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Pricing and hedging of variable annuities with path-dependent guarantee in Wishart stochastic volatility models

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ABSTRACT

This paper presents the pricing of a path-dependent guaranteed minimum maturity benefit in the Wishart multidimensional stochastic volatility model and the Wishart affine stochastic correlation model. We derive a closed-form solution for the option price in these two models, requiring only the computation of a one-dimensional integration. Thanks to the remarkable analytical properties of these models, we also compute all sensitivities of the option price to the model parameters. An implementation illustrates the results, confirms that pricing is fast and accurate, and provides a framework for pricing and risk management of this product in Wishart stochastic volatility models.

1. Introduction

Variable annuities play a significant role in the retirement market as essential insurance products. Their broad applicability and potential to improve retirement planning outcomes highlight their importance and have been widely studied in the literature previously along many different strands, see Feng et al. (2020) for an overview. In this article, we focus on variable annuities with Guaranteed Minimum Maturity Benefit (GMMB), which guarantees the policyholder an amount at maturity that is typically linked to an equity, provided the policyholder is alive at that time.

An important feature of the GMMB is the nature of the guarantee. In its simplest form, a fixed performance of the underlying equity is guaranteed, making this type of GMMB characteristically similar to a standard European call option. A slightly more complex guarantee is where the guaranteed performance grows at a constant rate over time, however the pricing of this product still simplifies to the previous case, the pricing of a standard European call option. More difficult to handle is the case where the guarantee is based on a geometric average of the equity-linked value over time, which introduces significant path dependency and makes pricing particularly challenging. Path-dependent style guarantees have been extensively studied under geometric Brownian motion assumptions, including ratchet-type guarantees (some early works include Windcliff et al. (2006) and Kijima and Wong (2007)), as well as under more sophisticated dynamics in Qian et al. (2010), Fan et al. (2015), Hieber (2017), and others. This extensive literature highlights that, due to the inherent path dependency of these insurance products, obtaining exact solutions remains difficult and, to the best of our knowledge, has not yet been achieved for Heston (1993)-like stochastic volatility dynamics.[1] Indeed, for this latter type of model, to the best of our knowledge, Cui et al. (2017b) is the only work that provides a framework to price the option but it is an approximation.

Another strand of literature, arguably (much) smaller, has focused on the pricing of guaranteed equity-linked annuities on multiple assets, see for example Ng and Li (2011), Ng and Li (2013), Da Fonseca and Ziveyi (2017) or Hartman et al. (2020). Due to the increase in dimensionality from the inclusion of multiple assets, the pricing of these guaranteed equity-linked annuities with multiple assets becomes much more tenuous, arising from the dependence between these assets. This problem in itself is difficult to account for in the modelling stage, and beyond this

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[1] Note that Kang et al. (2022) derive a closed-form solution for this type of product when the equity price follows a Markov switching model with multiple dynamics, one of which is the dynamic of Heston (1993), and as such it is a potential solution to the option pricing problem for this model. A comparison between their solution and ours is made later in our paper. The difficulty that comes with the lack of independence of price increments is explicitly mentioned in Fan et al. (2015, p. 71).

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introduces numerical implementation issues as well. These challenges make it quite difficult to extend on the multiple-asset side.

Although not systematically analysed in the aforementioned works, the problem of hedging or sensitivity analysis of these guaranteed equity-linked annuities is of practical importance to the issuers of these insurance products for risk management purposes. The calculation of these risk-management tools is scarce in the literature for ratchet-type guarantees, likely due to the difficulty of the path-dependency of the guaranteed equity-linked annuities. A work that provides these sensitivities for GMMBs includes Ignatieva et al. (2016) (albeit for a fixed guarantee) and for guaranteed minimum withdrawal benefits see Feng and Jing (2017). Obtaining sensitivities for a given insurance product often requires computing the derivatives of the moment generating function of the model state variables. Not all models allow such a calculation, making the construction of a risk management framework challenging.

Our contribution to the literature is threefold. First, we price the guaranteed equity-linked annuity with a guarantee that is a geometric average of the equity-linked value in the Wishart multidimensional stochastic volatility (WMSV) model of Da Fonseca et al. (2008), which is a single-equity multivariate stochastic volatility model whose volatility takes its values in the space of symmetric positive definite matrices. The WMSV model allows for the volatility to be driven by multiple factors that can have a general dependency structure between them. However, this additional flexibility in the volatility process does not come at the expense of tractability of the model, even for the path-dependent type guarantees, where we are able to obtain an exact pricing formula. Since the WMSV corresponds to the Heston (1993) model when its dimension is one, our work solves the problem for this model as well. Second, we also extend the literature on the pricing of the GMMBs on multiple assets by utilising the Wishart stochastic affine correlation (WASC) model of Da Fonseca et al. (2007). This is a correlated multi-asset, matrix-valued stochastic volatility model that is highly flexible and retains strong tractability due to the Wishart process, making it suitable to tackle the multiple-asset guaranteed equity-linked annuity problem, even for the path-dependent type guarantee, where once again we obtain a closed-form solution. Third, for both of these models, we showcase their risk management capabilities by computing the sensitivities of the GMMB products with respect to the model parameters. This can be achieved thanks to the specific structure of the moment generating function of the Wishart process, which allows us to use the derivative of the exponential map, a well-known result of Lie group analysis. Finally, a numerical implementation of the two models shows that, despite their matrix nature, they can be brought to an operational level.

The rest of the paper is structured as follows. In Section 2 we introduce the two models and develop all the analytical properties needed in this paper. Section 3 contains the pricing of the equity-linked annuity guaranteed with a geometric average as well as all sensitivities for this insurance product. An implementation and numerical experiments of our models are carried out in Section 4. Finally, in Section 5, we conclude and suggest further research directions.

2. The models

In this section we present the two stochastic volatility models used in this work, the first one is the single-asset multidimensional stochastic volatility model introduced in Da Fonseca et al. (2008) (the WMSV model) while the second one is a multi-asset stochastic volatility and correlation model introduced in Da Fonseca et al. (2007) (the WASC model). The WMSV and WASC models are respectively multifactor and multi-asset (and potentially multi-factor) extensions of the Heston stochastic volatility model. The WMSV and WASC models have previously been calibrated to options on major indicies in Da Fonseca and Grasselli (2011) and have demonstrated superior fit relative to the Heston model to these options due to the additional flexibility awarded to the models from its flexibility positive-definite matrix structure. The matrix structure allows us to accurately capture both multidimensional aspects of the volatility smile (WMSV) observed in the market as well as general dependence structure between the assets (WASC) whilst remaining highly tractable models, extensions based on the Heston model cannot achieve this, see Zhong et al. (2023) for an example of such limitations. New in this work are the derivatives with respect to the model parameters of the moment generating functions of the state variables for these two models, which are presented in Propositions 2.2-2.5 and Propositions 2.7-2.9. They play a crucial role in the development of risk management tools for the options we analyse and are based on the derivative of the exponential map.

2.1. The WMSV model

Let us first consider the WMSV proposed by Da Fonseca et al. (2008). In this model the volatility is described by the Wishart process, a matrix-valued stochastic process introduced by Bru (1991), and the dynamic of the equity price $s_t \in \mathbb{R}_+$ and its volatility that depends on $v_t \in \mathbb{S}_n^{++}$, with $\mathbb{S}_n^{++}$ set of positive definite matrices, under the risk-neutral measure $\mathbb{Q}$ is given by the stochastic differential equations (SDEs):

$$ds_t = s_t r dt + s_t \mathrm{tr}\left[\sqrt{v_t}\left(dw_t \rho^\top + db_t \sqrt{I_n - \rho\rho^\top}\right)\right], \tag{1}$$

$$dv_t = (\omega + m v_t + v_t m^\top)dt + \sqrt{v_t}\, dw_t \sigma + \sigma^\top dw_t^\top \sqrt{v_t}, \tag{2}$$

with $s_0 \in \mathbb{R}_+$ and $r \geq 0$, tr[.] is the trace operator, $(w_t)_{t\geq 0}$ and $(b_t)_{t\geq 0}$ are two matrix Brownian motions with values in $\mathsf{M}(n)$ (the set of square $n \times n$ real matrices) under the risk-neutral measure $\mathbb{Q}$. A matrix Brownian motion with values in $\mathsf{M}(n)$ is composed of n^2 independent Brownian motions. The correlation between the equity and its matrix volatility is controlled by $\rho \in \mathsf{M}(n)$ while $\cdot^\top$ stands for the matrix transposition. The process $(v_t)_{t\geq 0}$ is an $n \times n$ matrix that belongs to the set of positive definite matrices $\mathbb{S}_n^{++}$. The set of positive semi-definite matrices is denoted $\mathbb{S}_n^{+}$ and the set of symmetric matrices is denoted $\mathbb{S}_n$. We assume that $I_n - \rho\rho^\top \in \mathbb{S}_n^{++}$ so that the (matrix) square root is well defined, see Golub and Van Loan (1996, p. 149, Section 4.2.10). For the matrix coefficients, $\omega \in \mathbb{S}_n^{++}$ and satisfies certain constraints involving $\sigma^\top\sigma$ to ensure the positiveness of the matrix process v_t. The matrix $m \in \mathsf{M}(n)$ is such that $\{\Re(\lambda_i^m) < 0; i = 1, \ldots, n\}$ where $\lambda_i^m \in \mathrm{Spec}(m)$ for $i = 1, \ldots, n$ and $\mathrm{Spec}(m)$ is the spectrum of the matrix m while $\Re(\cdot)$ stands for the real part. The matrix σ belongs to $\mathsf{GL}_n(\mathbb{R})$ the general linear group over $\mathbb{R}$ (*i.e.*, the set of real invertible matrices). Thanks to the invariance of the law of the Brownian motion to rotations and the polar decomposition of σ, we can assume that $\sigma \in \mathbb{S}_n^{++}$. We denote by e_{ij} the basis of $\mathsf{M}(n)$, it is the $n \times n$ matrix with 1 in the (i, j) place and zero elsewhere. Lastly I_n stands for the $n \times n$ identity matrix while 0_n is the $n \times n$ null matrix.

The Wishart process was initially defined and analyzed in Bru (1991) under the assumption that

$$\omega = \beta\sigma^2, \tag{3}$$

with $\beta \in \mathbb{R}_+$ such that $\beta \geq n + 1$ to ensure that $v_t \in \mathbb{S}_n^{++}$. Hereafter, this specification will be referred to as the Bru case. It was later extended in Mayerhofer et al. (2011) (see also Cuchiero et al. (2011)) to the case $\omega \in \mathbb{S}_n^{++}$ and proved that if

$$\omega \geq \beta\sigma^2, \tag{4}$$

with $\beta \geq n + 1$ (where (4) means that $\omega - \beta\sigma^2 \in \mathbb{S}_n^{++}$) then $v_t \in \mathbb{S}_n^{++}$. For an overview of the Wishart process both from a theoretical point of view and numerical point of view we refer the reader to Alfonsi (2015).

In this model the instantaneous variance of the equity log-returns is associated to the trace of the Wishart matrix, that is: $d\langle \ln s_., \ln s_. \rangle_t = \mathrm{tr}[v_t]dt$, which alone is not Markovian and constitutes also a multivariate extension of the Heston (1993) model as it can be checked by considering the case $n = 1$ in (2). Computing the expectation of this trace using a partial differential approach would also require consideration of the full state variable v. The moment generating function of

$(x_t = \ln s_t, v_t)$ ($x_t \in \mathbb{R}$ and $\ln$ stands for the logarithm of a scalar number) is known explicitly thanks to the affine property of the WMSV model.

Following Da Fonseca et al. (2008) the infinitesimal generator of $(x_t = \ln s_t, v_t)$ is:

$$\mathcal{G} = \left(r - \frac{1}{2}\mathrm{tr}[v]\right)\partial_x + \frac{1}{2}\mathrm{tr}[v]\partial^2_{xx} + 2\mathrm{tr}[v\rho\sigma D]\partial_x \\ + \mathrm{tr}[(\omega + mv + vm^\top)D + 2vD\sigma^2 D], \tag{5}$$

where $\partial_x = \frac{\partial}{\partial x}$ and D is the $n \times n$ matrix operator $D_{ij} := \partial_{v_{ij}}$.

