Pricing VIX Futures with Stochastic Volatility and Random Jumps

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QMF 2009, Sydney

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Outline

- **Background**

  The introduction to VIX and VIX futures

  Literature review

  The motivation and outcomes

- **Analytically pricing VIX futures**

  Definitions of VIX and VIX futures

  The Heston stochastic volatility and random jumps model

  Pricing VIX futures and discussions

- **Concluding remarks**

- **Background**

  The introduction to VIX and VIX futures

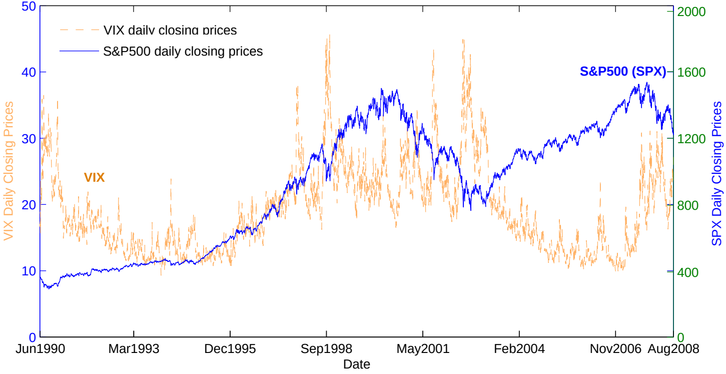
  Literature review

  The motivation and outcomes

- **Volatility Index (VIX) in CBOE**

  "The CBOE Volatility Index (VIX) is a key measure of market expectations of near-term volatility conveyed by $S\&P500$ stock index option prices."

  – Website of CBOE



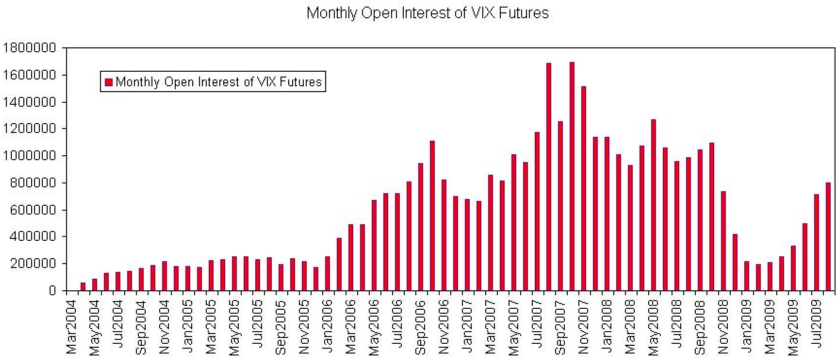
- VIX rises when investors are anxious or uncertain about the market and falls during times of confidence.

Background - The story of VIX

- 1993 The VIX Index was introduced by Professor Robert E. Whaley.
- 2003 The VIX methodology was revised.
- 2004 On March 26, 2004, the VIX Index began trading on the CBOE.
- 2006 VIX options were launched in February 2006.
- 2008 Binary options on VIX began trading.
- 2009 Mini-VIX futures were launched.

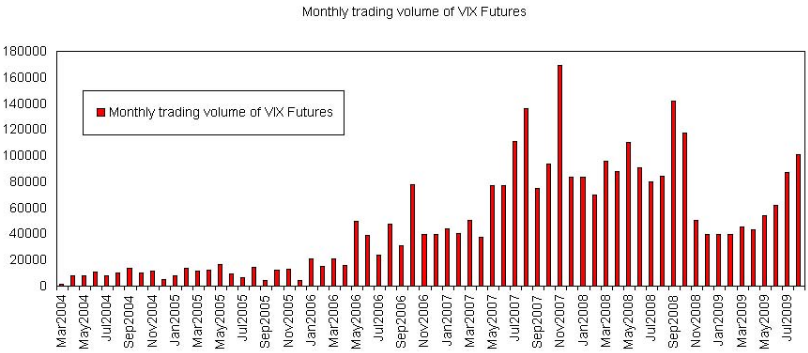
Background - Trading VIX Futures

- The open interests of VIX futures in CBOE



Background - Trading VIX Futures

- The trading volume of VIX futures in CBOE



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The financial models

- **Heston (1993):**
  Stochastic volatility without jumps (SV model).
- **Duffie et al. (2000), Eraker (2004):**
  Stochastic volatility with jumps in asset return and variance process (SVJJ model).
  Received considerable attention, e.g., SVJJ explains the volatility smile for short maturity (Pan 2002).

