Blocking Efficiency and Competitive Equilibria in Economies with Asymmetric Information

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Declaration

I hereby declare that this submission is my own work and that, to the best of my knowledge and belief, it contains no material previously published or written by another person (except where explicitly defined in the acknowledgements), nor material which to a substantial extent has been submitted for the award of any other degree or diploma of a university or other institution of higher learning.

Auckland, April 2013

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Abstract

In this thesis, two most fundamental problems in economic theory, namely the existence and the optimality of Walrasian equilibrium, are studied. It is assumed that there is uncertainty about the realized state of nature in an economy and different agents may have different information. Such an economy is called an *economy with asymmetric information.* Considering a pure exchange asymmetric information economy with finitely many states of nature, an atomless measure space of agents and a Banach lattice as the commodity space, it is shown that the private core and the set of Walrasian allocations coincide. The feasibility in this result is taken as free disposal. This optimality is known as the core-Walras equivalence theorem. When the feasibility is defined without free disposal, then it is shown that if a feasible allocation is not in the private core then it is privately blocked by a coalition of any given measure less than that of the grand coalition. This theorem not only gives the full answer to a question in [72], but also provides a sharper characterization of Walrasian allocations.

In addition to the above optimality, some other characterizations of Walrasian allocations by the veto power of the grand coalition are also established. One of them deals with robustly efficient allocations in a pure exchange mixed economy with asymmetric information whose commodity space is an ordered separable Banach space having an interior point in its positive cone. This gives a solution to the question posed in [48]. Other two characterizations are restricted to a discrete economy with a Banach lattice as the commodity space. First one claims that a feasible allocation is a Walrasian allocation if and only if it is Aubin non-dominated, whereas the other one is interpreted in terms of privately non-dominated allocations in suitable associated economies. These yield a partial solution to a question in [41]. The feasibility in all of these results is defined as free disposal.

In a pure exchange asymmetric information economy whose space of agents is a finite measure space, space of states of nature is a probability space with a complete measure, and commodity space is defined as the Euclidean space,

the existence of a maximin rational expectations equilibrium is established. So a solution to a question in [32] is obtained.

Chapter 1

Introduction

1.1 Historical Background

One of the central paradigms in economic theory is Walras's general equilibrium theory, refer to [85]. As mentioned in [12], Walras formulated an equilibrium concept of an economic system at any fixed time as the solution to a system of simultaneous equations representing the demand for goods by consumers, the supply of goods by producers, and the demand is equal to the supply on every market. It was also stated that consumers and producers were price takers, each consumer acted so as to maximize his utility and

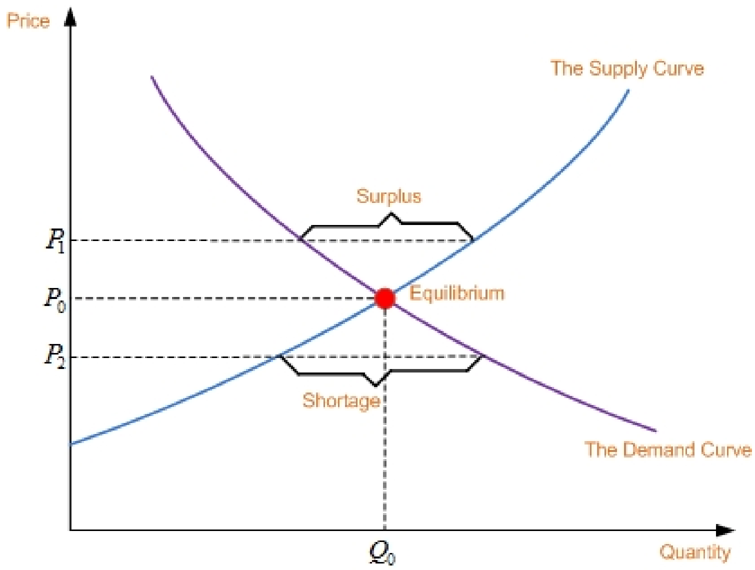

Figure 1.1: Demand and Supply

each producer acted so as to maximize his profit. A price system such that the above conditions are satisfied is known as the *equilibrium price* or the *market clearing price*. It is well known that if a price system is taken below the equilibrium price then there is a shortage in supply, but if it is taken above the equilibrium price then there is a surplus of supply in the market. This can be visualized in Figure 1.1.

Arrow and Debreu in [12], and McKenzie in [61] employed an economic model similar to Walras, which consists of finitely many consumers, producers and commodities. In that model, a commodity was a good or service, and was characterized by its physical characteristics, the location at which it would be available and the date on which it would be available. Under certain appropriate assumptions, they showed that there exists an equilibrium price for the economy. Any such price together with the commodity bundles demanded by consumers and the commodity bundles supplied by producers at that price is called a *Walrasian* (*competitive*) *equilibrium*. This result is called the *existence theorem* of Walrasian equilibrium in the literature. The economic model in [12, 61] is known as the *Arrow-Debreu-McKenzie model*. Note that the Walrasian equilibrium is a non-cooperative solution concept, since agents' actions are independent of each other. The proof of the existence theorem totally depends on mathematical arguments. On the one hand, Arrow and Debreu used an extension of Nash equilibrium theorem [62] in an abstract economy (see [27]). On the other hand, McKenzie provided a more direct and simple proof by using the Brouwer fixed point theorem. It is also worth to point out that the reversibility of production plans was allowed in [61], which

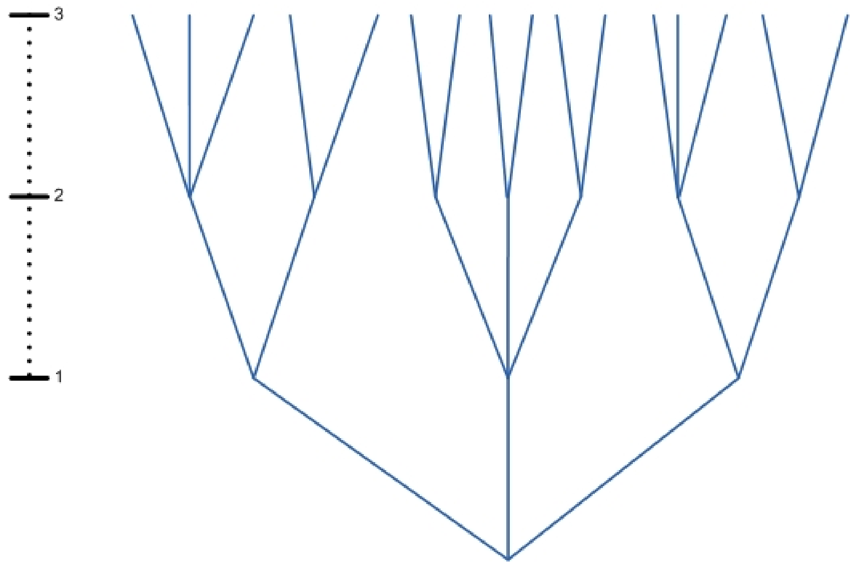

Figure 1.2: Uncertainty

was not the case in [12]. Since this model is purely deterministic, Arrow in [11] and Debreu in [29] introduced the notion of uncertainty about the states of nature by adding contingent claims into this classical model. The uncertainty on any date makes a family of events, and a commodity on any date is also characterized by an event. They assumed that all agents have the same information about the realized state of nature, and showed that the deterministic existence result still holds in this modified model.

Consider a pure exchange economy (an economy without production). One example of this type of economy is the international trade market where several countries exchange goods at fixed terms of trade. In a pure exchange economy, one of the important questions is the optimality of a Walrasian equilibrium. The *core* of an economy is one of the optimality notions, where it is assumed that agents (consumers) are free to cooperate and bargain among themselves. Roughly speaking, a core is a list of commodity bundles, one for each agent, such that no coalition of agents can redistribute their initial endowments among themselves and make themselves better off in the sense that every member of the coalition prefers the new bundle than the old ones. In [34], Edgeworth considered a two agent pure exchange economy with two commodities, which can be described geometrically. This geometric representation is given in Figure 1.1 and is known as the *Edgeworth box* corresponding to that economy. Edgeworth assumed that each of the two agents initially possessed certain amounts of each commodity. Let a denote the total initial endowment and correspond to the point B in the xy-plane. Suppose that P is the initial endowment of the first agent with respect to the xy-plane. Then, P also denotes the initial endowment of the second agent with respect to the st-plane. The consumption set of the first agent corresponds to the xy-plane and that of the second agent corresponds to the st-plane. The dotted curve represents the indifference curve of the first agent, that is, any two points on this curve are equally preferable to the first agent. Likewise, the undotted curve is the indifference curve of the second agent. A *Pareto optimal allocation* is a point at which the indifference curves of the two agents are tangent to each other. The set of Pareto optimal allocations is called the *contract curve* and is the curve joining the points O and B. The intersection of this curve with the shaded region is the core of this economy and a Walrasian equilibrium is a point of the core. In this study, Edgewoth also introduced the expanded economy consisting of $2n$ agents of two types. Agents having the same type means that agents have the same initial endowment and preference. He also noticed that as n becomes larger, the core becomes smaller, and finally only the Walrasian allocations remain. Edgeworth's idea was based on the graphical pictures and so it is not applicable to the case where there

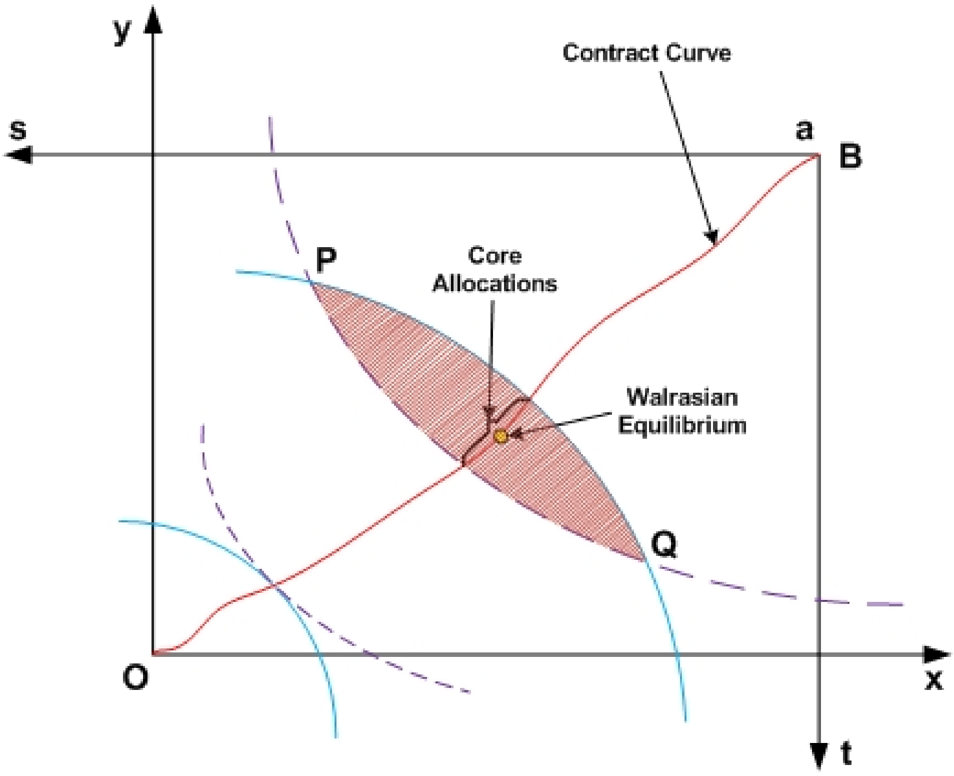

Figure 1.3: The Edgeworth Box

are arbitrarily finitely many agents and commodities, refer to [30]. To answer this question, Debreu and Scarf in [30] repeated the same analysis using advanced mathematical techniques. It was also showed that the agents having the same type are assigned the same consumption in a core allocation, and the intersection of the core of all expanded (replica) economies is called the *Edgeworth equilibria*. Finally, they proved that there are price systems for which Edgeworth equilibria become Walrasian equilibria. This result together with the non-emptiness of the core (see [77]) gives an alternative proof of the existence of a Walrasian equilibrium without using the equations of supply and demand of commodities.

In [15], Aumann remarked that in an economy consisting of a finite number of agents, the influence of an individual is not negligible. This means that any economy with finitely many agents is not perfectly competitive, and agents are price takers. Mathematically, the influence of an individual agent is not negligible unless the number of agents is infinite. To avoid this difficulty, Aumann introduced the notion of a continuum of agents in a pure exchange economy, where the unit closed interval [0,1] was used to represent the set of agents. The reason for doing this is that one can integrate over a continuum, and any change of the integrand at a single point (even a set of points with measure zero) does not affect the value of the integral. Hence, the action of an individual agent (even a set of agents with measure zero) is negligible. Employing standard assumptions, Aumann showed that the set of Walrasian allocations is non-empty and that the core of such an economy coincides with the set of Walrasian equilibrium allocations, refer to [15, 17]. In addition to this extension, there have been some other extensions appearing in general equilibrium theory. Two major extensions of them are an economy with infinitely many commodities and an economy with asymmetric information. In the line of research of infinitely many commodities, the initial work of Bewley in [19], Debreu in [28], and Peleg and Yaari in [66] has been much more appreciated. There are many aspects of motivation to consider infinite dimensional commodity spaces. For example, one can put commodities in discrete time periods over an infinite time horizon, or infinitely many states of nature, or infinitely many variations in any of the characteristics describing commodities. Meanwhile, it was Radner [73] who first extended the analysis of Arrow [11] and Debreu [29] to the case in which different agents have different information about the states of nature and introduced the concept of a Walrasian equilibrium in his model. This model is known as an *economy with asymmetric information*. In this model, each agent is characterized by a state dependent utility function, a random initial endowment, a private information set and a prior belief. In this framework, agents make contingent contracts for trading commodities before they obtain any information about the realized state of nature.

Aumann's core-Walras equivalence theorem is one of the most interesting results in economic theory. Many extensions of this result have been obtained in the literature. Firstly, an extension of this result to an economy with an atomless measure space of agents and finitely many commodities can be found in [51]. In the context of infinite dimensional commodity space, the relation between the core and the set of Walrasian allocations are more interesting, since preferences and endowments are more diverse, and thus blocking become more difficult, refer to [45]. Rustichini and Yannelis [76] extended this result to an economy whose commodity space is a separable Banach lattice. In [15], Aumann also pointed out that many real markets are indeed far from being perfect; such a market is probably best represented by a mixed model, in which some agents are points in a continuum and others are individually significant. One of the key results on the equivalence between the core and the set of Walrasian allocations in a mixed economy was established by Shitovitz in [79]. To be precise, he showed that if there exist at least two large agents and all of them have the same initial endowment and preference, then the core coincides with the set of Walrasian allocations. Similar results in mixed economies also came out in [33, 37, 42]. In all of these results, the feasibility was defined without free disposal and in terms of Bochner integrable functions. Due to asymmetry of information and communication opportunities among agents, several alternative core concepts have been introduced, refer to [86, 87]. In [87], Yannelis introduced the notion of *private core* based on the fact that agents have no access to the communication system, that is, each member of the coalition uses only his private information whenever a coalition blocks an allocation. It is also essential to mention that under standard assumptions, the private core is non-empty, Bayesian incentive compatible and rewards the information superiority of agents (see [58, 87]). Dealing with asymmetric information, Einy et al. [35] first extended Aumann's equivalence theorem to the case of the private core and the set of Walrasian allocations, where the free disposal feasibility assumption was used. Later, this result was further generalized to an asymmetric information economy with an atomless measure space of agents and an ordered separable Banach space having an interior point in its positive cone as the commodity space in [36]. In addition to these equivalence results with free disposal, Angeloni and Martins-da-Rocha [9] obtained an equivalence result between the private core and the set of Walrasian allocations in an atomless economy with finitely many commodities and without free disposal feasibility assumption. It is worth to point out that in all of these results, commodity spaces are separable and for the case of asymmetric information, there are only finitely many states. In contrast to the so far mentioned positive results, Podczeck [68] and Tourky and Yannelis [81] constructed counterexamples of economies to show that the classical core-Walras equivalence theorem in [15] may fail under desirable assumptions when the commodity space is a non-separable ordered Banach space and the feasibility is defined by Bochner integrable functions. However, when feasibility is defined in terms of Pettis integral, Podczeck [69] obtained a positive result for a ceratin class of commodity spaces without requiring that those commodity spaces are separable.

In 1972, three notes in the same issue of *Econometrica* gave sharper characterizations of the core of an atomless economy, where the feasibility was defined without free disposal. Firstly, Schmeidler [78] proved that if an allocation f is blocked by a coalition via some allocation g, then there is a coalition of arbitrarily small measure which blocks f via the allocation g. Schmeidler's result was further generalized by Grodal in [44] by restricting the set of coalitions to those consisting of finitely many arbitrarily small sets of agents with similar characteristics, which are presumably easier to form and also interpret. Finally, Vind [84] showed that if some coalition blocks a feasible allocation then there is a blocking coalition with any measure less than the measure of the grand coalition. These results imply that, for a finite-dimensional commodity space, the set of Walrasian allocations of an atomless economy coincides with the set of feasible allocations that are not blocked by coalitions of arbitrarily given measure less than that of the grand coalition. It is well known that similar results do not hold if one restricts the economy to finitely many agents. Besides this impossibility, Khan [56] showed that restrictions placed on the formation of coalitions as in [44, 78, 84] do not enlarge the core "very much" if the number of agents in the finite economy is large enough. Since these results rely heavily on Lyapunov's convexity theorem, which does not hold in an infinite dimensional setting, the exact extension of either Schmeidler's or Grodal's result is not possible, as mentioned in [45]. Indeed, Núñez [63] gave an example of an atomless economy with infinitely many commodities, where an allocation f is blocked by the grand coalition via an allocation g, but there is no other different coalition blocking f via the same allocation g. Despite this impossibility, Hervés-Beloso et al. [45] obtained a slightly weaker version of Grodal's result in a continuum economy whose commodity space is the space of real bounded sequences with the Mackey topology. Recently, Evren and Hüsseinov [36] further extended the results of Schmeidler, Grodal and Vind to economies whose commodity spaces are ordered Banach spaces having interior points in their positive cones. In addition to the above deterministic results, it was mentioned in the appendix of [36] that Schmeidler's, Grodal's and Vind's theorems in [36] can be further extended to a framework with asymmetric information. On the other hand, extensions of Vind's theorem to asymmetric information economies with the equal treatment setting and the Euclidean space or ℓ^∞ as commodity spaces were provided in [46, 47] respectively. Recently, using the notion of information sharing rule, Hervér-Beloso et al. [49] established some results similar to those in [78, 84] in asymmetric information economies. Interestingly, the afore-mentioned results in [36, 46, 47, 49] were obtained under the free disposal condition.

In addition to the characterizations of Walrasian allocations in terms of the core, some other characterizations by the veto power of coalitions have been proposed in the literature. Addressing complete information economies with finitely many agents and commodities, Aubin [13] introduced the ponder veto concept and showed that the core obtained by this veto mechanism coincides with the set of Walrasian allocations. Aubin's approach was employed by Evren and Hüsseinov in [36], Graziano and Meo in [41] and Hervés-Beloso et al. in [47] to characterize Walrasian allocations in asymmetric information economies. In fact, Graziano and Meo [41] showed that the Aubin private core provides a complete characterization of Walrasian allocations in an asymmetric information economy with a complete measure space of agents, finitely many states of nature and an ordered separable Banach space having an interior point in its positive cone as the commodity space. For an asymmetric information economy with finitely many agents and finitely many states of nature, Evren and Hüsseinov [36] and Hervés-Beloso et al. [47] obtained characterizations of Walrasian allocations sharper than those given in [41]. They proved that the set of Walrasian allocations coincides with the set of Aubin non-dominated feasible allocations. The commodity spaces in their work were an ordered separable Banach space whose positive cone has an interior point and ℓ_∞ with the Mackey topology. The other characterization theorems in [36, 41, 46, 47] enable one to obtain the first and second welfare theorems as easy corollaries. Instead of using infinitely many coalitions, only the veto power of the grand coalition was used in these theorems, but exercised in a family of economies obtained by perturbing the agents' initial endowments. The main result in [41] claims that a feasible allocation is a Walrasian allocation in an asymmetric information economy with a complete measure space of agents, finitely many states of nature and an ordered separable Banach space whose positive cone has an interior point as the commodity space if and only if it is privately non-dominated in suitable associated economies. This result is an extension of those provided in [36, 46, 47]. In [48], Hervés-Beloso and Moreno-García introduced the notion of *robustly efficient allocations* and depicted that it characterized the set of Walrasian allocations in a complete information economy with a continuum of non-atomic agents. Precisely, Hervés-Beloso and Mareno-García proved that the set of Walrasian allocations in a continuum economy with finitely many commodities are those that are non-dominated in any economy obtained by a slight perturbation of the real endowments of the agents belonging to either arbitrarily small coalitions, arbitrarily large coalitions, or coalitions of a given measure less than that of the grand coalition. This is a kind of core-Walras equivalence theorem in which one does not consider the veto power of infinitely many coalitions but the veto power of a single coalition in infinitely many economies. In addition, applying this theorem to a continuum economy with n different types of agents, Hervés-Beloso and Mareno-García obtained the characterization of the Walrasian allocations showed in [46] as a particular case. In the last section of [48], they pointed out that this result can be extended to an asymmetric information economy with the space of real bounded sequences as the commodity space, and questioned whether this result can be extended to economies with other commodity spaces. This question was partially tackled in [18]. In fact, Basile and Graziano [18] considered the concept of personalized equilibria introduced in [3] and showed that it coincides with the set of robustly efficient allocations in a discrete economy with an ordered topological vector space with locally convex topology as the commodity space.

When traders enter a market with different information about the items to be traded, the resulting market prices may reveal to some traders information originally available only to others. The possibility for such inferences rests upon traders having expectations of how equilibrium prices are related to initial information. This endogenous relationship was considered by Radner in his seminal paper [74], where he introduced the concept of a rational expectations equilibrium by imposing on agents the Bayesian (subjective expected utility) decision doctrine. Under the Bayesian decision making, agents maximize their subjective expected utilities conditioned on the combination of their own private information and also on the information that the equilibrium prices generate. The resulting equilibrium allocations are measurable with respect to the combination of the private information of each individual and also with respect to the information the equilibrium prices generate and clear the market for every state of nature. In papers [4, 5] and [74], conditions on the existence of a Bayesian rational expectations equilibrium (REE) were studied and some generic existence results were proved. However, Kreps [59] provided an example that shows that a Bayesian REE may not exist universally. In addition, a Bayesian REE may fail to be fully Pareto optimal and incentive compatible and may not be implementable as a perfect Bayesian equilibrium of an extensive form game, refer to [40] for more details. It was pointed out in [54] that the market hypothesis fails if the space of states of nature is of a dimension higher than that of the price simplex. Thus, in generic existence theorems of Allen [4, 5] and Radner [74], the assumption on the space of states of nature being finite or of sufficiently low dimension relative to the dimension of price simplex is essential. However, it was shown in [53] that if the space of states of nature is of a dimension strictly higher than that of the price simplex, then for a residual set of economies there is a rational expectations equilibrium which is given by a two-to-one and almost discontinuous price function. When the dimensions of both spaces coincide, as mentioned in [6], the existence of an equilibrium fails in finite economies. If the space of agents is a unit interval consisting of imperfectly and perfectly informed agents, under the hypothesis of suitably disperse forecasts, it was shown in [6] that for each state of nature the aggregate excess demand is continuous on the price simplex and satisfies Walras's law. This fact allowed Allen to apply a fixed point theorem to obtain the market clearing price vector for each state of nature and obtain the existence of an ε-rational expectations equilibrium for all $\varepsilon > 0$. The convergence as $\varepsilon \to 0$ holds for some cases in which open counterexamples to the existence of rational expectations equilibria are known. In the same year, Allen [7] also considered two types of agents (informed and uninformed) and prices carried only incomplete information, when prices conveyed some information from informed agents to uninformed agents. By applying a fixed point theorem, she obtained a new approximate non-revealing rational expectations equilibrium in the sense that the total discrepancy between demand and supply is small. Allen [8] further showed the existence of a rational expectations equilibrium with (strong) ε-market clearing in the sense that the discrepancy between demand and supply is zero for all but one commodity for which the value can be made arbitrarily small. In a recent paper [31], de Castro et al. introduced a new notion of REE by a careful examination of Krep's example of the nonexistence of a Bayesian REE. In this formulation, the Bayesian decision making adopted in the papers of [4] and [74] was abandoned and replaced by the maximin expected utility (MEU) (see [39]). In this new setup, agents maximize their MEU conditioned on their own private information and also on the information the equilibrium prices have generated. Contrary to a Bayesian REE, the resulting maximin REE may not be measurable with respect to the private information of each individual or the information that the equilibrium prices generate. Although Bayesian REE and maximin REE coincide in some special cases (e.g., fully revealing Bayesian REE and maximin REE), these two concepts are in general not equivalent. Nonetheless, the introduction of the MEU into the general equilibrium modeling enables de Castro et al. to prove that the universal existence of a maximin REE under the standard continuity and concavity assumptions on the utility functions of agents. Furthermore, they showed that a maximin REE is incentive compatible and efficient. Note that in the economic model considered in [31], it is assumed that there are finitely many states of nature and finitely many agents, and the commodity space is finite-dimensional.

1.2 Research Questions

It is clear from the historical background that the following research questions are still unsolved. Most of these questions have been posed as open problems in the literature.

Question 1.2.1. *For an asymmetric information economy with an atomless measure space of agents and an ordered separable Banach space as the commodity space, does the private core coincide with the set of Walrasian allocations?*

Question 1.2.2. [41] *For an asymmetric information economy, do the characterizations of Walrasian allocations in* [41] *hold when the positive cone of the commodity space has no interior points?*

Question 1.2.3. [72] *Considering the feasibility without free disposal, is it possible to obtain a characterization of the private core similar to that in* [84]*?*

Question 1.2.4. [48] *For an asymmetric information economy, do the characterizations of Walrasian allocations in terms of robustly efficient allocations hold for a commodity space other than the Euclidean space and the space of real bounded sequences?*

Question 1.2.5. [32] *Does the existence theorem of a maximin rational expectations equilibrium in* [32] *still hold whenever the space of agents is an atomless measure space, and the space of states of nature is an arbitrary probability space?*

1.3 Thesis Contributions and Organization

The contributions of this thesis can be expressed in answering the questions given in Section 1.2. These answers are included in subsequent chapters, which are organized as follows.

Chapter 2: This chapter is allocated to design the mathematical preliminaries and economic models in order to study the research questions proposed previously. Since the work of Arrow and Debreu in [12] and McKenzie in [61], several mathematical concepts and techniques have been employed in general equilibrium theory to model different scenarios. Here some notions and results of Set Theory, Topology, Metric Spaces, Functional Analysis, and Measure Theory and Integration are presented, which will be used frequently in Chapters 3-6. The last section of this chapter describes some economic concepts and different economic models.

Chapter 3: The intention of this chapter is to establish a relation between the private core and the set of Walrasian allocations in a pure exchange economy with asymmetric information, finitely many states of nature and the free disposal feasibility condition. In fact, it is shown that the private core and the set of Walrasian allocations coincide whenever the space of agents is an atomless complete finite positive measure space and the commodity space is a separable Banach lattice having a quasi-interior point in its positive cone. This gives a partial solution to Question 1.2.1. If the commodity space is not separable, then the equivalence theorem fails even the positive cone of the commodity space has an interior point and the economy satisfies the standard assumptions, refer to [68, 81]. Despite of this impossibility, a positive result is obtained for an equal treatment continuum economy with a Banach lattice having an interior point in its positive cone as the commodity space. It is also depicted that a similar result holds for an asymmetric information economy whose commodity space is an arbitrary Banach lattice if one confines to find the relationship between the above two notions for equal treatment allocations only.

Chapter 4: This chapter is committed to generalize Vind's theorem in [84] to a pure exchange economy with asymmetric information, finitely many states of nature and an infinite dimensional commodity space. The major achievement is a direct extension of Vind's result to an economy whose commodity space is an ordered Banach space with an interior point in its positive cone. As a particular case of this result, a solution to Question 1.2.3 is derived. In addition to this, similar results on the (strong) fine core are also obtained. For a continuum economy with the equal treatment property, a Banach lattice as the commodity space and the free disposal feasibility condition, it is also shown that an equal treatment allocation not in the private core can be similarly characterized.

Chapter 5: The aim of this chapter is to extend some existing characterizations of Walrasian allocations to a pure exchange asymmetric information economy having finitely many states of nature and an infinite dimensional commodity space. Firstly, an extension of Theorem 2 in [83] to an asymmetric information economy is obtained. This theorem is used as a tool to establish a characterization of Walrasian allocations in terms of robustly efficient allocations in a mixed economy with asymmetric information and an ordered separable Banach space whose positive cone has an interior point as the commodity space. This result yields a solution to Question 1.2.4. This chapter concludes with highlighting some characterizations of Walrasian allocations in a discrete economy by the veto power of the grand coalition, which are just partial solutions to Question 1.2.2.

Chapter 6: In this chapter, a general model of a pure exchange asymmetric information economy is studied. The space of states of nature is a probability space, and the space of agents is a measure space with a finite measure, and the commodity space is the Euclidean space. Under appropriate and standard assumptions on agents' characteristics, results on continuity and measurability of agents' aggregate preferred correspondence in the sense of Aumann in [17] are established. With these results and the assumption that the space of states of nature is complete, it is proved that a maximin rational expectations equilibrium (maximin REE) exists in this economic model. This existence result gives a solution to Question 1.2.5.

Chapter 7: This is the last chapter of the thesis and is devoted to present the conclusions and some potential research directions.

1.4 Bibliographic Notes

PhD research findings are eminent not only for the purpose of contributing to the relevant field itself, but also to deliver the new knowledge to a wider scope of audiences. This thesis was aimed to achieve those objectives via producing publications in leading journals, as well as participating in local and international conferences. The following papers are also listed as [22, 20, 21, 23] in Bibliography and have been written during Bhowmik's PhD study.

[**A**] A. Bhowmik, J. Cao, Blocking efficiency in mixed economies with asymmetric information, J. Math. Econ. **48** (2012), 396-403.

[**B**] A. Bhowmik, J. Cao, Robust efficiency in an economy with asymmetric information, J. Math. Econ. **49** (2013), 49-57.

[**C**] A. Bhowmik, J. Cao, On the core and Walrasian expectations equilibrium in infinite dimensional commodity spaces, Econ. Theory, DOI 10.1007/s00199-012-0703-5.

[**D**] A. Bhowmik, J. Cao, N.C. Yannelis, Aggregate preferred correspondence and the existence of a Maximin REE, submitted to J. Math. Anal. Appl.

Some initial ideas were demonstrated by Bhowmik in NZMS 2010, University of Otago. In 2011, Bhowmik attended the 11^{th} SAET Conference which took place in Portugal. This was his first international conference and he presented the paper [**C**]. Besides that, a part of Chapter 3 was delivered by Bhowmik to the conference participants in the 2011 NZMS Colloquium, held at the University of Auckland. Lastly, Bhowmik has exhibited the difficulties and novelties of the results of the paper [**A**] in the 12^{th} SAET Conference, held at the University of Queensland. All of these research discoveries provided a platform to enhance the research activities currently conducted at the university.

Chapter 2

Mathematical and Economics Preliminaries

In this chapter, some mathematical and economics terminologies and preliminaries are introduced. These include basic notations, definitions and many important facts, which will be used in the subsequent chapters.

2.1 Mathematics

In this section, some mathematical concepts and results, which are needed in the study of chapters 3-6, are presented. Most of these are taken from [1]. In addition, [14, 50, 52, 82, 88] are also used.

2.1.1 Set Theory

A *set* is a collection of objects, and objects constituting a set are called *elements* of the set. Typically, the uppercase letters $X, Y, Z, ...$ are used to denote sets and those representing elements are the lowercase letters $x, y, z, ...$ The symbols $\mathbb{N}, \mathbb{Q}$ and $\mathbb{R}$ represent the sets of *positive integers*, *rational numbers* and *real numbers* respectively. In addition, $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$ denotes the set of non-negative real numbers and $\mathbb{R}^\star = \mathbb{R} \cup \{\infty, -\infty\}$ is the set of *extended real numbers*, where ∞ and $-\infty$ can be interpreted as $-\infty < x < \infty$ for any real number x. The symbol ∞ is called the *infinity*. For any $a, b \in \mathbb{R}$ with $a < b$, define

$$[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\} \text{ and } (a, b) = \{x \in \mathbb{R} : a < x < b\}.$$

Here $[a, b]$ and (a, b) are called the *closed interval* and the *open interval.* In some instance, the term *family* is used instead of set. As usual, $\emptyset$ refers to the set containing no element and is known as the *empty set.* The notation $x \in X$ indicates that x is an element of X. If x is not an element of X, the notation $x \notin X$ is employed. For any two sets X and Y, let $X \setminus Y = \{x \in X : x \notin Y\}$. The expression $X \subseteq Y$ means that $x \in X$ implies $x \in Y$. In this case, X is called a *subset* of Y. The term *subfamily* is applied in an appropriate place. If $Y \subseteq X$, then $X \setminus Y$ is termed as the *complement* of Y in X. If $X \subseteq Y$ and $Y \subseteq X$, then X and Y are said to be *identical* and written as $X = Y$. Further, if X and Y are not identical, then the notation $X \neq Y$ is used. In addition, $X \subset Y$ denotes the situation "$X \subseteq Y$ and $X \neq Y$".

The *power set* of a set X, denoted by $\mathscr{P}(X)$, is the family of all subsets of X. For any $\{A_j : j \in J\} \subseteq \mathscr{P}(X)$, define

$$\bigcup_{j \in J} A_j = \{x \in X : x \in A_j \text{ for some } j \in J\},$$

and

$$\bigcap_{j \in J} A_j = \{x \in X : x \in A_j \text{ for all } j \in J\}.$$

The notations $\bigcup_{j \in J} A_j$ and $\bigcap_{j \in J} A_j$ are sometimes written as $\bigcup\{A_j : j \in J\}$ and $\bigcap\{A_j : j \in J\}$ respectively. Here $\bigcup_{j \in J} A_j$ and $\bigcap_{j \in J} A_j$ are termed as the *union* and the *intersection* of the family $\{A_j : j \in J\}$. The notation $\prod_{j \in J} A_j$ refers to the *Cartesian product* of $\{A_j : j \in J\}$, which is defined by

$$\prod_{j \in J} A_j = \{(x_j : j \in J) : x_j \in A_j \text{ for all } j \in J\}.$$

In particular, in the case of two sets A and B, notations $A \cup B$, $A \cap B$ and $A \times B$ are utilized instead to denote the union, the intersection and the Cartesian product of A and B, respectively. Two sets A and B are *disjoint* if $A \cap B = \emptyset$, and a family $\{A_j : j \in J\}$ is called *pairwise disjoint* if A_i and A_j are disjoint for all $i, j \in J$ with $i \neq j$. The *symmetric difference* between two sets A and B is defined by $A \Delta B = (A \setminus B) \cup (B \setminus A)$. A *partition* of a non-empty set X is a family $\{A_j : j \in J\}$ of non-empty pairwise disjoint subsets of X satisfying $\bigcup_{j \in J} A_j = X$. Let $\{\mathscr{B}_i : 1 \leq i \leq \ell\}$ be a finite family of partitions of X, then

$$\left\{\bigcap_{i=1}^{\ell} B_i : B_i \in \mathscr{B}_i \text{ for all } 1 \leq i \leq \ell\right\}$$

is also a partition of X. This is called the *refinement* of $\{\mathscr{B}_i : 1 \leq i \leq \ell\}$.

Let X and Y be two sets. A *relation* between elements of X and Y is a subset of $X \times Y$. If $X = Y$, then such a relation is also termed as the *binary relation* on X. A binary relation $\succeq$ on X is said to be *reflexive* if $(x, x) \in \succeq$ for all $x \in X$, *transitive* if $(x, y) \in \succeq$ and $(y, z) \in \succeq$ imply $(x, z) \in \succeq$, and *anti-symmetric* if $(x, y) \in \succeq$ and $(y, x) \in \succeq$ together imply $x = y$. A *partial order* on X is a binary relation on X which is reflexive, transitive and anti-symmetric. If $\succeq$ is a partial order on a non-empty set X then $(X, \succeq)$ is termed as a *partially ordered set*. Further, a binary relation $\succeq$ on X is called *complete* if for any $x, y \in X$, either $(x, y) \in \succeq$ or $(y, x) \in \succeq$ or both. The notation $(x, y) \in \succeq$ is also written as $x \succeq y$. A *correspondence* F from X to Y is defined as associating to each $x \in X$ a subset $F(x)$ of Y and is denoted by $F : X \rightrightarrows Y$. The *graph* of F, denoted by Gr_F, is defined as

$$\mathrm{Gr}_F = \{(x, y) \in X \times Y : y \in F(x), x \in X\}.$$

By identifying F with its graph, one can treat F as a relation between elements of X and Y. Here $F(x)$ is called the *image of F at x*. The *domain* of F is defined by $\mathrm{Dom}(F) = \{x \in X : F(x) \neq \emptyset\}$ and F is called *non-empty valued* if $\mathrm{Dom}(F) = X$. There are two ways to define the inverse image by F of a subset U of Y:

$$F^-(U) = \{x \in X : F(x) \cap U \neq \emptyset\} \text{ and } F^+(U) = \{x \in X : F(x) \subseteq U\}.$$

Here $F^-(U)$ and $F^+(U)$ are called the *lower and upper inverses* of U by F. If Z is a set and $G : Y \rightrightarrows Z$ then the *composition correspondence* $G \circ F : X \rightrightarrows Z$ is defined as

$$(G \circ F)(x) = \bigcup \{G(y) : y \in F(x)\}.$$

If $F(x)$ is a singleton for each $x \in X$, then F is called a *function*. The lower case letters such as $f, g, h, \ldots$ are employed to denote functions. A function $f : X \to \mathbb{R}$ is called the *real-valued*. The *support* of a real-valued function $f : X \to \mathbb{R}$ is defined by $\mathrm{supp}(f) = \{x \in X : f(x) \neq 0\}$. A function $f : X \to Y$ is said to be *one-one* if $f(x) \neq f(y)$ for $x \neq y$, $x, y \in X$ and *onto* if for each $y \in Y$ there is some $x \in X$ such that $f(x) = y$. A *bijection* is a one-one and onto function.

Axiom of Choice. If $\{A_j : j \in J\}$ is a non-empty family of non-empty sets, then there is a function $f : J \to \bigcup_{j \in J} A_j$ satisfying $f(j) \in A_j$ for each $j \in J$. In other words, the Cartesian product of a non-empty family of non-empty sets is non-empty.