The infinitesimal generator gives the moment generating function of $(\ln s_t, v_t)$ that allows us to perform the option pricing valuation. The proof of this result appears in Da Fonseca et al. (2008) and Da Fonseca and Grasselli (2011), for completeness we provide a proof on the online supplementary appendix. Compared to these references, we give additional details regarding some calculus of matrix functions that are involved in the derivation of the moment generating function.

Proposition 2.1. *(See Da Fonseca et al. (2008) and Da Fonseca and Grasselli (2011)). Let $(s_t, v_t)_{t\geq 0}$ a process following a WMSV dynamic (1)-(2). Given a scalar $z \in \mathbb{R}$ and a matrice $\theta_1 \in \mathbb{S}_n$, the joint moment generating function of $(x_t = \ln s_t, v_t) \in \mathbb{R} \times \mathbb{S}^{++}_n$ is given by*

$$G_{WMSV}(t, z, \theta_1, x_0, v_0) := \mathbb{E}\left[e^{zx_t + \mathrm{tr}[\theta_1 v_t]}\right], \\ = e^{zx_0 + zrt + \mathrm{tr}[a(t)v_0] + c(t)}, \tag{6}$$

where the deterministic matrix function $a(t)$ with values in $\mathsf{M}(n)$ and the scalar function $c(t)$ satisfies the following ordinary differential equations (ODEs) where we omit the time variable t:

$$a' = a(m + z\sigma\rho^\top) + (m + z\sigma\rho^\top)^\top a + 2a\sigma^2 a + \frac{z(z-1)}{2}I_n, \tag{7}$$

$$c' = \mathrm{tr}[\omega a], \tag{8}$$

with initial conditions $a(0) = \theta_1$, $c(0) = 0$. The symbol $.'$ stands for the time derivative. The solution is explicitly given by:

$$a(t) = (\theta_1 A_{12}(t) + A_{22}(t))^{-1}(\theta_1 A_{11}(t) + A_{21}(t)), \tag{9}$$

with

$$\begin{pmatrix} A_{11}(t) & A_{12}(t) \\ A_{21}(t) & A_{22}(t) \end{pmatrix} = e^{t\mathcal{A}}, \tag{10}$$

and

$$\mathcal{A} = \begin{pmatrix} m + z\sigma\rho^\top & -2\sigma^2 \\ \frac{z(z-1)}{2}I_n & -(m + z\sigma\rho^\top)^\top \end{pmatrix}, \tag{11}$$

while $c(t) = \int_0^t \mathrm{tr}[\omega a(u)]du$.

If we further assume that ω in (2) is such that $\omega = \beta\sigma^2$ with $\beta \in \mathbb{R}$ and $\beta \geq n+1$, i.e., the Bru condition (3) is satisfied, then the function $c(t)$ solution of (8) can be explicitly integrated to

$$c(t) = -\frac{\beta}{2}\mathrm{tr}[\log(\theta_1 A_{12}(t) + A_{22}(t)) + t(m + z\sigma\rho^\top)], \tag{12}$$

with $\log(.)$ the logarithm of a matrix, see Faraut (2008, p. 25, Section 2.2). Since $e^{c(t)}$ is involved in the moment generating function, the relation $\det(e^u) = e^{\mathrm{tr}[u]}$ for $u \in \mathsf{M}(n)$, where $\det(.)$ stands for the determinant of a matrix, implies

$$e^{c(t)} = \frac{e^{-\frac{\beta t}{2}\mathrm{tr}[m + z\sigma\rho^\top]}}{\det(\theta_1 A_{12}(t) + A_{22}(t))^{\beta/2}}. \tag{13}$$

We may denote the functions $a(t)$ and $c(t)$ by $a(t, z, \theta_1)$ and $c(t, z, \theta_1)$ when the explicit dependency to the parameters is necessary.

Remark 2.1. The proof of the proposition clearly shows that when $\omega \neq \beta\sigma^2$ then (8) cannot be explicitly integrated, or at least it is an open question.

In order to compute the sensitivity of the option prices, we need to be able to differentiate the moment generating function, which is not straightforward as it is a matrix function. Thanks to the analytical property of the Wishart process, and therefore of the WMSV model, the derivative of the moment generating function can be computed explicitly.

Proposition 2.2. *Let $(s_t, v_t)_{t\geq 0}$ be a process following a WMSV dynamic (1)–(2). Let ν be one of the model's real-valued parameters; specifically, ν could be any entry m_{ij} (for $i, j = 1, \ldots, n$), or entry σ_{ij}, or ρ_{ij}. We denote by $G_{WMSV}(t, z, \theta_1, x_0, v_0)$ the moment generating function of $(x_t = \ln s_t, v_t) \in \mathbb{R} \times \mathbb{S}^{++}_n$ (see (6)), where $z \in \mathbb{R}$ and $\theta_1 \in \mathbb{S}_n$. Assuming that θ_1 also depends on ν, the directional derivative (Fréchet derivative) of G_{WMSV} with respect to ν is given by:*

$$\partial_\nu G_{WMSV}(t, z, \theta_1, x_0, v_0) := (\mathrm{tr}[a_\nu(t)v_0] + c_\nu(t))G_{WMSV}(t, z, \theta_1, x_0, v_0), \tag{14}$$

with

$$a_\nu(t) = -(\theta_1 A_{12}(t) + A_{22}(t))^{-1}(\theta_{1,\nu}A_{12}(t) + \theta_1 A_{12,\nu}(t) + A_{22,\nu}(t))a(t) \\ + (\theta_1 A_{12}(t) + A_{22}(t))^{-1}(\theta_{1,\nu}A_{11}(t) + \theta_1 A_{11,\nu}(t) + A_{21,\nu}(t)), \tag{15}$$

where $a(t)$, $A_{11}(t)$, $A_{12}(t)$ are given in Proposition 2.1, $\theta_{1,\nu}$ is the derivative of θ_1 to ν, and

$$\begin{pmatrix} A_{11,\nu}(t) & A_{12,\nu}(t) \\ A_{21,\nu}(t) & A_{22,\nu}(t) \end{pmatrix} = (D\exp)_{t\mathcal{A}}(t\mathcal{H}), \tag{16}$$

with

$$(D\exp)_{t\mathcal{A}}(t\mathcal{H}) = \left.\frac{d}{d\nu}e^{t\mathcal{A} + \nu t\mathcal{H}}\right|_{\nu=0}, \tag{17}$$

the derivative of the exponential map at $t\mathcal{A}$ in the direction $t\mathcal{H}$ that is parameter dependent, see Hall (2015, Theorem 5.4) and Faraut (2008, Theorem 2.1.4). For each parameter with respect to which the derivation is performed corresponds a matrix $\mathcal{H}$, they are given by

ν	$\mathcal{H}$
m_{ij}	$\begin{pmatrix} e_{ij} & 0_n \\ 0_n & -e_{ij}^\top \end{pmatrix}$
σ_{ii}	$\begin{pmatrix} ze_{ii}\rho^\top & -2(\sigma e_{ii} + e_{ii}\sigma) \\ 0_n & -(ze_{ii}\rho^\top)^\top \end{pmatrix}$
σ_{ij} $(j \neq i)$	$\begin{pmatrix} z(e_{ij} + e_{ji})\rho^\top & -2(\sigma(e_{ij} + e_{ji}) + (e_{ij} + e_{ji})\sigma) \\ 0_n & -(z(e_{ij} + e_{ji})\rho^\top)^\top \end{pmatrix}$
ρ_{ij}	$\begin{pmatrix} z\sigma e_{ij}^\top & 0_n \\ 0_n & -(z\sigma e_{ij}^\top)^\top \end{pmatrix}$

while $c_\nu(t) = \int_0^t \mathrm{tr}[\omega a_\nu(u)]du$. We may denote $a_\nu(t, z, \theta_1, \theta_{1,\nu})$ instead of $a_\nu(t)$ in (15) and $c_\nu(t, z, \theta_1, \theta_{1,\nu})$ instead of $c_\nu(t)$ when convenient to avoid any ambiguity.

To implement the derivative of the exponential map, the following representation is useful.

Proposition 2.3. *Let the derivative of the exponential map given by (17), then the following representation holds*

$$\mathrm{vec}\left((D\exp)_{t\mathcal{A}}(t\mathcal{H})\right) = (I_{2n} \otimes e^{t\mathcal{A}})\varphi\left((I_{2n} \otimes t\mathcal{A}) - (t\mathcal{A}^\top \otimes I_{2n})\right)\mathrm{vec}(t\mathcal{H}), \tag{18}$$

with

$$\varphi(z) = \frac{1 - e^{-z}}{z}, \tag{19}$$

extended to $z = 0$ by $\varphi(0) = 1$, $\mathrm{vec}(X)$ the application defined for $X \in \mathsf{M}(2n)$ with values in $\mathbb{R}^{(2n)^2}$ the vec operator that stacks vertically the columns of X and $\otimes$ is the Kronecker product.

Assuming $\omega=\beta\sigma^2$, $c(t)$ of (6), which solves (8), is known and given by (12) or (13). Its derivative with respect to the model parameters can also be calculated as shown in the following proposition.

Proposition 2.4. *Let $(s_t,v_t)_{t\geq 0}$ be a process following a WMSV dynamic (1)–(2) with $\omega=\beta\sigma^2$ and $\beta\geq n+1$ (thus satisfying the Bru condition (3)). Let ν be one of the model's real-valued parameters, specifically ν can be any entry m_{ij} (for $i,j=1,\dots,n$), or else σ_{ij} or ρ_{ij}. We denote by $c(t)$ the function introduced in Proposition 2.2 that is given by (12). Then the derivative of $c(t)$ with respect to ν, denoted $c_\nu(t)$, is given as follows. In particular, when $\nu=m_{ij}$,*

$$-\frac{\beta t}{2}\mathrm{tr}[e_{ij}]-J_\nu\,, \tag{20}$$

when ν is σ_{ii} by

$$-\frac{\beta t}{2}\mathrm{tr}[ze_{ii}\rho^\top]-J_\nu\,, \tag{21}$$

when ν is σ_{ij} with $i\neq j$ by

$$-\frac{\beta t}{2}\mathrm{tr}[z(e_{ij}+e_{ji})\rho^\top]-J_\nu\,, \tag{22}$$

and when ν is ρ_{ij} by

$$-\frac{\beta t}{2}\mathrm{tr}[z\sigma e_{ij}^\top]-J_\nu\,, \tag{23}$$

with

$$J_\nu=\frac{\beta}{2}\mathrm{tr}\left[(\theta_1A_{12}(t)+A_{22}(t))^{-1}(\theta_{1,\nu}A_{12}(t)+\theta_1A_{12,\nu}(t)+A_{22,\nu}(t))\right]\,, \tag{24}$$

where in the above equation the term is computed using the appropriate terms $A_{12,\nu}(t)$ and $A_{22,\nu}(t)$ given by (16) with the corresponding $\mathcal{H}$.

The derivative of the moment generating function with respect to $v_{0,ij}$ or β is straightforward to compute and therefore the proof is omitted.

Proposition 2.5. *Let $(s_t,v_t)_{t\geq 0}$ a process following a WMSV dynamic (1)-(2) then the derivative with respect to one element of v_0 (i.e., $v_{0,ij}$ for $i=1,\dots,n$ and $j=1,\dots,n$) of the moment generating function of $(x_t=\ln s_t,v_t)$ given for $z\in\mathbb{R}$ and $\theta_1\in\mathbb{S}_n$ (assuming that θ_1 does not depend on $v_{0,ij}$) by $G_{WMSV}(t,z,\theta_1,x_0,v_0)$ of (6) is*

$$\partial_{v_{0,ij}}G_{WMSV}(t,z,\theta_1,x_0,v_0)=(1+\delta_{ij})a_{ij}(t)G_{WMSV}(t,z,\theta_1,x_0,v_0)\,, \tag{25}$$

with $\delta_{ij}=1$ if $i=j$ and 0 otherwise. If $\omega=\beta\sigma^2$ with $\beta\geq n+1$, i.e., so that $c(t)$ in (6) is given by (12), then the derivative with respect to β of this moment generating function, assuming that θ_1 does not depend on β, is

$$\partial_\beta G_{WMSV}(t,z,\theta_1,x_0,v_0)=c_\beta G_{WMSV}(t,z,\theta_1,x_0,v_0)\,, \tag{26}$$

with $c_\beta=-\frac{1}{2}\mathrm{tr}[\log(\theta_1A_{12}(t)+A_{22}(t))+t(m+z\sigma\rho^\top)]$.

