\int\limits_{t_j}^{t_{j+1}} X(s, y^1(t_j))ds + \right.$$
$$\left. + \frac{1}{\gamma(\varepsilon)}\sum_{t_j \le \tau_i < t_{j+1}} I_i(y^1(t_j))\right) < \eta_1. \quad (112)$$

Hence, there exist measurable selector $u_j(t) \in X(t, y^1(t_j))$ and vectors $p_{ij} \in I_i(y^1(t_j))$ such that

$$\left\|v_i - \frac{1}{\gamma(\varepsilon)}\left(\int\limits_{t_j}^{t_{j+1}} u_j(s)ds + \sum_{t_j \le \tau_i < t_{j+1}} p_{ij}\right)\right\| < \eta_1. \quad (113)$$

Consider the family of functions

$$x^1(t) = x^1(t_j) + \varepsilon\int\limits_{t_j}^{t} u_j(s)ds + \varepsilon\sum_{t_j \le \tau_i < t} p_{ij},$$
$$t \in (t_j, t_{j+1}],\ x^1(0) = x_0. \quad (114)$$

From (104), (113) and (114) using that $x^1(0) = y^1(0)$ follows that for $j = \overline{1, m}$

$$\|x^1(t_j) - y^1(t_j)\| \le \|x^1(t_{j-1}) - y^1(t_{j-1})\| + \eta_1\varepsilon\gamma(\varepsilon) \le ... \le$$
$$\le j\eta_1\varepsilon\gamma(\varepsilon) \le L\eta_1. \quad (115)$$

As for $t \in (t_j, t_{j+1}]$ we have

$$\|x^1(t) - x^1(t_j)\| \le M(1 + d)\varepsilon\gamma(\varepsilon) = M_1\varepsilon\gamma(\varepsilon),$$

taking into account the inequality (106) we get

$$\|x^1(t) - y^1(t)\| \le L\eta_1 + 2M_1\varepsilon\gamma(\varepsilon), \quad (116)$$

$$\|x^1(t) - y^1(t_j)\| \le L\eta_1 + M_1\varepsilon\gamma(\varepsilon).$$

Let us show that there exists a solution $x(t)$ of the inclusion (99) that is sufficiently close to $x^1(t)$.

Let $\theta_1, ..., \theta_p$ be the moments of impulses τ_i, that get into the semiinterval $(t_j, t_{j+1}]$. For convenience denote by $\theta_0 = t_j$, $\theta_{p+1} = t_{j+1}$. Let $\mu_k^+ = \|x^1(\theta_k + 0) - x(\theta_k + 0)\|$, $\mu_k^- = \|x^1(\theta_k) - x(\theta_k)\|$, $k = \overline{0, p}$.

Using the Lipschitz condition we have

$$\rho\left(\dot{x}^1(t), \varepsilon X(t, x^1(t))\right) \le h\left(\varepsilon X(t, y^1(t_j)), \varepsilon X(t, x^1(t))\right) \le$$
$$\le \varepsilon\lambda\|x^1(t) - y^1(t_j)\| \le \varepsilon\lambda(M_1\varepsilon\gamma(\varepsilon) + L\eta_1) = \eta^*,$$
$$\rho\left(\Delta x^1|_{t=\theta_k}, \varepsilon I_i(x^1(\theta_k))\right) \le h\left(\varepsilon I_i(y^1(t_j)), \varepsilon I_i(x^1(\theta_k))\right) \le$$
$$\le \varepsilon\lambda\|y^1(t_j) - x^1(\theta_k)\| \le \varepsilon\lambda(M_1\varepsilon\gamma(\varepsilon) + L\eta_1) = \eta^*.$$

According to A.F. Filippovs theorem [21] between the impulse points there exists a solution $x(t)$ of the inclusion (99) such that for $t \in (\theta_k, \theta_{k+1}]$ the estimate holds

$$\|x(t) - x^1(t)\| \le \mu_k^+ e^{\varepsilon\lambda(t-\theta_k)} + \varepsilon\int\limits_{\theta_k}^{t} e^{\varepsilon\lambda(t-s)}\eta^* ds.$$

