

Volterra versus Baire spaces?

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What is a Volterra space?

A space X is called *a Volterra space* if for any pair of functions $f, g : X \rightarrow \mathbb{R}$ such that $C(f)$ and $C(g)$ are dense in X , $C(f) \cap C(g)$ are also dense in X (Gauld and Piotrowski, 1993).

If $f : \mathbb{R} \rightarrow \mathbb{R}$ is a function such that $C(f)$ and $D(f)$ are dense, there is no function $g : X \rightarrow Y$ with $C(f) = D(g)$ and $D(f) = C(g)$ (Volterra, 1881).

If only $C(f) \cap C(g) \neq \emptyset$ is required, then X is called a *weakly Volterra* space.

Characterization: A space X is Volterra (resp. weakly Volterra) iff for any two dense G_δ -sets G and H in X , $G \cap H$ is dense (resp. nonempty).

Consequence: (i) Baire spaces are Volterra.

(ii) Spaces of the second category are weakly Volterra.

Of course, every Volterra space is weakly Volterra.

Must every Volterra space be Baire?

The answer is "No". In 1996, Gault, Greenwood and Piotrowski discovered:

- $\mathbb{R}^- \cup \mathbb{Q}^+$ is a space of the second category which is not Volterra;
- a T_1 Volterra space which is of the first category.

More examples were given by Gruenhage and Lutzer in 2000. For example,

- there is a countable regular space that is Volterra but not Baire.

Resolvable spaces, irresolvable spaces

A space X is called *resolvable* if it contains two disjoint dense sets, and X is called *irresolvable* if it is not resolvable.

- If every open subspace of X is irresolvable, then X is a Volterra space.

When is a Volterra space Baire?

Theorem 1: Let X be a stratifiable space. If X is Volterra, then it is Baire.

A space X is stratifiable if X is regular and one can assign a sequence of open sets $\{G(H, n) : n \in \mathbb{N}\}$ to each closed set $H \subseteq X$ such that

- (i) $H = \bigcap_{n \in \mathbb{N}} G(H, n) = \bigcap_{n \in \mathbb{N}} \overline{G(H, n)}$,
- (ii) $G(H, n) \subset G(H, n + 1)$ for all $n \in \mathbb{N}$,
- (iii) if $H \subseteq K$, $G(H, n) \subseteq G(K, n)$ for all $n \in \mathbb{N}$.

In 2000, Gruenhage and Lutzer proved that metric Volterra spaces are Baire, and asked: Must X be Baire if it is stratifiable and Volterra? Theorem 1 answers this question affirmatively.

A subset A of a space X is **simultaneously separated** if for each $x \in A$ there is an open set $U(x) \ni x$ such that $\{U(x) : x \in A\}$ is disjoint family.

$$\lambda(X) := \bigcup \{A^d : A \text{ is simultaneously separated}\}.$$

Sharm-Sharm Theorem: **Let X be a dense-in-itself and Hausdorff space. If $\lambda(X)$ is dense in X , then X is resolvable (Sharm and Sharm, 1988).**

A space X is **monotonically normal** if one can assign to each pair (H, K) of disjoint closed sets an open set $U(H, K)$ with $H \subseteq U(H, K) \subseteq \overline{U(H, K)} \subseteq X \setminus K$, so that $H \subseteq H'$ and $K \supseteq K'$ implies $U(H, K) \subseteq U(H', K')$.

Fact: **If X is monotonically normal, then $\lambda(X) = X$ (Dow, Tkachenko, Tkachuk and Wilson, 2002).**

Homogeneous spaces

It is known that if X is a homogeneous space, then X is Baire iff it is of the second category.

Similarly, if X is a homogeneous space, then X is Volterra if and only if it is weakly Volterra (Cao and Gauld, 2005).

Theorem 2: Let X be a locally convex topological vector space. If X is Volterra, then it is Baire.

Corollary 1: Let E be a normed linear space. Then (E, weak) is Volterra iff E is finite dimensional.

Corollary 2: For any Tychonoff space X , $C_p(X)$ (resp. $C_k(X)$) is Volterra iff $C_p(X)$ (resp. $C_k(X)$) is Baire.

The proof of Theorem 2 depends on a consequence of the **Hahn-Banach theorem**.

If X is a locally convex tvs, then there exists a **non-trivial** $f \in X^*$. It follows that

$$X \approx \ker(f) \times X/\ker(f) \approx \ker(f) \times \mathbb{R}$$

Now, we need the following simple, but important

Lemma: If Z is a space of the first category and M is a dense-in-itself separable metric space, then $Z \times M$ is not weakly Volterra.

Now, suppose that X is a Volterra space. Then $\ker(f)$ is of the second category. Thus, by a classical result, X must be of the second category.

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