2.2. The WASC model

The WASC model of Da Fonseca et al. (2007) consists of a n-dimensional risky asset $s_t=(s_{1,t},..,s_{n,t})^\top\in\mathbb{R}^n_+$ whose dynamic is given under the risk-neutral measure $\mathbb{Q}$ by

$$ds_t=\mathrm{diag}[s_t]\left(r\mathbf{1}dt+\sqrt{v_t}(dw_t\rho+\sqrt{1-\rho^\top\rho}db_t)\right), \tag{27}$$

$$dv_t=(\omega+mv_t+v_tm^\top)dt+\sqrt{v_t}dw_t\sigma+\sigma dw_t^\top\sqrt{v_t}, \tag{28}$$

where $b_t\in\mathbb{R}^n$ is a vector Brownian motion, $\mathbf{1}$ is a $n\times 1$ vector of ones, $\mathrm{diag}[z]$ is a $n\times n$ matrix with $z\in\mathbb{R}^n$ on its diagonal, $\rho\in\mathbb{R}^n$ with $\rho\in[-1,1]^n$ and $\rho^\top\rho\leq 1$, $(v_t)_{t\geq 0}$ is a Wishart process similar to (2) (so that $(w_t)_{t\geq 0}$ is a matrix Brownian motion). As $(b_t)_{t\geq 0}$ and $(w_t)_{t\geq 0}$ are independent then $dw_t\rho+\sqrt{1-\rho^\top\rho}db_t$ is a vector Brownian motion (with values in $\mathbb{R}^n$).

Remarkably, such a correlation structure is the only one which is compatible with the affine property of the model, see Da Fonseca et al. (2007). Note also that the returns' variance-covariance matrix is by construction v_t.

Following Da Fonseca et al. (2007) the infinitesimal generator of $(x_t=\ln s_t,v_t)\in\mathbb{R}^n\times\mathbb{S}_n^{++}$, with $x_t=(x_{1,t},\dots,x_{n,t})^\top=(\ln s_{1,t},\dots,\ln s_{n,t})^\top$ where ln is the natural logarithm of a scalar so that $x_t\in\mathbb{R}^n$, is:

$$\begin{aligned}\mathcal{G}=&\nabla_x\left(r\mathbf{1}-\frac{1}{2}\mathrm{vec}[(v_{ii})]\right)+\frac{1}{2}\nabla_xv\nabla_x^\top+2\mathrm{tr}[D\sigma\rho\nabla_xv]\\&+\mathrm{tr}[(\omega+mv+vm^\top)D+2vD\sigma^2D]\,,\end{aligned} \tag{29}$$

with $\nabla_x=(\partial_{x_1},\dots,\partial_{x_n})$, D as previously defined, $\mathrm{vec}[(v_{ii})]\in\mathbb{R}^n$ is a vector that contains the diagonal elements of a matrix v.

Remark 2.2. Note that in the WASC model $\rho\in\mathbb{R}^n$ whilst in the WMSV model $\rho\in\mathsf{M}(n)$.

As for the WMSV, the infinitesimal generator gives the moment generating function of $(x_t=\ln s_t,v_t)$ that allows us to perform the option pricing valuation. The proof of the following result appears in Da Fonseca et al. (2007) and is very similar to that of Proposition 2.1, and the Remark 2.1 applies here as well.

Proposition 2.6. *(See Da Fonseca et al. (2007)). Let $(s_t,v_t)_{t\geq 0}$ a process following a WASC dynamic (27)-(28). Given $z\in\mathbb{R}^n$ and a matrice $\theta_1\in\mathbb{S}_n$, then the joint moment generating function of $(x_t=\ln s_t,v_t)\in\mathbb{R}^n\times\mathbb{S}_n^{++}$ is*

$$\begin{aligned}G_{WASC}(t,z,\theta_1,x_0,v_0)&:=\mathbb{E}\left[e^{z^\top x_t+\mathrm{tr}[\theta_1v_t]}\right]\\&=e^{z^\top x_0+z^\top\mathbf{1}rt+\mathrm{tr}[a(t)v]+c(t)},\end{aligned} \tag{30}$$

with the deterministic matrix function $a(t)$ with values in $\mathsf{M}(n)$ and the scalar function $c(t)$ satisfying the ODEs:

$$a'=a(m+\sigma\rho z^\top)+(m+\sigma\rho z^\top)^\top a+2a\sigma^2a+\frac{1}{2}(zz^\top-\mathrm{diag}[z]), \tag{31}$$

$$c'=\mathrm{tr}[\omega a], \tag{32}$$

with initial conditions $a(0)=\theta_1$, $c(0)=0$. The solution is explicitly given by:

$$a(t)=(\theta_1A_{12}(t)+A_{22}(t))^{-1}(\theta_1A_{11}(t)+A_{21}(t)), \tag{33}$$

with

$$\begin{pmatrix}A_{11}(t)&A_{12}(t)\\A_{21}(t)&A_{22}(t)\end{pmatrix}=e^{t\mathcal{A}}, \tag{34}$$

with

$$\mathcal{A}=\begin{pmatrix}m+\sigma\rho z^\top&-2\sigma^2\\\frac{1}{2}\left(zz^\top-\mathrm{diag}[z]\right)&-(m+\sigma\rho z^\top)^\top\end{pmatrix}, \tag{35}$$

while $c(t)=\int_0^t\mathrm{tr}[\omega a(u)]du$.

If we further assume that ω in (28) is such that $\omega=\beta\sigma^2$ with $\beta\in\mathbb{R}$ and $\beta\geq n+1$, i.e., the Bru condition (3) is satisfied, then the function $c(t)$ solution of (32) can be explicitly integrated to

$$c(t)=-\frac{\beta}{2}\mathrm{tr}[\log(\theta_1A_{12}(t)+A_{22}(t))+t(m+\sigma\rho z^\top)], \tag{36}$$

with $\log(.)$ the logarithm of a matrix, see Faraut (2008, p. 25, Section 2.2). Since $e^{c(t)}$ is involved in the moment generating function, as for the WMSV we can reach

$$e^{c(t)}=\frac{e^{-\frac{\beta t}{2}\mathrm{tr}[m+\sigma\rho z^\top]}}{\det(\theta_1A_{12}(t)+A_{22}(t))^{\beta/2}}. \tag{37}$$

Again, we may denote the functions $a(t)$ and $c(t)$ by $a(t,z,\theta_1)$ and $c(t,z,\theta_1)$ when the explicit dependency to the parameters is necessary.

As for the WMSV, the derivative of the moment generating function can be computed explicitly as the following proposition shows. Since the proof of this proposition is similar to that of Proposition 2.2, it is omitted.

Proposition 2.7. *Let $(s_t, v_t)_{t\geq 0}$ be a process following a WASC dynamic (27)–(28). Let ν be one of the model's real-valued parameters: specifically, ν could be any entry m_{ij} (for $i,j=1,\dots,n$), or else σ_{ij}, or ρ_i. We denote by $G_{WASC}(t,z,\theta_1,s_0,v_0)$ the moment generating function of $(x_t=\ln s_t,\, v_t)\in\mathbb{R}^n\times\mathbb{S}_n^{++}$ from (30), where $z\in\mathbb{R}^n$ and $\theta_1\in\mathbb{S}_n$. Assuming θ_1 depends on ν, the derivative of G_{WASC} with respect to ν is given by*

$$\partial_\nu G_{WASC}(t,z,\theta_1,x_0,v_0) := (\mathrm{tr}[a_\nu(t)v_0]+c_\nu(t))G_{WASC}(t,z,\theta_1,x_0,v_0)\,, \tag{38}$$

with $a_\nu(t)$ having the same form as in (15) with $a(t)$, $A_{11}(t)$, $A_{12}(t)$ and $A_{22}(t)$ given in Proposition 2.6, $\theta_{1,\nu}$ the derivative of θ_1 to ν, $A_{11,\nu}(t)$, $A_{12,\nu}(t)$, $A_{21,\nu}(t)$ and $A_{22,\nu}(t)$ given by (16) with $\mathcal{A}$ as in (35) whilst $\mathcal{H}$ is parameter dependent. For each parameter corresponds a matrix $\mathcal{H}$, they are given by

ν	$\mathcal{H}$
m_{ij}	$\begin{pmatrix} e_{ij} & 0_n \\ 0_n & -e_{ij}^\top \end{pmatrix}$
σ_{ii}	$\begin{pmatrix} e_{ii}\rho z^\top & -2(\sigma e_{ii}+e_{ii}\sigma) \\ 0_n & -(e_{ii}\rho z^\top)^\top \end{pmatrix}$
$\sigma_{ij}\ (j\neq i)$	$\begin{pmatrix} (e_{ij}+e_{ji})\rho z^\top & -2(\sigma(e_{ij}+e_{ji})+(e_{ij}+e_{ji})\sigma) \\ 0_n & -((e_{ij}+e_{ji})\rho z^\top)^\top \end{pmatrix}$
ρ_i	$\begin{pmatrix} \sigma e_i z^\top & 0_n \\ 0_n & -(\sigma e_i z^\top)^\top \end{pmatrix}$

while function $c_\nu(t)=\int_0^t \mathrm{tr}[\omega a_\nu(u)]du$. Similar to Proposition 2.2, the functions $a_\nu(t)$ and $c_\nu(t)$ may be denoted by $a_\nu(t,z,\theta_1,\theta_{1,\nu})$ and $c_\nu(t,z,\theta_1,\theta_{1,\nu})$ respectively when explicit dependence on the parameters is required.

The following proposition is the equivalent of Proposition 2.4 for the WASC, the proof is also omitted.

Proposition 2.8. *Let $(s_t,v_t)_{t\geq 0}$ a process following a WASC dynamic (27)-(28) with $\omega=\beta\sigma^2$ with $\beta\geq n+1$ then the derivative with respect to ν of $c(t)$, denoted $c_\nu(t)$ in Proposition 2.7, is given when ν is m_{ij} by*

$$-\frac{\beta t}{2}\mathrm{tr}[e_{ij}]-J_\nu, \tag{39}$$

when ν is σ_{ii} by

$$-\frac{\beta t}{2}\mathrm{tr}[e_{ii}\rho z^\top]-J_\nu, \tag{40}$$

when ν is σ_{ij} with $i\neq j$ by

$$-\frac{\beta t}{2}\mathrm{tr}[(e_{ij}+e_{ji})\rho z^\top]-J_\nu, \tag{41}$$

and when ν is ρ_i by

$$-\frac{\beta t}{2}\mathrm{tr}[\sigma e_i z^\top]-J_\nu, \tag{42}$$

with J_ν given by (24) but with the appropriate terms $A_{12}(t)$, $A_{22}(t)$ given by Proposition 2.6, while $A_{12,\nu}(t)$ and $A_{22,\nu}(t)$ are given by Proposition 2.7 with the corresponding $\mathcal{H}$.

As for the WMSV, the derivative of the moment generating function with respect to β and $v_{0,ij}$ is very simple to compute as the following proposition shows (and no proof is provided).

Proposition 2.9. *Let $(s_t,v_t)_{t\geq 0}$ a process following a WASC dynamic (27)-(28) and let ν be an element of v_0 (i.e., $v_{0,ij}$ for $i=1,\dots,n$ and $j=1,\dots,n$), then the derivative with respect to ν of the moment generating function of $(x_t=\ln s_t, v_t)\in\mathbb{R}^n\times\mathbb{S}_n^{++}$ given for $z\in\mathbb{R}^n$ and $\theta_1\in\mathbb{S}_n$ by $G_{WASC}(t,z,\theta_1,x_0,v_0)$ of (30) is*

$$\partial_{v_{0,ij}}G_{WASC}(t,z,\theta_1,x_0,v_0)=(1+\delta_{ij})a_{ij}(t)G_{WASC}(t,z,\theta_1,x_0,v_0). \tag{43}$$

If $\omega=\beta\sigma^2$ with $\beta\geq n+1$ so that $c(t)$ in (30) is given by (36), then the derivative with respect β of this moment generating function, assuming that θ_1 does not depend on β, is

$$\partial_\beta G_{WASC}(t,z,\theta_1,x_0,v_0)=c_\beta G_{WASC}(t,z,\theta_1,x_0,v_0), \tag{44}$$

with $c_\beta=-\frac{1}{2}\mathrm{tr}[\log(\theta_1A_{12}(t)+A_{22}(t))+t(m+\sigma\rho z^\top)]$.