Denote by $\gamma_k = \theta_{k+1} - \theta_k \le \gamma(\varepsilon),\ \gamma_0 + ... + \gamma_p = \gamma(\varepsilon)$. Then

$$\mu_{k+1}^- \le \mu_k^+ e^{\varepsilon\lambda\gamma_k} + \frac{\eta^*}{\lambda}\left(e^{\lambda\varepsilon\gamma(\varepsilon)} - 1\right). \tag{117}$$

When getting over the impulse point we have

$$\mu_{k+1}^+ \le \mu_{k+1}^- + \varepsilon h\left(I_i(y^1(t_j)), I_i(x(\theta_{k+1}))\right) \le$$

$$\le \mu_{k+1}^- + \varepsilon h\left(I_i(x^1(\theta_{k+1})), I_i(x(\theta_{k+1}))\right) + \varepsilon h\left(I_i(y^1(t_j)), I_i(x^1(\theta_{k+1}))\right) \le$$

$$\le \mu_{k+1}^- + \varepsilon\lambda\mu_{k+1}^- + \varepsilon h\left(I_i(y^1(t_j)), I_i(x^1(\theta_{k+1}))\right) \le$$

$$\le (1+\varepsilon\lambda)\mu_{k+1}^- + \eta^*. \tag{118}$$

From (117) and (118) follows that

$$\mu_{k+1}^+ \le (1+\varepsilon\lambda)e^{\varepsilon\lambda\gamma_k}\mu_k^+ + \beta, \quad \beta = \frac{\eta^*}{\lambda}(1+\varepsilon\lambda)\left(e^{\lambda\varepsilon\gamma(\varepsilon)} - 1\right) + \eta^*.$$

Hence

$$\mu_1^+ \le (1+\varepsilon\lambda)e^{\lambda\varepsilon\gamma_0}\mu_0^+ + \beta \le$$

$$\le (1+\varepsilon\lambda)e^{\lambda\varepsilon\gamma(\varepsilon)}\mu_0^+ + \beta,$$

$$\mu_2^+ \le (1+\varepsilon\lambda)e^{\varepsilon\lambda\gamma_1}\mu_1^+ + \beta \le$$

$$\le (1+\varepsilon\lambda)^2 e^{\varepsilon\lambda(\gamma_0+\gamma_1)}\mu_0^+ + \beta(1+\varepsilon\lambda)e^{\varepsilon\lambda\gamma_1} + \beta \le$$

$$\le (1+\varepsilon\lambda)^2 e^{\lambda\varepsilon\gamma(\varepsilon)}\mu_0^+ + \beta\left((1+\varepsilon\lambda)e^{\lambda\varepsilon\gamma(\varepsilon)} + 1\right)$$

etc.

$$\mu_{k+1}^+ \le (1+\varepsilon\lambda)^{k+1} e^{\varepsilon\lambda\gamma(\varepsilon)}\mu_0^+ +$$

$$+\beta\left(e^{\lambda\varepsilon\gamma(\varepsilon)}((1+\varepsilon\lambda)^k + ... + (1+\varepsilon\lambda)) + 1\right) =$$

$$= (1+\varepsilon\lambda)^{k+1} e^{\lambda\varepsilon\gamma(\varepsilon)}\mu_0^+ + \beta\left(e^{\lambda\varepsilon\gamma(\varepsilon)}\frac{(1+\varepsilon\lambda)^k - 1}{\varepsilon\lambda}(1+\varepsilon\lambda) + 1\right) \le$$

$$\le e^{\lambda(1+d)\varepsilon\gamma(\varepsilon)}\mu_0^+ + \eta^*\left(\frac{1+\varepsilon\lambda}{\lambda}(e^{\lambda\varepsilon\gamma(\varepsilon)} - 1) + 1\right)\times$$

$$\times\left(e^{\lambda\varepsilon\gamma(\varepsilon)}\frac{e^{\lambda d\varepsilon\gamma(\varepsilon)} - 1}{\varepsilon\lambda}(1+\varepsilon\lambda) + 1\right) =$$

$$= \alpha\mu_0^+ + \beta_1,$$

where $\alpha = e^{\lambda\varepsilon\gamma(\varepsilon)(1+d)}$,

$$\beta_1 = (\varepsilon\gamma(\varepsilon)M_1 + L\eta_1)\left(\frac{1+\varepsilon\lambda}{\lambda}(e^{\lambda\varepsilon\gamma(\varepsilon)} - 1) + 1\right)\times$$

$$\times\left(e^{\lambda\varepsilon\gamma(\varepsilon)}\left(e^{\lambda d\varepsilon\gamma(\varepsilon)} - 1\right)(1+\varepsilon\lambda) + \varepsilon\lambda\right).$$