Suppose that $(X, \geq)$ is a partially ordered set. A *sequence* in X is just a function from $\mathbb{N}$ to X. Normally, the notation $\{x_n : n \geq 1\}$ is used to denote a sequence rather than the function $x : \mathbb{N} \to X$. Recall that $\{x_n : n \geq 1\}$ is *monotonically increasing* (resp. *monotonically decreasing*) if $x_{n+1} \geq x_n$ (resp. $x_n \geq x_{n+1}$) for all $n \geq 1$. If $\{x_n : n \geq 1\} \subseteq \mathbb{R}$ and $x_{n+1} > x_n$ (resp. $x_n > x_{n+1}$) for all $n \geq 1$, then the sequence is called *strictly increasing* (resp. *strictly decreasing*). A *subsequence* of a sequence $\{x_n : n \geq 1\}$ is a sequence $\{x_{n_k} : k \geq 1\}$, where $\{n_k : k \geq 1\}$ is strictly increasing. For any $x, y \in X$, the notation $x > y$ means $x \geq y$ and $x \neq y$. Sometimes $x \geq y$ (resp. $x > y$) is written as $y \leq x$ (resp. $y < x$). An element y is called an *upper bound* (resp. a *lower bound*) of a subset A of X if $y \geq x$ (resp. $x \geq y$) for all $x \in A$. The *supremum* of a subset A of X is a unique upper bound y such that $z \geq y$ for any upper bound z of A. Likewise, the *infimum* of a subset A of X is a unique lower bound y such that $y \geq z$ for any lower bound z of A. The supremum and the infimum of a subset A may not exist, and if they exist, then $\sup A$ and $\inf A$ are employed to denote the supremum and the infimum of A respectively. In particular, for a set $\{x, y\}$ containing only two points, special notations $x \vee y$ and $x \wedge y$ are employed to denote the supremum and the infimum respectively. Recall that a *lattice* is a partially ordered set in which every pair of elements $\{x, y\}$ has $x \vee y$ and $x \wedge y$.

Next, the notion of size is defined, and is called the *cardinality.* A set A has the *same cardinality* as B if there is a bijection between A and B. The cardinality of a set A is denoted by $|A|$. Further, B has *cardinality at least as large* as A if there is a bijection from A onto a subset of B. Sets of the same cardinality as $\{1, ..., n\}$ for any $n \in \mathbb{N}$ are *finite*, those have the same cardinality as $\mathbb{N}$ are known as *countably infinite.* Sets that are finite or countably infinite are called *countable* and sets that are not countable are called *uncountable.* $\mathbb{N}$ and $\mathbb{Q}$ have the same cardinality. The symbols $\aleph_0$ and $\mathfrak{c}$ are used to denote the cardinality of $\mathbb{N}$ and $\mathbb{R}$ respectively. The *Continuum Hypothesis* claims that $\mathfrak{c}$ is the smallest uncountable cardinal number.

2.1.2 Topology

A *topology* $\mathscr{T}$ on a set X is a subset of $\mathscr{P}(X)$ such that

(i) $\emptyset, X \in \mathscr{T}$;

(ii) $\mathscr{T}$ is closed under finite intersection, that is, if $\{G_j : 1 \leq j \leq \ell,\ \ell \in \mathbb{N}\} \subseteq \mathscr{T}$ then $\bigcap_{j=1}^{\ell} G_j \in \mathscr{T}$; and

(iii) $\mathscr{T}$ is closed under arbitrary union, that is, if $\{G_j : j \in J\} \subseteq \mathscr{T}$ then $\bigcup_{j \in J} G_j \in \mathscr{T}$.

A non-empty set X equipped with a topology $\mathscr{T}$ is called a *topological space*, and it is denoted by $(X, \mathscr{T})$. Sets in $\mathscr{T}$ are called *$\mathscr{T}$-open sets* or simply *open sets* in $(X, \mathscr{T})$. The complement of a $\mathscr{T}$-open set is called a *$\mathscr{T}$-closed set* or simply a *closed set*. Two trivial topologies on a set X are the *indiscrete topology*, which consists of only $\emptyset$ and X, and the *discrete topology*, which is $\mathscr{P}(X)$. Given two topologies $\mathscr{T}, \mathscr{T}'$ on a set X, $\mathscr{T}$ is said to be *stronger* or *finer* than $\mathscr{T}'$ if $\mathscr{T}' \subseteq \mathscr{T}$. In this case, $\mathscr{T}'$ is called *weaker* or *coarser* than $\mathscr{T}$. A subfamily $\mathscr{B}$ of $\mathscr{T}$ is called a *base* for a topological space $(X, \mathscr{T})$ if each element of $\mathscr{T}$ can be represented as the union of elements of $\mathscr{B}$. Conversely, if $\mathscr{B}$ is a family of sets that is closed under finite intersection and $X = \bigcup \mathscr{B}$, then the family of all unions of sets in $\mathscr{B}$ forms a topology in which $\mathscr{B}$ is a base. Such a topology is known as the *topology generated* by $\mathscr{B}$. If Y is a subset of $(X, \mathscr{T})$, then the family $\mathscr{T}_Y = \{V \cap Y : V \in \mathscr{T}\}$ forms a topology on Y. This topology is known as the *relative topology* or the *topology induced by* $\mathscr{T}$ on Y. Here $(Y, \mathscr{T}_Y)$ is called a *topological subspace* of $(X, \mathscr{T})$, and any set in $\mathscr{T}_Y$ is called (*relatively*) *open* in Y.

Let $(X, \mathscr{T})$ be a topological space and $E \subseteq Y \subseteq X$. The *interior* of E in Y, denoted by $\mathscr{T}_Y$-int$E$, is the largest (with respect to "$\subseteq$") open set in $(Y, \mathscr{T}_Y)$ contained in E. The *closure* of E in $(Y, \mathscr{T}_Y)$, denoted by $\mathscr{T}_Y$-cl$E$, is the smallest closed set in $(Y, \mathscr{T}_Y)$ containing E. When $Y = X$, without any confusion, the notations $\mathscr{T}$-int$E$ or int$E$, and $\mathscr{T}$-cl$E$ or cl$E$ are used instead. A *neighborhood* of a point x in a topological space $(X, \mathscr{T})$ is any set U containing x in its interior. In such a situation, x is called an *interior point* of U. A topological space is called *Hausdorff* if any two distinct points have disjoint neighborhoods. Note that a point $x \in \text{cl}E$ is equivalent to the fact that $U \cap E \neq \emptyset$ for every neighborhood U of x. A point x is said to be a *limit point* of a set E if for any neighborhood U of x, $(U \setminus \{x\}) \cap E \neq \emptyset$. A subset E of a topological space $(X, \mathscr{T})$ is *dense* in X if cl$E = X$, and $(X, \mathscr{T})$ is called *separable* if there is a countable dense subset of X. Let $A, B, C \subseteq X$. Then $\mathscr{T}$-int$A \cap C \subseteq \mathscr{T}_C$-int$(A \cap C)$. It is easy to show that if $B \in \mathscr{T}$, $\mathscr{T}$-cl$A \subseteq \mathscr{T}$-cl$B$ and C is dense in X, then $\mathscr{T}_C$-cl$(A \cap C) \subseteq \mathscr{T}_C$-cl$(B \cap C)$.

A *directed set* is a non-empty set D with a reflexive and transitive binary relation $\succeq$ in which for each pair of elements $\alpha, \beta \in D$ there is some element $\gamma \in D$ such that $\gamma \succeq \alpha$ and $\gamma \succeq \beta$. A *net* in a topological space $(X, \mathscr{T})$ is a function $x : D \to X$, and it is denoted by $\{x_\alpha : \alpha \in D\}$. Note that every sequence in a topological space is also a net where $D = \mathbb{N}$ with the usual ordering. A net $\{x_\alpha : \alpha \in D\}$ in X converges to a point x if for each neighborhood U of x there is some $\alpha_0 \in D$ such that $x_\alpha \in U$ for all $\alpha \succeq \alpha_0$. A function $f : X \to Y$ between two topological spaces X and Y is called *continuous* if $f^{-1}(U)$ is open in X for every open set U in Y. Recall that a subset E of a topological space is said to be *compact* if every family $\{U_j : j \in J\}$ of open sets satisfying $E \subseteq \bigcup_{j \in J} U_j$ has a finite subfamily $\{U_{j_1}, ..., U_{j_m}\}$ such that $E \subseteq \bigcup_{i=1}^{m} U_{j_i}$. Let $\{(X_j, \mathscr{T}_j) : j \in J\}$ be a family of topological spaces, and assume $\hat{X} = \prod_{j \in J} X_j$. The family of sets of the type $U = \prod_{j \in J} U_j$ forms a base for some topology on $\hat{X}$, where $U_j \in \mathscr{T}_j$ and $U_j = X_j$ for all but finitely many j. This topology is called the *Tychonoff product topology.*

Tychonoff Product Theorem. $\hat{X}$ is compact if and only if X_j is compact for all $j \in J$.

2.1.3 Metric Spaces

In mathematics, a metric space is a non-empty set where a concept of *distance* or *metric* between any two elements is defined. Formally, a *metric* on a non-empty set X is a function $\varrho : X \times X \to \mathbb{R}_+$ satisfying:

(i) $\varrho(x, x) = 0$ for all $x \in X$;

(ii) $\varrho(x, y) = 0$ implies $x = y$;

(iii) $\varrho(x, y) = \varrho(y, x)$ for all $x, y \in X$; and

(iv) $\varrho(x, y) \leq \varrho(x, z) + \varrho(z, y)$ for all $x, y, z \in X$.

If ϱ is a metric on a non-empty set X, then the pair (X, ϱ) is called a *metric space.* A *pseudometric* on X is a function $\varrho : X \times X \to \mathbb{R}_+$ such that (**i**), (**iii**) and (**iv**) hold, and in this case, (X, ϱ) is said to be a *pseudometric space.* Note that

$$\mathbb{R}^\ell = \{(x^1, ..., x^\ell) : x^i \in \mathbb{R} \text{ for all } 1 \leq i \leq \ell\}$$

with

$$\varrho(x, y) = \sqrt{\sum_{i=1}^{\ell} (x^i - y^i)^2}$$

is a metric space. In particular, when $\ell = 1$ then $\varrho(x, y)$ gives the *absolute difference* between x and y, and it is usually denoted by a special natation $|x - y|$. For a non-empty subset E of a metric space (X, ϱ), the *diameter* of E is defined by

$$\operatorname{diam} E = \sup\{\varrho(x, y) : x, y \in E\}.$$

A set E is *bounded* if $\text{diam}E < \infty$, and is *unbounded* if $\text{diam}E = \infty$. The *open ball* of radius $\varepsilon > 0$ centered at a point $x \in X$ in a pseudometric space (X, ϱ) is

$$B_\varepsilon(x) = \{y \in X : \varrho(x, y) < \varepsilon\}.$$

In a metric space (X, ϱ), the topology generated by

$$\{B_\varepsilon(x) : x \in X, \varepsilon > 0\} \cup \{\emptyset\}$$

is called the *metric topology* or *the topology induced by* ϱ on (X, ϱ). An interesting property of metric spaces claims that every topological subspace of a separable metric space is separable.

Heine-Borel Theorem. For each $\ell \geq 1$, a subset of $\mathbb{R}^\ell$ is compact if and only if it is closed and bounded.

The $(\ell - 1)$-simplex of $\mathbb{R}^\ell$ is defined as

$$\Im^\ell = \left\{ x = (x^1, ..., x^\ell) \in \mathbb{R}^\ell : x^i \geq 0 \text{ for all } 1 \leq i \leq \ell \text{ and } \sum_{i=1}^{\ell} x^i = 1 \right\}.$$

Brouwer Fixed Point Theorem. For any continuous function $f : \Im^\ell \to \Im^\ell$ there is a point $x \in \Im^\ell$ such that $f(x) = x$.

Throughout the rest of this subsection, suppose (X, ϱ) is a metric space. Recall that a sequence $\{x_n : n \geq 1\}$ converges to x in (X, ϱ) if and only if $\{\varrho(x_n, x) : n \geq 1\}$ converges to 0. The notation $\lim_{n\to\infty} x_n = x$ is used to represent that $\{x_n : n \geq 1\}$ converges to x in (X, ϱ). A *Cauchy sequence* in (X, ϱ) is a sequence $\{x_n : n \geq 1\}$ such that for each $\varepsilon > 0$ there is some $N \geq 1$ such that $\varrho(x_n, x_m) < \varepsilon$ for all $n, m \geq N$. In addition, (X, ϱ) is *complete* if every Cauchy sequence in X converges in X. Every compact metric space is complete. A very important fact of metric spaces is that a metric space is compact if and only if every sequence has a convergent subsequence. If (Y, d) is a metric space, then the following conditions are equivalent:

(i) A function $f : (X, \varrho) \to (Y, d)$ is continuous.

(ii) If $\{x_n : n \geq 1\}$ converges to x, then $\{f(x_n) : n \geq 1\}$ converges to $f(x)$.

Let (Y, d) be a metric space. A sequence $\{f_n : n \geq 1\} : X \to Y$ converges *pointwise* to a function $f : X \to Y$ if $\{f_n(x) : n \geq 1\}$ converges to $f(x)$ for all $x \in X$. In this case,

f is named as the *pointwise limit* of $\{f_n : n \geq 1\}$. A sequence $\{f_n : n \geq 1\} : X \to Y$ converges *uniformly* to a function $f : X \to Y$ if for each $\varepsilon > 0$ there is some $N \geq 1$ such that $d(f_n(x), f(x)) < \varepsilon$ for all $x \in X$ and $n \geq N$. Let $\mathscr{K}_0(Y)$ denote the family of non-empty compact subsets of Y. For any $A, B \in \mathscr{K}_0(\mathbb{R}^\ell)$, define

$$H(A,B) = \sup\left\{\sup_{a\in A} \operatorname{dist}(a,B),\ \sup_{b\in B} \operatorname{dist}(b,A)\right\},$$

where

$$\operatorname{dist}(a,B) = \inf_{b\in B} \varrho(a,b).$$

It can be readily checked that $H : \mathscr{K}_0(\mathbb{R}^\ell) \times \mathscr{K}_0(\mathbb{R}^\ell) \to \mathbb{R}_+$ is a metric on $\mathscr{K}_0(\mathbb{R}^\ell)$, called the *Hausdorff metric*. For any $A \subseteq \mathbb{R}^\ell$ and $\varepsilon > 0$, let

$$N_\varepsilon(A) = \left\{x \in \mathbb{R}^\ell : \operatorname{dist}(x,A) < \varepsilon\right\}.$$

The metric H can also be expressed as

$$H(A,B) = \sup\left\{|\operatorname{dist}(x,A) - \operatorname{dist}(x,B)| : x \in \mathbb{R}^\ell\right\},$$

or

$$H(A,B) = \inf\{\varepsilon > 0 : A \subseteq N_\varepsilon(B) \text{ and } B \subseteq N_\varepsilon(A)\}$$

for $A, B \in \mathscr{K}_0(\mathbb{R}^\ell)$. The topology $\mathscr{T}_H$ on $\mathscr{K}_0(\mathbb{R}^\ell)$, induced by H, is called the *Hausdorff metric topology*. If $F : X \to \big(\mathscr{K}_0(\mathbb{R}^\ell), \mathscr{T}_H\big)$ is continuous, then $F : X \rightrightarrows \mathbb{R}^\ell$ is called *Hausdorff continuous*. Recall that if F is Hausdorff continuous, then $F^-(C)$ is closed for every closed subset C of $\mathbb{R}^\ell$. Let $\{A_n : n \geq 1\} \subseteq \mathscr{P}(\mathbb{R}^\ell) \setminus \{\emptyset\}$. A point $x \in \mathbb{R}^\ell$ is called a *limit point* of $\{A_n : n \geq 1\}$ if there exist $N \geq 1$ and points $x_n \in A_n$ for each $n \geq N$ such that $\{x_n : n \geq N\}$ converges to x. The set of limit points of $\{A_n : n \geq 1\}$ is denoted by $\mathrm{Li}A_n$. Similarly, a point $x \in \mathbb{R}^\ell$ is called a *cluster point* of $\{A_n : n \geq 1\}$ if there exist positive integers $n_1 < n_2 < \cdots$ and for each k a point $x_k \in A_{n_k}$ such that $\{x_k : k \geq 1\}$ converges to x. The set of cluster points of $\{A_n : n \geq 1\}$ is denoted by $\mathrm{Ls}A_n$. It is clear that $\mathrm{Li}A_n \subseteq \mathrm{Ls}A_n$, and both $\mathrm{Ls}A_n$ and $\mathrm{Li}A_n$ are closed (possibly empty) sets. If $\mathrm{Ls}A_n \subseteq \mathrm{Li}A_n$ then $\mathrm{Li}A_n = \mathrm{Ls}A_n = A$ is called the *limit* of the sequence $\{A_n : n \geq 1\}$. Note that $\mathrm{Ls}A_n = \mathrm{Ls}\ \mathrm{cl}A_n$ and $\mathrm{Li}A_n = \mathrm{Li}\ \mathrm{cl}A_n$. If A and all A_n's are closed and contained in a compact subset M of $\mathbb{R}^\ell$, then it is well known that $\mathrm{Li}A_n = \mathrm{Ls}A_n = A$ if and only if $\{A_n : n \geq 1\}$ converges to A in the Hausdorff metric topology on $\mathscr{K}_0(M)$.

2.1.4 Functional Analysis

A *vector space* or a *linear space* $(X, +, \cdot)$ (over $\mathbb{R}$) is a non-empty set X with two algebraic operations " $+$ " and " $\cdot$ " such that for all $x, y, z \in X$ and $\alpha, \beta \in \mathbb{R}$ the following properties hold:

(i) $x + y \in X$;

(ii) $(x + y) + z = x + (y + z)$;

(iii) $x + 0 = x$, where 0 is the zero element of X;

(iv) $x + (-1)x = 0$, where 1 is the *unity* of $\mathbb{R}$;

(v) $1x = x$;

(vi) $\alpha(\beta x) = (\alpha\beta)x$;

(vii) $\alpha(x + y) = \alpha x + \alpha y$; and

(viii) $(\alpha + \beta)x = \alpha x + \beta x$.

For instance, $(\mathbb{R}^\ell, +, \cdot)$ is a vector space, where $+$ and $\cdot$ are defined as

$$(x^1, ..., x^\ell) + (y^1, ..., y^\ell) = (x^1 + y^1, ..., x^\ell + y^\ell)$$

and

$$\alpha(x^1, ..., x^\ell) = (\alpha x^1, ..., \alpha x^\ell).$$

Such operations are known as the *pointwise addition* and the *pointwise scalar multiplication.* A finite set $\{x_1, ..., x_m\}$ of vectors in X is called *linearly dependent* if there exists a set $\{a_1, ..., a_m\}$ of scalars, not all zero, such that $\sum_{i=1}^m a_i x_i = 0$. If the set $\{x_1, ..., x_m\}$ is not linearly dependent then it is called *linearly independent.* Recall that a *Hamel base* or simply a *base* of X is a set B such that every finite subset of B is linearly independent and each non-zero $x \in X$ has a unique representation $x = \sum_{i=1}^m b_i x_i$, where $\{x_1, ..., x_m\} \subseteq B$ and $\{b_1, ..., b_m\}$ is a set of non-zero scalars. It is known that any two Hamel bases on a non-empty set have the same cardinality. The *dimension* of X is the cardinality of any of its Hamel base.

A subset of a vector space is called a *vector subspace* or a *linear subspace* whenever it is a vector space in its own right under the induced operations. If A, B are two vector subspaces of X and α, β are real numbers, then $\alpha A + \beta B = \{\alpha x + \beta y : x \in A, y \in B\}$ is also a vector subspace of X. A subset S of a vector space is said to be *balanced* or *circled* if $x \in S$ and $0 \leq \alpha \leq 1$ imply $\alpha x \in S$. A function $f : X \to Y$ between two vector spaces X and Y is *linear* if

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$$

for all $x, y \in X$ and $\alpha, \beta \in \mathbb{R}$. If $Y = \mathbb{R}$, then f is said to be a *linear functional.* A *linear topology* on a vector space is a topology that makes $(x, y) \mapsto x + y$ and $(\alpha, x) \to \alpha x$ continuous functions. A *topological vector space* is a pair $(X, \mathscr{T})$, where X is a vector space and $\mathscr{T}$ is a linear topology on X. Recall that the *topological dual* of a topological vector space $(X, \mathscr{T})$, denoted by $(X, \mathscr{T})^*$ or simply by X^*, is a vector space consisting of all $\mathscr{T}$-continuous linear functionals on X. A subset E of $(X, \mathscr{T})$ is *convex* if for any $x, y \in E$ and $0 \leq \lambda \leq 1$, one has $\lambda x + (1-\lambda)y \in E$. Remember that if $x \in \text{int}E$, $y \in \text{cl}E$ and E is convex then $\lambda x + (1 - \lambda)y \in \text{int}E$ for all $0 < \lambda \leq 1$. A topological vector space is *locally convex* if every neighborhood of zero contains a convex neighborhood of zero. If $(X, \mathscr{T})$ is a locally convex topological vector space then $\mathscr{T}$ is termed as a *locally convex topology.* If E is a convex subset of X then a function $f : E \to \mathbb{R}$ is *concave* whenever

$$f(\lambda x + (1 - \lambda)y) \geq \lambda f(x) + (1 - \lambda)f(y)$$

for all $x, y \in E$ and $0 \leq \lambda \leq 1$. If the previous inequality holds with the sign "$>$", then such a function is known as *strictly concave.* A *cone* is a set that contains every non-negative multiples of each of its elements. For any open neighborhood U of 0 in X and $v \in X$, the *open cone spanned by* $v + U$ is $\{\alpha(v + U) : \alpha > 0\}$.

If X is a vector space and $\geq$ is a partial order on X, then the pair $(X, \geq)$ is called an *ordered vector space* whenever for any $x, y, z \in X$ and any positive real number α, $x \geq y$ implies that $\alpha x + z \geq \alpha y + z$. Recall that a *Riesz space* or a *vector lattice* is an ordered vector space that is also a lattice. A Riesz space $(X, \geq)$ is *order complete* or *Dedekind complete* if every non-empty subset of X that is upper (resp. lower) bounded has the supremum (resp. the infimum). For any element x of a Riesz space, $|x|$ stands for the *absolute value* of x and is defined by $|x| = x^+ + x^-$, where

$$x^+ = x \vee 0 \text{ and } x^- = (-x) \vee 0$$

are *positive part* and *negative part* of x respectively. Note that $x = x^+ - x^-$ and $x^+ \wedge x^- = 0$. An element $x \in X$ is called a *positive element* of X if $x \geq 0$ and $X_+ = \{x \in X : x \geq 0\}$. If $x \in X_+ \setminus \{0\}$, then the notation $x > 0$ is used. For any set S and any Riesz space X, X^S denotes the set of functions from S into X. For $f, g \in X^S$,

$\alpha \in \mathbb{R}$ and $A \subseteq S$, define $f+g : S \to X$ by $(f+g)(s) = f(s) + g(s)$ for all $s \in S$, $\alpha f : S \to X$ by $(\alpha f)(s) = \alpha f(s)$ for all $s \in S$ and $f|_A : A \to X$ by $f|_A(s) = f(s)$ for all $s \in A$. A function $f : S \to \mathbb{R}$ is *bounded* if $\sup\{|f(s)| : s \in S\} < \infty$. The notation $f \leq g$ (resp. $f = g$) means $f(s) \leq g(s)$ (resp. $f(s) = g(s)$) for all $s \in S$. The supremum and the infimum of f, g are defined as

$$(f \vee g)(s) = \sup\{f(s), g(s)\} \text{ and } (f \wedge g)(s) = \inf\{f(s), g(s)\}.$$

Riesz Decomposition Property. If X is a Riesz space and $0 \leq y \leq \sum_{i=1}^n x_i$, then there exist $y_1, ..., y_n \in X_+$ such that $y = \sum_{i=1}^n y_i$ and $y_i \leq x_i$ for all $1 \leq i \leq n$.

A *dual system* $\langle X, X' \rangle$ is a pair of vector spaces X and X' together with a function $(x, x') \mapsto \langle x, x' \rangle$, from $X \times X'$ to $\mathbb{R}$, satisfying the following properties:

(i) $x' \mapsto \langle x, x' \rangle$ is linear for each $x \in X$;

(ii) $x \to \langle x, x' \rangle$ is linear for each $x' \in X'$;

(iii) $\langle x, x' \rangle = 0$ for all $x' \in X'$ implies $x = 0$; and

(iv) $\langle x, x' \rangle = 0$ for all $x \in X$ implies $x' = 0$.

A locally convex topology $\mathscr{T}$ on X is said to be *compatible* with the dual system $\langle X, X' \rangle$ if $(X, \mathscr{T})^* = X'$ holds. This means that if $f \in (X, \mathscr{T})^*$ then there is a $x' \in X'$ such that $f(x) = \langle x, x' \rangle$ and vise-versa. Thus, the notation $\langle x, f \rangle$ is also used for any $f \in (X, \mathscr{T})^*$ instead of $f(x)$. The *weak topology* $\sigma(X, X')$ is a locally convex topology on X of pointwise convergence on X'. That is, $\{x_\alpha : \alpha \in D\}$ converges to x in $\sigma(X, X')$ if and only if $\{\langle x_\alpha, x' \rangle : \alpha \in D\}$ converges to $\langle x, x' \rangle$ for each $x' \in X'$. Analogously, the locally convex topology $\sigma(X', X)$ on X', known as the *weak* topology*, such that $\{x'_\alpha : \alpha \in D\}$ converges to x' in $\sigma(X', X)$ if and only if $\langle x, x'_\alpha \rangle$ converges to $\langle x, x' \rangle$ for all $x \in X$. The *Mackey topology* $\mathscr{T}(X, X')$ is a locally convex topology on X of uniform convergence on $\sigma(X', X)$-compact, convex and balanced subsets of X'. This means that $\{x_\alpha : \alpha \in D\}$ converges to x in $\mathscr{T}(X, X')$ if and only if $\left\{y_\alpha^A : \alpha \in D\right\}$ converges to 0 for all $\sigma(X', X)$-compact, convex and balanced subset A of X', where

$$y_\alpha^A = \sup\left\{|\langle x_\alpha - x, x' \rangle| : x' \in A\right\}.$$

A subset A of a Riesz space is called *solid* if $|x| \leq |y|$ and $y \in A$ imply $x \in A$. Recall that a linear topology on a Riesz space is said to be *locally solid* if it has a base at zero consisting of solid neighborhoods. A solid vector subspace of a Riesz space is called an *ideal*. Let X be a Riesz space. For any $x \in X$, the *principal ideal* generated by x is defined by

$$L(x) = \{y \in X : |y| \leq n|x| \text{ for some } n \in \mathbb{N}\}.$$

If $x \leq y$ in X, then an *order interval* is of the form $[x, y] = \{z \in X : x \leq z \leq y\}$. A subset A of X is called *order bounded* if $A \subseteq [x, y]$ for some $x, y \in X$. A linear functional $f : X \to \mathbb{R}$ on a Riesz space X is called *order bounded* if $f(A)$ is bounded in $\mathbb{R}$ for any order bounded subset A of X. Recall that an *order dual* $\widetilde{X}$ of a Riesz space X is an ordered vector space consisting of all order bounded linear functionals on X under the usual algebraic operations and the order $f \geq g$ such that $\langle x, f\rangle \geq \langle x, g\rangle$ for all $x \in X_+$. The order dual $\widetilde{X}$ of any Riesz space X is an order complete Riesz space, and its lattice operations are given by

$$\langle x, f \vee g\rangle = \sup\{\langle y, f\rangle + \langle z, g\rangle : y, z \in X_+ \text{ and } y + z = x\}$$

and

$$\langle x, f \wedge g\rangle = \inf\{\langle y, f\rangle + \langle z, g\rangle : y, z \in X_+ \text{ and } y + z = x\}$$

for all $f, g \in \widetilde{X}$, and $x \in X_+$. These two equalities are called the *Riesz-Kantorovich formulas*.

Lemma 2.1.1. [67] *Let Y be a vector space endowed with a Hausdorff, locally convex topology and let U, V be convex subsets of Y such that U is open and $U \cap V \neq \emptyset$. Let $y \in V \cap \mathrm{cl}U$. Suppose that f is a linear functional (not necessarily continuous) on Y with $\langle y, f\rangle \leq \langle y', f\rangle$ for all $y' \in U \cap V$. Then, there exist linear functionals f_1 and f_2 on Y such that f_1 is continuous, $\langle y, f_1\rangle \leq \langle u, f_1\rangle$ for all $u \in U$, $\langle y, f_2\rangle \leq \langle v, f_2\rangle$ for all $v \in V$ and $f = f_1 + f_2$.*

Lemma 2.1.2. [67] *Let Y be a Riesz space endowed with a Hausdorff, locally convex topology. If $L(z)$ is dense in Y, then $L(z)_+$ is dense in Y_+.*

Given a vector space X, a function $\|\cdot\| : X \to \mathbb{R}_+$ satisfying

(i) $\|x\| = 0$ if and only if $x = 0$;

(ii) $\|\alpha x\| = |\alpha|\|x\|$ for all $x \in X$ and $\alpha \in \mathbb{R}$; and

(iii) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$, is called a *norm* on X, and $(X, \|\cdot\|)$ is called a *normed space*. It can be readily checked that a normed space $(X, \|\cdot\|)$ is also a metric space with the metric $\varrho : X \times X \to \mathbb{R}_+$ defined by $\varrho(x, y) = \|x - y\|$, and the topology induced by this metric is called a *norm topology*. A *Banach space* is a normed space whose norm induces a complete metric. If $(X, \|\cdot\|)$ is a normed space, its *norm dual* X^* is a Banach space equipped with the norm $\|\cdot\|$ defined by

$$\|f\| = \sup\{|\langle x, f\rangle| : x \in X, \|x\| \leq 1\}.$$

A point $x \in X_+$ is called *strictly positive* element of a Banach space X, denoted by $x \gg 0$, if $\langle x, f\rangle > 0$ for all $f \in X_+^* \setminus \{0\}$. Let $X_{++} = \{x \in X_+ : x \gg 0\}$.

A norm is called a *lattice norm* if $|x| < |y|$ implies $\|x\| \leq \|y\|$. A *normed Riesz space* is a Riesz space with a lattice norm. A complete normed Riesz space is called a *Banach lattice*. For a Banach lattice X, a point $x \in X_+$ is strictly positive if and only if $L(x)$ is dense in X. In this case, x is also called a *quasi-interior point* of X_+. In particular, if $L(x) = X$, then $x \in X_+$ is called an *order unit* of X. An order unit is a quasi-interior point, but in general, the converse is not true. If $\text{int}X_+ \neq \emptyset$, then $\text{int}X_+ = X_{++}$. A lattice norm on a Riesz space is termed as an *M-norm* if

$$\|x \vee y\| = \max\{\|x\|, \|y\|\}$$

for any $x, y \geq 0$. An *M-space* is a normed Riesz space with an M-norm. A norm complete M-space is called an *AM-space*. Note that if $(X, \|\cdot\|)$ is a Banach lattice, then $L(x)$ with the norm

$$\|y\|_x = \inf\{\lambda > 0 : |y| \leq \lambda|x|\}$$

is an *AM-space* with x as an order unit.

Let X be a Banach space and S a finite set. For any $x, y \in X^S$, the order $x \leq y$ if and only if $x(s) \leq y(s)$ for all $s \in S$ is called the *pointwise order* on X^S. Analogously, the *product norm* on X^S is defined by

$$\|(x(s) : s \in S)\| = \sum_{s \in S} \|x(s)\|.$$

Note that X^S with the pointwise algebraic operations, the pointwise order and the product norm is a Banach space. If $x \in (X, \|\cdot\|)^S$ (that is, X^S is equipped with the $\|\cdot\|^S$-topology) and $g \in ((X, \|\cdot\|)^S)^*$, then there is an element $f \in ((X, \|\cdot\|)^*)^S$ such that

$$\langle x, g\rangle = \sum_{s\in S}\langle x(s), f(s)\rangle$$

and vise-versa. If $(X, \|\cdot\|)$ is a Banach lattice, then $X^* = \widetilde{X}$ and $\|\cdot\|$-topology is the finest locally solid topology on X. The following two theorems are crucial.

Hahn-Banach Theorem. Let X be a normed space and A a subspace of X. If $f \in A^*$ then there is some $g \in X^*$ such that $g|_A = f$ and $\|f\| = \|g\|$.

Separation Theorem. For any two non-empty disjoint convex subsets A and B of a Banach space X, if either A or B has non-empty interior, then there exists a non-zero $f \in X^*$ that separates A and B, that is, $\langle x, f\rangle \leq \langle y, f\rangle$ for all $x \in A$ and $y \in B$.

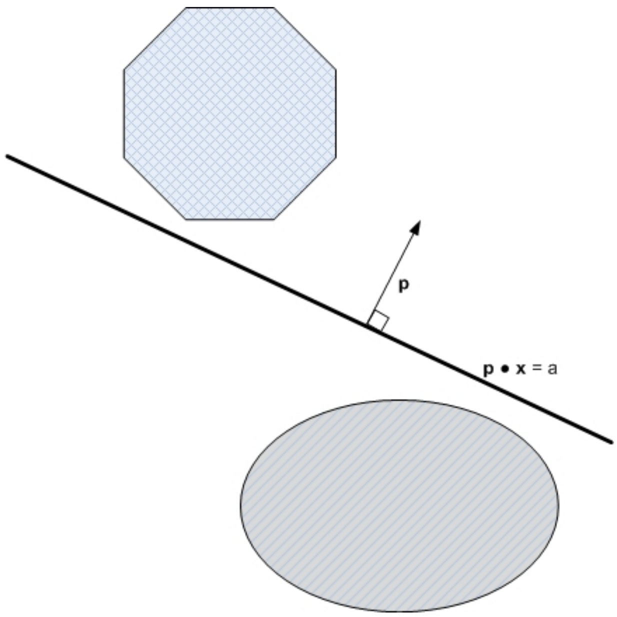

Figure 2.1: Separation of Two Convex Sets

This subsection concludes with some basic examples of ordered Banach spaces and a table containing some properties of these spaces. However, some of these spaces require additional concepts of measure theory, which are presented in the next subsection.

(i) $\mathbb{R}^\ell$: the ℓ-dimensional Euclidean space;

(ii) ℓ_∞: the space of real bounded sequences with the supremum norm

$$\|\{x_n : n \geq 1\}\|_\infty = \sup\{|x_n| : n \geq 1\};$$

(iii) $L_\infty(S, \mathscr{S}, \mu)$: the space of essentially bounded, measurable functions on a measure space $(S, \mathscr{S}, \mu)$ with the essential supremum norm

$$\|f\|_\infty = \inf\{M > 0 : |f(s)| \leq M \ \mu\text{-almost everywhere}\},$$

where for convention, $\inf \emptyset = \infty$ is assumed;

(iv) $C[a, b]$: the space of real-valued continuous functions on a closed interval $[a, b]$ with the supremum norm

$$\|f\|_\infty = \sup\{|f(x)| : x \in [a, b]\};$$

(v) c: the space of convergent sequences endowed with the norm defined in **(ii)**.

(vi) $C(K)$: the space of real-valued continuous functions on a compact Hausdorff space K with the supremum norm similar to that in **(iv)**;

(vii) ℓ_p: the space of real sequences $\{x_n : n \geq 1\}$ equipped with the norm

$$\|\{x_n : n \geq 1\}\|_p = \left(\sum_{n \geq 1} |x_n|^p\right)^{\frac{1}{p}},$$

where $1 \leq p < \infty$;

(viii) $L_p(S, \mathscr{S}, \mu)$: the space of measurable functions f on a measure space $(S, \mathscr{S}, \mu)$ equipped with the norm

$$\|f\|_p = \left(\int_S |f|^p d\mu\right)^{\frac{1}{p}},$$

where $1 \leq p < \infty$.

Y	$\text{int}Y_+ \neq \emptyset$	$Y_{++} \neq \emptyset$	Y^*
$\mathbb{R}^\ell$	+	+	$\mathbb{R}^\ell$
ℓ_∞	+	+	$ba(2^{\mathbb{N}})$
$L_\infty(S, \mathscr{S}, \mu)$	+	+	$ba(\mathscr{S})$
$C(K)$ (K: a compact Hausdorff space)	+	+	$ca_r(\mathfrak{B}(K))$
ℓ_1	-	+	ℓ_∞
ℓ_p $(1 < p < \infty)$	-	+	ℓ_q $\frac{1}{p} + \frac{1}{q} = 1$
$L_1(S, \mathscr{S}, \mu)$	-	+	$L_\infty(S, \mathscr{S}, \mu)$
$L_p(S, \mathscr{S}, \mu)$ $(1 < p < \infty)$	-	+	$L_q(S, \mathscr{S}, \mu)$ $\frac{1}{p} + \frac{1}{q} = 1$

Table 2.1: Some Banach Spaces

2.1.5 Measure Theory and Integration

A *σ-algebra* is a family $\mathscr{S}$ of subsets of a fixed set S satisfying the following properties:

(i) $S \in \mathscr{S}$;

(ii) if $A \in \mathscr{S}$ then $S \setminus A \in \mathscr{S}$; and

(iii) if $\{A_i : i \geq 1\} \subseteq \mathscr{S}$ then $\bigcup_{i \geq 1} A_i \in \mathscr{S}$.

The pair $(S, \mathscr{S})$ is called a *measurable space*, and an element of $\mathscr{S}$ is called a *measurable set*. It is clear that the intersection of any non-empty family of σ-algebras is a σ-algebra, and any non-empty family $\mathscr{C}$ of subsets of S is contained in the σ-algebra $\mathscr{P}(S)$. So the intersection of all σ-algebras containing $\mathscr{C}$ is the smallest σ-algebra containing $\mathscr{C}$ and is known as the *σ-algebra generated by $\mathscr{C}$*. Usually, the notation $\sigma(\mathscr{C})$ is used to denote the σ-algebra generated by $\mathscr{C}$. A *separable σ-algebra* is a σ-algebra that can be generated by a countable family of sets. If $(X, \mathscr{T})$ is a topological space, then $\sigma(\mathscr{T})$ is called *Borel σ-algebra* and an element of $\sigma(\mathscr{T})$ is termed as a *Borel measurable set* or simply a *Borel set*. A special symbol $\mathscr{B}(X)$ represents the Borel σ-algebra of X. Recall that a function $\mu : \mathscr{S} \to \mathbb{R}^\star$ is called a *signed charge* whenever it assumes at most one of the values of $-\infty$ and ∞, $\mu(\emptyset) = 0$, and for any finite family $\{A_i : 1 \leq i \leq \ell\}$ of pairwise disjoint sets in $\mathscr{S}$, the equality

$$\mu\left(\bigcup_{i=1}^{\ell} A_i\right) = \sum_{i=1}^{\ell} \mu(A_i)$$

holds. A *signed measure* $\mu : \mathscr{S} \to \mathbb{R}^\star$ is a signed charge with the additional assumption that for any countable family $\{A_i : i \geq 1\}$ of pairwise disjoint sets in $\mathscr{S}$, one has

$$\mu\left(\bigcup_{i\geq 1} A_i\right) = \sum_{i\geq 1} \mu(A_i).$$

A *signed Borel charge* (resp. *measure*) is a signed charge (resp. measure) defined on the Borel σ-algebra of a topological space. For any signed charge μ, its *total variation* is defined by

$$V_\mu = \sup\left\{\sum_{i=1}^{\ell} |\mu(A_i)| : \{A_1, ..., A_\ell\} \subseteq \mathscr{S} \text{ is a partition of } S\right\}.$$

A signed charge is of *bounded variation* if $V_\mu < \infty$. The family of signed charges having bounded variation is called the *space of charges* on $\mathscr{S}$, and is denoted by $ba(\mathscr{S})$. Note that $ba(\mathscr{S})$ with the algebraic operations " $+$ " and " $\cdot$ ", defined by

$$(\mu + \nu)(A) = \mu(A) + \nu(A) \text{ and } (\alpha \cdot \mu)(A) = \alpha\mu(A) \text{ for all } A \in \mathscr{S},$$

the partial order " $\leq$ ", defined by $\mu \leq \nu$ if $\mu(A) \leq \nu(A)$ for all $A \in \mathscr{S}$, and the norm $\|\cdot\|$, defined by $\|\mu\| = V_\mu$ for any $\mu \in ba(\mathscr{S})$, is a Banach lattice. The space of all signed measures in $ba(\mathscr{S})$ is also a Banach lattice and is denoted by $ca(\mathscr{S})$.