3. Valuation and risk management of the guaranteed equity-linked annuity

The objective of this work is to price a guaranteed minimum maturity benefit using Wishart-based stochastic volatility models. The payoff of such a product, conditional on the annuity holder being alive at maturity, is given by

$$\phi(s_{t_k},g_{t_k}) := \max(s_{t_k},g_{t_k})\,, \tag{45}$$

with t the current time, t_k the maturity of the contract, s_{t_k} the equity value at time t_k and guarantees g_{t_k} that is either

$$\begin{cases} g & \text{guarantee fixed,} \\ ge^{\delta t_k} & \text{guarantee is rolled at the rate } \delta, \\ \left(\prod_{i=0}^k s_{t_i}\right)^{1/(k+1)} & \text{guarantee is a geometric average,} \end{cases}$$

with $t=t_0<t_1<\ldots<t_k$ and $t_i-t_{i-1}=\tau$. The payoff can be rewritten as

$$\phi(s_{t_k},g_{t_k})=s_{t_k}+(g_{t_k}-s_{t_k})_+\,, \tag{46}$$

with $(x)_+=\max(x,0)$ for $x\in\mathbb{R}$. As a result pricing the time t guaranteed equity-linked annuity that expires at t_k with guarantee g_{t_k} amounts to pricing the below when accounting for mortality risk.

$$\mathbb{E}_t[1_{\{\tau_x>t_k\}}e^{-r(t_k-t)}\phi(s_{t_k},g_{t_k})], \tag{47}$$

where $\mathbb{E}_t[.]$ is the time t conditional expectation and τ_x is the lifetime of the annuity holder. If mortality risk and the equity-linked dynamics of the annuity are independent, this simplifies to an exchange option pricing problem. To streamline the derivation, we assume the annuitant survives until expiry and that mortality and market risks are independent. Thus, (47) becomes

$$\mathbb{E}_t[1_{\{\tau_x>t_k\}}]\mathbb{E}_t[e^{-r(t_k-t)}\phi(s_{t_k},g_{t_k})]. \tag{48}$$

This assumption is inconsequential, as any desired mortality model can be incorporated without affecting the core structure of the results listed in this section when using the WMSV or WASC model as observed in (48), and henceforth we exclusively focus on the second expectation in (48). Mortality risk, for example, can be incorporated using mortality tables, assuming the insurance portfolio is sufficiently large to diversify idiosyncratic mortality risk and that the tables are reliable and arbitrage-free (see Hieber (2017), Section 3.3.2, for further details). If the guarantee is fixed or rolled up to time t_k at a rate δ then it is a standard (*i.e.*, vanilla) put option. For the WMSV it is done in Da Fonseca et al. (2008) while for the WASC it is done in Da Fonseca et al. (2007), see also Da Fonseca and Grasselli (2011) for a calibration of those models. When the guarantee is a geometric average the guaranteed equity-linked annuity is far more challenging to price. Hieber (2017) shows how the pricing can be done in a Markov switching model where all the dynamics between which the equity dynamic can switch have the property of equity log-price independent increments, which is crucial to derive the results, see in particular the proof of Hieber (2017, Theorem 4). In

Kang et al. (2022), the authors show how this can be done for a Markov switching model with an equity dynamic that can switch between different specifications, one of which is the Heston (1993) model, while the other two have the property of equity log-price independent increments. This latter property is important because, when it is not satisfied, it creates significant difficulties in pricing certain path-dependent options. In some cases, the only remaining pricing methodology is the Monte Carlo method (see Fan et al. (2015, p. 71)).

We show that such a product can also be priced within Wishart-based stochastic volatility models and present two approaches. The first approach, which we illustrate using the WMSV, is based on Girsanov's theorem and the law of iterated expectations, it is much in the spirit of Kang et al. (2022) but there are few differences whose consequences we will analyse. Notice that the linearization of the Riccati equation performed in Kang et al. (2022) is the main tool used to compute moment generating functions of the WMSV in Da Fonseca et al. (2008) and the WASC in Da Fonseca et al. (2007) and was introduced for the Wishart process alone in Grasselli and Tebaldi (2008). The other approach, which is illustrated with the WASC, does not rely on Girsanov's theorem but only on the Mellin transform of the payoff, which we can trace in Hurd and Zhou (2010), and the law of iterated expectations. As the WASC is a multi-asset model, we take the opportunity to consider a more complex payoff that involves two assets, that is we consider the two-stock WASC model $s_t = (s_{1,t}, s_{2,t})^\top$ of Section 2.2 while the GMMB is on the asset $s_{1,t}$ with a geometric average guarantee that is computed on the second asset $s_{2,t}$, it translates into a payoff of the form

$$\begin{aligned}\phi(s_{1,t_k}, g_{2,t_k}) &= \max(s_{1,t_k}, g_{2,t_k}),\\ &= s_{1,t_k} + (g_{2,t_k} - s_{1,t_k})_+ ,\end{aligned} \tag{49}$$

with $g_{2,t_k} = \left(\prod_{i=0}^{k} s_{2,t_i}\right)^{1/(k+1)}$. Although the pricing of multi-asset guarantees has been much less analysed in the literature, there are still few papers dealing with this problem, see Ng and Li (2011), Ng and Li (2013), Da Fonseca and Ziveyi (2017) and Hartman et al. (2020) just to name a few.

3.1. Pricing in the WMSV model

The payoff of the guaranteed equity-linked annuity with a geometric average guarantee is similar to an option to exchange two assets and the standard technique to deal with this case is to consider one of the asset as numeraire as it allows the reduction of the problem dimension from two to one and can be stated as follows.

Proposition 3.1. *Let $(s_t, v_t)_{t\geq 0}$ a process following a WMSV dynamic (1)-(2) then the guaranteed equity-linked annuity price at time t with maturity t_k is given by*

$$\mathbb{E}_t\left[e^{-r(t_k-t)}(g_{t_k} - s_{t_k})_+\right] = s_t \bar{\mathbb{E}}_t\left[\left(\frac{g_{t_k}}{s_{t_k}} - 1\right)_+\right], \tag{50}$$

with $\bar{\mathbb{E}}_t[.]$ the conditional expectation under $\bar{\mathbb{Q}}$ a probability measure equivalent to $\mathbb{Q}$ such that

$$\frac{d\bar{\mathbb{Q}}}{d\mathbb{Q}} = e^{-rt}\left.\frac{s_t}{s_0}\right|_{\mathcal{F}_t}, \tag{51}$$

and under $\bar{\mathbb{Q}}$ the process $(x_t = \ln s_t, v_t)$ follows the dynamic

$$dx_t = \left(r + \frac{1}{2}\mathrm{tr}[v_t]\right)dt + \mathrm{tr}\left[\sqrt{v_t}\left(d\bar{w}_t\rho^\top + d\bar{b}_t\sqrt{I_n - \rho\rho^\top}\right)\right], \tag{52}$$

$$\begin{aligned}dv_t &= (\omega + (m + \sigma\rho^\top)v_t + v_t(m + \sigma\rho^\top)^\top)dt\\ &\quad + \sqrt{v_t}d\bar{w}_t\sigma + \sigma d\bar{w}_t^\top\sqrt{v_t},\end{aligned} \tag{53}$$

with $(\bar{b}_t)_{t\geq 0}$ and $(\bar{w}_t)_{t\geq 0}$ two matrix Brownian motions (that are independent).

Remark 3.1. To compute the moment generating function of $(x_t = \ln s_t, v_t)_{\geq 0}$ for a given t under $\bar{\mathbb{Q}}$ then the matrix $\mathcal{A}$ in (11) that appears in (10) has to be replaced with

$$\mathcal{A} = \begin{pmatrix} m + (z+1)\sigma\rho^\top & -2\sigma^2 \\ \frac{z(z+1)}{2}I_n & -(m + (z+1)\sigma\rho^\top)^\top \end{pmatrix}, \tag{54}$$

and $e^{c(t)}$ given by (13) in Proposition 2.1 rewrites

$$e^{c(t)} = \frac{e^{-\frac{\beta t}{2}\mathrm{tr}[m + (z+1)\sigma\rho^\top]}}{\det(\theta_1 A_{12}(t) + A_{22}(t))^{\beta/2}}, \tag{55}$$

with (A_{12}, A_{22}) given by (10) with $\mathcal{A}$ as in (54).

In order to price the guaranteed equity-linked annuity the next step is to compute the expectation (50) and the moment generating function of the variable g_{t_k}/s_{t_k} is needed. To this end, we use the law of iterated expectations that we apply to (x_t, v_t) and it gives the following proposition.

Proposition 3.2. *Let $(s_t, v_t)_{t\geq 0}$ a process following a WMSV dynamic (52)-(53) under $\bar{\mathbb{Q}}$ then the moment generating function of $y_{t_k} = \ln\left(\frac{g_{t_k}}{s_{t_k}}\right) \in \mathbb{R}$ (conditional to time $t = t_0$) is given for $z \in \mathbb{R}$ by*

$$\begin{aligned}\Phi^1(t, t_k, z, x_t, v_t) &:= \bar{\mathbb{E}}_t\left[e^{zy_{t_k}}\right],\\ &= \exp\left(\mathrm{tr}[\bar{\theta}_k v_{t_0}] + \sum_{l=1}^{k}\left(\frac{l}{k+1} - 1\right)zr\tau + \sum_{l=1}^{k} c_l\right),\end{aligned} \tag{56}$$

with $x_t = \ln s_t$ and for $l = 1, \ldots k$

$$\bar{\theta}_l = a(\tau, (l/(k+1) - 1)z, \bar{\theta}_{l-1}), \tag{57}$$

$$c_l = c(\tau, (l/(k+1) - 1)z, \bar{\theta}_{l-1}), \tag{58}$$

and $\bar{\theta}_0 = 0_n$ while $a(.,.,.)$ and $c(.,.,.)$ are given by Proposition 2.1 with the matrix $\mathcal{A}$ in Proposition 2.1 adjusted to the dynamic under $\bar{\mathbb{Q}}$ given by (52) and (53), that is $\mathcal{A}$ is given by (54).

The moment generating function of $y_{t_k} = \ln\left(\frac{g_{t_k}}{s_{t_k}}\right) \in \mathbb{R}$ being known we can price the option (50) using a Mellin transform.

Proposition 3.3. *The expectation (50) is given by*

$$\bar{\mathbb{E}}_t\left[\left(\frac{g_{t_k}}{s_{t_k}} - 1\right)_+\right] = \frac{1}{\pi \mathrm{i}}\int_{\gamma+\mathrm{i}0}^{\gamma+\mathrm{i}\infty} \Re\left(\frac{\Phi^1(t, t_k, (z+1), x_t, v_t)}{z(z+1)}\right)dz, \tag{59}$$

with $x_t = \ln s_t$, so that $x_t \in \mathbb{R}$, $\gamma > 0$ and $\Phi^1(t, t_k, ., .)$ given by (56).

Remark 3.2. In Kang et al. (2022), the authors price this option using a Markov-switching model in which the dynamic of the equity switches between three different specifications, one of which is the Heston (1993) model (the other two being the Black-Scholes model and the Kou and Wang (2004) model). In their proof, the authors do not use the joint moment generating function of (x_t, v_t), but successively the moment generating function of x_t and the moment generating function of v_t (for another value of the time variable t). Since the WMSV corresponds to the Heston (1993) model when $n = 1$, our proof shows that to properly price this option one needs the joint moment generating function of (x_t, v_t), which cannot be recovered from the moment generating function of x_t and the moment generating function of v_t. In the Heston (1993) and WMSV models, the equity price log-returns computed over disjoint, possibly distant, time intervals are not independent, making the problem much more complicated. Note that even if only the moment generating function of x_{t_k} is needed in the internal expectation in (92) (or equivalently (93)), since the system is Markovian with respect to (x_t, v_t) the resulting conditional expectation depends on (x_t, v_t), and therefore for the subsequent conditional expectations the joint moment generating function of (x_t, v_t) is needed, as (95) and (96) clearly show.