So

$$\delta_{j+1}^+ = \|x(t_{j+1}) - x^1(t_{j+1})\| \le \alpha\delta_j^+ + \beta_1.$$

We obtain the sequence of the inequalities

$$\delta_0^+ = 0,\ \delta_1^+ \le \beta_1,\ \delta_2^+ \le \alpha\beta_1 + \beta_1 = (\alpha+1)\beta_1, ...,$$

$$\delta_{j+1}^+ \le (\alpha^j + ... + 1)\beta_1 = \frac{\alpha^{j+1} - 1}{\alpha - 1}\beta_1 \le$$

$$\le \frac{e^{\lambda L(1+d)} - 1}{e^{\lambda(1+d)\varepsilon\gamma(\varepsilon)} - 1}(M_1\varepsilon\gamma(\varepsilon) + L\eta_1)\left(\frac{1+\varepsilon\lambda}{\lambda}(e^{\lambda\varepsilon\gamma(\varepsilon)} - 1) + 1\right)\times$$

$$\times\left(e^{\lambda\varepsilon\gamma(\varepsilon)}\left(e^{\lambda d\varepsilon\gamma(\varepsilon)} - 1\right)(1+\varepsilon\lambda) + \varepsilon\lambda\right).$$

As

$$\lim_{\varepsilon\downarrow 0}\left(\frac{1+\varepsilon\lambda}{\lambda}(e^{\lambda\varepsilon\gamma(\varepsilon)} - 1) + 1\right) = 1$$

and

$$\lim_{\varepsilon\downarrow 0}\frac{e^{\lambda\varepsilon\gamma(\varepsilon)}\left(e^{\lambda d\varepsilon\gamma(\varepsilon)} - 1\right)(1+\varepsilon\lambda) + \varepsilon\lambda}{e^{\lambda(1+d)\varepsilon\gamma(\varepsilon)} - 1} =$$

$$= \lim_{\varepsilon\to 0}\frac{e^{\lambda\varepsilon\gamma(\varepsilon)}\frac{e^{\lambda d\varepsilon\gamma(\varepsilon)}-1}{\lambda\varepsilon\gamma(\varepsilon)} + \frac{1}{\gamma(\varepsilon)}}{\frac{e^{\lambda(1+d)\varepsilon\gamma(\varepsilon)}-1}{\lambda\varepsilon\gamma(\varepsilon)}} = \frac{d}{1+d},$$

then

$$\delta_{j+1}^+ \le C(M_1\varepsilon\gamma(\varepsilon) + L\eta_1)$$

for $\varepsilon \le \varepsilon_2$.

Therefore for $t \in (t_j, t_{j+1}]$ the inequality holds

$$\|x(t) - x^1(t)\| \le \|x(t) - x(t_j)\| + \|x(t_j) - x^1(t_j)\| + \|x^1(t) - x^1(t_j)\| \le$$

$$\le M(1+d)\varepsilon\gamma(\varepsilon) + M_1\varepsilon\gamma(\varepsilon) + C(M_1\varepsilon\gamma(\varepsilon) + L\eta_1) =$$

$$= M_1(2+C)\varepsilon\gamma(\varepsilon) + CL\eta_1. \tag{119}$$

In view of the inequalities (111),(116) and (119) we get that $\|x(t) - y(t)\|$ can be done less than any preassigned η by means of choosing $\varepsilon \le \varepsilon_0$ and η_1.

The second statement of the theorem is proved similarly.

The corollary of the given theorem is the following statement:

Theorem 11. *[6, 24]. Let in the domain Q the conditions 1), 2) of the theorem 10 hold and besides*

3) *for any $X_0 \subset D' \subset D$ and $t \ge 0$ the $R-$solutions of the inclusion (101) together with a $\rho-$neighborhood belong to the domain D.*

Then for any $\eta \in (0, \rho]$ and $L > 0$ there exists $\varepsilon^0(\eta, L) > 0$ such that for all $\varepsilon \in (0, \varepsilon^0]$ and $t \in [0, L\varepsilon^{-1}]$ the following statements fulfill:

1) *for any $R-$solution $R(t)$ of the inclusion (101) there exists an $R-$solution $X(t)$ of the inclusion (99) such that*

$$h(X(t), R(t)) < \eta; \tag{120}$$

2) *for any $R-$solution $X(t)$ of the inclusion (99) there exists an $R-$solution $R(t)$ of the inclusion (101) such that the inequality (120) holds.*

The averaging on the infinite interval. Consider the initial inclusion (99) and the averaged inclusion (101).