If a signed measure assumes only nonnegative values, then it is called a *measure*. In this case, the triple $(S, \mathscr{S}, \mu)$ is called a *measure space*, and if $\mu(S) = 1$, then $(S, \mathscr{S}, \mu)$ is called a *probability space*. Let $(X, \mathscr{T})$ be a topological space. A measure μ on $\mathscr{B}(X)$ is called an *outer measure* if

$$\mu(A) = \inf\{\mu(U) : U \in \mathscr{T} \text{ and } A \subseteq U\}$$

for all $A \in \mathscr{B}(X)$. Analogously, a measure μ on $\mathscr{B}(X)$ is *tight* if

$$\mu(A) = \sup\{\mu(K) : K \in \mathscr{B}(X) \text{ is compact and } K \subseteq A\}$$

for all $A \in \mathscr{B}(X)$. A measure μ is said to be *regular* if it is a tight outer measure such that $\mu(K) < \infty$ for every compact set $K \in \mathscr{B}(X)$. The space $ca_r(\mathscr{B}(X))$ of regular signed measure is a subspace of $ca(\mathscr{B}(X))$, and is a Banach lattice.

Given a measure space $(S, \mathscr{S}, \mu)$, the measure μ is *positive* if $\mu(A) > 0$ for some $A \in \mathscr{S}$, and *complete* whenever $A \subseteq B \in \mathscr{S}$ and $\mu(B) = 0$ imply $A \in \mathscr{S}$. In such cases, $(S, \mathscr{S}, \mu)$ is known as a *positive* (*complete*) *measure space*. Further, μ and $(S, \mathscr{S}, \mu)$ are said to be *finite* if $\mu(S) < \infty$. The phrase μ*-almost everywhere* or *almost all* $s \in S$ means "everywhere except possibly for a set A with $\mu(A) = 0$". A measurable set A is called an *atom* if $\mu(A) > 0$ and for every measurable subset B of A either $\mu(B) = 0$ or $\mu(A \setminus B) = 0$ holds. If $\mathscr{S}$ has no atom, then $(S, \mathscr{S}, \mu)$ is called an *atomless measure space*. A standard result claims that if μ is finite then $S = S_0 \cup S_1$, where S_0 is atomless and S_1 is the union of countably many pairwise disjoint atoms.

Let $(S, \mathscr{S}, \mu)$ be a measure space. If $R \in \mathscr{S}$, define $\mathscr{S}_R = \{A \in \mathscr{S} : A \subseteq R\}$ and $\mu_R : \mathscr{S}_R \to \mathbb{R}_+ \cup \{\infty\}$ by $\mu_R(A) = \mu(A)$. The notation $A \sim B$ is used if $\mu(A \Delta B) = 0$. If $\{\mathscr{Q}_i : 1 \leq i \leq \ell\}$ is a family of partitions of S, then $\bigvee_{i=1}^{\ell} \mathscr{Q}_i$ denotes the refinement of $\{\mathscr{Q}_i : 1 \leq i \leq \ell\}$. Analogously, an element $A \in \bigwedge_{i=1}^{\ell} \mathscr{Q}_i$ if and only if for any $x, y \in A$ there is a finite set $\{x_j : 1 \leq j \leq n\}$ such that for all $1 \leq j \leq n-1$, x_j and x_{j+1} belong to the same atom of $\mathscr{Q}_i$ for some $1 \leq i \leq \ell$, refer to [64]. Similar to union or intersection of sets, for any two partitions $\mathscr{Q}_1, \mathscr{Q}_2$, the notations $\mathscr{Q}_1 \vee \mathscr{Q}_2$ and $\mathscr{Q}_1 \wedge \mathscr{Q}_2$ are used instead. The notation $\mathscr{Q}_1 \vee \mathscr{Q}_2$ also denotes the smallest σ-algebra containing the σ-algebras $\mathscr{Q}_1$ and $\mathscr{Q}_2$. If $\{(S_i, \mathscr{S}_i) : 1 \leq i \leq \ell\}$ is a family of measurable spaces, then the *product σ-algebra* $\bigotimes_{i=1}^{\ell} \mathscr{S}_i$ is the σ-algebra defined on $\prod_{i=1}^{\ell} S_i$ and is generated by

$$\prod_{i=1}^{\ell} \mathscr{S}_i = \left\{ \prod_{i=1}^{\ell} A_i : A_i \in \mathscr{S}_i \text{ for all } 1 \leq i \leq \ell \right\}.$$

Symbolically, $\bigotimes_{i=1}^{\ell} \mathscr{S}_i = \sigma(\prod_{i=1}^{\ell} \mathscr{S}_i)$. For convenience, the symbol $\mathscr{S}_1 \otimes \mathscr{S}_2$ is used to denote the product σ-algebra of $\mathscr{S}_1$ and $\mathscr{S}_2$. If $A \in \mathscr{S}_1 \otimes \mathscr{S}_2$, then the *projection* of A on S_1 is defined as $\{s_1 \in S_1 : (s_1, s_2) \in A \text{ for some } s_2 \in S_2\}$. In addition, if μ_1 and μ_2 are two measures defined on $\mathscr{S}_1$ and $\mathscr{S}_2$ respectively, then the product measure, denoted by $\mu_1 \times \mu_2$, is defined by $(\mu_1 \times \mu_2)(A_1 \times A_2) = \mu_1(A_1)\mu_2(A_2)$ for all $A_1 \in \mathscr{S}_1$ and $A_2 \in \mathscr{S}_2$.

Projection Theorem. Let $(S, \mathscr{S}, \mu)$ be a complete measure space and X a separable metric space. If $A \in \mathscr{S} \otimes \mathscr{B}(X)$, then its projection on S belongs to $\mathscr{S}$.

Suppose that $(S, \mathscr{S})$ and $(R, \mathscr{R})$ are two measurable spaces. A function $\varphi : S \to R$ is said to be $(\mathscr{S}, \mathscr{R})$*-measurable* if $\varphi^{-1}(A) \in \mathscr{S}$ for all $A \in \mathscr{R}$. If $\mathscr{S}, \mathscr{R}$ are understood, then an $(\mathscr{S},\mathscr{R})$-measurable function is simply said to be a *measurable function*. In particular, if Z is a metric space, then an $(\mathscr{S},\mathscr{B}(Z))$-measurable function is simply called an *$\mathscr{S}$-measurable function*. Furthermore, if $\mathscr{S} = \sigma(\mathscr{Q})$ for some partition $\mathscr{Q}$ of S, then the $\mathscr{Q}$-measurability of a function means the $\mathscr{S}$-measurability of the same function. Note that if f, g are $(\mathscr{S},\mathscr{R})$-measurable and $\alpha \in \mathbb{R}$, then $f+g$ and αf are $(\mathscr{S},\mathscr{R})$-measurable. A very useful result in measure theory states that the pointwise limit of a sequence of measurable functions from a measure space into a metric space is measurable. If X is a set and $\psi : X \to S$ then $\sigma(\psi) = \{\psi^{-1}(A) : A \in \mathscr{S}\}$ is the smallest σ-algebra of subsets of X such that ψ is measurable. Such a σ-algebra is called the *σ-algebra induced by ψ*.

Let X be a separable metric space. If $f : (S,\mathscr{S},\mu) \to X$ is $(\mathscr{S},\mathscr{B}(X))$-measurable, then the graph of f is $\mathscr{S} \otimes \mathscr{B}(X)$-measurable. The converse is true whenever μ is complete. Let $(R,\mathscr{R})$ be a measurable space and $\varphi : S \times R \to X$ a function. A $\mathscr{S} \otimes \mathscr{R}$-measurable function is also known as *jointly measurable*. For all $s \in S$, define a function $\varphi(s,\cdot) : R \to X$ by $\varphi(s,\cdot)(r) = \varphi(s,r)$. The function $\varphi(\cdot,r)$ can be defined analogously for each $r \in R$. Recall that $\mathscr{S} \otimes \mathscr{R}$-measurability of φ implies $\mathscr{S}$-measurability and $\mathscr{R}$-measurability of $\varphi(\cdot,r)$ for all $r \in R$ and $\varphi(s,\cdot)$ for all $s \in S$ respectively. Let Z be a metric space. A function $\psi : S \times X \to Z$ is called *Carathéodory* wherever for each $x \in X$, $\psi(\cdot,x)$ is $\mathscr{S}$-measurable and for each $s \in S$, $\psi(s,\cdot)$ is continuous. It is known that every Carathéodory function $\psi : S \times X \to Z$ is $\mathscr{S} \otimes \mathscr{B}(X)$-measurable. In addition, if $g : S \to X$ is measurable, then $s \mapsto \varphi(s,g(s))$ is measurable. This fact can be proved using an argument similar to that in Lemma 8.2.3 of [14].

Throughout the rest of this subsection, let $(S,\mathscr{S},\mu)$ be a finite measure space and $(X,\|\cdot\|)$ a Banach space. A function $\varphi : S \to X$ that assumes only a finite number of non-zero values, say $x_1,...,x_n$, is called a *simple function* if $A_i = \varphi^{-1}(x_i) \in \mathscr{S}$ for each $1 \le i \le n$. As usual, the formula $\varphi = \sum_{i=1}^n x_i \mathbf{1}_{A_i}$ is the standard representation of φ, where

$$\mathbf{1}_{A_i}(t) = \begin{cases} 1, & \text{if } t \in A_i; \\ 0, & \text{otherwise,} \end{cases}$$

is called the *characteristic function* of A_i on S. The *integral* of φ is defined by

$$\int_S \varphi d\mu = \sum_{i=1}^n x_i \mu(A_i).$$

Moreover, a function $f : S \to X$ is *strongly $\mathscr{S}$-measurable* if there exists a sequence $\{\varphi_n : n \geq 1\}$ of simple functions such that for almost all $s \in \mathscr{S}$,

$$\lim_{n\to\infty} \|f(s) - \varphi_n(s)\| = 0.$$

If f is strongly $\mathscr{S}$-measurable then there exist a set $N \in \mathscr{S}$ and a separable closed linear subspace Z of X such that $\{f(s) : s \in S \setminus N\} \subseteq Z$ and $\mu(N) = 0$. If f is strongly $\mathscr{S}$-measurable then so is $\|f\|$, where $\|f\|$ is defined by $\|f\|(s) = \|f(s)\|$ for all $s \in S$. Recall that if f is a strongly $\mathscr{S}$-measurable function, then the *equivalent class* of f, denoted by $[f]$, is the set of all strongly $\mathscr{S}$-measurable functions which are equal to f μ-almost everywhere. An important result of measure theory states that if X is separable, a function $f : S \to X$ is strongly $\mathscr{S}$-measurable if and only if it is $(\mathscr{S}, \mathscr{B}(X))$-measurable.

A measurable function $f : S \to \mathbb{R}_+$ is said to *Lebesgue integrable* if

$$B = \sup\left\{\int_S \varphi d\mu : 0 \leq \phi \leq f \text{ and } \varphi \text{ is simple}\right\} < \infty$$

and then $\int_S f d\mu = B$ is called the *Lebesgue integral* of f over S. A strongly $\mathscr{S}$-measurable function $f : S \to X$ is called a *Bochner integrable* if there exists a sequence $\{\varphi_n : n \geq 1\}$ of simple functions such that $\|f - \varphi_n\|$ is Lebesgue integrable for each $n \geq 1$ and

$$\lim_{n\to\infty} \int_S \|f - \varphi_n\| d\mu = 0.$$

In this case, for each $E \in \mathscr{S}$, the *Bochner integral* of f over E is defined by

$$\int_E f d\mu = \lim_{n\to\infty} \int_E \varphi_n d\mu,$$

where the last limit is in the norm topology on X. The notation $L_1(\mu, X)$ denotes the space of (equivalent classes of) Bochner integrable functions from S into X. For any two $f, g \in L_1(\mu, X)$ and $\alpha \in \mathbb{R}$, the following properties hold:

(i) $\int_S (f + g) d\mu = \int_S f d\mu + \int_S g d\mu$;

(ii) $\int_S \alpha f d\mu = \alpha \int_S f d\mu$;

(iii) if $f = 0$ μ-almost everywhere then $\int_S f d\mu = 0$;

(iv) $f \leq g$ μ-almost everywhere implies $\int_S f d\mu \leq \int_S g d\mu$;

(v) if $A, B \in \mathscr{S}$ with $A \cap B = \emptyset$ then $\int_{A\cup B} f d\mu = \int_A f d\mu + \int_B g d\mu$; and

(vi) $\| \int_S f d\mu \| \leq \int_S \|f\| d\mu$.

The *absolute continuity of Bochner integral* asserts that $\int_A f d\mu$ converges to 0 whenever $\mu(A)$ converges to 0. The proof of the following proposition can be also found in [89, p.131].

Proposition 2.1.3. *Let X be separable. Suppose $f : S \to X$ is measurable and that $\{x_n : n \geq 1\}$ is a dense subset of $f(S)$. Then there exists a sequence $\{f_k : k \geq 1\}$ of measurable functions converging uniformly to f on S, where $f_k(S) \subseteq \{x_n : n \geq 1\}$ for all $k \geq 1$.*

Proof. Since $f - x_n$ is measurable, so is $\|f - x_n\|$ for all $n \geq 1$. Note that the set

$$B_{(n,k)} = \left\{ s \in S : \|f(s) - x_n\| < \frac{1}{k} \right\}$$

is measurable for all $k, n \geq 1$, and $\bigcup_{n \geq 1} B_{(n,k)} = S$ for each $k \geq 1$. For all $k \geq 1$, let

$$A_{(1,k)} = B_{(1,k)} \text{ and } A_{(n,k)} = B_{(n,k)} \setminus \bigcup_{i<n} B_{(i,k)}$$

if $n \geq 2$. So for each $k \geq 1$, $\left\{A_{(n,k)} : n \geq 1\right\}$ is a sequence of pairwise disjoint measurable sets and $\bigcup_{n \geq 1} A_{(n,k)} = S$. Define a function $f_k : S \to X$ by $f_k(s) = x_n$ if $s \in A_{(n,k)}$. Obviously, $\lim_{k \to \infty} \|f(s) - f_k(s)\| = 0$ uniformly on S. This completes the proof. □

Remark 2.1.1. Under the hypothesis of Proposition 2.1.3, there exists a monotonically increasing sequence $\{\psi_k : k \geq 1\}$ of simple functions converging pointwise to f μ-almost everywhere. Further, if f is Bochner integrable then

$$\lim_{k \to \infty} \int_S \|f - \psi_k\| d\mu = 0.$$

Indeed, since

$$\sum_{n \geq 1} \mu\left(A_{(n,k)}\right) < \infty,$$

there is a monotonically increasing sequence $\{n_k : k \geq 1\}$ such that

$$\sum_{n \geq n_k + 1} \mu\left(A_{(n,k)}\right) < \frac{1}{k}.$$

Then, the function $\psi_k : S \to X$ such that $\psi_k = \sum_{n=1}^{n_k} x_n \mathbf{1}_{A_{(n,k)}}$ is simple and

$$\lim_{k\to\infty} \|f(s) - \psi_k(s)\| = 0$$

μ-almost everywhere. Note that if f is Bochner integrable, then

$$\int_S \|f - \psi_k\| d\mu = \sum_{n \geq n_k+1} \int_{A_{(n,k)}} \|f\| d\mu.$$

By the absolute continuity of Bochner integral, $\int_S \|f - \psi_k\| d\mu$ converges to 0.

Remark 2.1.2. If $(R, \mathscr{R}, \nu)$ is a finite measure space and $f : (S, \mathscr{S}, \mu) \times (R, \mathscr{R}, \nu) \to X$ is an $\mathscr{S} \otimes \mathscr{R}$-measurable function, then $\int_S f(\cdot, \cdot) d\mu$ is $\mathscr{R}$-measurable.

If $(S, \mathscr{S}, \mu)$ is atomless, then $\mathrm{cl}\left\{\int_E f d\mu : E \in \mathscr{S}\right\}$ is convex. This result is known as an infinite dimensional extension of Lyapunov convexity theorem, refer to [82].

Lemma 2.1.4. [21] *Assume that $(S, \mathscr{S}, \mu)$ is atomless. If $A, B \in \mathscr{S}$ such that $f \in L_1(\mu_A, X)$ and $\mu(A \cap B) \neq 0$, then*

$$H = \mathrm{cl}\left\{\left(\mu(R \cap B), \int_R f d\mu\right) : R \in \mathscr{S}_A\right\}$$

is a convex set. Moreover, for any $0 < \delta < 1$, there is a sequence $\{C_n : n \geq 1\} \subseteq \mathscr{S}_A$ such that $\mu(C_n \cap B) = \delta\mu(A \cap B)$ for all $n \geq 1$ and

$$\lim_{n\to\infty} \int_{C_n} f d\mu = \delta \int_A f d\mu.$$

Proof. To see the convexity of H, define a function $h : A \to \mathbb{R}_+$ by

$$h(t) = \begin{cases} 1, & \text{if } t \in A \cap B; \\ 0, & \text{otherwise.} \end{cases}$$

So for all $R \in \mathscr{S}_A$, one has $\int_R h d\mu = \mu(R \cap B)$, and

$$H = \mathrm{cl}\left\{\left(\int_R h d\mu, \int_R f d\mu\right) : R \in \mathscr{S}_A\right\}$$

is convex. Now, let $\{A_n : n \geq 1\}$ be a sequence of elements in $\mathscr{S}_A$ such that

$$\lim_{n\to\infty} \left(\mu(A_n \cap B), \int_{A_n} f d\mu\right) = \delta\left(\mu(A \cap B), \int_A f d\mu\right).$$

If $\mu(A_n \cap B) \geq \delta\mu(A \cap B)$, then select any $B_n \subseteq A_n \cap B$ such that $\mu(B_n) = \delta\mu(A \cap B)$ and put $C_n = (A_n \setminus B) \cup B_n$; otherwise, choose $B_n \subseteq (A \cap B) \setminus (A_n \cap B)$ with

$$\mu(B_n) = \delta\mu(A \cap B) - \mu(A_n \cap B)$$

and put $C_n = A_n \cup B_n$. As a result, one has $\mu(C_n \cap B) = \delta\mu(A \cap B)$ for all $n \geq 1$. Since $\lim_{n\to\infty} \mu(C_n \Delta A_n) = 0$, one obtains

$$\lim_{n\to\infty} \int_{C_n} f d\mu = \delta \int_A f d\mu.$$

This completes the proof. □

Corollary 2.1.5. *Suppose that $(S, \mathscr{S}, \mu)$ is atomless and $0 < \delta < 1$. If $A \in \mathscr{S}$ and $f \in L_1(\mu_A, X)$ then there is a sequence $\{C_n : n \geq 1\} \subseteq \mathscr{S}_A$ such that $\mu(C_n) = \delta\mu(A)$ for all $n \geq 1$ and*

$$\lim_{n\to\infty} \int_{C_n} f d\mu = \delta \int_A f d\mu.$$

A correspondence $F : (S, \mathscr{S}, \mu) \rightrightarrows X$ is *lower $\mathscr{S}$-measurable* if $F^-(U) \in \mathscr{S}$ for every open subset U of X. It is also termed as *lower measurable* if the underlying measure space is understood. Note that if $(S, \mathscr{S}, \mu)$ is complete, X is separable, and $\mathrm{Gr}_F \in \mathscr{S} \otimes \mathscr{B}(X)$, then $F : (S, \mathscr{S}, \mu) \rightrightarrows X$ is lower measurable. As usual, a correspondence $F : S \rightrightarrows X$ is said to be *compact-valued* (resp. *closed-valued*, *convex-valued*) if $F(s)$ is compact (resp. closed, convex) for all $s \in S$. Given $F : S \rightrightarrows X$, define $(\mathrm{cl}F) : S \rightrightarrows X$ by $(\mathrm{cl}F)(x) = \mathrm{cl}F(x)$. It is well known that F is lower measurable if and only if $\mathrm{cl}F$ is lower measurable. A compact-valued correspondence $F : S \rightrightarrows \mathbb{R}^\ell$ is lower measurable if and only if $F^-(C) \in \mathscr{S}$ for all closed subset C of X. Given $\{F_n : n \geq 1\} : S \rightrightarrows \mathbb{R}^\ell$, define $\bigcap_{n\geq 1} F_n : S \rightrightarrows \mathbb{R}^\ell$ by

$$\left(\bigcap_{n\geq 1} F_n\right)(s) = \bigcap_{n\geq 1} F_n(s).$$

According to Theorem 4.1 in [52], if F_n is lower measurable closed-valued for each $n \geq 1$ and at least one of F_n's is compact-valued, then $\bigcap_{n\geq 1} F_n$ is lower measurable. Recall that a non-empty compact-valued correspondence $F : S \rightrightarrows \mathbb{R}^\ell$ is lower measurable if and only if $F : S \to (\mathscr{K}_0(\mathbb{R}^\ell), \mathscr{T}_H)$ is measurable. A *measurable selection* of $F : S \rightrightarrows X$ is a measurable function $f : S \to X$ such that $f(s) \in F(s)$ for almost all $s \in S$.

Kuratowski-Ryll-Nardzewski Measurable Selection Theorem. If $F : S \rightrightarrows \mathbb{R}^\ell$ is non-empty closed-valued and lower measurable, then there is a measurable function $f : S \to \mathbb{R}^\ell$ such that $f(s) \in F(s)$ for all $s \in S$.

The following theorem provides some useful characterizations of lower measurability.

Theorem 2.1.6. [14] *Suppose that $F : S \rightrightarrows \mathbb{R}^\ell$ is non-empty closed-valued. Then the following properties are equivalent:*

(i) *F is lower measurable.*

(ii) *For all $x \in \mathbb{R}^\ell$, $\operatorname{dist}(x, F(\cdot))$ is measurable.*

(iii) *There exists a sequence $\{f_n : n \geq 1\}$ of measurable selections of F such that $F(s) = \operatorname{cl}\{f_n(s) : n \geq 1\}$ for all $s \in S$.*

Recall that an *integrable selection* of $F : S \rightrightarrows X$ is a measurable selection which is Bochner integrable as well. The integration of F in the sense of Aumann in [16] is defined by

$$\int_S F d\mu = \left\{ \int_S f d\mu : f \text{ is an integrable selection of } F \right\}.$$

If $(S, \mathscr{S}, \mu)$ is atomless, then $\operatorname{cl} \int_S F d\mu$ is convex, see [88] and

$$\operatorname{cl}\left(\bigcup \left\{ \int_R F d\mu : R \in \mathscr{S}, \mu(R) > 0 \right\} \right)$$

is convex, refer to [36]. Let $(S, \mathscr{S}, \mu)$ be complete and X be separable. If $\varphi : S \times X \to \mathbb{R}$ is Carathéodory, then according to [50],

$$\inf \left\{ \int_S \varphi(\cdot, f(\cdot)) d\mu : f \text{ is an integrable section of } F \right\} = \int_S \inf_{x \in F(\cdot)} \varphi(\cdot, x) d\mu.$$

Recall that $F : S \rightrightarrows \mathbb{R}^\ell_+$ is called *integrably bounded* if there is a function $\varphi : S \to \mathbb{R}_+$ such that $\|x\| \leq \varphi(s)$ for all $x \in F(s)$ and $s \in S$. If F is closed-valued and integrably bounded, then $\int_S F d\mu$ is compact, refer to [51, p.73]. If all of $F_n : S \rightrightarrows \mathbb{R}^\ell_+$ $(n \geq 1)$ are integrably bounded by the same function, then

$$\operatorname{Ls} \int_S F_n d\mu \subseteq \int_S \operatorname{Ls} F_n d\mu \text{ and } \int_S \operatorname{Li} F_n d\mu \subseteq \operatorname{Li} \int_S F_n d\mu.$$

2.2 Economics

In this section, some economic concepts, and different economic models are presented. These models are required in the forthcoming chapters.

2.2.1 Economic Concepts

This subsection is devoted to study some basic concepts in an economic system. For more details, refer to [2, 29, 51, 73, 75].

2.2.1.1 Commodity Space and Price System

Commodities and *prices* are formed a dual system. A commodity is a *good* or a *service*. There are different kinds of goods. For instance, goods of one kind are wheat, iron ore, water, or gas etc., and those of another kind are trucks, machine tools, cranes, or sheeps etc. Some examples of services are illustrated by human labor, the use of a truck and a hotel room. A commodity is characterized by its physical characteristics, and the *date* and the *location* at which it will be available. A date can be a year, a month, a week etc., and a location is assumed to be an elementary region of the space over which the economic activity takes place. Thus, physical characteristics (e.g. wheat with specified type) available at either different dates or different locations are treated as different commodities.

As in [51], the quantity of a commodity is a real number. The physical characteristics are homogeneous, which means that equal quantities of the same commodity are interchangeable in all their uses. Moreover, it is assumed that the commodities are infinitely divisible (that is, a quantity of a commodity is any non-negative real number). Each commodity is associated with a real number, called a *price*. If an economy has only ℓ many commodities, then it is natural to consider $\mathbb{R}^\ell$ as the commodity space, and a point of the commodity space is termed as a *commodity bundle*. So, a price system for this economy is of the form $p = (p^1, ..., p^\ell)$. A price system p is said to *prevail* in an economy if the amount $\frac{p^j}{p^i}$ of commodity i is needed in order to obtain one unit of commodity j. So the economy is assumed to work without the help of a good (money) serving as the medium of exchange. Given a price p and a commodity x, the value of x at p is given by $\langle x, p\rangle = \sum_{h=1}^{\ell} p^h x^h$.

2.2.1.2 Agents and Their Characteristics

An *agent* or a *consumer* is typically an individual, and an individual may be a household, or even a large group with a common purpose. Normally, the uppercase letters

$I, N, T, \ldots$ are used to denote the set of agents. Given an agent, his character is to choose a consumption plan which specifies the quantities of his inputs (that is, the quantity of each commodity which he has to make available to him) and outputs (that is, the quantity of labor which he makes available). The quantities of inputs are represented by positive numbers and those of outputs are represented by negative numbers. Thus, any consumption plan of an agent is a point in the commodity space $\mathbb{R}^\ell$ and the set of such plans is called the *consumption set*. Due to several facts (physical, physiological, or institutional), the consumption set of an agent is always bounded from below. For instance, a total amount of labor of an individual more than 24 hours in a day is not a part of any consumption plan. Formally, a consumption set can be defined as follows.

Definition 2.2.1. A *consumption set* is a non-empty subset of the commodity space, and it is closed, convex and bounded from below.

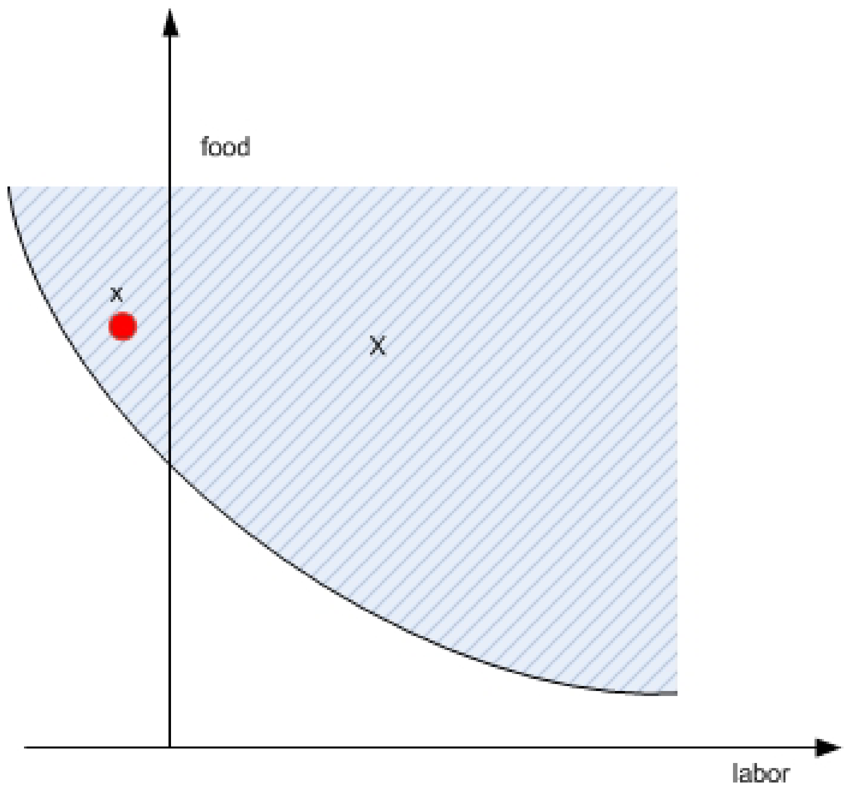

Figure 2.2: Consumption Set

Each agent is associated with some *taste* or *preference* in choosing consumption plans. Given an agent t, his consumption set X_t, and two commodity bundles $x, y \in X_t$,

t prefers x to y whenever he wants to select x if he got two alternatives x and y. For any $x, y \in X_t$, one and only one of the following alternatives is assumed: (**i**) x is preferred to y; (**ii**) x is *indifferent* from y; and (**iii**) y is preferred to x. Generally, the notation $\succ_t$ is employed to represent the *preference relation* of agent t, and $x \succ_t y$ means that x is preferred to y by agent t. Moreover, if two consumption bundles x and y are indifferent from each other, then the symbol $x \sim_t y$ is applied. Thus, $x \sim_t y$ is equivalent to $x \not\succ_t y$ and $y \not\succ_t x$. Further, the notation $y \succeq_t x$ is termed as the *preference-indifference relation* of agent t and it means that either $y \succ_t x$, or $x \sim_t y$ holds. It is presumed for all agent t and $x, y, z \in X_t$, (**i**) $x \not\succ_t x$; and (**ii**) if $y \succ_t x$ and $z \succeq_t y$, then $z \succ_t x$. A function $U_t : X_t \to \mathbb{R}$ is called a *utility function* representing $\succ_t$ if $y \succ_t x$ is equivalent to $U_t(y) > U_t(x)$ for all $x, y \in X_t$. If U_t is a utility function representing $\succ_t$ then so is $U_t + 5$. Thus, the utility function is not uniquely determined. Not every preference relation can be represented by a utility function. For instance, the lexic ordering $\succ$ on $\mathbb{R}^2$, that is, $(x', y') \succ (x, y)$ if (**i**) $x' > x$, or (**ii**) $x' = x$ and $y' > y$. The next theorem states that a very general class of preference relation can be represented by a utility function.

Theorem 2.2.1. [2] *In a topological space $(X, \mathscr{T})$ with a countable base, a preference relation $\succ$ can be represented by a continuous utility function if $\{y \in X : y \succ x\}$ and $\{y \in X : x \succ y\}$ are open for all $x \in X$.*

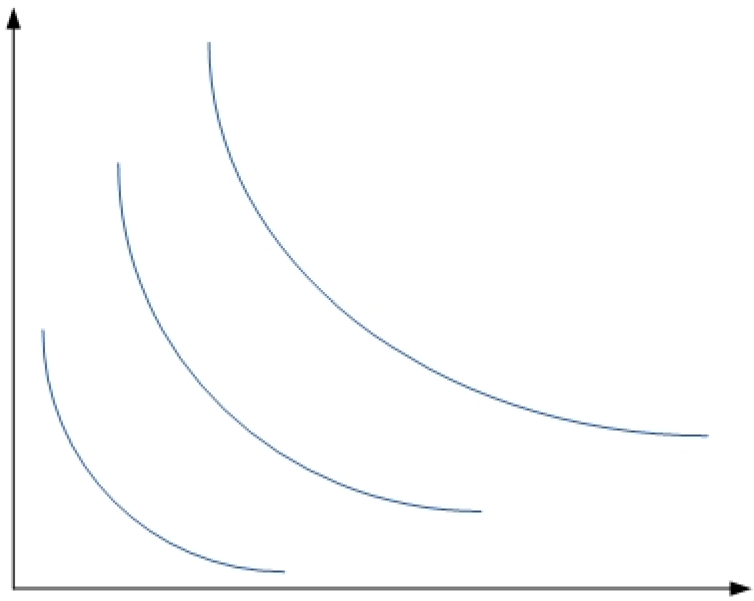

Figure 2.3: Preference-indifference Curve

Let $x \in X_t$. Then, the preference and preference-indifference relations can be expressed in term of correspondences $P_t : X_t \rightrightarrows X_t$ and $P_t^{\sim} : X_t \rightrightarrows X_t$ respectively, defined by

$$P_t(x) = \{y \in X_t : y \succ_t x\} \text{ and } P_t^{\sim}(x) = \{y \in X_t : y \succeq_t x\}.$$

Thus, P_t is also termed as the preference relation. Given $P_t : X_t \rightrightarrows X_t$, define the correspondence $P_t^{-1} : X_t \rightrightarrows X_t$ such that

$$P_t^{-1}(y) = \{x \in X : y \in P_t(x)\}.$$

The set $P_t^{-1}(y)$ is called the *lower section* of y under P_t.

An agent initially holds some amount of each commodity. The commodity bundle formed by these amount of commodities is called the *initial endowment* of that agent. It is further assumed that the initial endowment of each agent belongs to his consumption set. If $a(t)$ is the initial endowment of agent t, then given a price system p, the *budget set* of agent t is defined by

$$B(t,p) = \{x \in X_t : \langle x, p\rangle \leq \langle a(t), p\rangle\}.$$

The *demand set* of agent t at p is defined by $D(t,p) = \{x \in B(t,p) : P_t(x) \cap B(t,p) = \emptyset\}$.

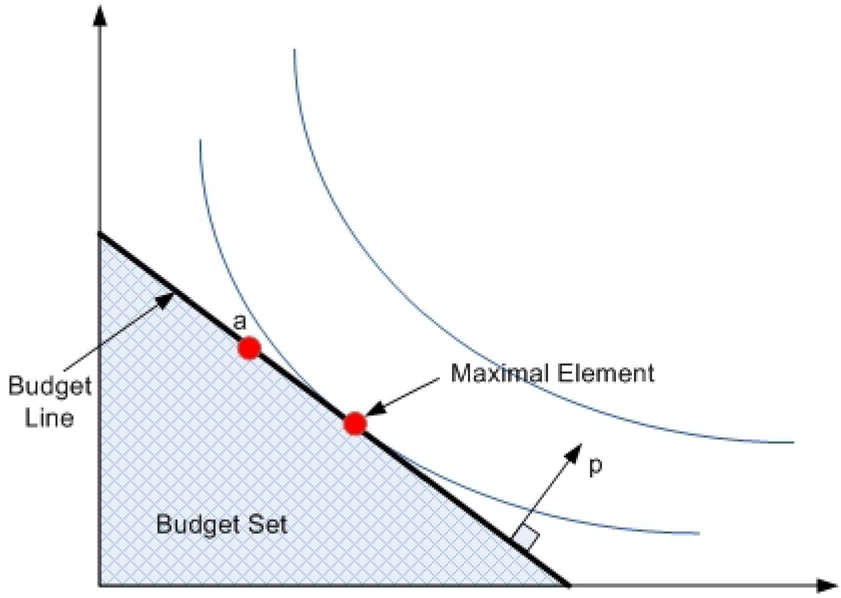

Figure 2.4: Budget Set

2.2.1.3 Pure Exchange Economies

A *pure exchange economy* is an economic system, where the exchange of commodities between agents takes place. Formally, a pure exchange economy can be defined as

$$\{(T, \Sigma, \nu); \mathbb{R}_+^\ell; (X_t, P_t, a(t)) : t \in T\},$$

where (T, Σ, μ) is the measure space of agents. The outcome of this exchange economy is a redistribution of the aggregate initial endowment. A detail discussion of a general pure exchange economy with asymmetric information can be found in Subsection 2.2.2.

2.2.1.4 Uncertainty and Asymmetric Information

In this subsection, the *environmental events* determine the consumption sets and the initial endowments of an economy. Commodities are distinguished not only by their physical characteristics, locations and dates of availability and/or usage, but also by the environmental events in which they are made available and/or used. For example, an umbrella made available at a particular location and a date in the rainy weather is different from the same umbrella made available at the same location and date if the weather is not rainy. Suppose that the activities of an economy extend over finitely many dates, denoted by $\{1, ..., T\}$. A complete history of the environmental variables from date 1 to date T is called a *state of nature*, and an *event* is a set of states of nature. Suppose that Ω denotes the set of states of nature, and $\mathscr{F}^\tau$ is the set of events at $1 \leq \tau \leq T$. To simplify the notation, let Y be the commodity space at each date τ. For any date τ, an agent t has some information $\mathscr{F}_t^\tau$ about the realized state of nature at T, where $\mathscr{F}_t^\tau$ is the σ-algebra generated by a partition Π_t^τ of Ω. It is assumed that $\mathscr{F}_t^\tau \subseteq \mathscr{F}^\tau$ for all $1 \leq \tau \leq T$ and $\mathscr{F}_t^\tau \subseteq \mathscr{F}_t^{\tau+1}$ for all $1 \leq \tau < T$. Different agents may have different information at each date. A consumption x_t^τ of an agent t at date τ is an $\mathscr{F}^\tau$-measurable function from Ω into Y, and $\left(x_t^1, ..., x_t^T\right)$ is a consumption plan of agent t. The set of such consumption plans is called the *consumption set* of t. According to Radner in [73], consumption plans of an agent t reflect the information available to agent t, which means that x_t^τ is $\mathscr{F}_t^\tau$-measurable for all τ and t. In particular, the initial endowment of each agent is one of the consumption plans of that agent.

In this thesis, a pure exchange economy is considered, which extends over three dates $\{0, 1, 2\}$. The date $\tau = 0$ is called the *ex ante stage*. At this stage, each agent knows what kind of states of nature will occur at $\tau = 2$, and his private information, the consumption set and the initial endowment. At stage $\tau = 1$, the *interim stage*, an agent t knows that the realized state of nature belongs to $\Pi_t^1(\omega^*)$, where ω^* is the true state at $\tau = 2$ and $\Pi_t^1(\omega^*)$ is the element of the partition Π_t^1 which generates the information to agent t. In these two stages, agents arrange contingent contracts for future delivery of goods. The stage $\tau = 2$ is said to be the *ex post stage*, and in this stage, agents execute the contracts and consumption takes place. It is also assumed that there is an exogenous enforcer (a government or a court), which makes sure that the agreements made ex ante are fulfilled ex post; otherwise, agents may renege on their ex ante contracts. Recently, de Castro et al. [32] mentioned that if the government or court will enforce the contracts ex post, one does not need to consider measurable contracts as in [73]. Suppose that $U_t : \Omega \times X_t \to \mathbb{R}$ denotes the *random utility function* and q_t is the *prior* of agent t. If Ω is finite, then agent t's *ex ante expected utility* is given by

$$V_t(x) = \sum_{\omega \in \Omega} U_t(\omega, x(\omega)) q_t(\omega)$$

and the *interim expected utility* is defined by

$$V_t(\omega, x) = \sum_{\omega' \in \Pi_t(\omega)} U_t(\omega', x(\omega')) \frac{q_t(\omega')}{q_t(\Pi_t(\omega))},$$

whenever ω is the true state of nature.