Since the error is rather subtle, we point out the problem in the proof of Kang et al. (2022). In the notation of our paper we have that the variables $\{x_{t_j} - x_{t_{j-1}}; j = 1, \dots, k\}$ are not independent due to their dependence on the volatility process v_t. If the volatility is constant, which is the case in the Black-Scholes model or the Kou and Wang (2004) model considered in Kang et al. (2022), then these variables are independent and this independence property is used to derive a characteristic function given by Kang et al. (2022, (B.11)) that will be used to price the option. Then Kang et al. (2022, (B.57)), which is the same equation as Kang et al. (2022, (B.11)) and therefore assumes that the equity price log-returns are independent, is applied to the Heston (1993) model (which is not correct). A consequence of this erroneous assumption is that the joint moment generating function of (x_t, v_t) is never used, but only the moment generating function of x_t given by Kang et al. (2022, (B.44)) and v_t given by Kang et al. (2022, (B.50)) (but for different values of the time variable t). Note that our criticism applies only to the Heston (1993) component of this work.

To the best of our knowledge, the only algorithm that can handle the difficulty introduced by a (diffusive) stochastic volatility process in the equity price log-returns is the one proposed in Cui et al. (2017b), but it is an approximation and not an exact formula as the one derived above. Extending this latter work to the WMSV model is an open question, but it would be an interesting one since Cui et al. (2017b) applies to very general path-dependence guarantees.

3.2. Pricing in the WASC model

The pricing of the multi-asset guaranteed equity-linked annuity requires the joint moment generating function of the two assets at different dates, this function is known thanks to the affine property of the WASC model.

Proposition 3.4. *Let $(s_t, v_t)_{t\geq 0}$ a process following a WASC dynamic (27)-(28) with $n = 2$ (i.e., there are two assets so that $s_t = (s_{1,t}, s_{2,t})^\top$). Let us denote $g_{2,t_k} = (\prod_{i=0}^k s_{2,t_i})^{1/(k+1)}$, $y_{2,t_k} = \ln g_{2,t_k}$ and $x_t = \ln s_t$ (i.e., $x_{1,t} = \ln s_{1,t}$ and $x_{2,t} = \ln s_{2,t}$). The moment generating function of $(x_{1,t_k}, y_{2,t_k}) \in \mathbb{R}^2$ is a function of $z = (z_1, z_2) \in \mathbb{R}^2$ given by*

$$\Phi^2(t, t_k, z, x_t, v_t) := \mathbb{E}_t\left[e^{z_1 x_{1,t_k} + z_2 y_{2,t_k}}\right], \\ = e^{\left(z_1 x_{1,t_0} + z_2 x_{2,t_0}\right) + \mathrm{tr}[\theta_k v_{t_0}] + \sum_{l=1}^k \left(z_1 + \frac{l z_2}{k+1}\right) r\tau + \sum_{l=1}^k c_l}, \quad (60)$$

with for $l = 1, \dots k$

$$\theta_l = a(\tau, (z_1, l z_2/(k+1)), \theta_{l-1}), \quad (61)$$

$$c_l = c(\tau, (z_1, l z_2/(k+1)), \theta_{l-1}), \quad (62)$$

and $\theta_0 = 0_n$ while $a(., ., .)$ and $c(., ., .)$ are given by Proposition 2.6.

Remark 3.3. Note that in the proof of the above proposition, even if in the first equality of (99) only the (conditional to time t_{k-1}) moment generating function of (x_{1,t_k}, x_{2,t_k}) is needed, for the subsequent (conditional) expectations the moment generating function of $(x_{1,t}, x_{2,t}, v_t)$, which includes the process v_t, is needed.

Proposition 3.5. *Let $(s_t, v_t)_{t\geq 0}$ a process following a WASC dynamic (27)-(28) with $n = 2$ (i.e., there are two assets). Let us denote $g_{2,t_k} = (\prod_{i=0}^k s_{2,t_i})^{1/(k+1)}$, $y_{2,t_k} = \ln g_{2,t_k}$ the guarantee which is a geometric average computed using the second asset while $x_{1,t} = \ln s_{1,t}$ is the log return of the first asset. The multi-asset guaranteed equity-linked annuity price at time t with maturity t_k between the asset $s_{1,t}$ and a geometric average guarantee computed on a second asset $s_{2,t}$ is given by*

$$\mathbb{E}_t\left[e^{-r(t_k - t)}(g_{2,t_k} - s_{1,t_k})_+\right] = \frac{e^{-r(t_k-t)}}{\pi \mathrm{i}} \int_{\gamma+\mathrm{i}0}^{\gamma+\mathrm{i}\infty} \Re\left(\frac{\Phi^2(t, t_k, (-s, s+1), x_t, v_t)}{s(s+1)}\right) ds, \quad (63)$$

with $\gamma > 0$ and $\Phi^2(., ., ., .)$ the function (60).

Note that we could have priced this option using the approach based on the change of numeraire used for the WMSV.

3.3. Risk management in the WMSV model

Once the option has been priced, risk management is required and relies on the sensitivity, or derivative, of the option price to the model parameters, see Feng et al. (2020). Since the Wishart stochastic volatility models utilize the Wishart process, which is a matrix process, the moment generating functions involve matrix functions, so in order to compute these sensitivities, the derivative of these functions must be determined for efficient implementation. The next propositions show how to deal with this difficulty.

Proposition 3.6. *The sensitivity of the guaranteed equity-linked annuity price given by (59) with respect to an element of m, σ and ρ, that we denote ν, is given by*

$$\partial_\nu \bar{\mathbb{E}}_t\left[\left(\frac{g_{t_k}}{s_{t_k}} - 1\right)_+\right] = \frac{1}{\pi \mathrm{i}} \int_{\gamma+\mathrm{i}0}^{\gamma+\mathrm{i}\infty} \Re\left(\frac{\Phi^1_\nu(t, t_k, (z+1), x_t, v_t)}{z(z+1)}\right) dz, \quad (64)$$

with $x_t = \ln s_t$, so that $x_t \in \mathbb{R}$, $\gamma > 0$ and $\Phi^1_\nu(.)$ given by

$$\Phi^1_\nu(t, t_k, z, x_t, v_t) = \left(\mathrm{tr}[\partial_\nu \bar{\theta}_k v_{t_0}] + \sum_{l=1}^k \partial_\nu c_l\right) \Phi^1(t, t_k, z, x_t, v_t), \quad (65)$$

with $x_t = \ln s_t$, $\Phi^1(.)$ given by (56) and for $l = 1, \dots k$

$$\partial_\nu \bar{\theta}_l = a_\nu(\tau, (l/(k+1) - 1)z, \bar{\theta}_{l-1}, \partial_\nu \bar{\theta}_{l-1}), \quad (66)$$

$$\partial_\nu c_l = c_\nu(\tau, (l/(k+1) - 1)z, \bar{\theta}_{l-1}, \partial_\nu \bar{\theta}_{l-1}), \quad (67)$$

and $\partial_\nu \bar{\theta}_0 = 0_n$, $\{\bar{\theta}_l; l = 0, \dots, n\}$ given by (57) while $a_\nu(.)$ and $c_\nu(.)$ are given in Proposition 2.2.

The sensitivities of the guaranteed equity-linked annuity price to $v_{t_0,ij}$ and β are much simpler as the following proposition shows.

Proposition 3.7. *The sensitivity of the guaranteed equity-linked annuity price given by (59) with respect to an element of $v_{t_0,ij}$, that we denote ν, amounts to replace $\Phi^1_\nu(t, t_k, z, x_t, v_t)$ in (65) with*

$$\Phi^1_\nu(t, t_k, z, x_t, v_t) = (1 + \delta_{ij})\bar{\theta}_{k,ij} \Phi^1(t, t_k, z, x_t, v_t), \quad (68)$$

with $\bar{\theta}_{k,ij}$ given by (57) and when considering β, if we assume that $\omega = \beta\sigma^2$, then

$$\Phi^1_\nu(t, t_k, z, x_t, v_t) = c_\beta \Phi^1(t, t_k, z, x_t, v_t), \quad (69)$$

with c_β given by Proposition 2.5.

3.4. Risk management in the WASC model

In this section we develop similar risk indicators, but for the WASC model and the product presented in section 3.2, and the results show that although it is a multi-asset model, it is as tractable as the WMSV model.

Proposition 3.8. *The sensitivity of the guaranteed equity-linked annuity price given by (63) with respect to an element of m, σ and ρ, that we denote ν, is given by*

$$\partial_v \mathbb{E}_t\left[e^{-r(t_k-t)}(g_{2,t_k}-s_{1,t_k})_+\right] = \\ = \frac{e^{-r(t_k-t)}}{\pi \mathrm{i}} \int_{\gamma+\mathrm{i}0}^{\gamma+\mathrm{i}\infty} \Re\left(\frac{\Phi_v^2(t,t_k,(-s,s+1),x_t,v_t)}{s(s+1)}\right) ds\,, \tag{70}$$

with $x_t = \ln s_t$, so that $x_t \in \mathbb{R}^2$, $\gamma > 0$ and $\Phi_v^2(.)$ given by

$$\Phi_v^2(t,t_k,(-s,s+1),x_t,v_t) = \\ = \left(\mathrm{tr}[\partial_v \theta_k v_{t_0}] + \sum_{l=1}^{k} \partial_v c_l\right) \Phi^2(t,t_k,(-s,s+1),x_t,v_t)\,, \tag{71}$$

with $\Phi^2(.)$ given by (60) and for $l = 1,\dots k$

$$\partial_v \theta_l = a_v(\tau,(z_1, l z_2/(k+1)), \theta_{l-1}, \partial_v \theta_{l-1}), \tag{72}$$

$$\partial_v c_l = c_v(\tau,(z_1, l z_2/(k+1)), \theta_{l-1}, \partial_v \theta_{l-1}), \tag{73}$$

and $\partial_v \bar{\theta}_0 = 0_n$, $\{\theta_l; l = 0,\dots,n\}$ given by (61) while $a_v(.)$ and $c_v(.)$ are given in Proposition 2.7.

As for the WMSV, the sensitivities of the guaranteed equity-linked annuity price to $v_{t_0,ij}$ and β are much simpler to compute.

Proposition 3.9. *The sensitivity of the guaranteed equity-linked annuity price given by (63) with respect to an element of $v_{t_0,ij}$, that we denote v, amounts to replace $\Phi_v^2(t,t_k,(-s,s+1),x_t,v_t)$ in (71) with*

$$\Phi_v^2(t,t_k,(-s,s+1),x_t,v_t) = (1+\delta_{ij})\theta_{k,ij}\Phi^2(t,t_k,(-s,s+1),x_t,v_t)\,, \tag{74}$$

with $\theta_{k,ij}$ given by (61) and when considering β, if we assume that $\omega = \beta\sigma^2$, then

$$\Phi_v^2(t,t_k,(-s,s+1),x_t,v_t) = c_\beta \Phi^2(t,t_k,(-s,s+1),x_t,v_t)\,, \tag{75}$$

with c_β given by Proposition 2.9.

Having established all the analytical results, the next section focuses on the implementation.

3.5. A note on GMxBs and multiple assets

The guaranteed minimum maturity benefit (GMMB) results derived in this paper for the WMSV and WASC can be readily extended to other forms of guaranteed minimum benefits, such as guaranteed minimum investment benefits (GMIB), guaranteed minimum death benefits (GMDB) and possibly even guaranteed minimum withdrawal benefits (GMWB). As noted in Section 4.3 of Kang et al. (2022), GMMB serves as a building block for these other variable annuities, suggesting that the methodology developed in this paper may be directly applicable to them as well. Moreover, the derivatives of the moment generating functions for WMSV and WASC, derived in Propositions 2.2/ 2.7, can be used for sensitivity analyses of these other variable annuity products.