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The research interest in pricing VIX futures

- **Carr and Wu (2006):**
  A lower bound and an upper bound
- **Zhang and Zhu (2006):**
  Exact pricing formula under the Heston (1993) SV model.
  Without paying attention to the jumps.
- **Lin (2007):**
  An approximation under the SVJJ model.
  Performs poorly.

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The outcomes of this research

- Exact pricing formula to price VIX futures in the general SVJJ model.

Models	Research in literature	Results
SV	Zhang and Zhu (2006)	Exact formula
SV	Brenner et al. (2007)	Approximation (third order)
SVJJ	Lin (2007)	Approximation (second order)
SVJJ	Found by our research	Exact formula

- Analyzing the accuracy of the approximations (Lin 2007, Brenner et al. 2007).

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Analytically Pricing VIX Futures

- **Background**
- **Analytically pricing VIX futures**

  Definitions of VIX and VIX futures

  The Heston stochastic volatility and random jumps model

  Pricing VIX futures and discussions

Analytically Pricing VIX Futures

- Volatility Index
- Modelling the $S\&P500$ and VIX
- Pricing formula and discussions

Analytically Pricing VIX Futures - Volatility Index

VIX, which is the underlying of VIX futures and options, is defined by means of VIX_t^2,

$$\text{VIX}_t^2 = \left(\frac{2}{\overline{\tau}} \sum_i \frac{\Delta K_i}{K_i^2} e^{r\overline{\tau}} Q(K_i) - \frac{1}{\overline{\tau}} [\frac{F}{K_0} - 1]^2 \right) \times 100^2$$

- $\overline{\tau} = \frac{30}{365}$,
- K_i is the strike price of the i-th out-of-the-money option written on the S&P500 in the calculation,
- F is the time-t forward S&P500 index level,
- $Q(K_i)$ denotes the time-t midquote price of the out-of-the-money option at strike K_i, K_0 is the first strike below the forward index level,
- r denotes the time-t risk-free rate with maturity $\overline{\tau}$.

Analytically Pricing VIX Futures - Volatility Index

This expression of the VIX squared can be presented in terms of the risk-neutral expectation of the log contract,

$$\text{VIX}_t^2 = -\frac{2}{\overline{\tau}} E^{\mathbb{Q}}[\ln(\frac{S_{t+\overline{\tau}}}{F})|F_t] \times 100^2 \tag{1}$$

- $\mathbb{Q}$ is the risk-neutral probability measure,
- $F = S_t e^{r\overline{\tau}}$ denotes the 30-day forward price of the underlying S&P500 with a risk-free interest rate r under the risk-neutral probability,
- F_t is the filtration up to time t.

An even more intuitive explanation.

- The VIX squared is the conditional risk-neutral expectation of the annualized realized variance of the S&P500 return over the next 30 calendar days

$$\text{VIX}_t^2 = E^{\mathbb{Q}}\left[\lim_{N\to\infty}\frac{1}{\tau}\sum_{i=1}^{N}\log^2\left(\frac{S_{t_i}}{S_{t_{i-1}}}\right)\right]\times 100^2 \qquad (2)$$

Analytically Pricing VIX Futures - The Model

Under the risk-neutral probability measure $\mathbb{Q}$, the dynamics processes of the S&P500 index and its variance

$$\begin{cases} dS_t = S_t r_t dt + S_t\sqrt{V_t}dW_t^S(\mathbb{Q}) + d(\displaystyle\sum_{n=1}^{N_t(\mathbb{Q})} S_{\tau_{n-}}[e^{Z_n^S(\mathbb{Q})} - 1]) - S_t\overline{\mu}^{\mathbb{Q}}\lambda dt \\ dV_t = \kappa^{\mathbb{Q}}(\theta^{\mathbb{Q}} - V_t)dt + \sigma_V\sqrt{V_t}dW_t^V(\mathbb{Q}) + d(\displaystyle\sum_{n=1}^{N_t(\mathbb{Q})} Z_n^V(\mathbb{Q})) \end{cases}$$