2.2.2 A Radner-type Model

In this subsection, a general model of an asymmetric information economy in the sense of Radner in [73] is proposed, and this model is used in Chapters 3-5. An interpretation via an atomless economy is also given in this section.

2.2.2.1 Description of a Model

Consider a pure exchange economy $\mathscr{E}$ with asymmetric information. The space of states of nature is a probability space $(\Omega, \mathscr{F}, \nu)$, where Ω is a finite set, and the space of agents is a complete, finite and positive measure space (T, Σ, μ). The commodity space of $\mathscr{E}$ is an ordered Banach space Y. Thus, $\mathscr{E}$ can be defined by

$$\mathscr{E} = \{(\Omega, \mathscr{F}, \nu);\ (T, \Sigma, \mu);\ Y_+;\ (\mathscr{F}_t, P_t, a(t, \cdot)) : t \in T\}.$$

Here, Y_+ is the *consumption set* in every state $\omega \in \Omega$ for every agent $t \in T$; $\mathscr{F}_t$ is the σ-algebra generated by a partition Π_t of Ω representing the *private information* of agent t; $P_t : Y_+^\Omega \rightrightarrows Y_+^\Omega$ is the *preference relation*[1] of agent t; and $a(t, \cdot) : \Omega \to Y_+$ is the

[1] Preference relations in this case are similar to those in Definition 5.4 in [9] and are used to capture

random initial endowment of agent t, assumed to be constant on elements of Π_t. The triple $(\mathscr{F}_t, P_t, a(t,\cdot))$ are called the *characteristics* of agent $t \in T$.

An *assignment* in $\mathscr{E}$ is a function $f : T \times \Omega \to Y_+$ such that $f(\cdot,\omega) \in L_1(\mu, Y)$ for all $\omega \in \Omega$. Throughout, a is taken to be an assignment. Put

$$L_t = \left\{x \in (Y_+)^\Omega : x \text{ is } \mathscr{F}_t\text{-measurable}\right\}.$$

An assignment f in $\mathscr{E}$ is called an *allocation* if $f(t,\cdot) \in L_t$ for almost all $t \in T$, and it is said to be *feasible* if

$$\int_T f(\cdot,\omega)d\mu \leq \int_T a(\cdot,\omega)d\mu$$

for all $\omega \in \Omega$. This feasibility condition is also known as *feasibility with free disposal.* However, if the last inequality is replaced with an equality, then it is named as *feasibility without free disposal* or an *exactly feasibility.* Any set $S \in \Sigma$ with $\mu(S) > 0$ is called a *coalition* of $\mathscr{E}$. If S and S' are two coalitions of $\mathscr{E}$ with $S' \subseteq S$, then S' is called a *sub-coalition* of S. A coalition S *privately blocks an allocation* f in $\mathscr{E}$ if there is an assignment g such that $g(t,\cdot) \in L_t$ and $g(t,\cdot) \in P_t(f(t,\cdot))$ for almost all $t \in S$, and

$$\int_S g(\cdot,\omega)d\mu \leq \int_S a(\cdot,\omega)d\mu$$

for all $\omega \in \Omega$. Following [87], the *private core* of $\mathscr{E}$, denoted by $\mathscr{PC}(\mathscr{E})$, is the set of all feasible allocations which are not privately blocked by any coalition. Further, an allocation f of $\mathscr{E}$ is *privately non-dominated* whenever there does not exist any feasible allocation g such that $g(t,\cdot) \in P_t(f(t,\cdot))$ for almost all $t \in T$. A feasible allocation f of $\mathscr{E}$ is called *privately Pareto optimal* if it is privately non-dominated. A *price system* is a non-zero function $\pi : \Omega \to Y_+^*$. The *budget set* of agent t with respect to a price system π is defined by

$$B_t(\pi) = \left\{x \in L_t : \sum_{\omega\in\Omega} \langle x(\omega), \pi(\omega)\rangle \leq \sum_{\omega\in\Omega} \langle a(t,\omega), \pi(\omega)\rangle\right\}.$$

A *Walrasian quasi-equilibrium* of $\mathscr{E}$ in the sense of Radner is a pair (f,π), where f is a feasible allocation and π is a price system such that

(**2.1**) for almost all $t \in T$, $f(t,\cdot) \in B_t(\pi)$ and $P_t(f(t,\cdot)) \cap B_t(\pi) = \emptyset$ whenever

the case of the (strong) fine core. The other results deal with private measurability condition, and so one can also use the preference relation $P_t : L_t \rightrightarrows L_t$ and obtains similar results. To avoid notational difficulty, only one type of preference relations is used here.

$\sum_{\omega\in\Omega}\langle a(t,\omega),\pi(\omega)\rangle \neq 0$ and

(2.2) $\sum_{\omega\in\Omega}\left\langle \int_T f(\cdot,\omega)d\mu,\ \pi(\omega)\right\rangle = \sum_{\omega\in\Omega}\left\langle \int_T a(\cdot,\omega)d\mu,\ \pi(\omega)\right\rangle.$

If there is some coalition S such that $\sum_{\omega\in\Omega}\langle a(t,\omega),\pi(\omega)\rangle \neq 0$ for all $t\in S$, then (f,π) is called *non-trivial*. Moreover, if both

(2.1′) for almost all $t\in T$, $f(t,\cdot)\in B_t(\pi)$ and $P_t(f(t,\cdot))\cap B_t(\pi)=\emptyset$,

and **(2.2)** hold, (f,π) is called a *Walrasian equilibrium* of $\mathscr{E}$ in the sense of Radner. In this case, f is called a *Walrasian allocation* and the set of such allocations is denoted by $\mathscr{W}(\mathscr{E})$. If (f,π) is a Walrasian quasi-equilibrium, then the feasibility of f and **(2.2)** imply

$$\left\langle \int_T f(\cdot,\omega)d\mu,\ \pi(\omega)\right\rangle = \left\langle \int_T a(\cdot,\omega)d\mu,\ \pi(\omega)\right\rangle$$

for all $\omega\in\Omega$. Note that every Walrasian equilibrium is a non-trivial Walrasian quasi-equilibrium. But the converse is satisfied if the following condition holds: For each feasible allocation f of $\mathscr{E}$ and any two disjoint coalitions S_1, S_2 of $\mathscr{E}$ with $S_1\cup S_2=T$, there is an assignment g such that $g(t,\cdot)\in L_t$ and $g(t,\cdot)\in P_t(f(t,\cdot))$ for almost all $t\in S_2$, and

$$\int_{S_1} a(\cdot,\omega)d\mu + \int_{S_2} f(\cdot,\omega)d\mu \geq \int_{S_2} g(\cdot,\omega)d\mu$$

for each $\omega\in\Omega$. This condition is called *irreducibility* and any economy satisfying this condition is called *irreducible*. This condition was introduced by McKenzie in [61] for economies with finitely many agents and was further extended to an atomless economy by Hildenbrand in [51], refer to [35].

Proposition 2.2.2. *If $\mathscr{E}$ is irreducible, every non-trivial Walrasian quasi-equilibrium of $\mathscr{E}$ is a Walrasian equilibrium.*

Proof. Let (f,π) be a non-trivial Walrasian quasi-equilibrium of $\mathscr{E}$ and

$$S_2 = \left\{ t\in S : \sum_{\omega\in\Omega}\langle a(t,\omega),\pi(\omega)\rangle \neq 0 \right\}.$$

Then $\mu(S_2)>0$. If $\mu(S_2)=\mu(T)$, there is nothing to verify. Otherwise, let $S_1 = T\setminus S_2$. By irreducibility, there is an assignment g such that $g(t,\cdot)\in L_t$ and $g(t,\cdot)\in P_t(f(t,\cdot))$ for almost all $t\in S_2$, and

$$\int_{S_1} a(\cdot,\omega)d\mu + \int_{S_2} f(\cdot,\omega)d\mu \geq \int_{S_2} g(\cdot,\omega)d\mu$$

for each $\omega \in \Omega$. Then

$$\int_{S_2} \sum_{\omega\in\Omega} \langle f(\cdot,\omega), \pi(\omega)\rangle d\mu \geq \int_{S_2} \sum_{\omega\in\Omega} \langle g(\cdot,\omega), \pi(\omega)\rangle d\mu.$$

By (**2.1**), one has

$$\sum_{\omega\in\Omega} \langle f(t,\omega), \pi(\omega)\rangle < \sum_{\omega\in\Omega} \langle g(t,\omega), \pi(\omega)\rangle$$

for almost all $t \in S_2$, which is a contradiction. □

The following assumptions are employed in Chapters 3-5.

($\mathbf{A}_1$) For any assignment f, and any separable closed linear subspace Z of Y^Ω with $f(T,\cdot) \subseteq Z$, the graph of $P^f_Z : T \rightrightarrows Z$ is in $\Sigma \otimes \mathscr{B}(Z)$, where $P^f_Z(t) = Z \cap P_t(f(t,\cdot))$.

($\mathbf{A}_2$) For all $t \in T$ and $x \in Y^\Omega_+$, $P_t(x)$ and $P_t^{-1}(x)$ are norm-open in Y^Ω_+.

($\mathbf{A}_3$) For all $t \in T$, P_t is monotone, that is, $x + y \in P_t(x)$ if $x, y \in Y^\Omega_+$ with $y \gg 0$.

($\mathbf{A}'_3$) For all $t \in T$, P_t is strictly monotone, that is, $x + y \in P_t(x)$ if $x, y \in Y^\Omega_+$ with $y > 0$.

($\mathbf{A}_4$) For all $(t,\omega) \in T \times \Omega$, $a(t,\omega) \gg 0$ whenever Y_+ has a quasi-interior point.

Note that under ($\mathbf{A}_2$) and ($\mathbf{A}_3$), $y \in P^{\sim}_t(x)$ if and only if $y \in \mathrm{cl}P_t(x)$. If $\mathrm{int}Y_+ \neq \emptyset$, then irreducibility follows from ($\mathbf{A}_4$), refer to [35]. Let $\mathfrak{P}$ denote the family of all partitions of Ω, and $T_{\mathscr{Q}} = \{t \in T : \Pi_t = \mathscr{Q}\}$ for all $\mathscr{Q} \in \mathfrak{P}$. Suppose that $T_{\mathscr{Q}} \in \Sigma$ for all $\mathscr{Q} \in \mathfrak{P}$. Since $L_t = L_{t'}$ if $t, t' \in T_{\mathscr{Q}}$, $L_{\mathscr{Q}}$ is used to denote the common value of L_t for any $t \in T_{\mathscr{Q}}$. For any coalition S, put

$$\mathfrak{P}_S = \{\mathscr{Q} \in \mathfrak{P} : S \cap T_{\mathscr{Q}} \neq \emptyset\} \text{ and } \mathfrak{P}(S) = \{\mathscr{Q} \in \mathfrak{P}_S : \mu(S \cap T_{\mathscr{Q}}) > 0\}.$$

For any allocation f, define a correspondence $P_f : T \rightrightarrows Y^\Omega_+$ by $P_f(t) = L_t \cap P_t(f(t,\cdot))$. If Y is separable, by ($\mathbf{A}_1$), one has

$$\text{(2.3)} \qquad \mathrm{Gr}_{P_f} = \bigcup_{\mathscr{Q}\in\mathfrak{P}_T} (T_{\mathscr{Q}} \times L_{\mathscr{Q}}) \cap \mathrm{Gr}_{P^f_{Y^\Omega}} \in \Sigma \otimes \mathscr{B}(Y^\Omega).$$

If T is a finite set, a special notation N is used to distinguish the set of agents from an arbitrary measure space (T,Σ,μ). Such an economy is called a *discrete economy.* In this case, an assignment is of the form $x = (x_1, ..., x_n)$, where n denotes the number of agents in N. In addition, the concept of "almost all $t \in T$" and "a measurable set with positive measure" are replaced by "all $i \in N$" and "a non-empty set" respectively. As usual, "summation" is used instead of "integration" in all corresponding places appeared previously. To deal with discrete economies, some additional assumptions are needed. First, $(\mathbf{A}_2)$ and $(\mathbf{A}_3')$ are renamed by $(\mathbf{B}_1)$ and $(\mathbf{B}_2)$ respectively.

$(\mathbf{B}_3)$ For each $i \in N$ and $x \in Y_+^\Omega$, $P_i(x)$ is convex.

$(\mathbf{B}_4)$ For each $\omega \in \Omega$, $\sum_{i \in N} a_i(\omega) \gg 0$.

$(\mathbf{B}_4')$ For each $\omega \in \Omega$, $\sum_{i \in N} a_i(\omega) > 0$.

$(\mathbf{B}_5)$ For each $i \in N$, $\inf\{a_i(\omega) : \omega \in \Omega\} > 0$.

$(\mathbf{B}_6)$ There exists an element $\hat{a} \in Y_+$ such that $L\left(\sum_{i \in N} a_i(\omega)\right) = L(\hat{a})$ for each $\omega \in \Omega$.

$(\mathbf{B}_7)$ $\mathscr{E}$ is irreducible.

The following welfare theorems can be found in [10, 28, 60], when Ω has only one element.

First Welfare Theorem. Every Walrasian allocation of $\mathscr{E}$ is (privately) Pareto optimal.

Second Welfare Theorem. Assume $(\mathbf{B}_2)$ and $(\mathbf{B}_3)$. If x is (privately) Pareto optimal allocation of $\mathscr{E}$, then $y \in P_i(x_i)$ implies

$$\sum_{\omega \in \Omega} \langle y(\omega), \pi(\omega) \rangle \geq \sum_{\omega \in \Omega} \langle x_i(\omega), \pi(\omega) \rangle.$$

The following fact is given in the proof of Lemma 3.3 of [33].

Lemma 2.2.3. *Let S be a coalition of $\mathscr{E}$ such that all agents in S have the same characteristics $(\mathscr{F}_S, P_S, a(S, \cdot))$, and L_S be the common value of L_t for $t \in S$. Assume $g : S \times \Omega \to Y_+$ is a function such that $g(t, \cdot) \in L_S$ and $g(t, \cdot) \in P_S(x)$ for all $t \in S$. Under $(\mathbf{A}_2)$, $(\mathbf{A}_3)$ and convexity of $\{y \in L_S : y \in P_S(x)\}$,*

$$\frac{1}{\mu(S)} \int_S g d\mu \in P_S(x).$$

Moreover, a similar result also holds if "P_S" is replaced by "$\mathrm{cl} P_S$".

2.2.2.2 An Interpretation via an Atomless Economy

As mentioned previously, T can be decomposed into two parts: one part is atomless and the other is the union of countably many pairwise disjoint atoms. That is, $T = T_0 \cup T_1$, where T_0 is the atomless part and T_1 is the union of countably many disjoint μ-atoms. Since each μ-atom is treated as an agent, if A is a μ-atom, then $A \in T_1$ is used, instead of writing $A \subseteq T_1$. Agents in T_0 are called *small agents* and those in T_1 are called *large agents*. Let $\mathscr{A} = \{A_i : i \geq 1\}$ be the family of μ-atoms in T.

Following [42], the economy $\mathscr{E}$ can be associated with an atomless economy $\mathscr{E}^*$. The space of agents of $\mathscr{E}^*$ is denoted by (T^*, Σ^*, μ^*), where $T^* = T_0 \cup T_1^*$ and T_1^* is an atomless measure space such that $\mu^*(T_1^*) = \mu(T_1)$ and $T_0 \cap T_1^* = \emptyset$. Suppose that (T^*, Σ^*, μ^*) is obtained by the direct sum of $(T_0, \Sigma_{T_0}, \mu_{T_0})$ and the measure space T_1^*. It is also assumed that each agent $A \in T_1$ one-to-one corresponds to a measurable subset A^* of T_1^* with $\mu^*(A^*) = \mu(A)$. Each agent $t \in A^*$ is characterized by the same characteristics as those of A. For an assignment f in $\mathscr{E}$, let $f^* = \Xi(f)$ be an assignment in $\mathscr{E}^*$ defined by

$$f^*(t,\omega) = \begin{cases} f(t,\omega), & \text{if } (t,\omega) \in T_0 \times \Omega; \\ f(A_i,\omega), & \text{if } (t,\omega) \in A_i^* \times \Omega. \end{cases}$$

Conversely, for an assignment f^* in $\mathscr{E}^*$, $f = \Phi(f^*)$ defined by

$$f(t,\omega) = \begin{cases} f^*(t,\omega), & \text{if } (t,\omega) \in T_0 \times \Omega; \\ \frac{1}{\mu^*(A_i^*)} \int_{A_i^*} f(\cdot,\omega) d\mu, & \text{if } \omega \in \Omega \text{ and } t = A_i \text{ for } i \geq 1, \end{cases}$$

is an assignment in $\mathscr{E}$. For a particular case when $T = N$, a continuum economy $\mathscr{E}_c$ is associated with $\mathscr{E}$ instead of $\mathscr{E}^*$, just like that in [38, 46, 47]. The space of agents of $\mathscr{E}_c$ is denoted by $(I, \mathscr{M}, \hat{\mu})$, where $I = [0,1]$, $\mathscr{M}$ is the family of Lebesgue measurable subsets of I with Lebesgue measure $\hat{\mu}$. Put $I = \bigcup_{i=1}^n I_i$, where $I_i = \left[\frac{i-1}{n}, \frac{i}{n}\right)$ for $i = 1, ..., n-1$, and $I_n = \left[\frac{n-1}{n}, 1\right]$. Each agent $t \in I_i$ has the same characteristics as those of $i \in N$. Further, I_i is known as the set of *type i agents*, and $\mathscr{E}_c$ is known as an economy with the *equal treatment property*. Similar to above, the assignments $f = \Xi(x)$ and $x = \Phi(f)$ are defined. The proof of the following result can be obtained in a way similar to that in [46].

Proposition 2.2.4. *Assume* $(\mathbf{B}_1)$ *and* $(\mathbf{B}_3)$. *If* (x, π) *is a non-trivial Walrasian quasi-equilibrium of* $\mathscr{E}$, *then* $(\Xi(x), \pi)$ *is a non-trivial Walrasian quasi-equilibrium of* $\mathscr{E}_c$. *Conversely, if* (f, π) *is a non-trivial Walrasian quasi-equilibrium of* $\mathscr{E}_c$, *then* $(\Phi(f), \pi)$ *is a non-trivial Walrasian quasi-equilibrium of* $\mathscr{E}$.

Remark 2.2.1. A similar conclusion holds if "non-trivial Walrasian quasi-equilibrium" is replaced with "Walrasian equilibrium". Both of these results are also true when N is replaced with T, refer to [72].

2.2.3 Maximin Formulation

In this subsection, another model of a pure exchange economy $\mathscr{E}$ with asymmetric information is considered. This is needed to introduce the main concept in Chapter 6, which is far different from that described in Subsection 2.2.2. In this framework, the space of states of nature is a probability space $(\Omega, \mathscr{F}, \nu)$ and the space of agents is a finite measure space (T, Σ, μ). The commodity space is $\mathbb{R}^\ell$ and $\mathbb{R}^\ell_+$ is the *consumption set* in every state $\omega \in \Omega$ for each agent $t \in T$. The *characteristics* of agent t are $(\mathscr{F}_t, U_t, a(t, \cdot), q_t)$, where $\mathscr{F}_t$ and $a(t, \cdot)$ are the same as before; $U_t : \Omega \times \mathbb{R}^\ell_+ \to \mathbb{R}$ is a *random utility function* of t; and q_t is a probability measure on Ω giving the *prior* of t. An *allocation* in $\mathscr{E}$ is a function $f : T \times \Omega \to \mathbb{R}^\ell_+$ such that $f(\cdot, \omega) \in L_1(\mu, \mathbb{R}^\ell)$ for all $\omega \in \Omega$. The following assumptions on agents' characteristics are essential in Chapter 6.

($\mathbf{C}_1$) a is $\Sigma \otimes \mathscr{F}$-measurable such that $\int_T a(\cdot, \omega) d\mu \gg 0$ for each $\omega \in \Omega$.

($\mathbf{C}_2$) $U_{\cdot}(\cdot, x)$ is $\Sigma \otimes \mathscr{F}$-measurable for all $x \in \mathbb{R}^\ell_+$ and $U_t(\omega, \cdot)$ is continuous for all $(t, \omega) \in T \times \Omega$.

($\mathbf{C}_3$) For each $(t, \omega) \in T \times \Omega$, $U_t(\omega, \cdot)$ is strictly monotone in the sense that if $x, y \in \mathbb{R}^\ell_+$ with $y > 0$, then $U_t(\omega, x + y) > U_t(\omega, x)$.

($\mathbf{C}_4$) For each $(t, \omega) \in T \times \Omega$, $U_t(\omega, \cdot)$ is concave.

Chapter 3

Edgeworth Equilibria with the Asymmetric Information

In this chapter, the economic model given in Subsection 2.2.2 is studied. When uncertain events occur in an economy and different agents have different information, the relationship between the private core and the set of Walrasian allocations is explored in this chapter. Section 3.1 deals with a separable Banach lattice as the commodity space. In this section, it is shown that the equivalence between the private core and the set of Walrasian allocations holds under some assumptions similar to those in [76]. It is well known that the above relationship may fail in an economy with a non-separable commodity space even if the economy has only one state of nature. The aim of Section 3.2 is to show that the private core coincides with the set of Walrasian allocations in an equal treatment continuum economy with a Banach lattice having an interior point in its positive cone as the commodity space. The commodity space in this case does not need to be separable. If the positive cone of a Banach lattice has the empty interior, then it is shown that an equal treatment allocation is a Walrasian allocation if and only if it is in the private core. The main results in Section 3.2 are taken from [22].

3.1 Edgeworth Equilibria in an Atomless Economy

This section is devoted to study the relationship between the private core and the set of Walrasian allocations in an asymmetric information economy with an atomless measure space of agents, finitely many states of nature, and a separable Banach lattice as the commodity space. Thus, $T = T_0$ and Y is a separable Banach lattice.

3.1.1 The Case When int$Y_+ \neq \emptyset$

In this subsection, it is assumed that int$Y_+ \neq \emptyset$. The following result was shown by Evren and Hüsseniov in [36]. Applying an approach given in [51], an alternative proof is provided. Note that the result is also valid for an asymmetric information economy with an ordered separable Banach space having an interior point in its positive cone as the commodity space.

Theorem 3.1.1. *Assume* $(\mathbf{A}_1)$-$(\mathbf{A}_4)$. *Then* $\mathscr{W}(\mathscr{E}) = \mathscr{PC}(\mathscr{E})$.

Proof. Let $f \in \mathscr{W}(\mathscr{E})$ and π be an equilibrium price associated with f. Suppose that there is a coalition S such that

$$\int_S g(\cdot,\omega)d\mu \leq \int_S a(\cdot,\omega)d\mu$$

for all $\omega \in \Omega$, and $g(t,\cdot) \in P_t(f(t,\cdot)) \cap L_t$ for almost all $t \in S$. This means that

$$\sum_{\omega\in\Omega}\langle g(t,\omega),\pi(\omega)\rangle > \sum_{\omega\in\Omega}\langle a(t,\omega),\pi(\omega)\rangle$$

for almost all $t \in S$. Consequently, one obtains

$$\int_S \sum_{\omega\in\Omega}\langle g(\cdot,\omega),\pi(\omega)\rangle d\mu > \int_S \sum_{\omega\in\Omega}\langle a(\cdot,\omega),\pi(\omega)\rangle d\mu,$$

which is a contradiction. Thus, $f \in \mathscr{PC}(\mathscr{E})$.

Conversely, suppose that $f \in \mathscr{PC}(\mathscr{E})$. Let $F : T \rightrightarrows Y_+^\Omega$ be a correspondence defined by

$$F(t) = \{x - a(t,\cdot) : x \in P_f(t)\} \cup \{0\},$$

where $P_f : T \rightrightarrows Y_+^\Omega$ is defined in Subsection 2.2.2. Obviously, $F(t) \neq \emptyset$ for all $t \in T$, and cl $\int_T F d\mu \cap -\text{int}Y_+^\Omega = \emptyset$. Since cl $\int_T F d\mu$ and $-\text{int}Y_+^\Omega$ are non-empty and convex, there is a non-zero element $\pi \in (Y_+^*)^\Omega$ such that $\sum_{\omega\in\Omega}\langle y(\omega),\pi(\omega)\rangle \geq 0$ for all $y \in \int_T F d\mu$. By $(\mathbf{2.3})$, $\text{Gr}_F \in \Sigma \otimes \mathscr{B}(Y^\Omega)$ and so

$$\int_T \inf_{z\in F(\cdot)} \sum_{\omega\in\Omega}\langle z(\omega),\pi(\omega)\rangle d\mu = \inf_{y\in\int_T F d\mu} \sum_{\omega\in\Omega}\langle y(\omega),\pi(\omega)\rangle \geq 0.$$

With this and the fact that $\inf_{z\in F(t)} \sum_{\omega\in\Omega}\langle z(\omega),\pi(\omega)\rangle \leq 0$, one can conclude that

$\inf_{z\in F(t)} \sum_{\omega\in\Omega}\langle z(\omega), \pi(\omega)\rangle = 0$ for almost all $t \in T$. Thus,

$$\sum_{\omega\in\Omega}\langle x(\omega), \pi(\omega)\rangle \geq \sum_{\omega\in\Omega}\langle a(t,\omega), \pi(\omega)\rangle \tag{3.1}$$

for all $x \in P_f(t)$ and almost all $t \in T$. By $(\mathbf{A}_3)$, one obtains

$$\sum_{\omega\in\Omega}\langle f(t,\omega), \pi(\omega)\rangle \geq \sum_{\omega\in\Omega}\langle a(t,\omega), \pi(\omega)\rangle$$

for almost all $t \in T$. Using the feasibility of f, it can be easily verified that for almost all $t \in T$,

$$\sum_{\omega\in\Omega}\langle f(t,\omega), \pi(\omega)\rangle = \sum_{\omega\in\Omega}\langle a(t,\omega), \pi(\omega)\rangle.$$

Thus,

$$\sum_{\omega\in\Omega}\left\langle \int_T f(\cdot,\omega)d\mu, \pi(\omega)\right\rangle = \sum_{\omega\in\Omega}\left\langle \int_T a(\cdot,\omega)d\mu, \pi(\omega)\right\rangle.$$

To complete the proof, one needs to verify that $P_t(f(t,\cdot)) \cap B_t(\pi) = \emptyset$ for almost all $t \in T$. By $(\mathbf{A}_4)$, $\sum_{\omega\in\Omega}\langle a(t,\omega), \pi(\omega)\rangle > 0$ for all $t \in T$. Select some $t \in T$ satisfying (**3.1**). It is claimed that for any such t, the inequality (**3.1**) holds with the sign " $>$ ". Indeed,

$$\sum_{\omega\in\Omega}\langle x(\omega), \pi(\omega)\rangle = \sum_{\omega\in\Omega}\langle a(t,\omega), \pi(\omega)\rangle$$

for some $x \in P_f(t)$ and $(\mathbf{A}_2)$ together yield $\lambda x \in P_f(t)$ and

$$\sum_{\omega\in\Omega}\langle \lambda x(\omega), \pi(\omega)\rangle < \sum_{\omega\in\Omega}\langle a(t,\omega), \pi(\omega)\rangle$$

for some $0 < \lambda < 1$, which contradicts with (**3.1**). So, (f, π) is a Walrasian equilibrium of $\mathscr{E}$. □

3.1.2 The Case When $Y_{++} \neq \emptyset$

In this subsection, the main equivalence theorem is established. To do this, some assumptions and techniques similar to those in [76] are employed.

Definition 3.1.1. [76] Let $v \in Y_+^\Omega \setminus \{0\}$ and V be an open convex solid neighborhood of 0 in Y^Ω. Suppose that K is the open cone spanned by $v + V$. The bundle v is called *extremely desirable* with respect to V if $x \in Y_+^\Omega$ and $y \in (x + K) \cap Y_+^\Omega$ together imply $y \in P_t(x)$ for almost all $t \in T$.

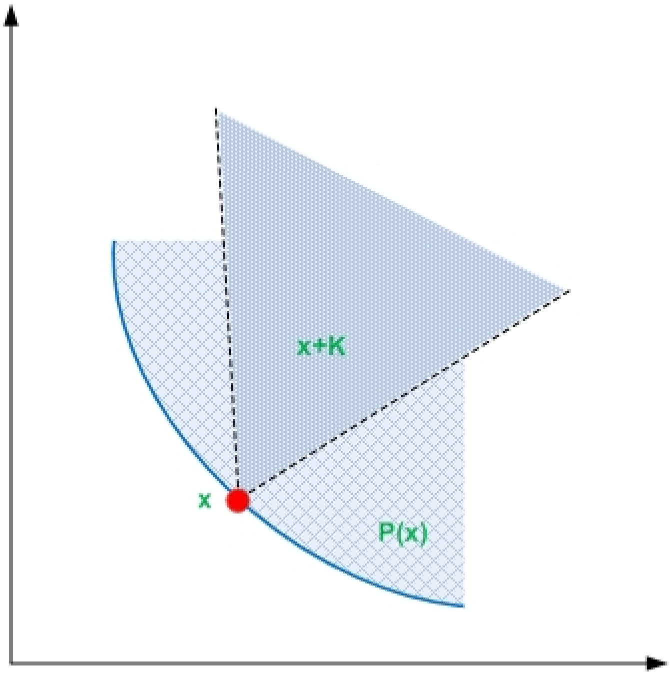

Figure 3.1: Extremely Desirable Bundle

($\mathbf{A}_5$) There is some $\bigwedge \mathfrak{P}(T)$-measurable element $v \in Y_+^\Omega \setminus \{0\}$ such that v is extremely desirable with respect to some open convex solid neighborhood U of 0 in Y^Ω.

($\mathbf{A}_6$) Suppose that $\delta_1, ..., \delta_m$ are positive numbers with $\sum_{i=1}^m \delta_i = 1$. If $x_i \in Y_+^\Omega$ and $x_i \notin \delta_i U$ for all $1 \leq i \leq m$, then $\sum_{i=1}^m x_i \notin U$.

Next lemma plays a key role in the proof of the equivalence theorem. To see this, let $\prod_{\omega\in\Omega} U(\omega) \subseteq U$ for some open ball $U(\omega)$ of 0 in Y. Put

$$W = \left(\frac{1}{|\Omega|} \bigcap_{\omega\in\Omega} U(\omega) \right)^\Omega,$$

and C and D be the open cones spanned by $v + W$ and $v + U$ respectively.

Lemma 3.1.2. *Assume* ($\mathbf{A}_2$), ($\mathbf{A}_3'$), ($\mathbf{A}_5$), ($\mathbf{A}_6$) *and that* $f \in \mathscr{P}\mathscr{C}(\mathscr{E})$. *Let* $g : S \to Y_+^\Omega$ *be a function such that* $g(t, \cdot) \in P_t(f(t, \cdot)) \cap L_t$ *and* $g = \sum_{i=1}^m y_i \chi_{S_i}$, *where*

(i) $\mu(S_i) = \eta$ *for all* $1 \leq i \leq m$;

(ii) *for each* $1 \leq i \leq m$ *there is some* $\mathscr{Q} \in \mathfrak{P}(S)$ *such that* $S_i \subseteq S \cap T_{\mathscr{Q}}$.

If $b = \sum_{i=1}^m \left(\frac{1}{\eta} \int_{S_i} a d\mu \right) \chi_{S_i}$ *then* $\int_S (g - b) d\mu \notin -C$.

Proof. Assume the contrary. Then $\sum_{i=1}^{m}(y_i - a_i)\eta \in -\alpha(v+W)$, where $a_i = \frac{1}{\eta}\int_{S_i} a d\mu$ and $\alpha > 0$. So there is an element $w \in \frac{\alpha}{\eta}W$ such that

$$\sum_{i=1}^{m} y_i + u + w = \sum_{i=1}^{m} a_i \geq 0,$$

where $u = \frac{\alpha}{\eta}v$. Since $\sum_{i=1}^{m} y_i + u \geq 0$ and W is solid, one has $\sum_{i=1}^{m} y_i + u \geq w^-$ and $w^- \in \frac{\alpha}{\eta}W$. For any m-tuple $\sigma = (\sigma_1, ..., \sigma_m)$ of positive real numbers with $\sum_{i=1}^{m} \sigma_i = 1$,

$$w^- \leq \sum_{i=1}^{m}(y_i + \sigma_i u).$$

By the Riesz decomposition property, one obtains a finite set $\{w_1^\sigma, ..., w_m^\sigma\}$ such that $w^- = \sum_{i=1}^{m} w_i^\sigma$ and $0 \leq w_i^\sigma \leq y_i + \sigma_i u$ for all $1 \leq i \leq m$. Let

$$I_{\mathscr{Q}} = \{i : S_i \subseteq S \cap T_{\mathscr{Q}}\}$$

for all $\mathscr{Q} \in \mathfrak{P}(S)$. Pick an $i \in I_{\mathscr{Q}}$ and note that $y_i + \sigma_i u$ is $\mathscr{Q}$-measurable. Define $d_i^\sigma : \Omega \to Y_+$ by

$$d_i^\sigma(\omega) = \sup\left\{w_i^\sigma(\omega') : \omega' \in \mathscr{Q}(\omega)\right\}.$$

Obviously, d_i^σ is $\mathscr{Q}$-measurable and $d_i^\sigma \leq y_i + \sigma_i u$. Let

$$z_i^\sigma = y_i + \sigma_i u - d_i^\sigma \text{ and } \delta_i^\sigma = \operatorname{dist}\left(z_i^\sigma, (D + y_i) \cap Y_+^\Omega\right).$$

Consider a continuous function $f : \Im^m \to \Im^m$ defined by

$$f(\sigma) = \left(\frac{\sigma_1 + \delta_1^\sigma}{1 + \sum_{j=1}^{m} \delta_j^\sigma}, ..., \frac{\sigma_m + \delta_m^\sigma}{1 + \sum_{j=1}^{m} \delta_j^\sigma}\right).$$

By the Brouwer fixed point theorem, one obtains a $\sigma^* = (\sigma_1^*, ..., \sigma_m^*) \in \Im^m$ satisfying $\delta_i^{\sigma^*} = \sigma_i^* \sum_{j=1}^{m} \delta_j^{\sigma^*}$ for all $1 \leq i \leq m$.

Claim. $\sum_{j=1}^{m} \delta_j^{\sigma^*} = 0$. If not, then $\delta_i^{\sigma^*} = 0$ is equivalent to $\sigma_i^* = 0$. Define the set $J = \{i : \delta_i^{\sigma^*} = 0\}$. Pick an $i \in J$. Then $z_i^{\sigma^*} = y_i - d_i^{\sigma^*}$. First assume that $d_i^{\sigma^*} > 0$. By $(\mathbf{A}_3')$, one has $y_i \in P_t(z_i^{\sigma^*}) \cap L_t$ for $t \in S_i$ and hence

$$z_i^{\sigma^*} \notin \operatorname{cl}\left((D + y_i) \cap Y_+^\Omega\right).$$

By definition, $\delta_i^{\sigma^*} > 0$, which is a contradiction with the fact that $i \in J$. Thus $d_i^{\sigma^*} = 0$

for all $i \in J$. Pick an $i \notin J$. Then $z_i^{\sigma^*} \notin D + y_i$ and so

$$d_i^{\sigma^*} \notin \frac{\sigma_i^* \alpha}{\eta} U$$

for all $i \notin J$. Consequently, $\sum_{i=1}^m d_i^{\sigma^*} \notin \frac{\alpha}{\eta} U$. Note that $d_i^{\sigma^*} \leq \sum_{\omega \in \Omega} w_i^{\sigma^*}(\omega) \mathbf{1}_\Omega$ and so

$$\sum_{i=1}^m d_i^{\sigma^*} \leq \sum_{\omega \in \Omega} w^-(\omega) \mathbf{1}_\Omega.$$

Since $\sum_{\omega \in \Omega} w^-(\omega) \mathbf{1}_\Omega \in \frac{\alpha}{\eta} U$ and $\frac{\alpha}{\eta} U$ is solid, $\sum_{i=1}^m d_i^{\sigma^*} \in \frac{\alpha}{\eta} U$, which is a contradiction. Thus the claim is verified.

It follows from the claim that $\delta_i^{\sigma^*} = 0$ for all $1 \leq i \leq m$. So $z_i^{\sigma^*} \in \operatorname{cl}\left((D + y_i) \cap Y_+^\Omega\right)$ for all $1 \leq i \leq m$. Using $y_i \in P_t(f(t, \cdot))$ and $(\mathbf{A}_5)$, one has $z_i^{\sigma^*} \in P_t(f(t, \cdot))$ for all $t \in S_i$ and $1 \leq i \leq m$. Define $h : T \times \Omega \to Y_+$ by

$$h(\cdot, \omega) = \sum_{i=1}^m z_i^{\sigma^*}(\omega) \mathbf{1}_{S_i}.$$

Clearly, $h(t, \cdot) \in P_f(t)$ for almost all $t \in S$ and

$$\int_S h d\mu \leq \eta \left(\sum_{i=1}^m y_i + u - w^- \right) \leq \eta \left(\sum_{i=1}^m y_i + u + w \right) = \eta \sum_{i=1}^m a_i = \int_S a d\mu.$$

This contradicts with the fact that $f \in \mathscr{PC}(\mathscr{E})$ and the proof is now completed. $\square$

Theorem 3.1.3. *Assume* $(\mathbf{A}_1)$, $(\mathbf{A}_2)$, $(\mathbf{A}_3')$ *and* $(\mathbf{A}_4)$-$(\mathbf{A}_6)$. *Then* $\mathscr{W}(\mathscr{E}) = \mathscr{PC}(\mathscr{E})$.

Proof. Similar to Theorem 3.1.1, one can obtain that $\mathscr{W}(\mathscr{E}) \subseteq \mathscr{PC}(\mathscr{E})$.