Theorem 12. *[6, 24]. Let in the domain Q the conditions 1) - 3) of the theorem 10 hold and besides*

4) *the R- solutions of the inclusion (101) are uniformly asymptotically stable.*

Then for any $\eta \in (0, \rho]$ there exists such $\varepsilon^0(\eta) > 0$ that for all $\varepsilon \in (0, \varepsilon^0]$ and $t \geq 0$ the inequality holds

$$h\left(R(t, \varepsilon), \bar{R}(\varepsilon t)\right) \leq \eta,$$

where $R(t, \varepsilon)$ is the R-solution of the inclusion (99), $\overline{R}(\varepsilon t)$ is the R-solution of the inclusion (101), $R(0, \varepsilon) = \bar{R}(0) = X_0$.

Theorem 13. *[6, 24]. Let in the domain Q the statements 1),2) of the theorem 10 hold and besides*

3) *the inclusion (101) has a periodic R- solution $\bar{R}(\tau)$, $\tau = \varepsilon t$, which trajectory C is asymptotically orbital stable and together with a $\rho-$neighborhood belongs to the domain D.*

Then for any $\eta \in (0, \rho]$ there exist such $\varepsilon^0 > 0$ and $\eta_0 \in (0, \eta]$ that for all $\varepsilon \in (0, \varepsilon^0]$, $\eta_1 \in (0, \eta_0]$ and $t \geq 0$ the inequality holds

$$h(R(t), C) \leq \eta,$$

where $R(t)$ is the R- solution of the inclusion (99), satisfying the initial condition $h(R(0), C) \leq \eta_1$.

Theorem 14. *[6, 24]. Let in the domain Q the statements 1),2) of the theorem 10 hold and besides*

3) *the inclusion (101) has an asymptotically stable equilibrium state $\bar{R}^0$, that together with a $\rho-$neighborhood belongs to the domain D.*

Then for any $\eta \in (0, \rho]$ there exist such $\varepsilon^0 > 0$ and $\eta_0 \in (0, \eta]$ that for all $\varepsilon \in (0, \varepsilon^0]$, $\eta_1 \in (0, \eta_0]$ and $t \geq 0$ the inequality holds

$$h(R(t), \bar{R}^0) \leq \eta,$$

where $R(t)$ is the R-solution of the inclusion (99), satisfying the initial condition $h(R(0), \bar{R}^0) \leq \eta_1$.

2.1.2. The partial averaging scheme

Along with the impulsive differential inclusion (99) we will consider the impulsive differential inclusion

$$\dot{y} \in \varepsilon \bar{X}(t, y), \ \ t \neq \nu_j, \ \ y(0) = x_0, \tag{121}$$

$$\Delta y|_{t=\nu_j} \in \varepsilon K_j(y),$$

where for any $(t, x) \in Q$ the limit

$$\lim_{T\to\infty} \frac{1}{T} h\left(\int\limits_t^{t+T} X(t, x)dt + \sum_{t \leq \tau_i < t+T} I_i(x),\right.$$

$$\left.\int\limits_t^{t+T} \bar{X}(t, x)dt + \sum_{t \leq \nu_j < t+T} K_j(x)\right) = 0 \tag{122}$$

exists.