- The two standard Brownian motions are correlated with $E[dW_t^S, dW_t^V] = \rho dt$;
- κ, θ and σ_V are respectively the mean-reverting speed parameter, long-term mean, and variance coefficient of the diffusion V_t;

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dV_t = \kappa^{\mathbb{Q}} (\theta^{\mathbb{Q}} - V_t) dt + \sigma_V \sqrt{V_t} dW_t^V(\mathbb{Q}) + d\left(\sum_{n=1}^{N_t(\mathbb{Q})} Z_n^V(\mathbb{Q}) \right)
\end{cases}
$$

- N_t is the independent Poisson process with intensity λ, i.e., $Pr\{N_{t+dt} - N_t = 1\} = \lambda dt$, $Pr\{N_{t+dt} - N_t = 0\} = 1 - \lambda dt$.
- $Z_n^V \sim \exp(\mu_V)$, and $Z_n^S | Z_n^V \sim N(\mu_S + \rho_J Z_n^V, \sigma_S^2)$;
- r_t is the constant spot interest rate;

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Analytically Pricing VIX Futures - The VIX

VIX squared is the conditional risk-neutral expectation of the log contract of the S&P500 over the next 30 calendar days.

$$\text{VIX}_t^2 = -\frac{2}{\overline{\tau}} E^{\mathbb{Q}}[\ln(\frac{S_{t+\overline{\tau}}}{F})|F_t] \times 100^2 \tag{3}$$

- Under the general specification Eq. (3), this expectation can be carried out explicitly in the form of,

$$\text{VIX}_t^2 = (aV_t + b) \times 100^2 \tag{4}$$

$$\begin{cases} a = \frac{1 - e^{-\kappa^{\mathbb{Q}}\overline{\tau}}}{\kappa^{\mathbb{Q}}\overline{\tau}}, \quad and \quad \overline{\tau} = 30/365 \\ b = (\theta^{\mathbb{Q}} + \frac{\lambda\mu_V}{\kappa^{\mathbb{Q}}})(1-a) + \lambda c \\ c = 2[\overline{\mu}^{\mathbb{Q}} - (\mu_S^{\mathbb{Q}} + \rho_J\mu_V)] \end{cases}$$

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The value of the VIX futures at time t with settlement at time T, $F(t,T)$, is calculated by

$$F(t,T) = E^Q[\mathrm{VIX}_T|F_t] = E^Q[\sqrt{aV_T + b}|F_t] \times 100 \tag{5}$$

Analytically Pricing VIX Futures - Formula

- The moment generating function, $f(\phi; t, \tau, V_t)$, of the stochastic variable V_T, conditional on the filtration F_t, with time to expiration $\tau = T - t$.

$$f(\phi; t, \tau, V_t) = E^{\mathbb{Q}}[e^{\phi V_T} | F_t] \tag{7}$$

- The characteristic function is just $f(\phi i; t, \tau, V_t)$.
- Feynman-Kac theorem implies that $f(\phi, \tau)$ must satisfy the following backward partial integro-differential equation (PIDE)

$$\begin{cases} -f_\tau + \kappa^{\mathbb{Q}}(\theta^{\mathbb{Q}} - V) f_V + \dfrac{1}{2}\sigma^2 V f_{VV} + \lambda E^{\mathbb{Q}}[f(V + Z^V) - f(V) | F_t] = 0 \\ f(\phi; t + \tau, 0, V) = e^{\phi V} \end{cases}$$