Conversely, let $f \in \mathscr{PC}(\mathscr{E})$. Consider the correspondence $F : T \rightrightarrows Y_+^\Omega$ defined by

$$F(t) = \{x - a(t, \cdot) : x \in P_f(t)\} \cup \{0\}.$$

Clearly, $F(t) \neq \emptyset$ for all $t \in T$. Note that if $\int_T F d\mu \cap -C = \emptyset$ then by the separation theorem, $\sum_{\omega \in \Omega} \langle y(\omega), \pi(\omega) \rangle \geq 0$ for all $y \in \int_T F d\mu$. By an argument similar to that in Theorem 3.1.1, one can show that $f \in \mathscr{W}(\mathscr{E})$. Thus, it is assumed that

$$\int_T F d\mu \cap -C \neq \emptyset$$

and a contradiction will be derived later. Suppose that $\int_T h d\mu \in \int_T F d\mu \cap -C$. Let

$$S = \{t \in T : h(t,\cdot) \neq 0 \text{ and } h(t,\cdot) = g(t,\cdot) - a(t,\cdot) \text{ for some } g(t,\cdot) \in P_f(t)\}.$$

Then S is measurable, $\mu(S) > 0$ and $\int_S (g-a)d\mu \in -C$. Without loss of generality, one can assume that $\mathfrak{P}(S) = \mathfrak{P}_S$. Pick an $\mathscr{Q} = \{A_1, ..., A_k\} \in \mathfrak{P}(S)$ and let $\omega_j \in A_j$ for all $1 \leq j \leq k$. Since $g(\cdot, \omega_j) \in L_1(\mu_{S\cap T_{\mathscr{Q}}}, Y_+)$, there is a monotonically increasing sequence $\left\{h_n^{(\mathscr{Q},\omega_j)} : n \geq 1\right\}$ of simple functions converging pointwise to $g(\cdot,\omega_j)$ such that

$$\lim_{n\to\infty} \int_{S\cap T_{\mathscr{Q}}} \left\| g(\cdot,\omega_j) - h_n^{(\mathscr{Q},\omega_j)}(\cdot) \right\| d\mu = 0.$$

Define the function $g_n : S \times \Omega \to Y_+$ such that $g_n(t,\omega) = h_n^{(\mathscr{Q},\omega_j)}(t)$ for all $(t,\omega) \in (S \cap T_{\mathscr{Q}}) \times A_j$. So, $\{g_n : n \geq 1\}$ is a monotonically increasing sequence of simple functions converging pointwise to g and

$$\lim_{n\to\infty} \int_S \|g(\cdot,\omega) - g_n(\cdot,\omega)\| d\mu = 0$$

for all $\omega \in \Omega$. Let

$$\widetilde{S}_n = \{t \in S : g_n(t,\cdot) \in P_f(t)\} \text{ and } S_n = \bigcup \{\widetilde{S}_n \cap T_{\mathscr{Q}} : \mathscr{Q} \in \mathfrak{P}(S_n)\}$$

By $(\mathbf{A}_2)$ and $(\mathbf{A}_3')$, $S_n \subseteq S_{n+1}$ for all $n \geq 1$ and $\bigcup_{n\geq 1} S_n \sim S$. Fix an $n \geq 1$. Let $g_n|_{S_n} = \sum_{i=1}^m y_i \chi_{R_i}$, where for all $1 \leq i \leq m$ there is some $\mathscr{Q} \in \mathfrak{P}(S_n)$ such that $R_i \subseteq S_n \cap T_{\mathscr{Q}}$ and $\mu(R_i) = \eta_n$. Since $g_n(t) > 0$ for all $t \in S_n$, $S_n = \bigcup_{i=1}^m R_i$. Assume that

$$a_n = \sum_{i=1}^m \left(\frac{1}{\eta_n} \int_{R_i} a d\mu \right) \chi_{R_i}.$$

By Lemma 3.1.2,

$$\int_{S_n} g_n d\mu - \int_{S_n} a_n d\mu \notin -C.$$

Note that

$$\left\| \int_{S_n} (g_n - a_n) d\mu - \int_S (g-a) d\mu \right\| \to 0 \text{ as } n \to \infty.$$

Since C is open, $\int_S (g-a)d\mu \notin -C$ and this completes the proof. $\square$

3.2 Edgeworth Equilibria in the Equal Treatment Setting

In this section, a relation between the private core and the set of Walrasian allocations is established in the setting of equal treatment. Throughout this section, $T = N$ and Y is a Banach lattice, which is not necessarily separable.

3.2.1 The Case When $\mathrm{int}Y_+ \neq \emptyset$

In this subsection, the equivalence between the private core and the set of Walrasian allocations is provided in an economy whose commodity space has an interior point in its positive cone. Since the commodity space Y is not necessarily separable, the negative result obtained in [68, 81] is not valid in every equal treatment continuum economy. In [70], some necessary and sufficient conditions were given for the core-Walras equivalence theorem in a deterministic economy with an atomless measure space of agents and a Banach lattice as the commodity space. In fact, Podczeck [70] obtained the equivalence between the core and the set of Walrasian allocations under some properties of the commodity space. In contrast, the equivalence theorem in this subsection does not require such properties. The following lemma is similar to Theorem 3.5 in [33], and is essential for the equivalence theorem.

Lemma 3.2.1. *Assume* $(\mathbf{B}_1)$-$(\mathbf{B}_3)$ *and* $a_i(\omega) \in \mathrm{int}Y_+$ *for all* $i \in N$ *and* $\omega \in \Omega$. *If* $f \in \mathscr{PC}(\mathscr{E}_c)$ *and* $x = \Phi(f)$, *then* $x_i \sim_i f(t, \cdot)$ *for almost all* $t \in I_i$ *and all* $i \in N$.

Proof. By ignoring a $\hat{\mu}$-null subset of I, one can choose a separable closed linear subspace Z of Y^Ω such that $f(I, \cdot) \subseteq Z$. Assume that there exist an $i_0 \in N$, a coalition $D \subseteq I_{i_0}$ such that $x_{i_0} \in P_{i_0}(f(t, \cdot))$ for all $t \in D$. For any $r \in \mathbb{Q} \cap (0, 1)$, define

$$D_r = \{t \in D : rx_{i_0} \in P_{i_0}(f(t, \cdot))\}.$$

Note that D_r is the projection of

$$\left(D \times \left\{x \in Y_+^\Omega : rx_{i_0} \in P_{i_0}(x)\right\}\right) \cap \{(t, f(t, \cdot)) : t \in D\}$$

on D. Thus D_r is Lebesgue measurable and $D = \bigcup\{D_r : r \in \mathbb{Q} \cap (0, 1)\}$. So, one can find a $r_1 \in \mathbb{Q} \cap (0, 1)$ and a sub-coalition $C \subseteq D$ such that $r_1 x_{i_0} \in P_{i_0}(f(t, \cdot))$ for all $t \in C$. Let $r_2 = n\hat{\mu}(C)$. Then $0 < r_2 \leq 1$. For each $\omega \in \Omega$, put

$$v(\omega) = r_1 r_2 \left(\int_I f(\cdot, \omega) d\hat{\mu} - \int_I a(\cdot, \omega) d\hat{\mu}\right) - r_2(1 - r_1) \int_{I_{i_0}} a_{i_0}(\omega) d\hat{\mu}.$$

Clearly, $\upsilon(\omega) \in -\text{int}Y_+$ for all $\omega \in \Omega$. So, there is an $\varepsilon > 0$ such that

$$\upsilon(\omega) + B(0, 2\varepsilon) \subseteq -\text{int}Y_+$$

for all $\omega \in \Omega$. Applying Corollary 2.1.5, one has a coalition E of $\mathscr{E}_c$ such that

$$\hat{\mu}(E) < \hat{\mu}(I \setminus I_{i_0}) \text{ and } \|d(\omega)\| < \varepsilon$$

for all $\omega \in \Omega$, where

$$d(\omega) = \int_E (f(\cdot,\omega) - a(\cdot,\omega))d\hat{\mu} - r_1 r_2 \int_{I \setminus I_{i_0}} (f(\cdot,\omega) - a(\cdot,\omega))d\hat{\mu}.$$

Let $S = C \cup E$. Then, $\hat{\mu}(S) < 1$. Pick an $u \in B(0,\varepsilon) \cap \text{int}Y_+$ and define $g : I \times \Omega \to Y_+$ by

$$g(t,\omega) = \begin{cases} f(t,\omega) + \frac{u}{\hat{\mu}(E)}, & \text{if } (t,\omega) \in E \times \Omega; \\ r_1 x_{i_0}, & \text{otherwise.} \end{cases}$$

Then, $g(t,\cdot) \in L_t$ and $g(t,\cdot) \in P_t(f(t,\cdot))$ for almost all $t \in S$, and

$$\int_S g(\cdot,\omega)d\hat{\mu} = \int_E f(\cdot,\omega)d\hat{\mu} + r_1 r_2 \int_{I_{i_0}} f(\cdot,\omega)d\hat{\mu} + u$$

for all $\omega \in \Omega$. It can be easily verified that for all $\omega \in \Omega$,

$$-\upsilon(\omega) + \int_S (g(\cdot,\omega) - a(\cdot,\omega))d\hat{\mu} = d(\omega) + u \in B(0, 2\varepsilon).$$

Hence,

$$\int_S a(\cdot,\omega)d\hat{\mu} - \int_S g(\cdot,\omega)d\hat{\mu} \gg 0$$

for all $\omega \in \Omega$, which contradicts with the fact that $f \in \mathscr{PC}(\mathscr{E}_c)$. Thus, $f(t,\cdot) \in \text{cl}P_i(x_i)$ for almost all $t \in I_i$ and all $i \in N$. Suppose that there is a coalition $R \subseteq I_{i'}$ for some $i' \in N$ such that $f(t,\cdot) \in P_{i'}(x_{i'})$ for all $t \in R$. By Lemma 2.2.3, one has

$$\frac{1}{\hat{\mu}(R)} \int_R f(\cdot,\cdot)d\hat{\mu} \in P_{i'}(x_{i'})$$

and

$$\frac{1}{\hat{\mu}(I_{i'} \setminus R)} \int_{I_{i'} \setminus R} f(\cdot,\cdot)d\hat{\mu} \in \text{cl}P_{i'}(x_{i'}).$$

Let $\delta = \frac{\hat{\mu}(R)}{\hat{\mu}(I_{i'})}$. Since

$$x_{i'} = \frac{\delta}{\hat{\mu}(R)} \int_R f(\cdot,\cdot) d\hat{\mu} + \frac{1-\delta}{\hat{\mu}(I_{i'} \setminus R)} \int_{I_{i'} \setminus R} f(\cdot,\cdot) d\hat{\mu},$$

$x_{i'} \in P_{i'}(x_{i'})$, which is a contradiction. Thus, $x_i \sim_i f(t,\cdot)$ for almost all $t \in I_i$ and all $i \in N$. □

Theorem 3.2.2. *Under the assumptions in Lemma 3.2.1, if $f \in \mathscr{PC}(\mathscr{E}_c)$, then (f,π) is a Walrasian equilibrium of $\mathscr{E}_c$ for some non-zero $\pi : \Omega \to Y_+^*$.*

Proof. Consider a correspondence $F : I \rightrightarrows Y_+^\Omega$ defined by

$$F(t) = \{g(t,\cdot) \in L_t : g(t,\cdot) \in P_t(f(t,\cdot))\}.$$

By $(\mathbf{B}_2)$, $F(t) \neq \emptyset$ for all $t \in I$. Note that

$$H = \|\cdot\|^\Omega\text{-cl}\left(\bigcup\left\{\int_S F d\hat{\mu} - \int_S a d\hat{\mu} : S \in \mathscr{M},\ \hat{\mu}(S) > 0\right\}\right)$$

is a convex subset of Y^Ω. Since $H \cap -\mathrm{int}Y_+^\Omega = \emptyset$, by the separation theorem, there is a non-zero element $\pi \in (Y_+^*)^\Omega$ such that for any coalition S,

$$\sum_{\omega\in\Omega} \langle y(\omega), \pi(\omega)\rangle \geq \sum_{\omega\in\Omega} \left\langle \int_S a(\cdot,\omega) d\hat{\mu}, \pi(\omega)\right\rangle$$

for all $y \in \int_S F d\hat{\mu}$. Let $x = \Phi(f)$. By Lemma 3.2.1, $x_i \sim_i f(t,\cdot)$ for almost all $t \in I_i$ and all $i \in N$. Pick an $i \in N$ and a $y_i \in P_i(x_i) \cap L_i$. If $y_i \in B_i(\pi)$, by $(\mathbf{B}_1)$, one can construct some $z_i \in B_i(\pi)$ such that $z_i \in P_i(x_i)$ and

$$\sum_{\omega\in\Omega} \left\langle \int_{I_i} z_i(\omega) d\hat{\mu}, \pi(\omega)\right\rangle < \sum_{\omega\in\Omega} \left\langle \int_{I_i} a_i(\omega) d\hat{\mu}, \pi(\omega)\right\rangle,$$

which is a contradiction. Thus,

$$\sum_{\omega\in\Omega} \langle y_i(\omega), \pi(\omega)\rangle > \sum_{\omega\in\Omega} \langle a_i(\omega), \pi(\omega)\rangle.$$

For almost all $t \in I_i$ and $i \in N$, by $(\mathbf{B}_2)$,

$$\sum_{\omega\in\Omega} \langle f(t,\omega), \pi(\omega)\rangle \geq \sum_{\omega\in\Omega} \langle a_i(\omega), \pi(\omega)\rangle$$

Using the feasibility of f, one can show that

$$\sum_{\omega \in \Omega} \langle f(t,\omega), \pi(\omega) \rangle = \sum_{\omega \in \Omega} \langle a_i(\omega), \pi(\omega) \rangle$$

for almost all $t \in I_i$ and all $i \in N$. Thus, (f, π) is a Walrasian equilibrium in $\mathscr{E}_c$. $\square$

Corollary 3.2.3. *Assume that* $\mathrm{int} Y_+ \neq \emptyset$. *Let x be a feasible allocation in $\mathscr{E}$ and $f = \Xi(x)$. Under* $(\mathbf{B}_1)$, $(\mathbf{B}_2)$ *and* $(\mathbf{B}_4)$, *if $f \in \mathscr{PC}(\mathscr{E}_c)$ then (f, π) is a non-trivial Walrasian quasi-equilibrium of $\mathscr{E}_c$ for some non-zero $\pi : \Omega \to Y_+^*$.*

3.2.2 The Case When $Y_{++} \neq \emptyset$

This section deals with an extension of Corollary 3.2.3 to an asymmetric information economy whose commodity space is a Banach lattice containing a quasi-interior point in its positive cone. The following properness assumption and the argument of getting continuity of the equilibrium price in the next theorem are similar to those in (A8) and Theorem 2 of [71]. The proof needs some additional construction because of the free disposal assumption. Note that the definition of ATY-properness is originally inherited from the M-properness introduced in [80].

Definition 3.2.1. [71] The preference relation P_i is called *ATY-proper* at $x_i \in L_i$ if there exists a convex subset $\widetilde{P}_i(x_i)$ of Y^Ω with non-empty $\|\cdot\|^\Omega$-interior such that

$$\widetilde{P}_i(x_i) \cap L_i = P_i(x_i) \cap L_i \text{ and } \left(\|\cdot\|^\Omega\text{-int}\widetilde{P}_i(x_i)\right) \cap L_i \neq \emptyset.$$

$(\mathbf{B}_8)$ If $(x_1, ..., x_n)$ is a privately Pareto optimal allocation in $\mathscr{E}$, then for each $i \in N$, P_i is ATY-proper at x_i.

Theorem 3.2.4. *Assume* $(\mathbf{B}_1)$-$(\mathbf{B}_4)$, $(\mathbf{B}_6)$ *and* $(\mathbf{B}_8)$. *Let x be a feasible allocation in $\mathscr{E}$ and $f = \Xi(x)$. If $f \in \mathscr{PC}(\mathscr{E}_c)$, then (f, π) is a non-trivial Walrasian quasi-equilibrium of $\mathscr{E}_c$ for some non-zero $\pi : \Omega \to Y_+^*$.*

Proof. Let $f \in \mathscr{PC}(\mathscr{E}_c)$ and $Z = L(\hat{a})$, where $\hat{a}$ is selected according to $(\mathbf{B}_6)$. Then, $(Z, \|\cdot\|_{\hat{a}})$ is an AM-space with $\hat{a}$ as an order unit. Note that $\hat{a} \in \|\cdot\|_{\hat{a}}$-int$Z_+$, Z_+ is $\|\cdot\|_{\hat{a}}$-closed in Z, and the $\|\cdot\|_{\hat{a}}$-closed unit ball of Z coincides with the order interval $[-\hat{a}, \hat{a}]$. Define a new economy $\hat{\mathscr{E}}$ which is identical with $\mathscr{E}$ except for the commodity space being Z equipped with the $\|\cdot\|_{\hat{a}}$-topology, each agent's consumption set being Z_+ in each state of nature $\omega \in \Omega$, and agent i's ex ante preference relation being

$\hat{P}_i : Z_+^\Omega \rightrightarrows Z_+^\Omega$, which is defined by $\hat{P}_i(z) = P_i(z) \cap Z^\Omega$ for all $z \in Z_+^\Omega$. If $(y_1, ..., y_n)$ is a feasible allocation of $\mathscr{E}$, then $y_i(\omega) \in Z_+$ for each $i \in N$ and $\omega \in \Omega$. Thus, $x_i(\omega) \in Z_+$ for each $i \in N$ and $\omega \in \Omega$. Since $\sum_{i \in N} a_i(\omega)$ is an order unit of Z,

$$\sum_{i \in N} a_i(\omega) \in \| \cdot \|_{\hat{a}}\text{-int} Z_+$$

for each $\omega \in \Omega$. Since $(Z, \| \cdot \|_{\hat{a}})$ is a Banach lattice, the $\| \cdot \|$-topology is weaker than the $\| \cdot \|_{\hat{a}}$-topology on Z. It follows that for any $x \in Z_+^\Omega$, $\hat{P}_i(x)$ and $\hat{P}_i^{-1}(x) = P_i^{-1}(x) \cap Z^\Omega$ are $\| \cdot \|_{\hat{a}}^\Omega$-open in Z_+^Ω. Thus, $\hat{\mathscr{E}}$ satisfies $(\mathbf{B}_1)$, $(\mathbf{B}_2)$ and $(\mathbf{B}_4)$, and $f \in \mathscr{PC}(\hat{\mathscr{E}}_c)$. By Corollary 3.2.3, there is a non-zero positive element $\hat{\pi} \in ((Z, \| \cdot \|_{\hat{a}})^*)^\Omega$ such that $(f, \hat{\pi})$ is a non-trivial Walrasian quasi-equilibrium of $\hat{\mathscr{E}}_c$. It is required to show that there is a non-zero positive element $\pi \in ((Z, \| \cdot \|)^*)^\Omega$ such that (f, π) is a non-trivial Walrasian quasi-equilibrium in $\mathscr{E}_c|_Z$, where $\mathscr{E}_c|_Z$ is identical with $\hat{\mathscr{E}}_c$ except for the commodity space being Z with the norm $\| \cdot \|$.

Since $(f, \hat{\pi})$ is a non-trivial Walrasian quasi-equilibrium of $\hat{\mathscr{E}}_c$, by Proposition 2.2.4, $(x, \hat{\pi})$ is a non-trivial Walrasian quasi-equilibrium of $\hat{\mathscr{E}}$. Thus, x is privately Pareto optimal in $\hat{\mathscr{E}}$, and also in $\mathscr{E}$. By $(\mathbf{B}_8)$ and Definition 3.2.1, there is a convex and $\| \cdot \|^\Omega$-open subset W_i of Y^Ω such that

$$\emptyset \neq W_i \cap L_i \subseteq P_i(x_i) \cap L_i \text{ and } \| \cdot \|^\Omega\text{-cl}(P_i(x_i) \cap L_i) \subseteq \| \cdot \|^\Omega\text{-cl} W_i.$$

Since $\sum_{i \in N} a_i(\omega)$ is a quasi-interior point of Y_+ for each $\omega \in \Omega$, Z is $\| \cdot \|$-dense in Y. By Lemma 2.1.2, Z_+ is $\| \cdot \|$-dense in Y_+. By definition, $L_i \cap Z_+^\Omega$ is $\| \cdot \|^\Omega$-dense in L_i. Thus, $W_i \cap L_i \cap Z_+^\Omega \neq \emptyset$. Let $Q_i = W_i \cap Z^\Omega$ and $\hat{L}_i = L_i \cap Z^\Omega$. Then, Q_i is convex and relatively $\| \cdot \|^\Omega$-open in Z^Ω. Further, $\emptyset \neq Q_i \cap \hat{L}_i \subseteq \hat{P}_i(x_i) \cap \hat{L}_i$ and

$$\| \cdot \|_Z^\Omega\text{-cl}(\hat{P}_i(x_i) \cap \hat{L}_i) \subseteq \| \cdot \|_Z^\Omega\text{-cl} Q_i.$$

By $(\mathbf{B}_2)$, $x_i \in \| \cdot \|_Z^\Omega\text{-cl}(\hat{P}_i(x_i) \cap \hat{L}_i)$, and so $x_i \in \| \cdot \|_Z^\Omega\text{-cl} Q_i$. For any $y_i \in Q_i \cap \hat{L}_i$, since $y_i \in P_i(x_i) \cap L_i$ and $(x, \hat{\pi})$ is a non-trivial Walrasian quasi-equilibrium,

$$\sum_{\omega \in \Omega} \langle y_i(\omega), \hat{\pi}(\omega) \rangle \geq \sum_{\omega \in \Omega} \langle x_i(\omega), \hat{\pi}(\omega) \rangle .$$

Since Ω is finite, $\hat{\pi} \in ((Z, \| \cdot \|_{\hat{a}})^\Omega)^*$. By Lemma 2.1.1, there exist an element $\pi_1^i \in$

$((Z, \|\cdot\|)^\Omega)^*$ and a linear functional π_2^i on $(Z, \|\cdot\|)^\Omega$ such that

$$\sum_{\omega\in\Omega} \left\langle x_i(\omega), \pi_1^i(\omega)\right\rangle \leq \sum_{\omega\in\Omega} \left\langle y_i(\omega), \pi_1^i(\omega)\right\rangle$$

for all $y_i \in Q_i$,

$$\sum_{\omega\in\Omega} \left\langle x_i(\omega), \pi_2^i(\omega)\right\rangle \leq \sum_{\omega\in\Omega} \left\langle y_i(\omega), \pi_2^i(\omega)\right\rangle$$

for all $y_i \in \hat{L}_i$, and $\hat{\pi} = \pi_1^i + \pi_2^i$. Since $\hat{L}_i$ is a cone, $\sum_{\omega\in\Omega} \left\langle x_i(\omega), \pi_2^i(\omega)\right\rangle = 0$. It follows that $\sum_{\omega\in\Omega} \left\langle y_i(\omega), \pi_2^i(\omega)\right\rangle \geq 0$ for all $y_i \in \hat{L}_i$. Hence, one obtains

(**3.2**) $\sum_{\omega\in\Omega} \langle x_i(\omega), \hat{\pi}(\omega)\rangle = \sum_{\omega\in\Omega} \left\langle x_i(\omega), \pi_1^i(\omega)\right\rangle$;

(**3.3**) $\sum_{\omega\in\Omega} \langle y_i(\omega), \hat{\pi}(\omega)\rangle \geq \sum_{\omega\in\Omega} \left\langle y_i(\omega), \pi_1^i(\omega)\right\rangle$ for all $y_i \in \hat{L}_i$.

In what follows, let $(Z, \|\cdot\|_{\hat{a}})^*$ be endowed with the dual order relative to the order of Z. Since each $y_i \in \hat{L}_i$ can be written as $y_i = \sum_{S\in\Pi_i} y_i^S \mathbf{1}_S$, where $y_i^S \in Z_+$, from (**3.2**) and (**3.3**), it can be verified that the following hold for all $S \in \Pi_i$:

(**3.4**) $\sum_{\omega\in S} \pi_1^i(\omega) \leq \sum_{\omega\in S} \hat{\pi}(\omega)$;

(**3.5**) $\sum_{\omega\in S} \langle x_i(\omega), \hat{\pi}(\omega)\rangle = \sum_{\omega\in S} \left\langle x_i(\omega), \pi_1^i(\omega)\right\rangle$.

Pick an arbitrary element $S \in \Pi_i$. Since $\hat{\pi}(\omega) \geq 0$ for all $\omega \in S$ and the order dual of Z is a Riesz space,

$$0 \vee \sum_{\omega\in S} \pi_1^i(\omega) \leq \sum_{\omega\in S} \hat{\pi}(\omega).$$

By the Riesz Decomposition Property, there is an element $\tilde{\pi}^i \in ((Z, \|\cdot\|)^*)^S$ such that $0 \leq \tilde{\pi}^i \leq \hat{\pi}$ on S and

$$\sum_{\omega\in S} \tilde{\pi}^i(\omega) = 0 \vee \sum_{\omega\in S} \pi_1^i(\omega).$$

It is claimed that for each $\omega \in S$,

$$\left\langle x_i(\omega), \tilde{\pi}^i(\omega)\right\rangle = \langle x_i(\omega), \hat{\pi}(\omega)\rangle.$$

To see this, let $x_i = \sum_{R\in\Pi_i} x_i^R \mathbf{1}_R$, where $x_i^R \in Z_+$. By (**3.5**),

$$\sum_{\omega\in S} \left\langle x_i(\omega), \tilde{\pi}^i(\omega)\right\rangle \geq \left\langle x_i^S, \sum_{\omega\in S} \pi_1^i(\omega)\right\rangle = \sum_{\omega\in S} \langle x_i(\omega), \hat{\pi}(\omega)\rangle.$$

Moreover,

$$\left\langle x_i(\omega), \tilde{\pi}^i(\omega)\right\rangle \leq \langle x_i(\omega), \hat{\pi}(\omega)\rangle$$

for each $\omega \in S$. So, one has

$$\left\langle x_i(\omega), \tilde{\pi}^i(\omega) \right\rangle = \langle x_i(\omega), \hat{\pi}(\omega) \rangle$$

for each $\omega \in S$, and the claim is verified. Since Π_i is a partition of Ω, there is an element $\tilde{\pi}^i \in ((Z, \|\cdot\|)^*)^\Omega$ such that $0 \leq \tilde{\pi}^i \leq \hat{\pi}$ on Ω and

$$\left\langle x_i(\omega), \tilde{\pi}^i(\omega) \right\rangle = \langle x_i(\omega), \hat{\pi}(\omega) \rangle$$

for each $\omega \in \Omega$. Let $N_0 = N \cup \{0\}$ and $\tilde{\pi}^0(\omega) = 0$ for each $\omega \in \Omega$. Since $\widetilde{Z}$ is a Riesz space, for each $\omega \in \Omega$, one can choose an element $\ddot{\pi}(\omega) \in \widetilde{Z}$ such that

$$\ddot{\pi}(\omega) = \sup \left\{ \tilde{\pi}^i(\omega) : i \in N \right\}.$$

Then $\ddot{\pi} \in ((Z, \|\cdot\|)^*)^\Omega$, and $\ddot{\pi} \leq \hat{\pi}$. Define $x_0 \in Z_+^\Omega$ such that

$$x_0(\omega) = \sum_{i \in N} a_i(\omega) - \sum_{i \in N} x_i(\omega)$$

for each $\omega \in \Omega$. By the Riesz-Kantorovich formulas, one obtains

$$\begin{aligned}
\left\langle \sum_{i \in N} a_i(\omega), \ddot{\pi}(\omega) \right\rangle &= \sup \left\{ \sum_{i \in N_0} \left\langle y_i, \tilde{\pi}^i(\omega) \right\rangle : y_i \in Z_+, \sum_{i \in N_0} y_i = \sum_{i \in N} a_i(\omega) \right\} \\
&\geq \sum_{i \in N_0} \left\langle x_i(\omega), \tilde{\pi}^i(\omega) \right\rangle = \sum_{i \in N} \left\langle x_i(\omega), \tilde{\pi}^i(\omega) \right\rangle \\
&= \left\langle \sum_{i \in N} x_i(\omega), \hat{\pi}(\omega) \right\rangle = \left\langle \sum_{i \in N} a_i(\omega), \hat{\pi}(\omega) \right\rangle
\end{aligned}$$

for all $\omega \in \Omega$. Applying $\ddot{\pi} \leq \hat{\pi}$, one has

$$\left\langle \sum_{i \in N} a_i(\omega), \ddot{\pi}(\omega) \right\rangle = \left\langle \sum_{i \in N} a_i(\omega), \hat{\pi}(\omega) \right\rangle$$

for each $\omega \in \Omega$. Note that $Z = L(\sum_{i \in N} a_i(\omega))$ for each $\omega \in \Omega$. Let $z \in Z_+$ be fixed. Choose $\delta > 0$ be such that $z \leq \delta \sum_{i \in N} a_i(\omega)$ for each $\omega \in \Omega$. Then,

$$\left\langle \left(\delta \sum_{i \in N} a_i(\omega) - z \right), \ddot{\pi}(\omega) \right\rangle \leq \left\langle \left(\delta \sum_{i \in N} a_i(\omega) - z \right), \hat{\pi}(\omega) \right\rangle,$$

and so $\langle z, \ddot{\pi}(\omega)\rangle \geq \langle z, \hat{\pi}(\omega)\rangle$ for each $\omega \in \Omega$. Consequently, $\ddot{\pi} \geq \hat{\pi}$ and therefore, $\ddot{\pi} = \hat{\pi}$. Thus, $\hat{\pi} \in ((Z, \|\cdot\|)^*)^\Omega$ and $(f, \hat{\pi})$ is a non-trivial Walrasian quasi-equilibrium of $\mathscr{E}_c|_Z$. By the Hahn-Banach theorem, one can choose a positive element $\pi \in ((Y, \|\cdot\|)^*)^\Omega$ such that π is an extension of $\hat{\pi}$. Since $L_t \cap Z_+^\Omega$ is $\|\cdot\|^\Omega$-dense in L_t and $P_t(f(t,\cdot))$ is $\|\cdot\|^\Omega$-open for each $t \in I$, one can show that

$$\sum_{\omega\in\Omega} \langle y(\omega), \pi(\omega)\rangle \geq \sum_{\omega\in\Omega} \langle a(t,\omega), \pi(\omega)\rangle$$

for all $y \in P_t(f(t,\cdot)) \cap L_t$. Further, if $\sum_{\omega\in\Omega} \langle a(t,\omega), \pi(\omega)\rangle > 0$, then similar to Theorem 3.1.1, one can show that (f, π) is a non-trivial Walrasian quasi-equilibrium of $\mathscr{E}_c$. This completes the proof. $\square$

3.2.3 The Case When $Y_{++} = \emptyset$

In this subsection, a further extension of Theorem 3.2.4 to an economy whose commodity space has no quasi-interior points in its positive cone is given. The following properness assumption and the argument of getting continuity of the equilibrium price in the next theorem are similar to those in (A8′) and Theorem 3 of [71].

Definition 3.2.2. [71] The relation $P_i : L_i \rightrightarrows L_i$ is called *strongly ATY-proper* at $x_i \in L_i$ if there is a convex subset $\widehat{P}_i(x_i)$ of Y^Ω with non-empty $\|\cdot\|^\Omega$-interior such that

$$\widehat{P}_i(x_i) \cap L_i = P_i(x_i) \cap L_i \text{ and } \left(\|\cdot\|^\Omega\text{-int}\widehat{P}_i(x_i)\right) \cap L_i \cap L\left(\sum_{i\in N} a_i\right) \neq \emptyset.$$

($\mathbf{B}_9$) If $(x_1, ..., x_n)$ is a privately Pareto optimal allocation in $\mathscr{E}$, then P_i is strongly ATY-proper at x_i for each $i \in N$.

Theorem 3.2.5. *Assume* ($\mathbf{B}_1$)-($\mathbf{B}_3$), ($\mathbf{B}_4'$), ($\mathbf{B}_6$) *and* ($\mathbf{B}_9$). *Let x be a feasible allocation in $\mathscr{E}$ and $f = \Xi(x)$. If $f \in \mathscr{PC}(\mathscr{E}_c)$, then (f, π) is a non-trivial Walrasian quasi-equilibrium of $\mathscr{E}_c$ for some non-zero $\pi : \Omega \to Y_+^*$.*

Proof. Let $f \in \mathscr{PC}(\mathscr{E}_c)$ and $Z = L(\hat{a})$, where $\hat{a}$ is selected according to ($\mathbf{B}_6$). Then, $(X, \|\cdot\|)$ equipped with the order of $(Y, \|\cdot\|)$ is a Banach lattice, where X denotes the $\|\cdot\|$-closure of Z in Y. Note that for any feasible allocation $(y_1, ..., y_n)$ of $\mathscr{E}$, $y_i(\omega)$ belongs to Z_+ for each $i \in N$ and each $\omega \in \Omega$. In particular, $x_i(\omega) \in Z_+$ for each $i \in N$ and each $\omega \in \Omega$. Clearly, for each $i \in N$, $P_i|_{X^\Omega}$ satisfies ($\mathbf{B}_1$)-($\mathbf{B}_3$), where $P_i|_{X^\Omega}$ is the preference relation of agent i on X^Ω defined by $P_i|_{X^\Omega}(x) = X^\Omega \cap P_i(x)$

for all $x \in X^\Omega$. Suppose that $(y_1, ..., y_n)$ is a privately Pareto optimal allocation in the economy $\mathscr{E}|_X$, which is identical with $\mathscr{E}$ except for the commodity space being X and agent i's preference being $P_i|_{X^\Omega}$. Then $(y_1, ..., y_n)$ is privately Pareto optimal in $\mathscr{E}$. Take $\widetilde{P}_i(y_i) = \widehat{P}_i(y_i) \cap X^\Omega$ for each $i \in N$, where $\widehat{P}_i(y_i)$ is chosen according to ($\mathbf{B}_9$) and Definition 3.2.2. So, for each $i \in N$, $\widetilde{P}_i(y_i)$ is convex with non-empty relative $\|\cdot\|^\Omega$-interior in X^Ω. Let $\hat{L}_i = L_i \cap X^\Omega$ for each $i \in N$. By ($\mathbf{B}_9$) and Definition 3.2.2, for each $i \in N$,

$$\widetilde{P}_i(y_i) \cap \hat{L}_i = P_i|_{X^\Omega}(y_i) \cap \hat{L}_i,$$

and $(\|\cdot\|_X^\Omega \mathrm{int}\widetilde{P}_i(y_i)) \cap \hat{L}_i \neq \emptyset$. Thus, ($\mathbf{B}_1$)-($\mathbf{B}_4$), ($\mathbf{B}_6$) and ($\mathbf{B}_8$) are satisfied by $\mathscr{E}|_X$. Note that $f \in \mathscr{PC}(\mathscr{E}_c|_X)$. By Theorem 3.2.4, there exists a non-zero positive element $\pi \in (X^*)^\Omega$ such that (f, π) is a non-trivial Walrasian quasi-equilibrium in $\mathscr{E}_c|_X$. Therefore, by Proposition 2.2.4, (x, π) is a non-trivial Walrasian quasi-equilibrium in $\mathscr{E}|_X$. By the Hahn-Banach theorem, there is a non-zero positive element $\hat{\pi} \in (Y^*)^\Omega$ which is an extension of π. Then, $(x, \hat{\pi})$ satisfies all conditions of non-trivial Walrasian quasi-equilibrium of $\mathscr{E}$ except for the fact that if $\sum_{\omega\in\Omega} \langle a_i(\omega), \hat{\pi}(\omega)\rangle \neq 0$, then

$$\sum_{\omega\in\Omega} \langle y_i(\omega), \hat{\pi}(\omega)\rangle > \sum_{\omega\in\Omega} \langle a_i(\omega), \hat{\pi}(\omega)\rangle$$

for all $y_i \in L_i \setminus \hat{L}_i$ satisfying $y_i \in P_i(x_i)$.

Since (x, π) is a non-trivial Walrasian quasi-equilibrium in $\mathscr{E}|_X$, x is privately Pareto optimal in $\mathscr{E}|_X$ and hence, in $\mathscr{E}$. Pick an $i \in N$. By ($\mathbf{B}_9$) and Definition 3.2.2, there is a convex and $\|\cdot\|^\Omega$-open subset Q_i of Y^Ω such that

$$\emptyset \neq Q_i \cap \hat{L}_i \subseteq P_i|_{X^\Omega}(x_i) \cap L_i \text{ and } \|\cdot\|^\Omega\text{-cl}(P_i(x_i) \cap L_i) \subseteq \|\cdot\|^\Omega\text{-cl}Q_i.$$

By ($\mathbf{B}_2$), $x_i \in \|\cdot\|^\Omega$-cl$(P_i(x_i) \cap L_i)$ and hence, $x_i \in \|\cdot\|^\Omega$-clQ_i. For any $y_i \in Q_i \cap \hat{L}_i$, since $y_i \in P_i|_{X^\Omega}(x_i) \cap L_i$ and $(x, \hat{\pi})$ is a non-trivial Walrasian quasi-equilibrium in $\mathcal{E}|_X$,

$$\sum_{\omega\in\Omega} \langle y_i(\omega), \hat{\pi}(\omega)\rangle \geq \sum_{\omega\in\Omega} \langle x_i(\omega), \hat{\pi}(\omega)\rangle .$$

Note that $\hat{L}_i$ is convex, $x_i \in \hat{L}_i$ and $\hat{\pi} \in (Y^\Omega)^*$. By an argument similar to that in Theorem 3.2.4, one can find an element $\pi_1^i \in (Y^\Omega)^*$ such that

$$\sum_{\omega\in\Omega} \left\langle x_i(\omega), \pi_1^i(\omega)\right\rangle = \sum_{\omega\in\Omega} \langle x_i(\omega), \hat{\pi}(\omega)\rangle ,$$

$$\sum_{\omega\in\Omega} \left\langle x_i(\omega), \pi_1^i(\omega)\right\rangle \le \sum_{\omega\in\Omega} \left\langle y_i(\omega), \pi_1^i(\omega)\right\rangle$$

for all $y_i \in Q_i$, and

$$\sum_{\omega\in\Omega} \left\langle y_i(\omega), \pi_1^i(\omega)\right\rangle \le \sum_{\omega\in\Omega} \left\langle y_i(\omega), \hat{\pi}(\omega)\right\rangle$$

for all $y_i \in \hat{L}_i$. Since π_1^i is $\|\cdot\|^\Omega$-continuous and

$$\|\cdot\|^\Omega\text{-cl}(P_i(x_i) \cap L_i) \subseteq \|\cdot\|^\Omega\text{-cl}Q_i,$$

one obtains

$$\sum_{\omega\in\Omega} \left\langle x_i(\omega), \pi_1^i(\omega)\right\rangle \le \sum_{\omega\in\Omega} \left\langle y_i(\omega), \pi_1^i(\omega)\right\rangle$$

for all $y_i \in P_i(x_i) \cap L_i$. Now, consider two elements $\pi_\star^i, \pi^\star \in (Y^\Omega)_+^*$ defined by

$$\left\langle y_i, \pi_\star^i\right\rangle = \sum_{\omega\in\Omega} \left\langle y_i^S(\omega), \pi_1^i(\omega)\right\rangle \text{ and } \left\langle y_i, \pi^\star\right\rangle = \sum_{\omega\in\Omega} \left\langle y_i^S(\omega), \hat{\pi}(\omega)\right\rangle,$$

where

$$y_i^S = \frac{1}{|S|}\sum_{\omega\in S} y_i(\omega)\mathbf{1}_S$$

for $S \in \Pi_i$. Then $\tilde{\pi}^i = \pi_\star^i + \hat{\pi} - \pi^\star \in (Y^\Omega)^*$, and it can be verified that

(3.6) $\sum_{\omega\in\Omega} \left\langle x_i(\omega), \tilde{\pi}^i(\omega)\right\rangle = \sum_{\omega\in\Omega} \left\langle x_i(\omega), \hat{\pi}(\omega)\right\rangle$;

(3.7) $\sum_{\omega\in\Omega} \left\langle x_i(\omega), \tilde{\pi}^i(\omega)\right\rangle \le \sum_{\omega\in\Omega} \left\langle y_i(\omega), \tilde{\pi}^i(\omega)\right\rangle$ for all $y_i \in P_i(x_i) \cap L_i$; and

(3.8) $\sum_{\omega\in\Omega} \left\langle z(\omega), \tilde{\pi}^i(\omega)\right\rangle \le \sum_{\omega\in\Omega} \left\langle z(\omega), \hat{\pi}(\omega)\right\rangle$ for all $z \in X_+^\Omega$.