Beyond this, multi-asset modelling is a crucial aspect of modern finance, as it enables a unified analysis of correlated assets and price variable annuities of several assets. Such problems have been studied in the actuarial literature in Ng and Li (2011, 2013), just to name a few. In the former using an analytical method and the multivariate Esscher transform of assets following geometric Brownian motion dynamics, and in the latter using a Monte Carlo method. The analytical work of Ng and Li (2011) was extended in Da Fonseca and Ziveyi (2017) to a multi-asset model based on independent simple Heston models. However, it is known that the WASC model is of considerable interest because its positive-definite matrix specification allows for a flexible and general correlation structure but is still affine, see Da Fonseca et al. (2007). This stands in contrast to the popular Duffie and Kan (1996) vector affine processes, which are subject to constraints on admissible correlations Duffie et al. (2003). Since the pricing formulas of Da Fonseca and Ziveyi (2017) essentially rely on the availability of the moment generating function of the assets, all the results of this work (and therefore Ng and Li (2011)) can be obtained for the WASC model as its moment generating function is also known (*i.e.* Eq. (30)). But compared to Da Fonseca and Ziveyi (2017), the very general correlation between the assets does not come at the cost of tractability as all the sensitivities can also be explicitly computed.

Table 1
Parameters for WMSV model.

Parameter	Value
s_0	1
v_0	$\begin{pmatrix} 0.0298 & 0.0090 \\ 0.0090 & 0.0108 \end{pmatrix}$
ω	$\begin{pmatrix} 0.0969 & 0.0577 \\ 0.0577 & 0.0625 \end{pmatrix}$
m	$\begin{pmatrix} -1.2479 & -0.0820 \\ -0.0820 & -1.1433 \end{pmatrix}$
σ	$\begin{pmatrix} 0.1417 & 0.0548 \\ 0.0548 & 0.1090 \end{pmatrix}$
ρ	$\begin{pmatrix} 0.2545 & -0.0322 \\ -0.0322 & 0.7658 \end{pmatrix}$

Note. Model parameters for the WMSV model that follows the dynamics (1)-(2).

4. Implementation

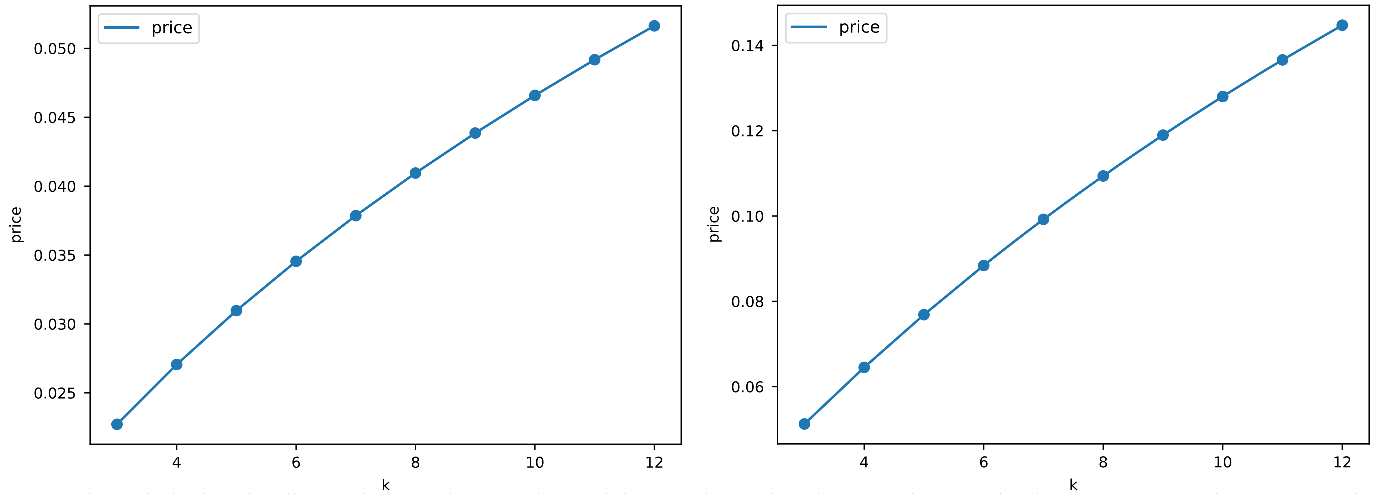
In this section we provide a numerical implementation of the WMSV and WASC models using the parameters of Table 1 for the WMSV and Table 2 for the WASC. These parameter values are consistent with those obtained in practice, see Da Fonseca and Grasselli (2011) for the calibration of these two models using option data. For the guaranteed equity-linked annuity, which we shall refer to as the option, we take $\tau = 1/12$ (*i.e.*, 1 month) and the number of periods in Proposition 3.3 and Proposition 3.5 is $k = 6$. The interest rate is $r = 0$ while in the Mellin transform the integration parameter is $\gamma = 1.1$. The option price is 3.4546×10^{-2} for the WMSV model and 8.8402×10^{-2} for the WASC model. The integrals in (59) and (63) are computed using the "quad" function of Scipy. The values are checked using a Monte Carlo method with 10000 paths, and a discretisation, using an Euler scheme, of the time interval of a trading day (*i.e.*, $dt = 1/250$, with 250 trading days a year). The confidence intervals at the 95% level for the prices are $[3.4225\times10^{-2}, 3.6018\times10^{-2}]$ for the WMSV model and $[8.5542\times10^{-2}, 9.0476\times\times10^{-2}]$ for the WASC model. The execution times to obtain the prices are for the WMSV model: 4.62 seconds for the analytical formula and 82.71 seconds for the Monte Carlo approach. For the WASC model, the corresponding times are 4.96 seconds and 82.16 seconds. In both cases, the Monte Carlo computation time is linear in $1/dt$ and in the number of paths.[2]

Since one important characteristic of the option is its rachet/cliquet property, and in particular the number of periods k, to quantify its impact on the option price, we let that parameter vary from $k = 3$ to $k = 12$ and report the results in Fig. 1. In both cases we find that the option price is increasing.

As the models are multi-dimensional, analysing the sensitivity of the option price to each parameter is tedious and lengthy. To provide a concise illustration of the impact of the (matrix) volatility parameter σ on the option price, we scale this parameter, the whole matrix, by a scalar $\lambda \in [0.9\,,1.1]$ (*i.e.*, $\pm10\%$), and price the option with the other parameters, either model parameters or option characteristics, unchanged. Fig. 2 and Fig. 3 report the results for the WMSV and WASC, respectively. In both cases, the option price is decreasing.[3]

[2] The codes are available upon request to the authors.

[3] The decreasing pattern can be understood using the analogy with the Asian option. The integrals (59) and (63) are similar to an Asian option with a floating strike. For the single stock case, increasing the volatility increases the distribu-



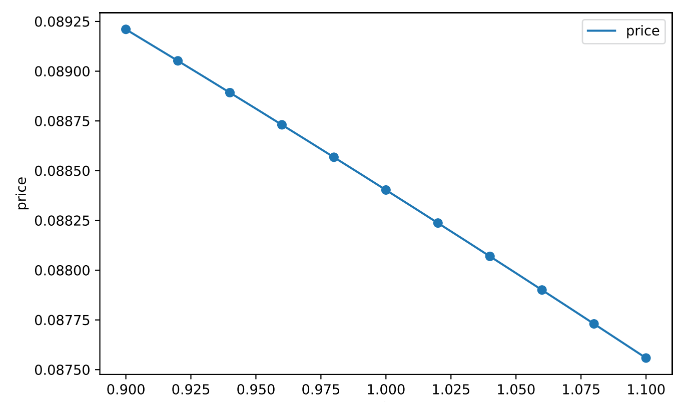
Note. Both panels displays the effect on the integrals (59) and (63) of changing the number of compounding periods k between $k = 3$ periods (3-months with $\tau = 1/12$) and $k = 12$ periods (1 year). The left panel displays the WMSV model whilst the right panel displays the WASC model.

Fig. 1. Effect of the number of compounds k.

Table 2
Parameters for WASC model.

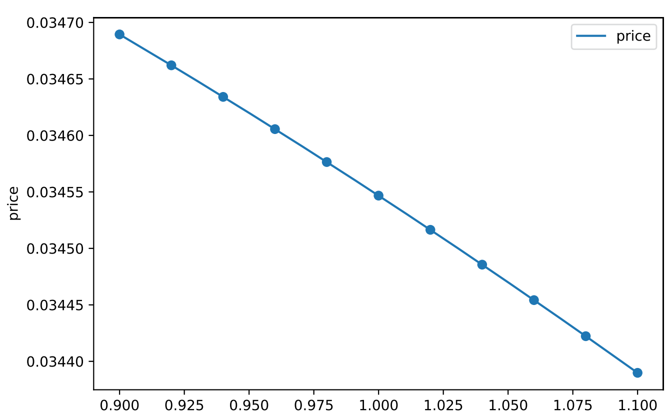
Parameter	Value
s_0	$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$
v_0	$\begin{pmatrix} 0.0446 & 0.0336 \\ 0.0336 & 0.0424 \end{pmatrix}$
ω	$\begin{pmatrix} 0.9817 & 0.7066 \\ 0.7066 & 0.6559 \end{pmatrix}$
m	$\begin{pmatrix} -0.7820 & -0.3772 \\ -0.0539 & -1.2497 \end{pmatrix}$
σ	$\begin{pmatrix} 0.4146 & 0.2151 \\ 0.2151 & 0.3154 \end{pmatrix}$
ρ	$\begin{pmatrix} -0.6407 \\ -0.1105 \end{pmatrix}$

Note. Model parameters for the WASC model that follows the dynamics (27)-(28).



Note. The figure displays the sensitivity of the integral (63) with respect to σ for the WASC model. We change the volatility σ by scaling it by $\lambda \in [0.9, 1.1]$.

Fig. 3. Sensitivity analyses for WASC.



Note. The figure displays the sensitivity of the integral (59) with respect to σ for the WMSV model. We change the volatility σ by scaling it by $\lambda \in [0.9, 1.1]$.

Fig. 2. Sensitivity analysis for the WMSV.

tion of the asset but its impact is weaker on the geometric average of the asset than on the terminal distribution of the asset, due to the smoothing effect of averaging, and since the payoff is the difference (if positive) between the geometric average of the asset and the asset value, the option price decreases. For the multiple-asset case the explanation is essentially the same.

To demonstrate that the sensitivity formulae presented in Propositions 3.6 and 3.8, which are based on the derivative of the exponential map, are operational, we report in Table 3 for the WMSV model and in Table 4 for the WASC model the sensitivities of the option price to the parameters m, σ and ρ. As for the option prices, the integrals are computed using the "quad" function of Scipy with same value for γ. We have checked the accuracy of the results using a simple finite difference method. These sensitivities are crucial for the risk management of the options, as they allow a "what if" scenario analysis, see Feng et al. (2020). They also show that, although the WMSV and WASC are multidimensional models, they are highly tractable.

5. Conclusion

In this paper, we price in the WMSV model of Da Fonseca et al. (2008) and the WASC model of Da Fonseca et al. (2007) an annuity with a guarantee given by a path-dependent, geometric average of the equity-linked value. Thanks to the strong analytical properties of these models, which are affine models with values in the space of symmetric positive definite matrices, we derive for both a closed-form solution for the option price involving only a one-dimensional integration. We also show how to compute the sensitivities of the guaranteed equity-linked annuity price to the parameters of the models. This is also possible thanks to these analytical properties, and crucially relies on the use of the derivative of the exponential map, a central result of Lie group analysis. We

Table 3
Option price sensitivity to a parameter of the WMSV model.

Parameter	$\partial_v \tilde{\mathbb{E}}_t \left[\left(g_{t_k}/s_{t_k} - 1 \right)_+ \right] \times 10^3$
m_{11}	4.7277
m_{12}	1.9255
m_{22}	2.52624
σ_{11}	−5.7774
σ_{12}	−6.6857
σ_{22}	−2.8898
ρ_{11}	−7.4743
ρ_{12}	2.8159
ρ_{22}	1.5988

Note. This table contains the gradient given in Proposition 3.6 of the option price to the parameter of interest evaluated at the model parameters given by Table 1 and geometric average parameters $\tau = 1/12$ and $k = 6$.

Table 4
Option price sensitivity to a parameter of the WASC model.