Analytically Pricing VIX Futures - Formula

The above PIDE (7) can be analytically solved and hence we obtain the moment generation function in the form of

$$f(\phi; t, \tau, V_t) = e^{C(\phi,\tau)+D(\phi,\tau)V_t+A(\phi,\tau)} \tag{7}$$

where

$$\begin{cases} A(\phi,\tau) = \dfrac{2\mu_V \lambda}{2\mu_V \kappa^{\mathbb{Q}} - \sigma_V^2} \ln\left(1 + \dfrac{\phi(\sigma_V^2 - 2\mu_V \kappa^{\mathbb{Q}})}{2\kappa^{\mathbb{Q}}(1-\mu_V \phi)}(e^{-\kappa^{\mathbb{Q}}\tau} - 1)\right) \\ C(\phi,\tau) = \dfrac{-2\kappa\theta}{\sigma_V^2} \ln\left(1 + \dfrac{\sigma_V^2 \phi}{2\kappa^{\mathbb{Q}}}(e^{-\kappa^{\mathbb{Q}}\tau} - 1)\right) \\ D(\phi,\tau) = \frac{2\kappa^{\mathbb{Q}}\phi}{\sigma_V^2\phi + (2\kappa^{\mathbb{Q}} - \sigma_V^2\phi)e^{\kappa^{\mathbb{Q}}\tau}} \end{cases}$$

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Analytically Pricing VIX Futures - Formula

The Fourier inversion of the characteristic function $f(\phi i; t, \tau, V_t)$ provides the required conditional density function $p^{\mathbb{Q}}(V_T|V_t)$

$$p^{\mathbb{Q}}(V_T|V_t) = \frac{1}{\pi}\int_0^{\infty} Re[e^{-i\phi V_T} f(i\phi; t, \tau, V_t)]d\phi \tag{8}$$

The price of a VIX future contract at time t is thus expressed in the form of

$$F(t,T) = E^{\mathbb{Q}}[\text{VIX}_T|F_t] = \int_0^{\infty} p^{\mathbb{Q}}(V_T|V_t)\sqrt{aV_T + b}dV_T \times 100 \tag{9}$$

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Analytically Pricing VIX Futures - Formula

This pricing formula can be further simplified by utilizing the expression

$$E[\sqrt{x}] = \frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{1 - E[e^{-sx}]}{s^{\frac{3}{2}}} ds \tag{10}$$

Invoking this identity, Formula (9) can be simplified as

$$F(t,T) = \frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{1 - e^{-sb} f(-sa; t, \tau, V_t)}{s^{\frac{3}{2}}} ds \tag{11}$$

where $f(\phi; t, \tau, V_t)$ is the moment generating function shown in Eq. (7).

The closed-form pricing formula for VIX futures

$$F(t,T) = \frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{1 - e^{-sb} f(-sa; t, \tau, V_t)}{s^{\frac{3}{2}}} ds \tag{12}$$

where $f(\phi; t, \tau, V_t)$ is the moment generating function shown in Eq. (7).

- A closed-form solution;
- Efficient and exact;
- Useful in empirical study: model calibration;

Analytically Pricing VIX Futures - Discussions

Numerical Results:

- the implementation our VIX futures pricing formula, Eq. (12).
- the Monte Carlo simulations to verify the correctness of our newly-found formula.
- the results obtained from the convexity correction approximations (e.g., Lin (2007) and Brenner et al. (2007)), to show the improvement in accuracy of our exact solution.

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Analytically Pricing VIX Futures - Discussions

For simplicity, we have employed the simple Euler-Maruyama discretization for the variance dynamics:

$$v_t = v_{t-1} + \kappa^{\mathbb{Q}}(\theta^{\mathbb{Q}} - v_{t-1})\Delta t + \sigma\sqrt{|v_{t-1}|}\sqrt{\Delta t}W_t + \sum_{n=1}^{N_t} Z_n^V \qquad (13)$$

- W_t is a standard normal random variables
- $Z_n^V \sim \exp(\mu_V)$,
- N_t is the independent Poisson process with intensity $\lambda\Delta t$.