Since Y is a locally solid Riesz space, Y^* is an ideal in the order dual of Y. Let $N_0 = N \cup \{0\}$ and $\tilde{\pi}^0(\omega) = 0$ for each $\omega \in \Omega$. Define $\ddot{\pi} \in (Y^*)^\Omega$ such that

$$\ddot{\pi}(\omega) = \sup\left\{\tilde{\pi}^i(\omega) : i \in N_0\right\}$$

for each $\omega \in \Omega$, and $x_0 \in Z_+^\Omega$ such that

$$x_0(\omega) = \sum_{i\in N} a_i(\omega) - \sum_{i\in N} x_i(\omega)$$

for each $\omega \in \Omega$. By the Riesz-Kantorovich formulas and techniques similar to those in

Theorem 3.2.4, one has

$$\left\langle \sum_{i\in N} a_i(\omega), \ddot{\pi}(\omega) \right\rangle \geq \sum_{i\in N} \left\langle x_i(\omega), \tilde{\pi}^i(\omega) \right\rangle$$

for each $\omega \in \Omega$. Using (**3.6**), one obtains

$$\sum_{\omega\in\Omega} \left\langle \sum_{i\in N} a_i(\omega), \ddot{\pi}(\omega) \right\rangle \geq \sum_{\omega\in\Omega} \left\langle \sum_{i\in N} a_i(\omega), \hat{\pi}(\omega) \right\rangle.$$

Further, the Riesz-Kantorovich formulas and (**3.8**) imply

$$\sum_{\omega\in\Omega} \langle z(\omega), \ddot{\pi}(\omega)\rangle \leq \sum_{\omega\in\Omega} \langle z(\omega), \hat{\pi}(\omega)\rangle$$

for all $z \in X_+^\Omega$. Since $Z^\Omega = L\left(\sum_{i\in N} a_i\right)$, one has $\ddot{\pi} \equiv \hat{\pi}$ on Z^Ω, combining with (**3.6**) and (**3.7**), one derives

$$\sum_{\omega\in\Omega} \langle x_i(\omega), \ddot{\pi}(\omega)\rangle \leq \sum_{\omega\in\Omega} \langle y_i(\omega), \ddot{\pi}(\omega)\rangle$$

for all $y_i \in P_i(x_i) \cap L_i$. Suppose that $\sum_{\omega\in\Omega} \langle a_i(\omega), \pi(\omega)\rangle \neq 0$. It follows from ($\mathbf{B}_2$) and the fact that $(x, \ddot{\pi})$ is a non-trivial Walrasian equilibrium in $\mathscr{E}|_X$,

$$\sum_{\omega\in\Omega} \langle x_i(\omega), \ddot{\pi}(\omega)\rangle = \sum_{\omega\in\Omega} \langle a_i(\omega), \ddot{\pi}(\omega)\rangle.$$

By ($\mathbf{B}_1$), one obtains

$$\sum_{\omega\in\Omega} \langle a_i(\omega), \ddot{\pi}(\omega)\rangle < \sum_{\omega\in\Omega} \langle y_i(\omega), \ddot{\pi}(\omega)\rangle$$

for all $y_i \in P_i(x_i) \cap L_i$. Thus, $(x, \ddot{\pi})$ is a non-trivial Walrasian quasi-equilibrium in $\mathscr{E}$. By Proposition 2.2.4, $(f, \ddot{\pi})$ is a non-trivial Walrasian quasi-equilibrium in $\mathscr{E}_c$. □

Chapter 4

Blocking Efficiency of the Core Solutions

In this chapter, the economic model given in Subsection 2.2.2 is further studied. Sharp interpretations of various core solutions are provided in economies with asymmetric information, atomless measure spaces of agents, finitely many states of nature and infinite dimensional commodity spaces. Firstly, an extension of a result in [47] to an economy with a Banach lattice as the commodity space is established in Section 4.1. The main result in this section is taken from [22]. Section 4.2 deals with a Vind-type theorem for the private core without free disposal condition in an asymmetric information economy whose commodity space is an ordered Banach space having an interior point in its positive cone. In particular, an answer to the question in Remark 1 of [72] is given. For particular interests, a variation of Grodal's theorem is also obtained in this section. In Section 4.3, similar investigations on the (strong) fine core without free disposal, introduced in [87], are continued. The main results in Sections 4.2 and 4.3 are taken from [20].

4.1 The Private Core with Equal Treatment

In this section, an extension of Vind's theorem is given to an asymmetric information economy with a continuum of agents having the equal treatment property, finitely many states of nature and a Banach lattice as the commodity space. This result can be treated as an extension of Proposition 3.1 in [46] and Theorem 3.3 in [47], and is established without using any convexity theorem on the commodity space. Throughout this section, $T = N$ and Y is a Banach lattice.

Lemma 4.1.1. *Assume* $(\mathbf{B}_1)$, $(\mathbf{B}_3)$ *and* $(\mathbf{B}_5)$. *Let* x *be an allocation and* $f = \Xi(x)$. *If* f *is privately blocked in* $\mathscr{E}_c$, *then it is privately blocked by a coalition* $S \subseteq I$ *via an assignment* g *such that* $g(t, \cdot) = y_i \in L_i$ *if* $t \in S \cap I_i$ *and* $i \in N$, *and*

$$\int_S (a(\cdot, \omega) - g(\cdot, \omega))d\hat{\mu} \geq z$$

for all $\omega \in \Omega$, *where* $z > 0$.

Proof. Since f is privately blocked in $\mathscr{E}_c$, there are a coalition $\hat{S} \subseteq I$ and an assignment $\hat{h}$ such that $\hat{h}(t, \cdot) \in L_t$ and $\hat{h}(t, \cdot) \in P_t(f(t, \cdot))$ for almost all $t \in \hat{S}$, and

$$\int_{\hat{S}} \hat{h}(\cdot, \omega)d\hat{\mu} \leq \int_{\hat{S}} a(\cdot, \omega)d\hat{\mu}$$

for all $\omega \in \Omega$. Let $\hat{S}_i = \hat{S} \cap I_i$ for each $i \in N$, $\hat{N} = \{i \in N : \hat{\mu}(\hat{S}_i) \neq 0\}$, and $S = \bigcup_{i \in \hat{N}} \hat{S}_i$. For each $i \in \hat{N}$ and $\omega \in \Omega$, put

$$z_i(\omega) = \frac{1}{\hat{\mu}(\hat{S}_i)} \int_{\hat{S}_i} \hat{h}(\cdot, \omega)d\hat{\mu}.$$

Then $z_i \in L_i$ for all $i \in \hat{N}$. Define $h : S \times \Omega \to Y_+$ by $h(t, \omega) = z_i(\omega)$ if $(t, \omega) \in \hat{S}_i \times \Omega$. Clearly, for each $\omega \in \Omega$,

$$\int_S h(\cdot, \omega)d\hat{\mu} \leq \int_S a(\cdot, \omega)d\hat{\mu},$$

equivalently,

$$\sum_{i \in \hat{N}} z_i(\omega)\hat{\mu}(\hat{S}_i) \leq \sum_{i \in \hat{N}} a_i(\omega)\hat{\mu}(\hat{S}_i).$$

Moreover, Lemma 2.2.3 implies that $z_i \in P_i(x_i)$ for all $i \in \hat{N}$. Choose a sequence $\{c_m : m \geq 1\} \subseteq (0, 1)$ converging to 0. For each $i \in \hat{N}$ and each integer $m \geq 1$, define a function $y_i^m : \Omega \to Y_+$ by

$$y_i^m(\omega) = (1 - c_m)z_i(\omega).$$

Then $y_i^m \in L_i$ for all $i \in \hat{N}$. For each $i \in \hat{N}$, since $z_i \in P_i(x_i)$ and $P_i(x_i)$ is $\| \cdot \|^\Omega$-open, then $y_i^m \in P_i(x_i)$ for all $i \in \hat{N}$ whenever m is sufficiently large. If one chooses such an m, then

$$\sum_{i \in \hat{N}} y_i^m(\omega)\hat{\mu}(\hat{S}_i) \leq (1 - c_m) \sum_{i \in \hat{N}} a_i(\omega)\hat{\mu}(\hat{S}_i),$$

and consequently,

$$\sum_{i\in\hat{N}} a_i(\omega)\hat{\mu}(\hat{S}_i) - \sum_{i\in\hat{N}} y_i^m(\omega)\hat{\mu}(\hat{S}_i) \geq c_m \sum_{i\in\hat{N}} a_i(\omega)\hat{\mu}(\hat{S}_i).$$

Let

$$z = \inf\left\{ c_m \sum_{i\in\widehat{N}} a_i(\omega)\hat{\mu}(\hat{S}_i) : \omega\in\Omega \right\}.$$

By ($\mathbf{B}_5$), one has $z > 0$. Define $g : S\times\Omega \to Y_+$ such that $g(t,\omega) = y_i^m(\omega)$ for all $(t,\omega)\in\hat{S}_i\times\Omega$; and $g(t,\omega) = a(t,\omega)$, otherwise. Then g is an allocation and $\int_S (a(\cdot,\omega) - g(\cdot,\omega))d\hat{\mu} \geq z$ for all $\omega\in\Omega$, which is required by the lemma. $\square$

Theorem 4.1.2. *Assume* ($\mathbf{B}_1$)*-*($\mathbf{B}_3$) *and* ($\mathbf{B}_5$)*. Let x be a feasible allocation in $\mathscr{E}$ and $f = \Xi(x)$. If $f \notin \mathscr{PC}(\mathscr{E}_c)$, then for any $0 < \varepsilon < 1$, there is a coalition S with $\hat{\mu}(S) = \varepsilon$ privately blocking f.*

Proof. Since $f \notin \mathscr{PC}(\mathscr{E}_c)$, by Lemma 4.1.1, there is a coalition $S \subseteq I$ that privately blocks f via an assignment g such that $\int_S (a(\cdot,\omega) - g(\cdot,\omega))d\hat{\mu} \geq z$ for all $\omega\in\Omega$ and $g(t,\cdot) = y_i \in L_i$ if $t\in\hat{S}_i$, where $\hat{S}_i = S\cap I_i$ for all $i\in N$. Choose a $\lambda\in(0,1)$. Since $\hat{\mu}$ is atomless, there is some $E_i \subseteq \hat{S}_i$ such that $\hat{\mu}(E_i) = \lambda\hat{\mu}(\hat{S}_i)$. Moreover, for any $t\in\hat{S}_i$, $a(t,\cdot) - g(t,\cdot) = a_i - y_i$. Hence,

$$\int_{E_i} (a(\cdot,\omega) - g(\cdot,\omega))d\hat{\mu} = (a_i(\omega) - y_i(\omega))\lambda\hat{\mu}(\hat{S}_i) = \lambda \int_{\hat{S}_i} (a(\cdot,\omega) - g(\cdot,\omega))d\hat{\mu}$$

for all $\omega\in\Omega$. Take $E = \bigcup_{i\in\hat{N}} E_i$. Then, $\hat{\mu}(E) = \lambda\hat{\mu}(S)$ and

$$\int_E (a(\cdot,\omega) - g(\cdot,\omega))d\hat{\mu} = \lambda \int_S (a(\cdot,\omega) - g(\cdot,\omega))d\hat{\mu}$$

for all $\omega\in\Omega$. Since $\lambda\int_S (a(\cdot,\omega) - g(\cdot,\omega))d\hat{\mu} > 0$ for any $\omega\in\Omega$, there is a coalition $E\subseteq S$ with $\hat{\mu}(E) = \lambda\hat{\mu}(S)$ privately blocking f via g. This proves the theorem for $\varepsilon \leq \hat{\mu}(S)$. If $\hat{\mu}(S) = 1$, the proof has been completed. Otherwise, $\hat{\mu}(I\setminus S) > 0$ and one needs to consider the case $\varepsilon > \hat{\mu}(S)$. Let $R = I\setminus S$ and

$$\lambda = 1 - \frac{\varepsilon - \hat{\mu}(S)}{\hat{\mu}(R)}.$$

Then $0 < \lambda < 1$ and define $g_\lambda : S\times\Omega \to Y_+$ such that

$$g_\lambda(t,\omega) = \lambda g(t,\omega) + (1-\lambda) f(t,\omega).$$

Note that $f(t,\cdot) \in \mathrm{cl}P_t(f(t,\cdot))$ for all $t \in S$. So, by $(\mathbf{B}_1)$, one has $g_\lambda(t,\cdot) \in P_t(f(t,\cdot))$ and $g_\lambda(t,\cdot) \in L_t$ for all $t \in S$. Put $R_i = R \cap I_i$ for all $i \in N$. Since $\hat{\mu}$ is atomless, for each $i \in N$, there is some $B_i \subseteq R_i$ such that $\hat{\mu}(B_i) = (1-\lambda)\hat{\mu}(R_i)$. Moreover, $a(t,\cdot) - f(t,\cdot) = a_i - x_i$ for any $t \in R_i$. Hence,

$$\int_{B_i} (a(\cdot,\omega) - f(\cdot,\omega))d\hat{\mu} = (1-\lambda)\int_{R_i} (a(\cdot,\omega) - f(\cdot,\omega))d\hat{\mu}$$

for all $\omega \in \Omega$. Let $B = \bigcup_{i\in N} B_i$. Then $\hat{\mu}(B) = (1-\lambda)\hat{\mu}(R)$ and

$$\int_{B} (a(\cdot,\omega) - f(\cdot,\omega))d\hat{\mu} = (1-\lambda)\int_{R} (a(\cdot,\omega) - f(\cdot,\omega))d\hat{\mu}$$

for all $\omega \in \Omega$. Define a function $h_\lambda : R \times \Omega \to Y_+$ by

$$h_\lambda(t,\omega) = f(t,\omega) + \frac{\lambda\hat{\mu}(S)}{\hat{\mu}(B)} z.$$

Since $f(t,\cdot) \in L_t$ and z is constant, $h_\lambda(t,\cdot) \in L_t$ for all $t \in R$. By $(\mathbf{B}_2)$, $h_\lambda(t,\cdot) \in P_t(f(t,\cdot))$ for all $t \in B$. Put $\widetilde{S} = S \cup B$. Then $\hat{\mu}(\widetilde{S}) = \varepsilon$. It is claimed that S privately blocks f. Define an allocation $y_\lambda : I \times \Omega \to Y_+$ such that

$$y_\lambda(t,\omega) = \begin{cases} g_\lambda(t,\omega), & \text{if } (t,\omega) \in S \times \Omega; \\ h_\lambda(t,\omega), & \text{otherwise.} \end{cases}$$

Clearly, $y_\lambda(t,\cdot) \in P_t(f(t,\cdot)) \cap L_t$ for all $t \in \widetilde{S}$. It remains to verify that y_λ is feasible for $\widetilde{S}$. Since $\int_S (a(\cdot,\omega) - g(\cdot,\omega))d\hat{\mu} \geq z$ for all $\omega \in \Omega$, one has

$$\begin{aligned} \int_{\widetilde{S}} (a(\cdot,\omega) - y_\lambda(\cdot,\omega))d\hat{\mu} \quad &\geq \quad \lambda z + (1-\lambda)\int_S (a(\cdot,\omega) - f(\cdot,\omega))d\hat{\mu} \\ &\quad + \int_B (a(\cdot,\omega) - f(\cdot,\omega))d\hat{\mu} - \lambda\hat{\mu}(S)z. \end{aligned}$$

On the other hand, $\lambda z - \lambda\hat{\mu}(S)z = \lambda\hat{\mu}(R)z > 0$, and

$$\int_{B} (a(\cdot,\omega) - f(\cdot,\omega))d\hat{\mu} = (1-\lambda)\int_{R} (a(\cdot,\omega) - f(\cdot,\omega))d\hat{\mu}.$$

Combining the previous inequalities and equalities, one obtains

$$\int_{\widetilde{S}} (a(\cdot,\omega) - y_\lambda(\cdot,\omega))d\hat{\mu} > (1-\lambda)\int_{I} (a(\cdot,\omega) - f(\cdot,\omega))d\hat{\mu} \geq 0$$

for all $\omega \in \Omega$, which verifies that $\widetilde{S}$ privately blocks f via y_λ. □

4.2 The Private Core without Free Disposal

When feasibility is defined with free disposal, Walrasian allocations may not be incentive compatible and contracts may not be enforceable, refer to [9]. Thus, to avoid this problem, it is desirable to consider a framework without free disposal. Recently, Angeloni and Martins-da-Rocha [9] showed that Aumann's equivalence theorem is still valid in a framework with asymmetric information and without free disposal. As a result, whether Vind's theorem is still valid in the same framework emerges as a question. Indeed, as mentioned in [72], whether there is a Vind-type theorem on the private core in an asymmetric information economy without free disposal even for a finite dimensional commodity space is still an open problem. In this section, a positive answer to this question is given for an asymmetric information economy with an ordered Banach space having an interior point in its positive cone as the commodity space. To achieve this goal, one needs several technical lemmas, and the proof of the following one can be found in that of Lemma 1 in [36]. For the sake of completeness, a full proof is provided here.

Lemma 4.2.1. *Assume* $(\mathbf{A}_1)$ *and* $(\mathbf{A}_2)$*. Let* f, h *be assignments and* S *be a coalition in* $\mathscr{E}$ *such that* $h(t,\cdot) \in P_t(f(t,\cdot))$ *for almost all* $t \in S$*. Suppose that* $\{c_m : m \geq 1\}$ *is a monotonically decreasing sequence in* $(0,1)$ *converging to 0 and* $h_m : S \times \Omega \to Y_+$ *is a function defined by* $h_m(t,\omega) = (1-c_m)h(t,\omega)$ *for all* $(t,\omega) \in S \times \Omega$*. Then there is a monotonically increasing sequence* $\{S_m : m \geq 1\} \subseteq \Sigma_S$ *such that* $h_m(t,\cdot) \in P_t(f(t,\cdot))$ *for almost all* $t \in S_m$ *and* $\lim_{m\to\infty} \mu(S \setminus S_m) = 0$*.*

Proof. By ignoring a μ-null subset of S, one can choose a separable closed linear subspace Z of Y^Ω such that

$$f(S,\cdot) \cup h(S,\cdot) \cup a(S,\cdot) \subseteq Z.$$

Define a correspondence $P^f : S \rightrightarrows Z$ by $P^f(t) = P^f_Z(t)$. By $(\mathbf{A}_1)$, $\mathrm{Gr}_{P^f} \in \Sigma_S \otimes \mathscr{B}(Z)$. For any $\varepsilon > 0$, define a correspondence $N_\varepsilon : S \rightrightarrows Z$ such that for each $t \in S$,

$$N_\varepsilon(t) = \{y \in Z : \|y - h(t,\cdot)\| < \varepsilon\}.$$

Then, $\mathrm{Gr}_{N_\varepsilon} \in \Sigma_S \otimes \mathscr{B}(Z)$. Choose ε_t such that

$$\varepsilon_t = \sup \left\{ \varepsilon > 0 : N_\varepsilon(t) \subseteq P^f(t) \right\}$$

for all $t \in S$. By $(\mathbf{A}_2)$, $\varepsilon_t > 0$ for almost all $t \in S$. Let $\beta > 0$. Then,

$$\{t \in S : \varepsilon_t < \beta\} = \bigcup_{r \in \mathbb{Q} \cap (0,\beta)} \left\{ t \in S : N_r(t) \cap \left(Z \setminus P^f(t) \right) \neq \emptyset \right\},$$

which is the projection of the set

$$\bigcup_{r \in \mathbb{Q} \cap (0,\beta)} (\mathrm{Gr}_{N_r} \cap ((S \times Z) \setminus \mathrm{Gr}_{P^f})) \in \Sigma_S \otimes \mathscr{B}(Z)$$

on S. By the projection theorem, $\{t \in S : \varepsilon_t < \beta\} \in \Sigma_S$, which means that the function $t \mapsto \varepsilon_t$ is measurable. For each $m \geq 1$, put

$$S_m = \{t \in S : \|h_m(t, \cdot) - h(t, \cdot)\| < \varepsilon_t\}.$$

It is clear that $S_m \in \Sigma_S$, $S_m \subseteq S_{m+1}$ for all $m \geq 1$ and $\bigcup_{m \geq 1} S_m \sim S$. Hence, $\lim_{m \to \infty} \mu(S \setminus S_m) = 0$. By the definition of ε_t, one concludes that $h_m(t, \cdot) \in P_t(f(t, \cdot))$ for almost all $t \in S_m$. $\square$

Definition 4.2.1. [87] A coalition S is said to be *NY-privately* [1] *blocking* an allocation f in $\mathscr{E}$ if there is an assignment g such that $g(t, \cdot) \in L_t$ and $g(t, \cdot) \in P_t(f(t, \cdot))$ for almost all $t \in S$, and

$$\int_S g(\cdot, \omega) d\mu = \int_S a(\cdot, \omega) d\mu$$

for all $\omega \in \Omega$. The *NY-private core* of $\mathscr{E}$, denoted by $\mathscr{PC}^{NY}(\mathscr{E})$, is the set of exactly feasible allocations which are not NY-privately blocked by any coalition.

Throughout the rest of this chapter, suppose that $T = T_0$, and $\mathrm{int}Y_+ \neq \emptyset$. The following result can be viewed as an infinite dimensional extension of Vind's theorem to the private core without free disposal.

Theorem 4.2.2. *Assume* $(\mathbf{A}_1)$-$(\mathbf{A}_4)$. *If f is an exactly feasible allocation and $f \notin \mathscr{PC}^{NY}(\mathscr{E})$, then for any $0 < \varepsilon < \mu(T)$ there exists a coalition S in $\mathscr{E}$ which NY-privately blocks f with $\mu(S) = \varepsilon$.*

[1] Here, NY is the abbreviation of Nicholas Yannelis, which is used to distinguish it from the concept of privately blocking.

Proof. Since $f \notin \mathscr{PC}^{NY}(\mathscr{E})$, there exist a coalition S and an assignment g such that $g(t,\cdot) \in L_t$ and $g(t,\cdot) \in P_t(f(t,\cdot))$ for almost all $t \in S$, and $\int_S g(\cdot,\omega)d\mu = \int_S a(\cdot,\omega)d\mu$ for all $\omega \in \Omega$. For all $\omega \in \Omega$ and $\mathscr{Q} \in \mathfrak{P}(S)$, let

$$e_{\mathscr{Q}}(\omega) = \frac{1}{\mu(S \cap T_{\mathscr{Q}})} \int_{S \cap T_{\mathscr{Q}}} a(\cdot,\omega)d\mu.$$

Choose an $e \gg 0$ such that $e \leq \frac{1}{3}e_{\mathscr{Q}}(\omega)$ for all $\omega \in \Omega$ and $\mathscr{Q} \in \mathfrak{P}(S)$. Next, choose an open ball U with center 0 such that $e - U \subseteq \mathrm{int}Y_+$. Let $\{c_m : m \geq 1\}$ be a monotonically decreasing sequence in $(0,1)$ converging to 0. Pick an arbitrary element $\mathscr{Q} \in \mathfrak{P}(S)$, and then define a function $g_m^{\mathscr{Q}} : (S \cap T_{\mathscr{Q}}) \times \Omega \to Y_+$ such that

$$g_m^{\mathscr{Q}}(t,\omega) = (1 - c_m)g(t,\omega) + c_m(e_{\mathscr{Q}}(\omega) - 2e).$$

By Lemma 4.2.1 and $(\mathbf{A}_3)$, there is a monotonically increasing sequence $\left\{S_m^{\mathscr{Q}} : m \geq 1\right\} \subseteq \Sigma_{S \cap T_{\mathscr{Q}}}$ such that $\lim_{m\to\infty}\left((S \cap T_{\mathscr{Q}}) \setminus S_m^{\mathscr{Q}}\right) = 0$ and $g_m^{\mathscr{Q}}(t,\cdot) \in P_t(f(t,\cdot))$ for almost all $t \in S_m^{\mathscr{Q}}$. By absolute continuity of the Bochner integral, there exists some $\delta > 0$ such that

$$\frac{2}{\mu(S \cap T_{\mathscr{Q}})} \int_{R_{\mathscr{Q}}} \left(g(\cdot,\omega) - e_{\mathscr{Q}}(\omega)\right) d\mu \in U$$

for all $\omega \in \Omega$ and $R_{\mathscr{Q}} \in \Sigma_{S \cap T_{\mathscr{Q}}}$ with $\mu(R_{\mathscr{Q}}) < \delta$ and all $\mathscr{Q} \in \mathfrak{P}(S)$. Choose any λ with $0 < \lambda < 1$. For each $\mathscr{Q} \in \mathfrak{P}(S)$, choose an $m_{\mathscr{Q}}$ such that

$$\mu\left(S_{m_{\mathscr{Q}}}^{\mathscr{Q}}\right) > \left(1 - \frac{\lambda}{2}\right)\mu(S \cap T_{\mathscr{Q}}) \text{ and } \mu\left((S \cap T_{\mathscr{Q}}) \setminus S_{m_{\mathscr{Q}}}^{\mathscr{Q}}\right) < \delta.$$

Let $m_0 = \max\{m_{\mathscr{Q}} : \mathscr{Q} \in \mathfrak{P}(S)\}$. Then, it follows that for all $\omega \in \Omega$ and $\mathscr{Q} \in \mathfrak{P}(S)$,

$$\frac{1}{\mu(S_{m_0}^{\mathscr{Q}})} \int_{(S \cap T_{\mathscr{Q}}) \setminus S_{m_0}^{\mathscr{Q}}} \left(g(\cdot,\omega) - e_{\mathscr{Q}}(\omega)\right) d\mu \in U.$$

Now, for each $\mathscr{Q} \in \mathfrak{P}(S)$ and $(t,\omega) \in S_{m_0}^{\mathscr{Q}} \times \Omega$, set

$$x(t,\omega) = e_{\mathscr{Q}}(\omega) - \frac{1}{\mu(S_{m_0}^{\mathscr{Q}})} \int_{(S \cap T_{\mathscr{Q}}) \setminus S_{m_0}^{\mathscr{Q}}} \left(g(\cdot,\omega) - e_{\mathscr{Q}}(\omega)\right) d\mu.$$

Consider a function $y^{\mathscr{Q}} : (S \cap T_{\mathscr{Q}}) \times \Omega \to Y_+$ defined by

$$y^{\mathscr{Q}}(t,\omega) = \begin{cases} (1 - c_{m_0})g(t,\omega) + c_{m_0}x(t,\omega), & \text{if } (t,\omega) \in S_{m_0}^{\mathscr{Q}} \times \Omega; \\ g(t,\omega), & \text{otherwise.} \end{cases}$$

Since $y^{\mathscr{Q}}(t,\omega) \gg g^{\mathscr{Q}}_{m_0}(t,\omega) + c_{m_0}e$ for all $(t,\omega) \in S^{\mathscr{Q}}_{m_0} \times \Omega$, by $(\mathbf{A}_3)$, $y^{\mathscr{Q}}(t,\cdot) \in P_t(f(t,\cdot))$ for almost all $t \in S^{\mathscr{Q}}_{m_0}$. Thus, $y^{\mathscr{Q}}(t,\cdot) \in L_t$ and $y^{\mathscr{Q}}(t,\cdot) \in P_t(f(t,\cdot))$ for almost all $t \in S \cap T_{\mathscr{Q}}$. Moreover,

$$\text{(4.1)} \qquad \int_{S\cap T_{\mathscr{Q}}} y^{\mathscr{Q}}(\cdot,\omega)d\mu = \int_{S\cap T_{\mathscr{Q}}} \left((1-c_{m_0})g(\cdot,\omega) + c_{m_0}a(\cdot,\omega)\right)d\mu$$

for all $\omega \in \Omega$. By Corollary 2.1.5, there exists a sequence $\{F^{\mathscr{Q}}_n : n \geq 1\} \subseteq \Sigma_{S\cap T_{\mathscr{Q}}}$ such that $\mu(F^{\mathscr{Q}}_n) = \lambda\mu(S \cap T_{\mathscr{Q}})$ and for all $\omega \in \Omega$,

$$\lim_{n\to\infty} \int_{F^{\mathscr{Q}}_n} \Big(y^{\mathscr{Q}}(\cdot,\omega) - a(\cdot,\omega)\Big)\,d\mu = \lambda \int_{S\cap T_{\mathscr{Q}}} \Big(y^{\mathscr{Q}}(\cdot,\omega) - a(\cdot,\omega)\Big)\,d\mu.$$

The function $b^{\mathscr{Q}}_n : \Omega \to Y_+$, defined by

$$b^{\mathscr{Q}}_n(\omega) = \lambda \int_{S\cap T_{\mathscr{Q}}} \Big(y^{\mathscr{Q}}(\cdot,\omega) - a(\cdot,\omega)\Big)\,d\mu - \int_{F^{\mathscr{Q}}_n} \Big(y^{\mathscr{Q}}(\cdot,\omega) - a(\cdot,\omega)\Big)\,d\mu,$$

is $\mathscr{Q}$-measurable for all $n \geq 1$ and $\lim_{n\to\infty} \|b^{\mathscr{Q}}_n(\omega)\| = 0$ for all $\omega \in \Omega$. Note that

$$\inf\Big\{\mu(F^{\mathscr{Q}}_n \cap S^{\mathscr{Q}}_{m_0}) : n \geq 1\Big\} \geq \frac{\lambda}{2}\,\mu(S \cap T_{\mathscr{Q}}).$$

Choose an $n_{\mathscr{Q}}$ such that $\frac{2}{\lambda\mu(S\cap T_{\mathscr{Q}})}\, b^{\mathscr{Q}}_{n_{\mathscr{Q}}}(\omega) \in c_{m_0}U$ for all $\omega \in \Omega$. Then, one has

$$c_{m_0}e + \frac{1}{\mu\left(F^{\mathscr{Q}}_{n_{\mathscr{Q}}} \cap S^{\mathscr{Q}}_{m_0}\right)}\, b^{\mathscr{Q}}_{n_{\mathscr{Q}}}(\omega) \gg 0$$

for all $\omega \in \Omega$. Define a function $g^{\mathscr{Q}} : F^{\mathscr{Q}}_{n_{\mathscr{Q}}} \times \Omega \to Y_+$ such that

$$g^{\mathscr{Q}}(t,\omega) = \begin{cases} y^{\mathscr{Q}}(t,\omega) + \dfrac{b^{\mathscr{Q}}_{n_{\mathscr{Q}}}(\omega)}{\mu\left(F^{\mathscr{Q}}_{n_{\mathscr{Q}}} \cap S^{\mathscr{Q}}_{m_0}\right)}, & \text{if } (t,\omega) \in \left(F^{\mathscr{Q}}_{n_{\mathscr{Q}}} \cap S^{\mathscr{Q}}_{m_0}\right) \times \Omega; \\ y^{\mathscr{Q}}(t,\omega), & \text{otherwise.} \end{cases}$$

By $(\mathbf{A}_3)$ and the fact that $g^{\mathscr{Q}}_{m_0}(t,\cdot) \in P_t(f(t,\cdot))$ for almost all $t \in F^{\mathscr{Q}}_{n_{\mathscr{Q}}} \cap S^{\mathscr{Q}}_{m_0}$, one has $g^{\mathscr{Q}}(t,\cdot) \in P_t(f(t,\cdot))$ for almost all $t \in F^{\mathscr{Q}}_{n_{\mathscr{Q}}} \cap S^{\mathscr{Q}}_{m_0}$. So, $g^{\mathscr{Q}}(t,\cdot) \in L_t$ and $g^{\mathscr{Q}}(t,\cdot) \in P_t(f(t,\cdot))$ for almost all $t \in F^{\mathscr{Q}}_{n_{\mathscr{Q}}}$. Furthermore,

$$\text{(4.2)} \qquad \int_{F^{\mathscr{Q}}_{n_{\mathscr{Q}}}} (g^{\mathscr{Q}}(\cdot,\omega) - a(\cdot,\omega))d\mu = \lambda \int_{S\cap T_{\mathscr{Q}}} \Big(y^{\mathscr{Q}}(\cdot,\omega) - a(\cdot,\omega)\Big)\,d\mu$$

for all $\omega \in \Omega$. Let $F = \bigcup\{F^{\mathscr{Q}}_{n_{\mathscr{Q}}} : \mathscr{Q} \in \mathfrak{P}(S)\}$. So $\mu(F) = \lambda\mu(S)$. Define a function

$h : T \times \Omega \to Y_+$ such that $h(t,\omega) = g^{\mathcal{Q}}(t,\omega)$ if $(t,\omega) \in F^{\mathcal{Q}}_{n_{\mathcal{Q}}} \times \Omega$; and $h(t,\omega) = g(t,\omega)$, otherwise. Then, h is an allocation and $h(t,\cdot) \in P_t(f(t,\cdot))$ for almost all $t \in F$. By (**4.1**)-(**4.2**), one obtains $\int_F (h(\cdot,\omega) - a(\cdot,\omega))d\mu = 0$ for all $\omega \in \Omega$. Thus, f is NY-privately blocked by F via h. This proves the theorem for $\varepsilon \leq \mu(S)$. If $\mu(S) = \mu(T)$, the proof has been completed. Otherwise, one needs to consider the case for $\varepsilon > \mu(S)$. Let $A = T \setminus S$, and let the λ be chosen such that

$$\lambda = 1 - \frac{\varepsilon - \mu(S)}{\mu(A)}.$$

Furthermore, let

$$S' = \bigcup \left\{ S^{\mathcal{Q}}_{m_0} : \mathcal{Q} \in \mathfrak{P}(S) \right\} \quad \text{and} \quad u = \frac{\lambda c_{m_0} \mu(S')}{2(\varepsilon - \mu(S))}\, e.$$

Again, pick an arbitrary element $\mathcal{Q} \in \mathfrak{P}(A)$. By Corollary 2.1.5, there exists a sequence $\{B^{\mathcal{Q}}_k : k \geq 1\} \subseteq \Sigma_{A \cap T_{\mathcal{Q}}}$ such that $\mu\left(B^{\mathcal{Q}}_k\right) = (1-\lambda)\mu(A \cap T_{\mathcal{Q}})$ and for all $\omega \in \Omega$,

$$\lim_{k\to\infty} \int_{B^{\mathcal{Q}}_k} (a(\cdot,\omega) - f(\cdot,\omega) - u)d\mu = (1-\lambda) \int_{A \cap T_{\mathcal{Q}}} (a(\cdot,\omega) - f(\cdot,\omega) - u)d\mu.$$

The function $d^{\mathcal{Q}}_k : \Omega \to Y_+$, defined by

$$d^{\mathcal{Q}}_k(\omega) = (1-\lambda) \int_{A \cap T_{\mathcal{Q}}} (a(\cdot,\omega) - f(\cdot,\omega) - u)d\mu - \int_{B^{\mathcal{Q}}_k} (a(\cdot,\omega) - f(\cdot,\omega) - u)\, d\mu,$$

is $\mathcal{Q}$-measurable for all $k \geq 1$ and $\lim_{k\to\infty} \|d^{\mathcal{Q}}_k(\omega)\| = 0$ for all $\omega \in \Omega$. Choose some $k_{\mathcal{Q}}$ such that

$$u - \frac{d^{\mathcal{Q}}_{k_{\mathcal{Q}}}(\omega)}{(1-\lambda)\mu(A \cap T_{\mathcal{Q}})} \gg 0$$

for each $\omega \in \Omega$, and then consider the function $f^{\mathcal{Q}} : B^{\mathcal{Q}}_{k_{\mathcal{Q}}} \times \Omega \to Y_+$ defined by

$$f^{\mathcal{Q}}(t,\omega) = f(t,\omega) + u - \frac{1}{(1-\lambda)\mu(A \cap T_{\mathcal{Q}})}\, d^{\mathcal{Q}}_{k_{\mathcal{Q}}}(\omega).$$

Obviously, $f^{\mathcal{Q}}(t,\cdot) \in L_t$ for almost all $t \in B^{\mathcal{Q}}_{k_{\mathcal{Q}}}$. By ($\mathbf{A}_3$), $f^{\mathcal{Q}}(t,\cdot) \in P_t(f(t,\cdot))$ for all $t \in B^{\mathcal{Q}}_{k_{\mathcal{Q}}}$. Furthermore, for each $\omega \in \Omega$,

$$\int_{B^{\mathcal{Q}}_{k_{\mathcal{Q}}}} \left(a(\cdot,\omega) - f^{\mathcal{Q}}(\cdot,\omega)\right) d\mu = (1-\lambda) \int_{A \cap T_{\mathcal{Q}}} (a(\cdot,\omega) - f(\cdot,\omega) - u)d\mu. \tag{4.3}$$

Let $B = \bigcup\{B^{\mathscr{Q}}_{k_{\mathscr{Q}}} : \mathscr{Q} \in \mathfrak{P}(A)\}$. Then, $\mu(B) = (1-\lambda)\mu(A)$. Now, define a function $f_\lambda : B \times \Omega \to Y_+$ such that $f_\lambda(t,\omega) = f^{\mathscr{Q}}(t,\omega)$ if $(t,\omega) \in B^{\mathscr{Q}}_{k_{\mathscr{Q}}} \times \Omega$, and for any $\mathscr{Q} \in \mathfrak{P}(S)$, consider the function $\hat{y}^{\mathscr{Q}} : (S \cap T_{\mathscr{Q}}) \times \Omega \to Y_+$ defined by

$$\hat{y}^{\mathscr{Q}}(t,\omega) = \begin{cases} y^{\mathscr{Q}}(t,\omega) - \frac{c_{m_0}}{2}e, & \text{if } (t,\omega) \in S^{\mathscr{Q}}_{m_0} \times \Omega; \\ y^{\mathscr{Q}}(t,\omega), & \text{otherwise.} \end{cases}$$

Recall that $y^{\mathscr{Q}}(t,\omega) \gg g^{\mathscr{Q}}_{m_0}(t,\omega) + c_{m_0}e$ for all $(t,\omega) \in S^{\mathscr{Q}}_{m_0} \times \Omega$ and $g^{\mathscr{Q}}_{m_0}(t,\cdot) \in P_t(f(t,\cdot))$ for almost all $t \in S^{\mathscr{Q}}_{m_0}$. By $(\mathbf{A}_3)$, $\hat{y}^{\mathscr{Q}}(t,\cdot) \in P_t(f(t,\cdot))$ for almost all $t \in S^{\mathscr{Q}}_{m_0}$. So, $\hat{y}^{\mathscr{Q}}(t,\cdot) \in L_t$ and $\hat{y}^{\mathscr{Q}}(t,\cdot) \in P_t(f(t,\cdot))$ for almost all $t \in S \cap T_{\mathscr{Q}}$. Take

$$\widehat{S} = \bigcup\{S \cap T_{\mathscr{Q}} : \mathscr{Q} \in \mathfrak{P}(S)\}.$$

Then, $\mu(\widehat{S}) = \mu(S)$. Define $y_\lambda : \widehat{S} \times \Omega \to Y_+$ by $y_\lambda(t,\omega) = \hat{y}^{\mathscr{Q}}(t,\omega)$ if $(t,\omega) \in (S \cap T_{\mathscr{Q}}) \times \Omega$. It can be checked that for each $\omega \in \Omega$,

$$\textbf{(4.4)} \qquad \int_{\widehat{S}} a(\cdot,\omega)d\mu - \int_{\widehat{S}} y_\lambda(\cdot,\omega)d\mu = \frac{c_{m_0}\mu(S')}{2}e.$$

For any given $\mathscr{Q} \in \mathfrak{P}(S)$, define the function $x^{\mathscr{Q}} : (S \cap T_{\mathscr{Q}}) \times \Omega \to Y_+$ by

$$x^{\mathscr{Q}}(t,\omega) = \begin{cases} g^{\mathscr{Q}}_{m_0}(t,\omega), & \text{if } (t,\omega) \in S^{\mathscr{Q}}_{m_0} \times \Omega; \\ g(t,\omega), & \text{otherwise,} \end{cases}$$

and consider another function $p^{\mathscr{Q}} : (S \cap T_{\mathscr{Q}}) \times \Omega \to Y_+$ defined by

$$p^{\mathscr{Q}}(t,\omega) = y^{\mathscr{Q}}(t,\omega) - x^{\mathscr{Q}}(t,\omega).$$