Parameter	$\partial_v \mathbb{E}_t \left[e^{-r(t_k - t)} (g_{2,t_k} - s_{1,t_k})_+ \right] \times 10^3$
m_{11}	17.6750
m_{12}	10.2880
m_{22}	−4.2073
σ_{11}	−22.4295
σ_{12}	2.1613
σ_{22}	1.7971
ρ_1	4.1145
ρ_2	1.4917

Note. This table contains the gradient given in Proposition 3.8 of the option price to the parameter of interest evaluated at the model parameters given by Table 2 and geometric average parameters $\tau = 1/12$ and $k = 6$.

illustrate how the models can be implemented using real data. To the best of our knowledge, this is the first exact and rigorous pricing formula for the WMSV and WASC models. As the WMSV model includes the Heston (1993) model as a special case, it is therefore the first exact and rigorous expression for the guaranteed equity-linked annuity with a guarantee given by a geometric average of the equity-linked value in the Heston (1993) model.

Looking ahead, there are more sophisticated guarantees than the path-dependent guarantee considered in this paper. Qian et al. (2010), Bernard and Li (2013), Hieber (2017) and Cui et al. (2017b) consider guarantees that depend on the performance of an equity, but this performance can be capped and/or floored. The pricing of path-dependent variable annuities remains a very important topic in the actuarial literature (see Ai et al. (2023), Günther and Hieber (2024b), and Goudenege et al. (2025)) and in the applied mathematical literature (see, for example, Zhang and Zhong (2024)), although the latter mainly focuses on improving numerical algorithms. The pricing of these more sophisticated options in the WMSV and WASC models is an open question. To the best of our knowledge, Cui et al. (2017a) is the only paper that provides a numerical approximation to these problems that works for stochastic volatility (of the diffuse type) models. The methodology appears to be flexible enough to deal with more sophisticated models, see Günther and Hieber (2024a). Whether this approach can be extended to the WMSV and WASC models is an open question, which is likely to be challenging given the multidimensional nature of these models. Pricing other products in Wishart stochastic volatility models, such as a guaranteed minimum withdrawal benefit, are also interesting problems. These challenges are left for future work.

CRediT authorship contribution statement

José Da Fonseca: Writing – review & editing, Writing – original draft, Methodology, Conceptualization. **Patrick Wong:** Writing – review & editing, Writing – original draft, Methodology, Conceptualization.

Declaration of generative AI in scientific writing

Not used.

Declaration of competing interest

None.

Appendix A

Proof of Proposition 2.2. Suppose we wish to compute the sensitivity of the moment generating function given by (6) to the parameter m_{ij} for a given $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, n\}$. Thanks to expression of (6) it amounts to compute the derivative of $a(t)$ given by (9) and therefore the derivative of $A_{11}(t)$, $A_{12}(t)$, $A_{21}(t)$ and $A_{22}(t)$ (we assume that $\theta_{1,v}$ is known). Since these terms are given by (10), computing the derivative to m_{ij} (for a given i and j) amounts to replace $\mathcal{A}$ in (10) with

$$\begin{pmatrix} m + v e_{ij} + z\sigma\rho^\top & -2\sigma^2 \\ \frac{z(z-1)}{2} I_n & -(m + v e_{ij} + z\sigma\rho^\top)^\top \end{pmatrix} = \mathcal{A} + v\mathcal{H}, \tag{76}$$

where $\mathcal{A}$ given by (11) and

$$\mathcal{H} = \begin{pmatrix} e_{ij} & 0_n \\ 0_n & -e_{ij}^\top \end{pmatrix},$$

so that we need to compute

$$\frac{d}{dv} e^{t(\mathcal{A}+v\mathcal{H})}|_{v=0}. \tag{77}$$

The above quantity is the derivative of the exponential map and is denoted as in (17). If $c(t) = \int_0^t \mathrm{tr}[\omega a(u)]du$ then $c_v(t) = \int_0^t \mathrm{tr}[\omega a_v(u)]du$.

Possibly a special care is needed for σ since it is supposed symmetric and appears as a square. Computing the derivative of the moment generating function to σ_{ii} for $i \in \{1, \dots, n\}$ amounts to replace $\mathcal{A}$ in (10) with

$$\begin{pmatrix} m + z(\sigma + v e_{ii})\rho^\top & -2(\sigma + v e_{ii})^2 \\ \frac{z(z-1)}{2} I_n & -(m + z(\sigma + v e_{ii})\rho^\top)^\top \end{pmatrix} = \mathcal{A} + v\mathcal{H} + v^2\mathcal{H}_2, \tag{78}$$

where $\mathcal{A}$ given by (11) and

$$\mathcal{H} = \begin{pmatrix} ze_{ii}\rho^\top & -2(\sigma e_{ii} + e_{ii}\sigma) \\ 0_n & -(ze_{ii}\rho^\top)^\top \end{pmatrix} \tag{79}$$

while $\mathcal{H}_2$ is a matrix that cancels when computing the derivative and evaluating it at $v = 0$ since this matrix is multiplied by v^2. Similar argument applies when considering the variable σ_{ij} for $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, n\}$ with $i \neq j$, it amounts to replace $\mathcal{A}$ in (10) with

$$\begin{pmatrix} m + z(\sigma + v(e_{ij} + e_{ji}))\rho^\top & -2(\sigma + v(e_{ij} + e_{ji}))^2 \\ \frac{z(z-1)}{2} I_n & -(m + z(\sigma + v(e_{ij} + e_{ji}))\rho^\top)^\top \end{pmatrix} = \mathcal{A} + v\mathcal{H} + v^2\mathcal{H}_2, \tag{80}$$

where $\mathcal{A}$ given by (11), $\mathcal{H}$ as in (79) but with e_{ii} replaced with $(e_{ij} + e_{ji})$ while $\mathcal{H}_2$ is a matrix that cancels when computing the derivative and evaluating it at $v = 0$ since this matrix is multiplied by v^2.

For the other parameters (*i.e.* ρ) we proceed similarly. For each parameter corresponds a matrix $\mathcal{H}$, which does not depend on v, and is reported in the proposition. □

Proof of Proposition 2.3. Starting from the standard formula of Hall (2015, Theorem 5.4) and Faraut (2008, Theorem 2.1.4), and assuming two matrices $t\mathcal{A}$ and $t\mathcal{H}$ of $\mathsf{M}(2n)$ then

$$(D\exp)_{t\mathcal{A}}(t\mathcal{H}) = \left.\frac{de^{t\mathcal{A}+vt\mathcal{H}}}{dv}\right|_{v=0} = e^{t\mathcal{A}}\frac{I_{2n} - e^{-\mathrm{ad}_{t\mathcal{A}}}}{\mathrm{ad}_{t\mathcal{A}}}(t\mathcal{H}) \tag{81}$$

$$= e^{t\mathcal{A}}\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)!}(\mathrm{ad}_{t\mathcal{A}})^k(t\mathcal{H}), \tag{82}$$

with $(\mathrm{ad}_{t\mathcal{A}})^k(t\mathcal{H}) = (\mathrm{ad}_{t\mathcal{A}})^{k-1}(\mathrm{ad}_{t\mathcal{A}}(t\mathcal{H}))$ and $\mathrm{ad}_{t\mathcal{A}}(t\mathcal{H}) = [t\mathcal{A}, t\mathcal{H}] = t^2\mathcal{A}\mathcal{H} - t^2\mathcal{H}\mathcal{A}$, where $[X, Y] = XY - YX$ for two matrices of adequate sizes is called the bracket/commutator, see Hall (2015, p. 56) or Faraut (2008, p. 38), while $\mathrm{ad}_X(Y) = [X, Y]$ is called the adjoint map, see Hall (2015, p. 51 Definition 3.7) or Faraut (2008, p. 22). Using the vec operator and the equalities $\mathrm{vec}(XY) = (I_{2n} \otimes X)\mathrm{vec}(Y) = (Y^\top \otimes I_{2n})\mathrm{vec}(X)$ for $X, Y \in \mathsf{M}(2n)$, see for example Lütkepohl (2005, p. 662, Eq.(3)), we get that

$$\begin{aligned}\mathrm{vec}(\mathrm{ad}_{t\mathcal{A}}(t\mathcal{H})) &= \mathrm{vec}([t\mathcal{A}, t\mathcal{H}]) \\ &= \mathrm{vec}(t^2\mathcal{A}\mathcal{H} - t^2\mathcal{H}\mathcal{A}]) \\ &= ((I_{2n} \otimes t\mathcal{A}) - (t\mathcal{A}^\top \otimes I_{2n}))\mathrm{vec}(t\mathcal{H}).\end{aligned} \tag{83}$$

Then, by recurrence we can prove that

$$\mathrm{vec}((\mathrm{ad}_{t\mathcal{A}})^k(t\mathcal{H})) = ((I_{2n} \otimes t\mathcal{A}) - (t\mathcal{A}^\top \otimes I_{2n}))^k\mathrm{vec}(t\mathcal{H}), \tag{84}$$

so that

$$\mathrm{vec}((D\exp)_{t\mathcal{A}}(t\mathcal{H})) = (I_{2n} \otimes e^{t\mathcal{A}})\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)!}\mathrm{vec}((\mathrm{ad}_{t\mathcal{A}})^k(t\mathcal{H})), \tag{85}$$

and obtain the result after defining the function $\varphi(.)$ as in (19) which has the series expansion $\varphi(z) = \sum_{k=0}^{\infty}\frac{(-1)^k z^k}{(k+1)!}$ and appears in (82) and (85). □

Proof of Proposition 2.4. If $\omega = \beta\sigma^2$, then $c(t)$ from (6) is given by (12). We first separate (12) into a linear trace term and a matrix-log trace term. For the latter, we use the fact that

$$\frac{\partial}{\partial v}\mathrm{tr}\left[\log(M(v))\right] = \mathrm{tr}\left[M(v)^{-1}\frac{\partial M(v)}{\partial v}\right],$$

and apply the chain rule to $\theta_1 A_{12}(t) + A_{22}(t)$. This yields

$$J_v = \frac{\beta}{2}\mathrm{tr}\left[\left(\theta_1 A_{12}(t) + A_{22}(t)\right)^{-1}\left(\theta_{1,v} A_{12}(t) + \theta_1 A_{12,v}(t) + A_{22,v}(t)\right)\right].$$

Combining both parts completes the proof, giving

$$c_v(t) = -\frac{\beta t}{2}\mathrm{tr}[\ldots] - J_v,$$

with the precise form of the linear term depending on which parameter v is considered. □

Proof of Proposition 3.1. Note that

$$\int_0^t \mathrm{tr}[v_u]du = \int_0^t \mathrm{tr}\left[(\sqrt{v_u}\rho)^\top\sqrt{v_u}\rho\right]du + \int_0^t \mathrm{tr}\left[(\sqrt{v_u}(I_n - \rho\rho^\top)^{1/2})^\top\sqrt{v_u}(I_n - \rho\rho^\top)^{1/2}\right]du, \tag{86}$$

so that if we denote $h_1 = \sqrt{v_u}\rho$ and $h_2 = \sqrt{v_u}(I_n - \rho\rho^\top)^{1/2}$ (we drop the dependence on the time parameter u for h_1 and h_2) we rewrite (51) as

$$\frac{d\bar{\mathbb{Q}}}{d\mathbb{Q}} = e^{-\frac{1}{2}\int_0^t \mathrm{tr}[v_u]du + \int_0^t \mathrm{tr}\left[\sqrt{v_u}\left(dw_u\rho^\top + db_u\sqrt{I_n - \rho\rho^\top}\right)\right]}, \tag{87}$$

$$= e^{\int_0^t \mathrm{tr}\left[h_1^\top dw_u\right] - \frac{1}{2}\int_0^t \mathrm{tr}[h_1^\top h_1]du}e^{\int_0^t \mathrm{tr}\left[h_2^\top db_u\right] - \frac{1}{2}\int_0^t \mathrm{tr}[h_2^\top h_2]du}, \tag{88}$$

and by Girsanov's theorem, see Donati-Martin et al. (2004, Remark 2.2), $\bar{w}_t = w_t - \int_0^t h_1 du$ and $\bar{b}_t = b_t - \int_0^t h_2 du$ are two matrix Brownian motions under $\bar{\mathbb{Q}}$ while the dynamic of $(x_t = \ln s_t, v_t)$ is given by (52) and (53). □