Theorem 15. *[6, 24]. Let in the domain Q the following hold:*

1) *the mappings $X, \bar{X} : Q \to conv(\mathbb{R}^n)$ $I_j, K_j : D \to conv(\mathbb{R}^n)$ are continuous, uniformly bounded and satisfy the Lipschitz condition in x;*
2) *uniformly with respect to t and x in the domain Q the limit (122) exists and*

$$\frac{1}{T}\, i(t, t+T) \leq d < \infty, \quad \frac{1}{T}\, j(t, t+T) \leq d < \infty,$$

   where $i(t, t+T)$ and $j(t, t+T)$ are the quantities of points of the sequences $\{\tau_i\}$ and $\{s_j\}$ on the interval $(t, t+T]$;
3) *for any $x_0 \in D' \subset D$, $\varepsilon \in (0, \sigma]$ and $t \geq 0$ the solutions of the inclusion (121) together with a $\rho-$neighborhood belong to the domain D.*

Then for any $\eta \in (0, \rho]$ and $L > 0$ there exists $\varepsilon^0(\eta, L) \in (0, \sigma]$ such that for all $\varepsilon \in (0, \varepsilon^0]$ and $t \in [0, L\varepsilon^{-1}]$ the following statements fulfill:

1) *for any solution $y(t)$ of the inclusion (121) there exists a solution $x(t)$ of the inclusion (99) such that*

$$\|x(t) - y(t)\| \leq \eta; \tag{123}$$

2) *for any solution $x(t)$ of the inclusion (99) there exists a solution $y(t)$ of the inclusion (121) such that the inequality (123) holds.*

Theorem 16. *[6, 24]. Let in the domain Q the statements 1), 2) of the theorem 15 hold and besides*

3) *for any $X_0 \subset D' \subset D$, $\varepsilon \in (0, \sigma]$ and $t \geq 0$ the $R-$solutions of the inclusion (121) together with a $\rho-$neighborhood belong to the domain D.*

Then for any $\eta \in (0, \rho]$ and $L > 0$ there exists $\varepsilon^0(\eta, L) \in (0, \sigma]$ such that for all $\varepsilon \in (0, \varepsilon^0]$ and $t \in [0, L\varepsilon^{-1}]$ the following statements fulfill:

1) *for any $R-$solution $Y(t)$ of the inclusion (121) there exists an $R-$solution $X(t)$ of the inclusion (99) such that*

$$h(X(t), Y(t)) \leq \eta, \tag{124}$$

   where $X(0) = Y(0) = X_0$;
2) *for any $R-$solution $X(t)$ of the inclusion (99) there exists an $R-$solution $Y(t)$ of the inclusion (121) such that the inequality (124) holds.*

2.2. Differential inclusions with impulses in non-fixed moments of time

Consider the differential inclusion with impulses in non-fixed moments of time

$$\dot{x} \in F^1(t, x, \varepsilon), \ \ x(0) = x_0, \ \ t \neq \varepsilon\tau_i^1(x), \ \ t \neq \sigma_i^1(x), \tag{125}$$

$$\Delta x|_{t=\varepsilon\tau_i^1(x)} \in \varepsilon I_i^1(x), \tag{126}$$

$$\Delta x|_{t=\sigma_p^1(x)} \in K_p^1(x). \tag{127}$$

Let us assign to the inclusion (125) - (127) the following differential inclusion

$$\dot{y} \in F^2(t, y, \varepsilon), \ \ y(0) = x_0, \ \ t \neq \varepsilon\tau_i^2(x), \ \ t \neq \sigma_i^2(y), \tag{128}$$

$$\Delta y|_{t=\varepsilon\tau_i^2(y)} \in \varepsilon I_i^2(y), \tag{129}$$

$$\Delta y|_{t=\sigma_p^2(y)} \in K_p^2(y), \qquad (130)$$

where $t \in [0, L]$ is time, $x \in D \subset \mathbb{R}^n$ is a phase vector, $\varepsilon > 0$ is a small parameter, the impulse surfases $\tau_i^j, \sigma_i^j : D \to \mathbb{R}$, the multivalued mappings $F^j : [0, L] \times D \times (0, \epsilon] \to conv(\mathbb{R}^n)$, $I_i^j, K_p^j : D \to conv(\mathbb{R}^n)$, $i = \overline{1,k}$, $p = \overline{1,r}$, $j = 1, 2$.