Note that $p^{\mathscr{Q}}(t,\omega) \gg c_{m_0}e$ for all $(t,\omega) \in S^{\mathscr{Q}}_{m_0} \times \Omega$, $p^{\mathscr{Q}}(t,\omega) = 0$ for all $(t,\omega) \in ((S \cap T_{\mathscr{Q}}) \setminus S^{\mathscr{Q}}_{m_0}) \times \Omega$ and $p^{\mathscr{Q}}(t,\cdot) \in L_t$ for almost all $t \in S \cap T_{\mathscr{Q}}$. Define a function $\delta^{\mathscr{Q}} : \Omega \to Y_+$ such that for each $\omega \in \Omega$,

$$\delta^{\mathscr{Q}}(\omega) = \lambda\left(\int_{S^{\mathscr{Q}}_{m_0}} p^{\mathscr{Q}}(\cdot,\omega)d\mu - \frac{c_{m_0}\mu(S^{\mathscr{Q}}_{m_0})}{2}e\right).$$

Then $\delta^{\mathscr{Q}}(\omega) \gg 0$ for each $\omega \in \Omega$ and $\delta^{\mathscr{Q}}$ is $\mathscr{Q}$-measurable. Choose an open neighborhood

W of 0 such that $\delta^{\mathscr{Q}}(\omega) - W \subseteq \mathrm{int}Y_+$ for all $\omega \in \Omega$. Note that

$$\int_{S\cap T_{\mathscr{Q}}} x^{\mathscr{Q}} d\mu \in \mathrm{cl} \int_{S\cap T_{\mathscr{Q}}} P_f d\mu,$$

where $P_f : T \rightrightarrows Y_+^{\Omega}$ is defined in Subsection 2.2.2. By ($\mathbf{A}_3$),

$$\int_{S\cap T_{\mathscr{Q}}} f d\mu \in \mathrm{cl} \int_{S\cap T_{\mathscr{Q}}} P_f d\mu.$$

Since $\mathrm{cl} \int_{S\cap T_{\mathscr{Q}}} P_f d\mu$ is convex,

$$\lambda \int_{S\cap T_{\mathscr{Q}}} x^{\mathscr{Q}} d\mu + (1-\lambda) \int_{S\cap T_{\mathscr{Q}}} f d\mu \in \mathrm{cl} \int_{S\cap T_{\mathscr{Q}}} P_f d\mu.$$

It follows that

$$\left(\lambda \int_{S\cap T_{\mathscr{Q}}} x^{\mathscr{Q}} d\mu + (1-\lambda) \int_{S\cap T_{\mathscr{Q}}} f d\mu + W^{\Omega}\right) \bigcap \int_{S\cap T_{\mathscr{Q}}} P_f d\mu \neq \emptyset.$$

So, there exist a $\mathscr{Q}$-measurable element $u^{\mathscr{Q}} \in W^{\Omega}$ and an integrable selection $v^{\mathscr{Q}}$ of P_f such that

$$\lambda \int_{S\cap T_{\mathscr{Q}}} x^{\mathscr{Q}} d\mu + (1-\lambda) \int_{S\cap T_{\mathscr{Q}}} f d\mu + u^{\mathscr{Q}} = \int_{S\cap T_{\mathscr{Q}}} v^{\mathscr{Q}} d\mu.$$

Define a function $h^{\mathscr{Q}} : (S \cap T_{\mathscr{Q}}) \times \Omega \to Y_+$ such that for all $(t,\omega) \in (S \cap T_{\mathscr{Q}}) \times \Omega$,

$$h^{\mathscr{Q}}(t,\omega) = v^{\mathscr{Q}}(t,\omega) + \frac{1}{\mu(S \cap T_{\mathscr{Q}})} \left(\delta^{\mathscr{Q}}(\omega) - u^{\mathscr{Q}}(\omega)\right).$$

By ($\mathbf{A}_3$), $h^{\mathscr{Q}}(t,\cdot) \in P_t(f(t,\cdot))$ and $h^{\mathscr{Q}}(t,\cdot) \in L_t$ for almost all $t \in S \cap T_{\mathscr{Q}}$, and

$$\int_{S\cap T_{\mathscr{Q}}} h^{\mathscr{Q}}(\cdot,\omega) d\mu = \lambda \int_{S\cap T_{\mathscr{Q}}} x^{\mathscr{Q}}(\cdot,\omega) d\mu + (1-\lambda) \int_{S\cap T_{\mathscr{Q}}} f(\cdot,\omega) d\mu + \delta^{\mathscr{Q}}(\omega).$$

Let $h_\lambda : \widehat{S} \times \Omega \to Y_+$ be defined by $h_\lambda(t,\omega) = h^{\mathscr{Q}}(t,\omega)$ if $(t,\omega) \in (S \cap T_{\mathscr{Q}}) \times \Omega$. Then, $h_\lambda(t,\cdot) \in L_t$ and $h_\lambda(t,\cdot) \in P_t(f(t,\cdot))$ for almost all $t \in \widehat{S}$. Finally, it can be simply verified that for each $\omega \in \Omega$,

$$\int_{\widehat{S}} h_\lambda(\cdot,\omega) d\mu = \lambda \int_{\widehat{S}} y_\lambda(\cdot,\omega) d\mu + (1-\lambda) \int_{\widehat{S}} f(\cdot,\omega) d\mu. \qquad (\mathbf{4.5})$$

Let $\widetilde{S} = \widehat{S} \cup B$. By the selection of λ, one has $\mu(\widetilde{S}) = \mu(S) + (1-\lambda)\mu(A) = \varepsilon$. It only remains to verify that $\widetilde{S}$ NY-privately blocks f. To this end, consider the function

$g_\lambda : T \times \Omega \to Y_+$ defined by

$$g_\lambda(t,\omega) = \begin{cases} h_\lambda(t,\omega), & \text{if } (t,\omega) \in \widehat{S} \times \Omega; \\ f_\lambda(t,\omega), & \text{if } (t,\omega) \in B \times \Omega; \\ g(t,\omega), & \text{otherwise.} \end{cases}$$

Obviously, g_λ is an allocation satisfying $g_\lambda(t,\cdot) \in P_t(f(t,\cdot))$ for almost all $t \in \widetilde{S}$. Furthermore, using (**4.3**)- (**4.5**), it can be simply verified that

$$\int_{\widetilde{S}} (a(\cdot,\omega) - g_\lambda(\cdot,\omega))d\mu = (1-\lambda)\int_T (a(\cdot,\omega) - f(\cdot,\omega))d\mu = 0$$

holds for all $\omega \in \Omega$. This completes the proof. □

This section concludes with an extension of Grodal's theorem in [44] to an asymmetric information economy with an ordered Banach space Y having an interior point in its positive cone as the commodity space and without free disposal. When $Y = \mathbb{R}^\ell$, an extension of Grodal's theorem to an asymmetric information economy is straightforward, where the number of sub-coalitions is at most $|\Omega|\ell$. In the case of infinitely many commodities, the situation is quite different, and there is no immediate extension of Grodal's theorem. As an application of Theorem 4.2.2, similar to Corollary 2 in [36], one has the following extension of Grodal's theorem.

Theorem 4.2.3. *Assume* $(\mathbf{A}_1)$-$(\mathbf{A}_4)$. *Let T be endowed with a pseudometric which makes T a separable topological space such that $\mathscr{B}(T) \subseteq \Sigma$, f an exactly feasible allocation and $f \notin \mathscr{PC}^{NY}(\mathscr{E})$. For any $\varepsilon, \delta > 0$, there exists a coalition R with $\mu(R) \le \varepsilon$ privately blocking f and $R = \bigcup_{i=1}^n R_i$ for a finite collection of coalitions $\{R_1, ..., R_n\}$ with diameter of R_i smaller than δ for all $i = 1, ..., n$.*

Proof. By Theorem 4.2.2, there exists a coalition S with $\mu(S) = \varepsilon$ privately blocking f via g. It is claimed that there exists a $\eta > 0$ such that any coalition $E \subseteq S$ satisfying $\mu(S \setminus E) < \eta$ privately blocks f via some assignment h. To verify the claim, applying an argument similar to that in Theorem 4.2.2, for each $\mathscr{Q} \in \mathfrak{P}(S)$, one can find a function $y^{\mathscr{Q}} : (S \cap T_{\mathscr{Q}}) \times \Omega \to Y_+$ such that $y^{\mathscr{Q}}(t,\cdot) \in L_t$ and $y^{\mathscr{Q}}(t,\cdot) \in P_t(f(t,\cdot))$ for almost all $t \in S \cap T_{\mathscr{Q}}$, and (**4.1**) holds. By absolute continuity of the Bochner integral, there is an $\eta_1 > 0$ such that

$$\frac{2}{\lambda\mu(S \cap T_{\mathscr{Q}})} \int_{E_{\mathscr{Q}}} \left(y^{\mathscr{Q}}(\cdot,\omega) - a(\cdot,\omega) \right) d\mu \in c_{m_0} U$$

for all $\omega \in \Omega$ and $E_{\mathscr{Q}} \in \Sigma_{S \cap T_{\mathcal{Q}}}$ satisfying $\mu(E_{\mathscr{Q}}) < \eta_1$ and all $\mathscr{Q} \in \mathfrak{P}(S)$. Further, for all $E_{\mathscr{Q}} \in \Sigma_{S \cap T_{\mathscr{Q}}}$ with $\mu(E_{\mathscr{Q}}) > \lambda \mu(S \cap T_{\mathscr{Q}})$, one has

$$\mu(E_{\mathscr{Q}} \cap S^{\mathscr{Q}}_{m_0}) > \frac{\lambda}{2} \mu(S \cap T_{\mathscr{Q}}).$$

Choose $\eta > 0$ such that

$$\eta = \min\{\eta_1, (1-\lambda)\mu(S \cap T_{\mathscr{Q}}) : \mathscr{Q} \in \mathfrak{P}(S)\}.$$

Pick any $E_{\mathscr{Q}} \in \Sigma_{S \cap T_{\mathscr{Q}}}$ such that $\mu((S \cap T_{\mathscr{Q}}) \setminus E_{\mathscr{Q}}) < \eta$. Then,

$$\mu(E_{\mathscr{Q}} \cap S^{\mathscr{Q}}_{m_0}) > \frac{\lambda}{2} \mu(S \cap T_{\mathscr{Q}}).$$

Let

$$b^{\mathscr{Q}}(\omega) = \frac{1}{\mu(E_{\mathscr{Q}} \cap S^{\mathscr{Q}}_{m_0})} \int_{(S \cap T_{\mathscr{Q}}) \setminus E_{\mathscr{Q}}} \Big(y^{\mathscr{Q}}(\cdot, \omega) - a(\cdot, \omega) \Big) \, d\mu$$

for each $\omega \in \Omega$. It follows that $c_{m_0} e + b^{\mathscr{Q}}(\omega) \gg 0$ for all $\omega \in \Omega$. Define a function $g^{\mathscr{Q}} : E_{\mathscr{Q}} \times \Omega \to Y_+$ such that

$$g^{\mathscr{Q}}(t, \omega) = \begin{cases} y^{\mathscr{Q}}(t,\omega) + b^{\mathscr{Q}}(\omega), & \text{if } (t,\omega) \in \big(E_{\mathscr{Q}} \cap S^{\mathscr{Q}}_{m_0}\big) \times \Omega; \\ y^{\mathscr{Q}}(t,\omega), & \text{otherwise.} \end{cases}$$

Recall that $y^{\mathscr{Q}}(t,\omega) \gg g^{\mathscr{Q}}_{m_0}(t,\omega) + c_{m_0} e$ and $g^{\mathscr{Q}}_{m_0}(t,\cdot) \in P_t(f(t,\cdot))$ for almost all $t \in E_{\mathscr{Q}} \cap S^{\mathscr{Q}}_{m_0}$. By $(\mathbf{A}_3)$, one has $g^{\mathscr{Q}}(t,\cdot) \in P_t(f(t,\cdot))$ for almost all $t \in E_{\mathscr{Q}} \cap S^{\mathscr{Q}}_{m_0}$. Thus, $g^{\mathscr{Q}}(t,\cdot) \in L_t$ and $g^{\mathscr{Q}}(t,\cdot) \in P_t(f(t,\cdot))$ for almost all $t \in E_{\mathscr{Q}}$, and

$$\int_{E_{\mathscr{Q}}} (g^{\mathscr{Q}}(\cdot,\omega) - a(\cdot,\omega)) d\mu = \int_{S \cap T_{\mathscr{Q}}} \Big(y^{\mathscr{Q}}(\cdot,\omega) - a(\cdot,\omega) \Big) \, d\mu$$

for all $\omega \in \Omega$. Now for any $E \in \Sigma_S$ with $\mu(S \setminus E) < \eta$, $\mu((S \cap T_{\mathscr{Q}}) \setminus (E \cap T_{\mathscr{Q}})) < \eta$ for all $\mathscr{Q} \in \mathfrak{P}(S)$. So, for each $\mathscr{Q} \in \mathfrak{P}(S)$, there is a function $g^{\mathscr{Q}} : (E \cap T_{\mathscr{Q}}) \times \Omega \to Y_+$ such that $g^{\mathscr{Q}}(t,\cdot) \in L_t$ and $g^{\mathscr{Q}}(t,\cdot) \in P_t(f(t,\cdot))$ for almost all $t \in E \cap T_{\mathscr{Q}}$, and

$$\int_{E \cap T_{\mathscr{Q}}} (g^{\mathscr{Q}}(\cdot,\omega) - a(\cdot,\omega)) d\mu = \int_{S \cap T_{\mathscr{Q}}} \Big(y^{\mathscr{Q}}(\cdot,\omega) - a(\cdot,\omega) \Big) \, d\mu$$

for all $\omega \in \Omega$. Thus, the assignment $h : T \times \Omega \to Y_+$, defined by $h(t,\omega) = g^{\mathscr{Q}}(t,\omega)$ if $(t,\omega) \in (E \cap T_{\mathscr{Q}}) \times \Omega$, $\mathscr{Q} \in \mathfrak{P}(S)$; and $h(t,\omega) = g(t,\omega)$, otherwise, satisfies the claim. The rest of the proof is the same as that of Corollary 2 in [36]. $\square$

4.3 The Fine Core and the Strong Fine Core

In this section, the concepts of the fine core and the strong fine core are introduced in an asymmetric information economy, where agents pool their information when they form a coalition. Thus, it is natural for one to ask whether a result similar to Theorem 4.2.2 holds for these cores. The main purpose of this section is to explore this issue.

Definition 4.3.1. [87] An allocation f in $\mathscr{E}$ is *fine blocked* by a coalition S if there is an assignment g such that $g(t,\cdot)$ is $\bigvee \mathfrak{P}_S$-measurable and $g(t,\cdot) \in P_t(f(t,\cdot))$ for almost all $t \in S$, and

$$\int_S g(\cdot,\omega)d\mu = \int_S a(\cdot,\omega)d\mu$$

for all $\omega \in \Omega$. The *fine core* of $\mathscr{E}$, denoted by $\mathscr{FC}(\mathscr{E})$, is the set of all exactly feasible allocations which are not fine blocked by any coalition of $\mathscr{E}$.

The next lemma is inspired by Lemma 1 in [36]. It is closely related to Theorem 9 of [25], whose finite dimensional version was proved by Grodal in [43]. Note that Grodal's result has been used in the proofs of the main results of [48] and [84].

Lemma 4.3.1. *Assume* $(\mathbf{A}_1)$, $(\mathbf{A}_2)$ *and* $(\mathbf{A}_4)$. *If an allocation f is fine blocked by a coalition S of $\mathscr{E}$, then there exists an assignment g such that*

(i) $\int_S (a(\cdot,\omega) - g(\cdot,\omega))d\mu \gg 0$ *for all* $\omega \in \Omega$;

(ii) $g(t,\cdot)$ *is* $\bigvee \mathfrak{P}_S$*-measurable and* $g(t,\cdot) \in P_t(f(t,\cdot))$ *for almost all* $t \in S$.

Proof. Suppose that f is fine blocked by S via h. Let $\{c_m : m \geq 1\}$ be a decreasing sequence in $(0,1)$ converging to 0. For each $m \geq 1$, define a function $h_m : S \times \Omega \to Y_+$ such that $h_m(t,\omega) = (1 - c_m)h(t,\omega)$. It is clear that $h_m(t,\cdot)$ is $\bigvee \mathfrak{P}_S$-measurable for almost all $t \in S$. By Lemma 4.2.1, there is a monotonically increasing sequence $\{S_m : m \geq 1\} \subseteq \Sigma_S$ such that $\lim_{m\to\infty} \mu(S \setminus S_m) = 0$ and $h_m(t,\cdot) \in P_t(f(t,\cdot))$ for almost all $t \in S_m$. Next, for each $m \geq 1$, consider the function $g_m : T \times \Omega \to Y_+$ defined by

$$g_m(t,\omega) = \begin{cases} h(t,\omega), & \text{if } (t,\omega) \in (T \setminus S_m) \times \Omega; \\ h_m(t,\omega), & \text{if } (t,\omega) \in S_m \times \Omega. \end{cases}$$

Then $g_m(t,\cdot)$ is $\bigvee \mathfrak{P}_S$-measurable and $g_m(t,\cdot) \in P_t(f(t,\cdot))$ for almost all $t \in S$. Now, for each $\omega \in \Omega$,

$$\int_S g_m(\cdot,\omega)d\mu = \int_{S\setminus S_m} (h(\cdot,\omega) - h_m(\cdot,\omega))d\mu + \int_S h_m(\cdot,\omega)d\mu.$$

In addition, $\int_S h_m(\cdot,\omega)d\mu = (1-c_m)\int_S a(\cdot,\omega)d\mu$ for all $\omega \in \Omega$. Consequently, one obtains

$$\int_S (a(\cdot,\omega) - g_m(\cdot,\omega))d\mu = c_m \left(\int_S a(\cdot,\omega)d\mu - \int_{S\setminus S_m} h(\cdot,\omega)d\mu \right).$$

By absolute continuity of the Bochner integral, one can choose an integer $m \geq 1$ sufficiently large such that

$$\int_S a(\cdot,\omega)d\mu - \int_{S\setminus S_m} h(\cdot,\omega)d\mu \gg 0$$

for all $\omega \in \Omega$. The proof is completed by letting $g = g_m$. □

Theorem 4.3.2. *Assume* $(\mathbf{A}_1)$-$(\mathbf{A}_4)$. *If f is an exactly feasible allocation and $f \notin \mathscr{FC}(\mathscr{E})$, then for any $0 < \varepsilon < \mu(T)$ there exists a coalition S in $\mathscr{E}$ which fine blocks f with $\mu(S) = \varepsilon$.*

Proof. By the definition of $\mathscr{FC}(\mathscr{E})$ and Lemma 4.3.1, there exist a coalition S and an assignment g such that (**i**) and (**ii**) in Lemma 4.3.1 hold. Define a function $z : \Omega \to Y_+$ such that for all $\omega \in \Omega$,

$$z(\omega) = \int_S (a(\cdot,\omega) - g(\cdot,\omega))d\mu. \tag{4.6}$$

Then $z(\omega) \gg 0$ for all $\omega \in \Omega$. Pick some δ with $0 < \delta < 1$. For any fixed $\mathscr{Q} \in \mathfrak{P}(S)$, by Corollary 2.1.5, there exists a sequence $\{F_n^{\mathscr{Q}} : n \geq 1\} \subseteq \Sigma_{S\cap T_{\mathscr{Q}}}$ such that for all $n \geq 1$, $\mu(F_n^{\mathscr{Q}}) = \delta\mu(S \cap T_{\mathscr{Q}})$ and

$$\lim_{n\to\infty} \int_{F_n^{\mathscr{Q}}} (a(\cdot,\omega) - g(\cdot,\omega))d\mu = \delta \int_{S\cap T_{\mathscr{Q}}} (a(\cdot,\omega) - g(\cdot,\omega))d\mu$$

for all $\omega \in \Omega$. For each $n \geq 1$, let

$$F_n = \left(\bigcup_{\mathscr{Q}\in\mathfrak{P}(S)} F_n^{\mathscr{Q}} \right) \bigcup \left(\bigcup_{\mathscr{Q}\in\mathfrak{P}_S\setminus\mathfrak{P}(S)} (S \cap T_{\mathscr{Q}}) \right).$$

Then, it can be easily verified that $\mu(F_n) = \delta\mu(S)$ for all $n \geq 1$ and

$$\lim_{n\to\infty} \int_{F_n} (a(\cdot,\omega) - g(\cdot,\omega))d\mu = \delta z(\omega)$$

for all $\omega \in \Omega$. Hence, there exists an n_0 such that $\int_{F_{n_0}}(a(\cdot,\omega) - g(\cdot,\omega))d\mu \gg 0$ for all $\omega \in \Omega$. Since $\bigvee \mathfrak{P}_{F_{n_0}} = \bigvee \mathfrak{P}_S$, the function $z_{n_0} : \Omega \to Y_+$, defined by

$$z_{n_0}(\omega) = \int_{F_{n_0}} (a(\cdot,\omega) - g(\cdot,\omega))d\mu$$

for all $\omega \in \Omega$, is $\bigvee \mathfrak{P}_{F_{n_0}}$-measurable. Define a function $\hat{g} : T \times \Omega \to Y_+$ such that

$$\hat{g}(t,\omega) = \begin{cases} g(t,\omega) + \frac{z_{n_0}(\omega)}{\delta\mu(S)}, & \text{if } (t,\omega) \in F_{n_0} \times \Omega; \\ g(t,\omega), & \text{otherwise.} \end{cases}$$

By $(\mathbf{A}_3)$, f is fine blocked by F_{n_0} via $\hat{g}$, which proves the theorem for $\varepsilon \leq \mu(S)$. If $\mu(S) = \mu(T)$, the proof has been completed. Otherwise, $\mu(T \setminus S) > 0$ and one needs to consider the case of $\varepsilon > \mu(S)$. Let $R = T \setminus S$. Again, by Corollary 2.1.5, there is a sequence $\{B_n^{\mathscr{Q}} : n \geq 1\} \subseteq \Sigma_{R \cap T_{\mathscr{Q}}}$ such that $\mu(B_n^{\mathscr{Q}}) = (1-\delta)\mu(R \cap T_{\mathscr{Q}})$ for all $n \geq 1$ and

$$\lim_{n\to\infty} \int_{B_n^{\mathscr{Q}}} (a(\cdot,\omega) - f(\cdot,\omega))d\mu = (1-\delta)\int_{R\cap T_{\mathscr{Q}}} (a(\cdot,\omega) - f(\cdot,\omega))d\mu$$

for all $\omega \in \Omega$. For each $n \geq 1$, let

$$B_n = \left(\bigcup_{\mathscr{Q}\in\mathfrak{P}(R)} B_n^{\mathscr{Q}}\right) \bigcup \left(\bigcup_{\mathscr{Q}\in\mathfrak{P}_R\setminus\mathfrak{P}(R)} (R \cap T_{\mathscr{Q}})\right).$$

For all $n \geq 1$, define a function $b_n : \Omega \to Y_+$ such that

$$b_n(\omega) = (1-\delta)\int_R (a(\cdot,\omega) - f(\cdot,\omega))d\mu - \int_{B_n} (a(\cdot,\omega) - f(\cdot,\omega))d\mu. \tag{4.7}$$

Then, b_n is $\bigvee \mathfrak{P}_{B_n}$-measurable for all $n \geq 1$, and $\lim_{n\to\infty} \|b_n(\omega)\| = 0$ for all $\omega \in \Omega$. Choose an $n_1 \geq 1$, an $x \gg 0$ and an open neighborhood W of 0 such that

$$\delta z(\omega) - b_{n_1}(\omega) - x - W \subseteq \mathrm{int}Y_+$$

for all $\omega \in \Omega$. Consider a correspondence $Q_f : S \rightrightarrows Y_+^\Omega$, defined by

$$Q_f(t) = \left\{y \in Y_+^\Omega : y \in P_t(f(t,\cdot)) \text{ and } y \text{ is} \bigvee \mathfrak{P}_S\text{-measurable}\right\}$$

for all $t \in S$. Note that $\int_S g d\mu \in \mathrm{cl}\int_S Q_f d\mu$, and by $(\mathbf{A}_3)$, $\int_S f d\mu \in \mathrm{cl}\int_S Q_f d\mu$. Since

$\mathrm{cl}\int_S Q_f d\mu$ is convex,

$$\delta \int_S g d\mu + (1-\delta) \int_S f d\mu \in \mathrm{cl} \int_S Q_f d\mu.$$

It follows that

$$\left(\delta \int_S g d\mu + (1-\delta) \int_S f d\mu + W^\Omega\right) \bigcap \int_S Q_f d\mu \neq \emptyset.$$

Hence, there exist a $\bigvee \mathfrak{P}_S$-measurable element $u \in W^\Omega$ and an integrable selection h of Q_f such that

$$\delta \int_S g d\mu + (1-\delta) \int_S f d\mu + u = \int_S h d\mu.$$

Consider the function $g_\delta : S \times \Omega \to Y_+$ defined by

$$g_\delta(t,\omega) = h(t,\omega) + \frac{1}{\mu(S)}(\delta z(\omega) - b_{n_1}(\omega) - x - u(\omega)).$$

Let $\widetilde{S} = S \cup B_{n_1}$. Note that $g_\delta(t,\cdot)$ is $\mathfrak{P}_{\widetilde{S}}$-measurable for almost all $t \in S$, and by $(\mathbf{A}_3)$, one has $g_\delta(t,\cdot) \in P_t(f(t,\cdot))$ for almost all $t \in S$. Furthermore, it can be simply checked that for each $\omega \in \Omega$,

$$\int_S g_\delta(\cdot,\omega) d\mu = \delta \int_S g(\cdot,\omega) d\mu + (1-\delta) \int_S f(\cdot,\omega) d\mu + \delta z(\omega) - b_{n_1}(\omega) - x. \tag{4.8}$$

Since $\mu(\widetilde{S}) = \mu(S) + (1-\delta)\mu(R)$, to have $\mu(\widetilde{S}) = \varepsilon$, δ can be chosen as

$$\delta = 1 - \frac{\varepsilon - \mu(S)}{\mu(R)}.$$

Thus, to complete the proof, it remains to verify that f is fine blocked by $\widetilde{S}$. To this end, define $y_\delta : T \times \Omega \to Y_+$ by

$$y_\delta(t,\omega) = \begin{cases} g_\delta(t,\omega), & \text{if } (t,\omega) \in S \times \Omega; \\ f(t,\omega) + \frac{1}{\mu(B_{n_1})} x, & \text{otherwise.} \end{cases}$$

By $(\mathbf{A}_3)$, $y_\delta(t,\cdot) \in P_t(f(t,\cdot))$ for almost all $t \in \widetilde{S}$. Furthermore, $y_\delta(t,\cdot)$ is $\bigvee \mathfrak{P}_{\widetilde{S}}$-measurable for almost all $t \in \widetilde{S}$. By (**4.6**)-(**4.8**), one can conclude that

$$\int_{\widetilde{S}} (a(\cdot,\omega) - y_\delta(\cdot,\omega)) d\mu = (1-\delta) \int_T (a(\cdot,\omega) - f(\cdot,\omega)) d\mu = 0$$

for all $\omega \in \Omega$. This completes the proof. $\square$

In the literature, there is a sightly different version of the core concept, namely the strong core, see [30, 36]. An extension of this concept to an asymmetric information economy was given in [9, 72]. Next, a variation of the fine core is introduced.

Definition 4.3.2. An allocation f in $\mathscr{E}$ is *strongly fine blocked* by a coalition S if there exist a sub-coalition S_0 of S and an assignment g such that $g(t, \cdot)$ is $\bigvee \mathfrak{P}_S$-measurable and $g(t, \cdot) \in P_t^{\sim}(f(t, \cdot))$ for almost all $t \in S$, $g(t, \cdot) \in P_t(f(t, \cdot))$ for almost all $t \in S_0$ and

$$\int_S g(\cdot, \omega) d\mu = \int_S a(\cdot, \omega) d\mu$$

for all $\omega \in \Omega$. The *strong fine core* of $\mathscr{E}$, denoted by $\mathscr{F}\mathscr{C}^s(\mathscr{E})$, is the set of all exactly feasible allocations which are not strongly fine blocked by any coalition.

A Vind-type theorem for the strong fine core directly follows from Theorem 4.3.2 and the following lemma.

Lemma 4.3.3. *Assume* $(\mathbf{A}_1)$, $(\mathbf{A}_2)$, $(\mathbf{A}_3')$ *and* $(\mathbf{A}_4)$. *If f is an exactly feasible allocation and $f \notin \mathscr{F}\mathscr{C}^s(\mathscr{E})$, then $f \notin \mathscr{F}\mathscr{C}(\mathscr{E})$.*

Proof. Suppose that f is strongly fine blocked by S. Then there are a sub-coalition S_0 of S and an assignment y such that $y(t, \cdot)$ is $\bigvee \mathfrak{P}_S$-measurable and $y(t, \cdot) \in P_t^{\sim}(f(t, \cdot))$ for almost all $t \in S$, $y(t, \cdot) \in P_t(f(t, \cdot))$ for almost all $t \in S_0$ and

$$\int_S y(\cdot, \omega) d\mu = \int_S a(\cdot, \omega) d\mu$$

for all $\omega \in \Omega$. If $\mu(S_0) = \mu(S)$, then there is nothing to verify. Assume that $\mu(S_0) < \mu(S)$. By $(\mathbf{A}_3')$ and the fact that $y(t, \cdot) \in P_t(f(t, \cdot))$ for almost all $t \in S_0$, there exist an atom A of $\bigvee \mathfrak{P}_S$ and a sub-coalition S_1 of S_0 such that $y(t, \omega) > 0$ for almost all $t \in S_1$ and all $\omega \in A$. Let $\{c_m : m \geq 1\}$ be a monotonically deceasing sequence in $(0, 1)$ converging to 0. For each $m \geq 1$, define a function $y_m : S_1 \times \Omega \to Y_+$ such that

$$y_m(t, \omega) = (1 - c_m) y(t, \omega).$$

Then $y_m(t, \cdot)$ is $\bigvee \mathfrak{P}_S$-measurable for almost all $t \in S_1$. By Lemma 4.2.1, there is a sub-coalition S_m of S_1 such that $y_m(t, \cdot) \in P_t(f(t, \cdot))$ for almost all $t \in S_m$. Define a function $b : \Omega \to Y_+$ such that $b(\omega) = c_m \int_{S_m} y(\cdot, \omega) d\mu$ for all $\omega \in \Omega$, and consider the function $\hat{y} : (S \setminus S_0) \times \Omega \to Y_+$ defined by

$$\hat{y}(t,\omega) = y(t,\omega) + \frac{b(\omega)}{\mu(S \setminus S_0)}.$$

Furthermore, define another function $h : T \times \Omega \to Y_+$ by

$$h(t,\omega) = \begin{cases} y_m(t,\omega), & \text{if } (t,\omega) \in S_m \times \Omega; \\ \hat{y}(t,\omega), & \text{if } (t,\omega) \in (S \setminus S_0) \times \Omega; \\ y(t,\omega), & \text{otherwise.} \end{cases}$$

Then, $\hat{y}(t,\cdot)$ is $\bigvee \mathfrak{P}_S$-measurable for almost all $t \in S \setminus S_0$. By $(\mathbf{A}_3')$, $\hat{y}(t,\cdot) \in P_t(f(t,\cdot))$ for almost all $t \in S \setminus S_0$. It follows that $h(t,\cdot)$ is $\bigvee \mathfrak{P}_S$-measurable and $h(t,\cdot) \in P_t(f(t,\cdot))$ for almost all $t \in S$, and $\int_S h(\cdot,\omega)d\mu = \int_S a(\cdot,\omega)d\mu$ for all $\omega \in \Omega$. This verifies that $f \notin \mathscr{F}\mathscr{C}(\mathscr{E})$. $\square$

Chapter 5

Veto Mechanism by the Grand Coalition

In this chapter, the economic model given in Subsection 2.2.2 is considered. Three different characterizations of Walrasian allocations are investigated by the veto power of the grand coalition in asymmetric information economies having infinite dimensional commodity spaces. To do these, some results on the privately blocking power of coalitions are established in Section 5.1. One of them is an extension of Theorem 2 in [83] to an asymmetric information economy with an ordered Banach space whose positive cone has an interior point as the commodity space. The first characterization theorem is presented in Section 5.2 and claims that an allocation is a Walrasian allocation if and only if it is robustly efficient in a mixed economy with an ordered separable Banach space admitting an interior point in its positive cone as the commodity space. In this section, it is also shown that a similar characterization in [41] is a particular case of the above characterization theorem under some restrictions on the space of agents. Section 5.3 is confined to a discrete economy. In this section, two characterizations of Walrasian allocations are given. The first one deals with the Aubin non-dominated allocations and the other is interpreted in terms of privately non-dominated allocations in economies with finitely many agents and Banach lattices as commodity spaces. The main results in the first two sections are taken from [21] and the main results in the last section are taken from [22]. Throughout this chapter excluding the last section, it is assumed that the positive cone of the commodity space has non-empty interior, that is, $\text{int}Y_+ \neq \emptyset$.

5.1 Some Technical Results

In this section, some technical results are provided.

Lemma 5.1.1. [36] *Assume* ($\mathbf{A}_1$)-($\mathbf{A}_4$). *Let* f^* *be a feasible allocation of* $\mathscr{E}^*$ *and* $0 < \varepsilon < \mu^*(T^*)$. *If* $f^* \notin \mathscr{PC}(\mathscr{E}^*)$, *then there is a coalition* S^* *with* $\mu^*(S^*) = \varepsilon$ *privately blocking* f^*.

The next result is similar to Lemma 1 in [36] and Lemma 4.3.1, and for the sake of completeness, a full proof is given for Lemma 5.1.2.

Lemma 5.1.2. *Assume* ($\mathbf{A}_1$)-($\mathbf{A}_4$). *Suppose that an allocation* f^* *in* $\mathscr{E}^*$ *is privately blocked by a coalition* S^*. *Then there exists an assignment* g^* *such that*

(i) $g^*(t,\cdot) \in L_t$ *and* $g^*(t,\cdot) \in P_t(f^*(t,\cdot))$ *for almost all* $t \in S^*$;

(ii) $\int_{S^*}(a(\cdot,\omega) - g^*(\cdot,\omega))d\mu^* \gg 0$ *for all* $\omega \in \Omega$;

(iii) $g^*(t,\omega) \gg 0$ *for all* $(t,\omega) \in S^* \times \Omega$.

Proof. Since f^* is privately blocked by S^*, there is an assignment h^* such that $h^*(t,\cdot) \in L_t$ and $h^*(t,\cdot) \in P_t(f^*(t,\cdot))$ for almost all $t \in S^*$, and

$$\int_{S^*} h^*(\cdot,\omega)d\mu^* \leq \int_{S^*} a^*(\cdot,\omega)d\mu^*$$

for all $\omega \in \Omega$. Choose a monotonically decreasing sequence $\{c_m : m \geq 1\} \subset (0,1)$ converging to 0. For each $m \geq 1$, define $h^*_m : S^* \times \Omega \to Y_+$ by $h^*_m(t,\omega) = (1-c_m)h^*(t,\omega)$. By Lemma 4.2.1, there is a monotonically increasing sequence $\{S^*_m : m \geq 1\} \subseteq \Sigma^*_{S^*}$ such that $h^*_m(t,\cdot) \in P_t(f^*(t,\cdot))$ for almost all $t \in S^*_m$ and $\lim_{m\to\infty} \mu^*(S^* \setminus S^*_m) = 0$. Define $g^*_m : T^* \times \Omega \to Y_+$ such that

$$g^*_m(t,\omega) = \begin{cases} h^*(t,\omega), & \text{if } (t,\omega) \in (T^* \setminus S^*_m) \times \Omega; \\ h^*_m(t,\omega), & \text{if } (t,\omega) \in S^*_m \times \Omega. \end{cases}$$

Then $g^*_m(t,\cdot) \in L_t$ and $g^*_m(t,\cdot) \in P_t(f^*(t,\cdot))$ for almost all $t \in S^*$. Since

$$\int_{S^*} g^*_m(\cdot,\omega)d\mu^* = \int_{S^* \setminus S^*_m} (h^*(\cdot,\omega) - h^*_m(\cdot,\omega))d\mu^* + \int_{S^*} h^*_m(\cdot,\omega)d\mu^*$$

and

$$\int_{S^*} h^*_m(\cdot,\omega)d\mu^* \leq (1-c_m)\int_{S^*} a(\cdot,\omega)d\mu^*$$

hold for all $\omega \in \Omega$, one obtains

$$\int_{S^*} (a(\cdot, \omega) - g_m^*(\cdot, \omega)) d\mu^* \geq c_m \left(\int_{S^*} a(\cdot, \omega) d\mu^* - \int_{S^* \setminus S_m^*} h^*(\cdot, \omega) d\mu^* \right).$$

By absolute continuity of the Bochner integral, one can choose an integer $m \geq 1$ sufficiently large such that

$$\int_{S^*} a(\cdot, \omega) d\mu^* - \int_{S^* \setminus S_m^*} h^*(\cdot, \omega) d\mu^* \gg 0$$

for all $\omega \in \Omega$. Pick some $b \in \text{int}Y_+$ such that $\int_{S^*} (a(\cdot, \omega) - g_m^*(\cdot, \omega)) d\mu^* \gg \mu(S^*) b$ for all $\omega \in \Omega$, and define a function $g^* : T^* \times \Omega \to \text{int}Y_+$ such that $g^*(t, \omega) = g_m^*(t, \omega) + b$ for all $(t, \omega) \in T^* \times \Omega$. Then, one can readily verify that the assignment g^* is desired. $\square$

Now, a common extension of Lemma 3.1 in [48] and Theorem 2 in [83] is provided.

Theorem 5.1.3. *Assume* $(\mathbf{A}_1)$-$(\mathbf{A}_3)$ *and that* f^* *is an allocation in* $\mathscr{E}^*$. *Suppose there exist a coalition* S^*, *a sub-coalition* R^* *of* S^* *and an assignment* g^* *such that* $\mathfrak{P}(S^*) = \mathfrak{P}(R^*)$, $g^*(t, \omega) \gg 0$ *for all* $(t, \omega) \in R^* \times \Omega$, *and* $g^*(t, \cdot) \in L_t$ *and* $g^*(t, \cdot) \in P_t(f^*(t, \cdot))$ *for almost all* $t \in S^*$. *For each* $0 < r < 1$, *there exists an assignment* h^* *such that* $h^*(t, \cdot) \in L_t$ *and* $h^*(t, \cdot) \in P_t(f^*(t, \cdot))$ *for almost all* $t \in S^*$, *and*

$$\int_{S^*} h^*(\cdot, \omega) d\mu^* = \int_{S^*} \left(r g^*(\cdot, \omega) + (1 - r) f^*(\cdot, \omega) \right) d\mu^*$$

for all $\omega \in \Omega$.