Analytically Pricing VIX Futures - Discussions

By using the convexity correction approximation proposed by Brockhaus and Long (2000), Lin (2007) was able to present the VIX futures approximation formula in the form of

$$F(t,T) = E^{\mathbb{Q}}[\text{VIX}_T|F_t] \approx \sqrt{E_t^{\mathbb{Q}}(\text{VIX}_T^2)} - \frac{var^{\mathbb{Q}}(\text{VIX}_T^2)}{8[E^{\mathbb{Q}}(\text{VIX}_T^2)]^{\frac{3}{2}}} \qquad (14)$$

- $var^{\mathbb{Q}}(\text{VIX}_T^2)/\{8[E^{\mathbb{Q}}(\text{VIX}_T^2)]^{\frac{3}{2}}\}$ is the convexity adjustment relevant to the VIX futures.
- This formula is indeed very easy to be implemented.
- But what about the accuracy?

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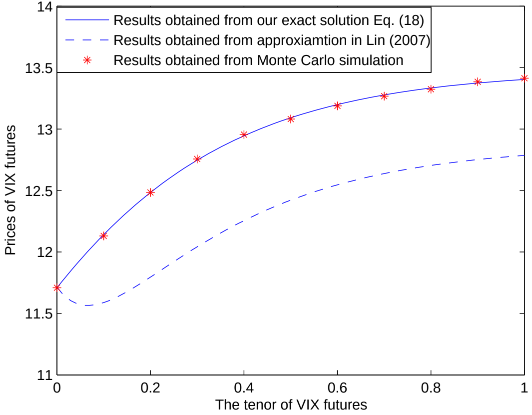
- Numerical Comparison of VIX futures prices



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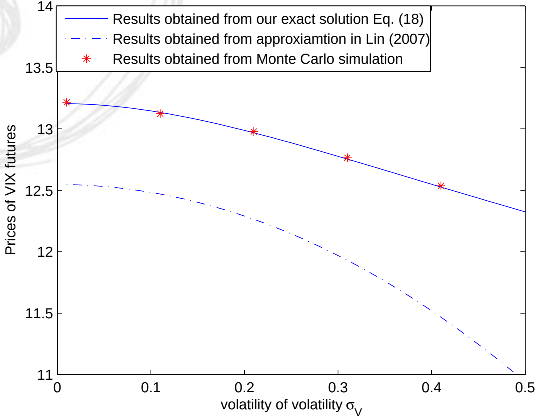

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Analytically Pricing VIX Futures - Discussions

- Lin (2007)'s convexity correction approximation is essentially a Taylor-series expansion of the square root function to the second order.
- Brenner et al. (2007) explored a third order Taylor expansion of the square root function and obtained an approximation formula for VIX futures, based on the Heston SV model.
- It is thereby quite interesting to examine whether their third-order approximation formula has improved the accuracy.

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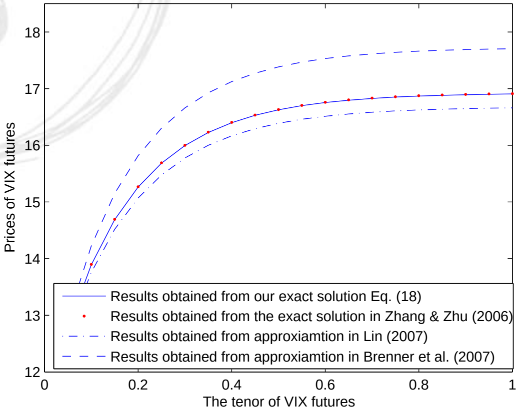

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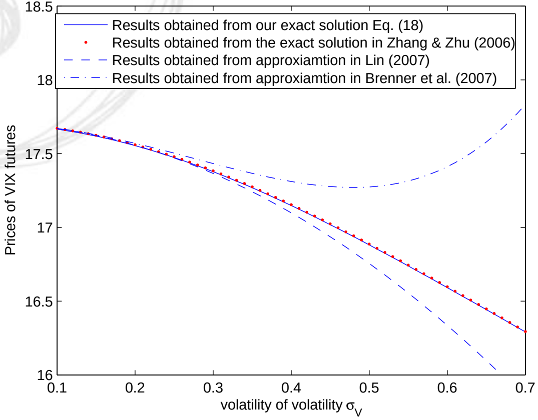

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- A closed-form exact solution is presented for the pricing of VIX futures, based on the SVJJ model. It has filled up a gap that there is no closed-form exact solution available in the SVJJ model.
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Thank you!

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