Proof of Proposition 3.2. Using the notation $x_t = \ln s_t$, and recall that $g_{t_k} = \left(\prod_{l=0}^k s_{t_l}\right)^{\frac{1}{k+1}}$, with $y_{t_k} = \ln(g_{t_k}/s_{t_k})$ and using the law of iterated expectations we have

$$\bar{\mathbb{E}}_t\left[e^{zy_{t_k}}\right] = \bar{\mathbb{E}}_t\left[e^{z\left(\frac{1}{k+1}\sum_{l=0}^k \ln s_{t_l} - \ln s_{t_k}\right)}\right] \tag{89}$$

$$= \bar{\mathbb{E}}_t\left[e^{z\left(\frac{1}{k+1}\sum_{l=0}^k x_{t_l} - x_{t_k}\right)}\right] \tag{90}$$

$$= \bar{\mathbb{E}}_t\left[e^{z\left(\frac{1}{k+1}\sum_{l=0}^{k-1} x_{t_l} + \left(\frac{1}{k+1}-1\right)x_{t_k}\right)}\right] \tag{91}$$

$$= \bar{\mathbb{E}}_t\left[\exp\left(\frac{z}{k+1}\sum_{l=0}^{k-1} x_{t_l}\right)\bar{\mathbb{E}}_{t_{k-1}}\left[e^{\left(\frac{1}{k+1}-1\right)zx_{t_k}}\right]\right], \tag{92}$$

and from (6), expressed under $\bar{\mathbb{Q}}$, we have that

$$\bar{\mathbb{E}}_{t_{k-1}}\left[e^{\left(\frac{1}{k+1}-1\right)zx_{t_k}}\right] = \exp\left(\left(\frac{1}{k+1}-1\right)zx_{t_{k-1}} + \mathrm{tr}[a(\tau,(1/(k+1)-1)z,0_n)v_{t_{k-1}}]\right) \times e^{(1/(k+1)-1)zr\tau + c(\tau,(1/(k+1)-1)z,0_n)}, \tag{93}$$

and from this equality we define $\bar{\theta}_1 = a(\tau,(1/(k+1)-1)z,\bar{\theta}_0)$ and $c_1 = c(\tau,(1/(k+1)-1)z,\bar{\theta}_0)$ with $\bar{\theta}_0 = 0_n$, and these terms give (57) and (58) for $l = 1$. Note that (93) requires the (conditional to $\mathcal{F}_{t_{k-1}}$) moment generating function of $x_{t_{k-1}}$ alone so that in (6) we take $\theta_1 = 0_n$. Replacing the above expectation into (92), we get

$$\bar{\mathbb{E}}_t\left[e^{zy_{t_k}}\right] = \bar{\mathbb{E}}_t\left[\exp\left(\frac{z}{k+1}\sum_{l=0}^{k-1} x_{t_l}\right)\exp\left(\left(\frac{1}{k+1}-1\right)zx_{t_{k-1}} + \mathrm{tr}[\bar{\theta}_1 v_{t_{k-1}}]\right)\right]e^{(1/(k+1)-1)zr\tau+c_1},$$

$$= \bar{\mathbb{E}}_t\left[\exp\left(\frac{z}{k+1}\sum_{l=0}^{k-2} x_{t_l}\right)\exp\left(\left(\frac{2}{k+1}-1\right)zx_{t_{k-1}} + \mathrm{tr}[\bar{\theta}_1 v_{t_{k-1}}]\right)\right]e^{(1/(k+1)-1)zr\tau+c_1},$$

and after conditioning again with respect to $\mathcal{F}_{t_{k-2}}$ that

$$\bar{\mathbb{E}}_t\left[e^{zy_{t_k}}\right] = \bar{\mathbb{E}}_t\left[\exp\left(\frac{z}{k+1}\sum_{l=0}^{k-2} x_{t_l}\right)\bar{\mathbb{E}}_{t_{k-2}}\left[e^{\left(\frac{2}{k+1}-1\right)zx_{t_{k-1}}+\mathrm{tr}[\bar{\theta}_1 v_{t_{k-1}}]}\right]\right]e^{(1/(k+1)-1)zr\tau+c_1}, \tag{94}$$

but since we have

$$\bar{\mathbb{E}}_{t_{k-2}}\left[e^{\left(\frac{2}{k+1}-1\right)zx_{t_{k-1}}+\mathrm{tr}[\bar{\theta}_1 v_{t_{k-1}}]}\right] = \exp\left(\left(\frac{2}{k+1}-1\right)zx_{t_{k-2}} + \mathrm{tr}[\bar{\theta}_2 v_{t_{k-2}}] + (2/(k+1)-1)zr\tau + c_2\right), \tag{95}$$

with $\bar{\theta}_2$ and c_2 given by (57) and (58), respectively, and the use of (6) expressed under $\bar{\mathbb{Q}}$, inserting (95) in (94) we conclude that

$$\bar{\mathbb{E}}_t\left[e^{zy_{t_k}}\right] = \bar{\mathbb{E}}_t\left[\exp\left(\frac{z}{k+1}\sum_{l=0}^{k-2} x_{t_l} + \left(\frac{2}{k+1}-1\right)zx_{t_{k-2}} + \mathrm{tr}[\bar{\theta}_2 v_{t_{k-2}}]\right)\right] \times e^{(2/(k+1)-1)zr\tau+(1/(k+1)-1)zr\tau+c_2+c_1}. \tag{96}$$

It is important to notice that this time the (conditional to $\mathcal{F}_{t_{k-2}}$) moment generating function of the pair $(x_{t_{k-1}}, v_{t_{k-1}})$ is used. Proceeding recursively until reaching the current time t we get the result. □

Proof of Proposition 3.3. For $y \in \mathbb{R}$ consider the Mellin transform of $l \to (e^y - l)_+$, it gives

$$\int_0^{+\infty} (e^y - l)_+ l^{z-1} dl = \frac{e^{(z+1)y}}{z(z+1)},$$

and requires $\Re(z) > 0$. Multiplying the above equality by the density of y_{t_k} under $\bar{\mathbb{Q}}$ and after integration gives

$$\int_0^{+\infty} \bar{\mathbb{E}}_t\left[\left(e^{y_{t_k}} - l\right)_+\right] l^{z-1} dl = \frac{\Phi^1(t, t_k, z+1, x_t, v_t)}{z(z+1)}.$$

Taking the inverse Mellin transform and taking into account the fact that the expectation is real gives

$$\bar{\mathbb{E}}_t\left[\left(e^{y_{t_k}} - l\right)_+\right] = \frac{1}{\pi \mathrm{i}}\int_{\gamma+\mathrm{i}0}^{\gamma+\mathrm{i}\infty} \Re\left(l^{-z}\frac{\Phi^1(t, t_k, (z+1), x_t, v_t)}{z(z+1)}\right)dz, \tag{97}$$

and choosing $l = 1$ leads to the result. Notice that if $z = (\gamma, s)$ with $\gamma > 0$ and $s \in \mathbb{R}$ then $s \to \frac{\Phi^1(t,t_k,(z+1),x_t,v_t)}{z(z+1)}$ is integrable as we have

$$\left|\frac{\Phi^1(t,t_k,(z+1),x_t,v_t)}{z(z+1)}\right| = \left|\frac{\bar{\mathbb{E}}_t[e^{(z+1)y_{t_k}}]}{z(z+1)}\right| \le \frac{\bar{\mathbb{E}}_t[e^{(\gamma+1)y_{t_k}}]}{|z(z+1)|}, \tag{98}$$

that is integrable. □

Proof of Proposition 3.4. We use the law of iterated expectations, that is

$$\begin{aligned}\mathbb{E}_t\left[e^{z_1 x_{1,t_k}+z_2 y_{2,t_k}}\right] &= \mathbb{E}_t\left[\exp\left(\frac{z_2}{k+1}\sum_{l=0}^{k-1}x_{2,t_l}\right)\mathbb{E}_{t_{k-1}}\left[e^{z_1 x_{1,t_k}+\frac{z_2}{k+1}x_{2,t_k}}\right]\right],\\ &= \mathbb{E}_t\left[\exp\left(\frac{z_2}{k+1}\sum_{l=0}^{k-1}x_{2,t_l}+z_1 x_{1,t_{k-1}}+\frac{z_2}{k+1}x_{2,t_{k-1}}+\mathrm{tr}[a(\tau,(z_1,z_2/(k+1)),0_n)v_{t_{k-1}}]\right)\right]\\ &\times e^{(z_1+z_2/(k+1))r\tau+c(\tau,(z_1,z_2/(k+1)),\theta_0)}.\end{aligned} \tag{99}$$

The above equation suggests to define $\theta_1 = a(\tau,(z_1,z_2/(k+1)),\theta_0)$, $c_1 = c(\tau,(z_1,z_2/(k+1)),\theta_0)$ with $\theta_0 = 0_n$ and $a(.,.,.)$ and $c(.,.,.)$ as in Proposition 2.6. Conditioning again with respect to $\mathcal{F}_{t_{k-2}}$ we get that the expectation above, denoted I, is equal to

$$\begin{aligned}I &= \mathbb{E}_t\left[\exp\left(\frac{z_2}{k+1}\sum_{l=0}^{k-2}x_{2,t_l}\right)\mathbb{E}_{t_{k-2}}\left[e^{z_1 x_{1,t_{k-1}}+\frac{2z_2}{k+1}x_{2,t_{k-1}}+\mathrm{tr}[\theta_1 v_{t_{k-1}}]}\right]\right],\\ &= \mathbb{E}_t\left[\exp\left(\frac{z_2}{k+1}\sum_{l=0}^{k-2}x_{2,t_l}+z_1 x_{1,t_{k-2}}+\frac{2z_2}{k+1}x_{2,t_{k-2}}+\mathrm{tr}[\theta_2 v_{t_{k-2}}]\right)\right]e^{(z_1+2z_2/(k+1))r\tau+c_2},\end{aligned}$$

and since

$$\mathbb{E}_{t_{k-2}}\left[e^{z_1 x_{1,t_{k-1}}+\frac{2z_2}{k+1}x_{2,t_{k-1}}+\mathrm{tr}[\theta_1 v_{t_{k-1}}]}\right] = \exp\left(z_1 x_{1,t_{k-2}}+\frac{2z_2}{k+1}x_{2,t_{k-2}}+\mathrm{tr}[\theta_2 v_{t_{k-2}}]\right)e^{(z_1+2z_2/(k+1))r\tau+c_2},$$

with θ_2 and c_2 given by (61) and (62), respectively, we get that

$$I = \mathbb{E}_t\left[\exp\left(\frac{z_2}{k+1}\sum_{l=0}^{k-2}x_{2,t_l}+z_1 x_{1,t_{k-2}}+\frac{2z_2}{k+1}x_{2,t_{k-2}}+\mathrm{tr}[\theta_2 v_{t_{k-2}}]\right)\right]e^{(z_1+2z_2/(k+1))r\tau+c_2}.$$

Inserting the above equation in (99) and proceeding recursively until reaching the current time t we get the result. □

Proof of Proposition 3.5. Consider for $(y_1, y_2) \in \mathbb{R}^2$ the function $(e^{y_2} - e^{y_1})_+ = e^{y_1}(e^{y_2-y_1} - 1)_+$ and compute the Mellin transform of $l \to e^{y_1}(e^{y_2-y_1} - l)_+$, it gives

$$\int_0^{+\infty} e^{y_1}(e^{y_2-y_1}-l)_+ l^{s-1}dl = \frac{e^{y_1}e^{(s+1)(y_2-y_1)}}{s(s+1)}. \tag{100}$$

Multiplying the above equality by the density of (x_{1,t_k}, y_{2,t_k}) under $\mathbb{Q}$ and integrating gives

$$\int_0^{+\infty} \mathbb{E}_t[(e^{y_{2,t_k}}-e^{x_{1,t_k}})_+]l^{s-1}dl = \frac{\mathbb{E}_t\left[e^{-sx_{1,t_k}+(s+1)y_{2,t_k}}\right]}{s(s+1)}. \tag{101}$$

Since $\mathbb{E}_t\left[e^{-sx_{1,t_k}+(s+1)y_{2,t_k}}\right] = \Phi^2(t,t_k,(-s,s+1),x_t,v_t)$ given by (60), using the inverse Mellin transform, taking into account the fact that the expectation is real, and choosing $l = 1$ leads to the result. □

Appendix B. Supplementary material

Supplementary material related to this article can be found online at https://doi.org/10.1016/j.insmatheco.2025.103114.

Data availability

No data was used for the research described in the article.

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