Let $T_D(x)$ be the Bouligard contingent cone for $x \in D$, i.e.

$$T_D(x) = \left\{ y \in \mathbb{R}^n : \lim_{s\downarrow 0} s^{-1} \inf_{z\in D} |x + sy - z| = 0 \right\}.$$

Theorem 17. *[6, 25]. Let in the domain $Q = [0, L]\times D\times(0, \varepsilon]$ the following conditions fulfill:*

1) *the mappings $F^j(t, x, \varepsilon), I_i^j(x)$ and functions $\tau_i^j(x)$ are continuous in t, satisfy the Lipschitz condition in x and $F^j(t, x, \varepsilon) \subset T_D(x)$, $x + I_i^j(x) \subset D$;*
2) *the mappings $F^j(t, x, \varepsilon)$ $I_i^j(x)$ are uniformly bounded with constant M;*
3) *uniformly with respect to $(t, x) \in Q$*

$$\lim_{\varepsilon\to 0} \frac{1}{\Delta} h\left(\int_t^{t+\Delta} F^1(t, x, \varepsilon)\, dt + \varepsilon \sum_{t<\varepsilon\tau_i^1(x)<t+\Delta} I_i^1(x)\,, \right.$$

$$\left. \int_t^{t+\Delta} F^2(t, x, \varepsilon)\, dt + \varepsilon \sum_{t<\varepsilon\tau_i^2(x)<t+\Delta} I_i^2(x) \right) = 0;$$

4) *the numbers $J_j(t, t + \Delta)$, $j = 1, 2$ of the asymptotically small impulses of the solutions of the inclusions (125), (126) and (128), (129) satisfy the following inequalities on the interval $(t, t + \Delta]$*
$$\frac{1}{\Delta} J_j(t, t + \Delta) \le \frac{A}{\varepsilon} < \infty;$$
5) *the surfaces $t = \varepsilon\tau_i^j(x)$ do not intersect each other and for all $x \in D$, $z \in \varepsilon I_i^j$, $j = 1, 2$, the following inequalities hold*
$$\tau_i^j(x) \ge \tau_i^j(x + z),\ |\tau_{i+1}^j(x) - \tau_i^j(x)| \le M;$$
6) *the mappings $K_i^j(x)$ and functions $\sigma_j^i(x)$ satisfy the Lipschitz condition with constant μ and $x + K_i^j(x) \subset D$, $x \in D$;*
7) *the surfaces $t = \sigma_j^i(x)$ do not intersect each other and for all $x \in D$, $z \in K_i^j$, $j = 1, 2$ the following inequalities hold $\sigma_j^i(x) \ge \sigma_j^i(x + z)$;*
8) $\mu M < 1$, $\quad h(K_i^1(x), K_i^2(x)) \le \xi$, $|\sigma_i^1(x) - \sigma_i^2(x)| \le \xi$.

Then for any $\eta > 0$ there exist $\xi > 0$ and $\delta > 0$ such that for any solution $x(t)$ of the inclusion (125) - (127) there exists a solution $y(t)$ of the inclusion (128) - (130) with the initial condition $\|y_0 - x_0\| \le \delta$ such that

$$\|x(t) - y(t)\| \le \eta,$$

$$t \in [0, L]\backslash \left\{ \bigcup_p [s_p^2 - \delta_p, s_p^2 + \delta_p] \bigcup_i [t_i^2 - \Delta_i, t_i^2 + \Delta_i] \right\},$$

where $s_p^2 = \sigma_p(y((s_p^2)), t_i^2 = \tau_i(y((t_i^2)), \sum_p \delta_p + \sum_i \Delta_i < C\eta$.

Remark 16. To obtain the classical N.N. Bogolubov's integral continuity condition we converge ε to zero and T to infinity in the condition 3). In this sense this theorem generalizes the first N.N. Bogolyubov's theorem for the method of partial averaging.

Remark 17. In this review we have not considered the V.A. Plotnikov's results devoted to the averaging of the differential equations with discontinuous right-hand side in case of the sliding mode [5, 26, 27], to the application of the averaging method and differential inclusions to the research of control problems [1, 2, 4, 7, 28, 29, 30, 31, 32, 33], to averaging of discrete inclusions [34], to questions of asymptotical researches of singularly perturbed differential inclusions and to the generalization of the A.N. Tikhonov theorem on differential inclusions [4, 35, 36, 37].

Conclusion

For now the questions of the construction of the higher approximations for R-solutions of differential inclusions, the possibility of application of the averaging method for differential inclusions with discontinuous right-hand side, differential equations and inclusions in semilinear metric spaces with fast and slow variables, etc. are open. It is caused by the basic difficulties connected with nonlinearity of the spaces.

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