Proof. Let $\{c_m : m \geq 1\}$ be a monotonically decreasing sequence in $(0, 1)$ converging to 0. For each $m \geq 1$, define a function $g_m^* : S^* \times \Omega \to Y_+$ such that

$$g_m^*(t, \omega) = (1 - c_m) g^*(t, \omega).$$

Then $g_m^*(t, \cdot) \in L_t$ for almost all $t \in S^*$ and $g_m^*(t, \omega) \gg 0$ for all $(t, \omega) \in R^* \times \Omega$. For any $\mathscr{Q} \in \mathfrak{P}(S^*)$, by Lemma 4.2.1, there is an increasing sequence $\left\{ S^*_{(m, \mathscr{Q})} : m \geq 1 \right\} \subseteq \Sigma^*_{S^* \cap T^*_{\mathscr{Q}}}$ such that $g_m^*(t, \cdot) \in P_t(f^*(t, \cdot))$ for almost all $t \in S^*_{(m, \mathscr{Q})}$ and

$$\lim_{m \to \infty} \mu^* \left((S^* \cap T^*_{\mathscr{Q}}) \setminus S^*_{(m, \mathscr{Q})} \right) = 0.$$

Choose an $m_{\mathscr{Q}}$ such that

$$\mu^* \left(R^* \cap S^*_{(m_{\mathscr{Q}}, \mathscr{Q})} \right) > 0,$$

and define the function $y^*_{\mathscr{Q}} : (S^* \cap T^*_{\mathscr{Q}}) \times \Omega \to Y_+$ by

$$y^*_{\mathscr{Q}}(t,\omega) = \begin{cases} g^*_{m_{\mathscr{Q}}}(t,\omega), & \text{if } (t,\omega) \in S^*_{(m_{\mathscr{Q}},\mathscr{Q})} \times \Omega; \\ g^*(t,\omega), & \text{otherwise.} \end{cases}$$

Obviously, $y^*_{\mathscr{Q}}(t,\cdot) \in L_t$ and $y^*_{\mathscr{Q}}(t,\cdot) \in P_t(f^*(t,\cdot))$ for almost all $t \in S^* \cap T^*_{\mathscr{Q}}$. Furthermore, for all $\omega \in \Omega$,

$$\textbf{(5.1)} \qquad \int_{S^*\cap T^*_{\mathscr{Q}}} y^*_{\mathscr{Q}}(\cdot,\omega)d\mu^* = \int_{S^*\cap T^*_{\mathscr{Q}}} g^*(\cdot,\omega)d\mu^* - c_{m_{\mathscr{Q}}} \int_{S^*_{(m_{\mathscr{Q}},\mathscr{Q})}} g^*(\cdot,\omega)d\mu^*.$$

Let $x_{\mathscr{Q}} \gg 0$ be chosen such that for all $\omega \in \Omega$,

$$\textbf{(5.2)} \qquad x_{\mathscr{Q}} \leq \frac{c_{m_{\mathscr{Q}}}}{2} \int_{S^*_{(m_{\mathscr{Q}},\mathscr{Q})}} g^*(\cdot,\omega)d\mu^*.$$

Given $0 < r < 1$, let $U(r,\mathscr{Q})$ be a neighborhood of 0 such that $rx_{\mathscr{Q}} - U(r,\mathscr{Q}) \subseteq \mathrm{int}Y_+$. By Corollary 2.1.5, there is a sequence $\left\{E^*_{(n,\mathscr{Q})} : n \geq 1\right\} \subseteq \Sigma^*_{S^*\cap T^*_{\mathscr{Q}}}$ such that

$$\lim_{n\to\infty} \left(\mu^*\left(E^*_{(n,\mathscr{Q})}\right), \int_{E^*_{(n,\mathscr{Q})}} \left(y^*_{\mathscr{Q}}(\cdot,\omega) - f^*(\cdot,\omega)\right) d\mu^*\right) = r\left(\mu^*\left(S^* \cap T^*_{\mathscr{Q}}\right),\ z^*_{\mathscr{Q}}(\omega)\right)$$

for each $\omega \in \Omega$, where $z^*_{\mathscr{Q}}(\omega) = \int_{S^*\cap T^*_{\mathscr{Q}}} (y^*_{\mathscr{Q}}(\cdot,\omega) - f^*(\cdot,\omega))d\mu^*$. Define a function $b^*_{(n,\mathscr{Q})} : \Omega \to Y$ such that

$$b^*_{(n,\mathscr{Q})}(\omega) = \int_{E^*_{(n,\mathscr{Q})}} (y^*_{\mathscr{Q}}(\cdot,\omega) - f^*(\cdot,\omega))d\mu^* - rz^*_{\mathscr{Q}}(\omega).$$

Then, $\lim_{n\to\infty} \|b^*_{(n,\mathscr{Q})}(\omega)\| = 0$ for all $\omega \in \Omega$. Thus, there exists an $n_{\mathscr{Q}}$ such that

$$\mu^*\left(E^*_{(n_{\mathscr{Q}},\mathscr{Q})}\right) < \mu^*(S^* \cap T^*_{\mathscr{Q}}) \text{ and } b^*_{(n_{\mathscr{Q}},\mathscr{Q})}(\omega) \in U(r,\mathscr{Q})$$

for all $\omega \in \Omega$. Now, consider the function $g^*_{\mathscr{Q}} : (S^* \cap T^*_{\mathscr{Q}}) \times \Omega \to Y_+$ defined by

$$g^*_{\mathscr{Q}}(t,\omega) = \begin{cases} y^*_{\mathscr{Q}}(t,\omega), & \text{if } (t,\omega) \in E^*_{(n_{\mathscr{Q}},\mathscr{Q})} \times \Omega; \\ f^*(t,\omega) + \frac{rx_{\mathscr{Q}}}{\mu^*\left((S^*\cap T^*_{\mathscr{Q}})\setminus E^*_{(n_{\mathscr{Q}},\mathscr{Q})}\right)}, & \text{otherwise.} \end{cases}$$

Note that $(\mathbf{A}_3)$ implies that $g^*_{\mathscr{Q}}(t,\cdot) \in P_t(f^*(t,\cdot))$ and $g^*_{\mathscr{Q}}(t,\cdot) \in L_t$ for almost all

$t \in S^* \cap T^*_{\mathscr{Q}}$. It can be easily verified that for all $\omega \in \Omega$,

$$\int_{S^*\cap T^*_{\mathscr{Q}}} (g^*_{\mathscr{Q}}(\cdot,\omega) - f^*(\cdot,\omega))d\mu^* = \int_{E^*_{(n_{\mathscr{Q}},\mathscr{Q})}} (y^*_{\mathscr{Q}}(\cdot,\omega) - f^*(\cdot,\omega))d\mu^* + rx_{\mathscr{Q}}.$$

On the other hand, since

$$\int_{E^*_{(n_{\mathscr{Q}},\mathscr{Q})}} (y^*_{\mathscr{Q}}(\cdot,\omega) - f^*(\cdot,\omega))d\mu^* - b^*_{(n_{\mathscr{Q}},\mathscr{Q})}(\omega) = r\int_{S^*\cap T^*_{\mathscr{Q}}} (y^*_{\mathscr{Q}}(\cdot,\omega) - f^*(\cdot,\omega))d\mu^*$$

for all $\omega \in \Omega$, one obtains

$$\int_{S^*\cap T^*_{\mathscr{Q}}} g^*_{\mathscr{Q}}(\cdot,\omega)d\mu^* \ll \int_{S^*\cap T^*_{\mathscr{Q}}} (ry^*_{\mathscr{Q}}(\cdot,\omega) + (1-r)f^*(\cdot,\omega))d\mu^* + 2rx_{\mathscr{Q}} \tag{5.3}$$

for each $\omega \in \Omega$. Combining (**5.1**)-(**5.3**), one further obtains that

$$\int_{S^*\cap T^*_{\mathscr{Q}}} g^*_{\mathscr{Q}}(\cdot,\omega)d\mu^* \ll \int_{S^*\cap T^*_{\mathscr{Q}}} (rg^*(\cdot,\omega) + (1-r)f^*(\cdot,\omega))d\mu^*$$

for each $\omega \in \Omega$. Consider a $\mathscr{Q}$-measurable function $d^*_{\mathscr{Q}} : \Omega \to Y_+$ defined by

$$d^*_{\mathscr{Q}}(\omega) = \frac{1}{\mu^*(S^*\cap T^*_{\mathscr{Q}})}\int_{S^*\cap T^*_{\mathscr{Q}}} \left(rg^*(\cdot,\omega) + (1-r)f^*(\cdot,\omega) - g^*_{\mathscr{Q}}(\cdot,\omega)\right)d\mu^*$$

for each $\omega \in \Omega$. It is clear that $d^*_{\mathscr{Q}}(\omega) \gg 0$ for each $\omega \in \Omega$. Define a function $h^*_{\mathscr{Q}} : (S^* \cap T^*_{\mathscr{Q}}) \times \Omega \to Y_+$ by

$$h^*_{\mathscr{Q}}(t,\omega) = g^*_{\mathscr{Q}}(t,\omega) + d^*_{\mathscr{Q}}(\omega).$$

Then, $h^*_{\mathscr{Q}}(t,\cdot) \in L_t$ and $h^*_{\mathscr{Q}}(t,\cdot) \in P_t(f^*(t,\cdot))$ for almost all $t \in S^* \cap T^*_{\mathscr{Q}}$, and

$$\int_{S^*\cap T^*_{\mathscr{Q}}} h^*_{\mathscr{Q}}(\cdot,\omega)d\mu^* = \int_{S^*\cap T^*_{\mathscr{Q}}} (rg^*(\cdot,\omega) + (1-r)f^*(\cdot,\omega))d\mu^*$$

for all $\omega \in \Omega$. Let $h^* : T^* \times \Omega \to Y_+$ be defined by

$$h^*(t,\omega) = \begin{cases} h^*_{\mathscr{Q}}(t,\omega), & \text{if } (t,\omega) \in (S^* \cap T^*_{\mathscr{Q}}) \times \Omega \text{ and } \mathscr{Q} \in \mathfrak{P}(S^*)\ ; \\ g^*(t,\omega), & \text{otherwise.} \end{cases}$$

It can be readily checked that h^* is the desired assignment. □

Remark 5.1.1. The conclusion of Lemma 5.1.2 also holds if (T^*, Σ^*, μ^*) is replaced by (T, Σ, μ). In addition, one should notice that Theorem 5.1.3 is still valid for (T, Σ, μ) if $S^* \subseteq T_0$. This fact will be used in the next section.

Remark 5.1.2. If a coalition S privately blocks an allocation f in an atomless economy and if $0 < r < 1$, in the light of Lemma 5.1.2 and Theorem 5.1.3, one has assignments g and h such that $g(t,\cdot),\ h(t,\cdot) \in L_t$, $g(t,\cdot) \in P_t(f(t,\cdot))$, $h(t,\cdot) \in P_t(f(t,\cdot))$ for almost all $t \in S$ and

$$\int_S h(\cdot,\omega)d\mu = \int_S (rg(\cdot,\omega) + (1-r)f(\cdot,\omega))d\mu$$

for all $\omega \in \Omega$.

Finally, an extension of a result in [42] to an infinite dimensional asymmetric information economy is established.

Lemma 5.1.4. *Assume* $(\mathbf{A}_1)$-$(\mathbf{A}_4)$ *and that* f^* *be an allocation in* $\mathscr{E}^*$ *privately blocked by a coalition* S^* *with* $\mu^*(S^* \cap T_1^*) \geq \varepsilon$ *for some* $\varepsilon > 0$. *Then there exist a coalition* R^* *and an assignment* g^* *such that*

(i) $\mu^*(R^* \cap T_1^*) = \varepsilon$ *and* $\int_{R^*}(a(\cdot,\omega) - g^*(\cdot,\omega))d\mu^* \gg 0$ *for all* $\omega \in \Omega$*;*

(ii) $g^*(t,\cdot) \in L_t$ *and* $g^*(t,\cdot) \in P_t(f^*(t,\cdot))$ *for almost all* $t \in R^*$.

Proof. If $\varepsilon = \mu^*(S^* \cap T_1^*)$, the conclusion directly follows from Lemma 5.1.2. Assume that $\varepsilon < \mu^*(S^* \cap T_1^*)$. Applying Lemma 5.1.2, one has an assignment g^* satisfying **(i)**-**(ii)** of Lemma 5.1.2. Let

$$\delta = \frac{\varepsilon}{\mu^*(S^* \cap T_1^*)}.$$

By Lemma 2.1.4, for each $\mathscr{Q} \in \mathfrak{P}(S^*)$, there exists a sequence $\left\{E^*_{(n,\mathscr{Q})} : n \geq 1\right\} \subseteq \Sigma^*_{S^* \cap T^*_{\mathscr{Q}}}$ such that for all $n \geq 1$,

$$\mu^*\left(E^*_{(n,\mathscr{Q})} \cap T_1^*\right) = \delta\mu^*(S^* \cap T^*_{\mathscr{Q}} \cap T_1^*)$$

and

$$\lim_{n\to\infty} \int_{E^*_{(n,\mathscr{Q})}} (a - g^*)d\mu^* = \delta \int_{S^* \cap T^*_{\mathscr{Q}}} (a - g^*)d\mu^*.$$

Let

$$E_n^* = \bigcup \left\{E^*_{(n,\mathscr{Q})} : \mathscr{Q} \in \mathfrak{P}(S^*)\right\}$$

for all $n \geq 1$. Then

$$\lim_{n\to\infty} \int_{E_n^*} (a - g^*) d\mu^* = \delta \int_{S^*} (a - g^*) d\mu^*.$$

Pick an n_0 such that $\int_{E_{n_0}^*} (a - g^*) d\mu^* \gg 0$, and put $R^* = E_{n_0}^*$. □

5.2 The Robust Efficiency

In this section, a characterization of a Walrasian allocation is obtained in a mixed economy by the private blocking power of the grand coalition. To do this, throughout this section, assume that all agents in T_1 having the same characteristics $(\mathscr{F}_{T_1}, P_{T_1}, a(T_1, \cdot))$. So for all $t \in T_1$, the common value of L_t is denoted by L_{T_1}.

The following concept was first introduced and studied in [48], where it was used to provide a new characterization of Walrasian allocations in a pure exchange economy with a continuum of non-atomic agents and finitely many commodities.

Definition 5.2.1. For any coalition S, allocation f in $\mathscr{E}$ and any $0 \leq r \leq 1$, suppose $\mathscr{E}(S, f, r)$ is an asymmetric information economy which coincides with $\mathscr{E}$ except for the initial endowment allocation that is given by

$$a(S, f, r)(t, \cdot) = \begin{cases} a(t, \cdot), & \text{if } t \in T \setminus S; \\ (1 - r)a(t, \cdot) + r f(t, \cdot), & \text{if } t \in S. \end{cases}$$

A feasible allocation f in $\mathscr{E}$ is said to be *robustly efficient* if f is not privately blocked by the grand coalition in every economy $\mathscr{E}(S, f, r)$.

For each feasible allocation f in $\mathscr{E}$, define the function $\mathbf{c}_f : \Omega \to Y_+$ such that

$$\mathbf{c}_f(\omega) = \frac{1}{\mu(T_1)} \int_{T_1} f(\cdot, \omega) d\mu$$

for all $\omega \in \Omega$. Let $\hat{f} : T \times \Omega \to Y_+$ be the allocation defined by

$$\hat{f}(t, \omega) = \begin{cases} f(t, \omega), & \text{if } (t, \omega) \in T_0 \times \Omega; \\ \mathbf{c}_f(\omega), & \text{if } (t, \omega) \in T_1 \times \Omega. \end{cases}$$

The following additional assumption will be needed in the sequel.

$(\mathbf{A}_7)$ For each feasible allocation f in $\mathscr{E}$ which is not privately blocked by the grand coalition in $\mathscr{E}(S, f, r)$ for any coalition S and real number r with $\mu(S) < \mu(T)$ and $0 \leq r \leq 1$, the set $\{x \in L_{T_1} : x \in P_{T_1}(\mathbf{c}_f)\}$ is convex.

The following example shows that $(\mathbf{A}_7)$ is weaker than the convexity of $P_{T_1}(x)$ for all $x \in Y_+^{\Omega}$.

Example 5.2.1. Consider a complete measure space (T, Σ, μ), where $T = [0,1] \cup \{2,3\}$, $[0,1]$ is endowed with the Lebesgue measure, and $\mu(2) = \mu(3) = 1$. For all $t \in [0,1]$, define $U_t : \mathbb{R}_+^2 \to \mathbb{R}$ by $U_t(x,y) = x + y$ for all $(x,y) \in \mathbb{R}_+^2$; and if $t \in \{2,3\}$, define $U_t : \mathbb{R}_+^2 \to \mathbb{R}$ by

$$U_t(x,y) = \begin{cases} \inf\{x+y, x^2+y^2\}, & \text{if } x+y \leq 2; \\ \\ x+y, & \text{otherwise.} \end{cases}$$

Further, for all $t \in T$, define $P_t : \mathbb{R}_+^2 \rightrightarrows \mathbb{R}_+^2$ by

$$P_t(x,y) = \left\{ (x',y') \in \mathbb{R}_+^2 : U_t\left(x',y'\right) > U_t\left(x,y\right) \right\}$$

for all $(x,y) \in \mathbb{R}_+^2$, and

$$A = \left\{ (x,y) \in \mathbb{R}_+^2 : x + y \leq 2 \right\}.$$

Note that for $t \in \{2,3\}$, $P_t(\frac{1}{2}, \frac{1}{2})$ is not convex and $P_t(x,y)$ is convex when $(x,y) \in \mathbb{R}_+^2 \setminus A$. Obviously, $(t,(x,y)) \mapsto U_t(x,y)$ is Carathéodory. Suppose that $\mathscr{E}$ is a deterministic economy whose space of agents is (T, Σ, μ) and commodity space is $\mathbb{R}^2$. It is assumed that the consumption set of each agent $t \in T$ is $\mathbb{R}_+^2$ and the preference relation of each agent $t \in T$ is P_t. By Remark 6 in [36], $(\mathbf{A}_1)$ is satisfied by $\mathscr{E}$. Let agent t's initial endowment be $a(t) = (16,16)$ if $t \in [0,1]$; and $a(t) = (8,8)$ if $t \in \{2,3\}$. Since $a \in \mathscr{W}(\mathscr{E})$, $\mathscr{W}(\mathscr{E}) \neq \emptyset$. Clearly, $\mathscr{E}$ satisfies assumptions $(\mathbf{A}_2)$-$(\mathbf{A}_4)$. It is claimed that if a feasible allocation f is not blocked by the grand coalition in $\mathscr{E}(S, f, r)$ for every coalition S with $\mu(S) < \mu(T)$ and $0 \leq r \leq 1$, then $f(2), f(3) \in \mathbb{R}_+^2 \setminus A$. Indeed, without loss of generality, suppose that $f(2) \in A$. Since f is not blocked by the grand coalition in $\mathscr{E}(f, S, 0)$ for any coalition S with $\mu(S) < \mu(T)$, one has

$$\int_{[0,1]} f d\mu + f(3) \gg (24, 24).$$

Choose $S = [0,1] \cup \{3\}$ and any $0 < r \leq 1$. Observe that $\int_T a(S,f,r) d\mu \gg (32,32)$, and thus f is blocked by the grand coalition in $\mathscr{E}(S, f, r)$. This is a contradiction, which verifies the claim. Since $\mathbb{R}^2_+ \setminus A$ is convex,

$$\mathbf{c}_f = \frac{1}{2}(f(2) + f(3)) \in \mathbb{R}^2_+ \setminus A.$$

Note that for all $t \in \{2, 3\}$,

$$\{(x, y) \in \mathbb{R}^2_+ : U_t(x, y) > U_t(\mathbf{c}_f)\} \subseteq \mathbb{R}^2_+ \setminus A.$$

Since U_t is concave on $\mathbb{R}^2_+ \setminus A$, then $P_t(x, y)$ is convex for all $t \in \{2, 3\}$. Hence, it is verified that $\mathscr{E}$ satisfies $(\mathbf{A}_7)$.

Theorem 5.2.2. *Assume* $(\mathbf{A}_1)$-$(\mathbf{A}_4)$, $(\mathbf{A}_7)$ *and* $|\mathscr{A}| \geq 1$. *If* f *is a robustly efficient allocation in* $\mathscr{E}$, *then* $\hat{f}(t, \cdot) \sim_{T_1} f(t, \cdot)$ *for all* $t \in T_1$.

Proof. Suppose that there exists a coalition $D \subseteq T_1$ such that $\hat{f}(t, \cdot) \in P_{T_1}(f(t, \cdot))$ for all $t \in D$. Note that the conclusion of Lemma 4.2.1 is also true if T_0 is replaced by D, and so one can find some $0 < r_1 < 1$ and a sub-coalition $C \subseteq D$ such that $r_1\hat{f}(t, \cdot) \in P_{T_1}(f(t, \cdot))$ for all $t \in C$. Let $r_2 = \frac{\mu(C)}{\mu(T_1)}$ and $r_3 = r_1 + \eta$ for some $\eta > 0$ such that $0 < r_3 < 1$. Then $0 < r_2 \leq 1$. Suppose that for each $\omega \in \Omega$,

$$\alpha(\omega) = r_2 r_3 \left(\int_T f(\cdot, \omega) d\mu - \int_T a(\cdot, \omega) d\mu \right) - r_2(1 - r_3) \int_{T_1} a(\cdot, \omega) d\mu.$$

Since $\alpha(\omega) \in -\text{int}Y_+$ for all $\omega \in \Omega$, one can choose an $\varepsilon > 0$ such that

$$\alpha(\omega) + B(0, 2\varepsilon) \subseteq -\text{int}Y_+$$

for all $\omega \in \Omega$. By Corollary 2.1.5, there is an $E_0 \in \Sigma_{T_0}$ with $\mu(E_0) < \mu(T_0)$ such that $\|d(\omega)\| < \varepsilon$ for all $\omega \in \Omega$, where

$$d(\omega) = \int_{E_0} (f(\cdot, \omega) - a(\cdot, \omega)) d\mu - r_2 r_3 \int_{T_0} (f(\cdot, \omega) - a(\cdot, \omega)) d\mu.$$

Let $S = C \cup E_0$. Then, $\mu(S) < \mu(T)$. Pick an $u \in B(0, \varepsilon) \cap \text{int}Y_+$, and define a function $g : T \times \Omega \to \text{int}Y_+$ by

$$g(t, \omega) = \begin{cases} \hat{f}(t, \omega) + \frac{u}{2\mu(E_0)}, & \text{if } (t, \omega) \in E_0 \times \Omega; \\ r_3\hat{f}(t, \omega) + \frac{u}{2\mu(C)}, & \text{otherwise.} \end{cases}$$

Then, $g(t,\cdot) \in L_t$ and $g(t,\cdot) \in P_t(f(t,\cdot))$ for almost all $t \in S$, and

$$\int_S g(\cdot,\omega)d\mu = \int_{E_0} f(\cdot,\omega)d\mu + r_2 r_3 \int_{T_1} f(\cdot,\omega)d\mu + u$$

for all $\omega \in \Omega$. Since all agents in T_1 have the same characteristics,

$$\int_C a(\cdot,\omega)d\mu = r_2 \int_{T_1} a(\cdot,\omega)d\mu$$

for all $\omega \in \Omega$. Then it can be easily verified that for all $\omega \in \Omega$,

$$-\alpha(\omega) + \int_S (g(\cdot,\omega) - a(\cdot,\omega))d\mu = d(\omega) + u \in B(0, 2\varepsilon).$$

It follows that $\int_S a(\cdot,\omega)d\mu - \int_S g(\cdot,\omega)d\mu \gg 0$ for all $\omega \in \Omega$. Select an $z \gg 0$ such that

$$\int_S a(\cdot,\omega)d\mu - \int_S g(\cdot,\omega)d\mu \gg z$$

for all $\omega \in \Omega$ and pick a $0 < r < 1$ such that $r_1 \hat{f}(t,\omega) \leq rg(t,\omega)$ for all $(t,\omega) \in C \times \Omega$. Consider the function $h_1 : C \times \Omega \to Y_+$ defined by $h_1(t,\omega) = r_1 \hat{f}(t,\omega)$. Note that $h_1(t,\cdot) \in L_{T_1}$ and $h_1(t,\cdot) \in P_{T_1}(f(t,\cdot))$ for all $t \in C$. By Theorem 5.1.3, there is an assignment $h_2 : T \times \Omega \to Y_+$ such that $h_2(t,\cdot) \in L_t$ and $h_2(t,\cdot) \in P_t(f(t,\cdot))$ for almost all $t \in E_0$, and

$$\int_{E_0} h_2(\cdot,\omega)d\mu = \int_{E_0} \left(rg(\cdot,\omega) + (1-r)f(\cdot,\omega)\right) d\mu$$

for all $\omega \in \Omega$. Let $h : S \times \Omega \to Y_+$ be defined by

$$h(t,\omega) = \begin{cases} h_1(t,\omega), & \text{if } (t,\omega) \in C \times \Omega; \\ h_2(t,\omega), & \text{otherwise.} \end{cases}$$

Then $h(t,\cdot) \in L_t$ and $h(t,\cdot) \in P_t(f(t,\cdot))$ for almost all $t \in S$, and

$$\int_S h(\cdot,\omega)d\mu \leq \int_S (rg(\cdot,\omega) + (1-r)f(\cdot,\omega))d\mu \text{ for all } \omega \in \Omega. \tag{5.4}$$

Define a function $y : T \times \Omega \to Y_+$ such that

$$y(t,\omega) = \begin{cases} h(t,\omega), & \text{if } (t,\omega) \in S \times \Omega; \\ f(t,\omega) + \frac{rz}{\mu(T \setminus S)}, & \text{if } (t,\omega) \in (T \setminus S) \times \Omega. \end{cases}$$

By $(\mathbf{A}_3)$, $y(t,\cdot) \in P_t(f(t,\cdot))$ for all $t \in T \setminus S$. Thus, y is an allocation and $y(t,\cdot) \in P_t(f(t,\cdot))$ for almost all $t \in T$. Furthermore, using $(\mathbf{5.4})$ and $\int_S (a(\cdot,\omega) - g(\cdot,\omega))d\mu \gg z$, one can simply verify that for each $\omega \in \Omega$,

$$\int_T (y(\cdot,\omega) - a(T \setminus S, f, r)(\cdot,\omega))d\mu \leq (1-r)\int_T (f(\cdot,\omega) - a(\cdot,\omega))d\mu \leq 0.$$

This means that f is privately blocked by the grand coalition in $\mathscr{E}(T \setminus S, f, r)$, which contradicts with the fact that f is robustly efficient. So $f(t,\cdot) \in \mathrm{cl}P_{T_1}(\hat{f}(t,\cdot))$ for all $t \in T_1$. Suppose that there is a coalition $W \subseteq T_1$ such that $f(t,\cdot) \in P_{T_1}(\hat{f}(t,\cdot))$ for all $t \in W$. Pick an arbitrary $t \in T_1$. Then Lemma 2.2.3 implies that

$$\frac{1}{\mu(W)}\int_W f(\cdot,\cdot)d\mu \in P_{T_1}(\hat{f}(t,\cdot))$$

and

$$\frac{1}{\mu(T_1 \setminus W)}\int_{T_1 \setminus W} f(\cdot,\cdot)d\mu \in \mathrm{cl}P_{T_1}(\hat{f}(t,\cdot)).$$

Let $\delta = \frac{\mu(W)}{\mu(T_1)}$. Since

$$\hat{f}(t,\cdot) = \frac{\delta}{\mu(W)}\int_W f(\cdot,\cdot)d\mu + \frac{1-\delta}{\mu(T_1 \setminus W)}\int_{T_1 \setminus W} f(\cdot,\cdot)d\mu,$$

$\hat{f}(t,\cdot) \in P_{T_1}(\hat{f}(t,\cdot))$, which is a contradiction. Thus, $f(t,\cdot) \sim_{T_1} \hat{f}(t,\cdot)$ for all $t \in T_1$. $\square$

Remark 5.2.1. Assume $(\mathbf{A}_1)$-$(\mathbf{A}_4)$, $(\mathbf{A}_7)$ and that f is a feasible allocation that is not privately blocked by the grand coalition in $\mathscr{E}(S, f, r)$ for any coalition S and real number r with $\mu(S) < \mu(T)$ and $0 \leq r \leq 1$. Note that under these assumptions, the conclusion of Theorem 5.2.2 still holds. It is claimed that if $\hat{f} \in \mathscr{W}(\mathscr{E})$ then $f \in \mathscr{W}(\mathscr{E})$. To see this, let $(\hat{f}, \pi)$ be a Walrasian expectations equilibrium of $\mathscr{E}$. Since $\int f d\mu = \int \hat{f} d\mu$, by $\hat{f}(t,\cdot) \sim_{T_1} f(t,\cdot)$ for all $t \in T_1$, one only needs to verify that $f(t,\cdot) \in B_t(\pi)$ for all $t \in T_1$. Choose a $y \in \mathrm{int}Y_+$ and a sequence $\{c_m : m \geq 1\} \subseteq \mathbb{R}_+$ converging to 0. By $(\mathbf{A}_3)$, one has $f(t,\cdot) + c_m y \in P_{T_1}(\hat{f}(t,\cdot))$ for all $t \in T_1$. Moreover, $f(t,\cdot) + c_m y \in L_{T_1}$

for all $t \in T_1$. It follows that for all $t \in T_1$,

$$\sum_{\omega\in\Omega} \langle f(t,\omega) + c_m y, \pi(\omega)\rangle > \sum_{\omega\in\Omega} \langle a(t,\omega), \pi(\omega)\rangle \geq \sum_{\omega\in\Omega} \langle \hat{f}(t,\omega), \pi(\omega)\rangle.$$

So, one obtains

$$\sum_{\omega\in\Omega} \langle f(t,\omega), \pi(\omega)\rangle \geq \sum_{\omega\in\Omega} \left\langle \hat{f}(t,\omega), \pi(\omega)\right\rangle$$

for all $t \in T_1$. Thus,

$$\sum_{\omega\in\Omega} \langle f(t,\omega), \pi(\omega)\rangle = \sum_{\omega\in\Omega} \left\langle \hat{f}(t,\omega), \pi(\omega)\right\rangle \leq \sum_{\omega\in\Omega} \langle a(t,\omega), \pi(\omega)\rangle$$

holds for all $t \in T_1$.

Next theorem deals with a characterization of Walrasian expectations allocations of $\mathscr{E}$ by the veto power of the grand coalition in $\mathscr{E}(S, f, r)$ with $\mu(S) < \mu(T)$ and $0 \leq r \leq 1$.

Theorem 5.2.3. *Assume* $(\mathbf{A}_1)$-$(\mathbf{A}_4)$, $(\mathbf{A}_7)$ *and that* Y *is separable. If* $|\mathscr{A}| = 0$ *or* $|\mathscr{A}| \geq 2$, *then a feasible allocation* $f \in \mathscr{W}(\mathscr{E})$ *if and only if* f *is not privately blocked by the grand coalition in every economy* $\mathscr{E}(S, f, r)$ *with* $\mu(S) < \mu(T)$ *and* $0 \leq r \leq 1$.

Proof. Let $f \in \mathscr{W}(\mathscr{E})$ and π be an equilibrium price associated with f. Suppose that there are a coalition S and a real number r with $\mu(S) < \mu(T)$ and $0 \leq r \leq 1$ such that f is privately blocked by the grand coalition in $\mathscr{E}(S, f, r)$. Then, there is an allocation $g : T \times \Omega \to Y_+$ such that $g(t,\cdot) \in P_t(f(t,\cdot))$ for almost all $t \in T$ and

$$\int_T g(\cdot,\omega)d\mu \leq \int_T a(S,f,r)(\cdot,\omega)d\mu$$

for all $\omega \in \Omega$. Using the definition of Walrasian equilibrium, one has

$$\sum_{\omega\in\Omega} \langle g(t,\omega), \pi(\omega)\rangle > \sum_{\omega\in\Omega} \langle a(t,\omega), \pi(\omega)\rangle \geq \sum_{\omega\in\Omega} \langle f(t,\omega), \pi(\omega)\rangle$$

for almost all $t \in T$. Then,

$$\sum_{\omega\in\Omega} \langle g(t,\omega), \pi(\omega)\rangle > \sum_{\omega\in\Omega} \langle (1-r)a(t,\omega) + rf(t,\omega), \pi(\omega)\rangle$$

for almost all $t \in S$. Consequently, one obtains

$$\int_T \sum_{\omega\in\Omega} \langle g(\cdot,\omega), \pi(\omega)\rangle d\mu > \int_T \sum_{\omega\in\Omega} \langle a(S,f,r)(\cdot,\omega), \pi(\omega)\rangle d\mu,$$

which is a contradiction. So, f is not privately blocked by the grand coalition in every economy $\mathscr{E}(S,f,r)$ with $\mu(S) < \mu(T)$ and $0 \leq r \leq 1$.

Conversely, let f be not privately blocked by the grand coalition in every economy $\mathscr{E}(S,f,r)$ with $\mu(S) < \mu(T)$ and $0 \leq r \leq 1$. Suppose that $f \notin \mathscr{W}(\mathscr{E})$. The rest of the proof is completed by considering the following two cases.

Case 1. $|\mathscr{A}| = 0$. By Theorem 3.1.1 and Lemma 5.1.1, f is blocked by a coalition S via g with $\mu(S) < \mu(T)$. Applying Lemma 5.1.2, one can choose g so that for all $\omega \in \Omega$,

$$\int_S (a(\cdot,\omega) - g(\cdot,\omega))d\mu \gg z$$

for some $z \in \mathrm{int}Y_+$ and $g(t,\omega) \gg 0$ for all $(t,\omega) \in S \times \Omega$. For any $0 < r < 1$, by Theorem 5.1.3, there is an assignment h such that $h(t,\cdot) \in L_t$ and $h(t,\cdot) \in P_t(f(t,\cdot))$ for almost all $t \in S$, and

$$\int_S h(\cdot,\omega)d\mu = \int_S (rg(\cdot,\omega) + (1-r)f(\cdot,\omega))d\mu$$

for all $\omega \in \Omega$. Similar to Theorem 5.2.2, one can show that f is privately blocked by the grand coalition via the allocation $y : T \times \Omega \to Y_+$ defined by

$$y(t,\omega) = \begin{cases} h(t,\omega), & \text{if } (t,\omega) \in S \times \Omega; \\ f(t,\omega) + \frac{rz}{\mu(T\setminus S)}, & \text{if } (t,\omega) \in (T \setminus S) \times \Omega \end{cases}$$

in $\mathscr{E}(T \setminus S, f, r)$. This is a contradiction. Thus, $f \in \mathscr{W}(\mathscr{E})$.

Case 2. $|\mathscr{A}| \geq 2$. By Remark 5.2.1, $\hat{f} \notin \mathscr{W}(\mathscr{E})$. Since $\hat{f}|_{T_0\times\Omega} = \hat{f}^*|_{T_0\times\Omega}$ and $\hat{f}(\cdot,\omega)$ is constant in T_1 for $\omega \in \Omega$, it is easy to see that $\hat{f}^* \notin \mathscr{W}(\mathscr{E}^*)$. By Theorem 3.1.1, $\hat{f}^* \notin \mathscr{PC}(\mathscr{E}^*)$. Pick any $A_0 \in T_1$ and let $\mu(A_0) = \varepsilon > 0$. According to Lemma 5.1.1, $\hat{f}^*$ is privately blocked by a coalition S^* of $\mathscr{E}^*$ with $\mu^*(S^*) = \mu^*(T_0) + \varepsilon$, which yields $\mu^*(S^* \cap T_1^*) \geq \varepsilon$. By Lemma 5.1.4, there exist a coalition R^* and an assignment g^* such that (**i**) and (**ii**) of Lemma 5.1.4 hold. Take a coalition E of $\mathscr{E}$ such that

$E = (R^* \cap T_0) \cup A_0$, and define a function $\tilde{g} : E \times \Omega \to Y_+$ by

$$\tilde{g}(t,\omega) = \begin{cases} g^*(t,\omega), & \text{if } (t,\omega) \in (R^* \cap T_0) \times \Omega; \\ \frac{1}{\varepsilon} \int_{R^* \cap T_1^*} g^*(\cdot,\omega) d\mu^*, & \text{otherwise.} \end{cases}$$

Further, define another function $\tilde{g}^* : E^* \times \Omega \to Y_+$ such that

$$\tilde{g}^*(t,\omega) = \begin{cases} \tilde{g}(t,\omega), & \text{if } (t,\omega) \in (R^* \cap T_0) \times \Omega; \\ \tilde{g}(A_0,\omega), & \text{if } (t,\omega) \in A_0^* \times \Omega. \end{cases}$$

Similar to Lemma 2.2.3, $\tilde{g}^*(t,\cdot) \in P_t(\hat{f}^*(t,\cdot))$ and $\tilde{g}^*(t,\cdot) \in L_t$ for almost all $t \in E^*$, and

$$\int_{E^*} (a(\cdot,\omega) - \tilde{g}^*(\cdot,\omega)) d\mu^* \gg 0$$

for all $\omega \in \Omega$. Select some $b \gg 0$ such that

$$\int_{E^*} (a(\cdot,\omega) - \tilde{g}^*(\cdot,\omega)) d\mu^* \gg b$$

for all $\omega \in \Omega$, and define a function $g_b^* : E^* \times \Omega \to Y_+$ by

$$g_b^*(t,\omega) = \tilde{g}^*(t,\omega) + \frac{b}{2\mu^*(E^*)}.$$

By $(\mathbf{A}_3)$, one has $g_b^*(t,\cdot) \in P_t(\hat{f}^*(t,\cdot))$ for almost all $t \in E^*$. Consider a function $g_b : T \times \Omega \to Y_+$ defined by

$$g_b(t,\omega) = \begin{cases} g_b^*(t,\omega), & \text{if } (t,\omega) \in (E \cap T_0) \times \Omega; \\ \tilde{g}(A_0,\omega) + \frac{b}{2\mu^*(E^*)}, & \text{if } t = A_0, \omega \in \Omega; \\ f(t,\omega), & \text{otherwise.} \end{cases}$$

Then, g_b is an allocation and $g_b(t,\omega) \gg 0$ for all $(t,\omega) \in E \times \Omega$. Choose a real number r satisfying $0 < r < 1$ and $\tilde{g}(A_0,\omega) \le r g_b(A_0,\omega)$ for each $\omega \in \Omega$. By Theorem 5.1.3, there is an assignment h_b such that $h_b(t,\cdot) \in L_t$ and $h_b(t,\cdot) \in P_t(f(t,\cdot))$ for almost all $t \in E \cap T_0$, and

$$\int_{E \cap T_0} h_b(\cdot,\omega) d\mu = \int_{E \cap T_0} (r g_b(\cdot,\omega) + (1-r) f(\cdot,\omega)) d\mu$$

for all $\omega \in \Omega$. Note that the function $h : E \times \Omega \to Y_+$, defined by

$$h(t,\omega) = \begin{cases} h_b(t,\omega), & \text{if } (t,\omega) \in (E \cap T_0) \times \Omega; \\ \tilde{g}(t,\omega), & \text{otherwise,} \end{cases}$$

is similar to that in the proof of Theorem 5.2.2. Thus, applying an argument similar to that just after the definition of h in the proof of Theorem 5.2.2, one can draw a contradiction. Hence, $f \in \mathscr{W}(\mathscr{E})$, and the proof is completed. □

Let $\mathscr{B}$ be the set of simple functions $\gamma : T \to [0,1]$ such that $\mu(\text{supp}(\gamma)) > 0$. For any feasible allocation f in $\mathscr{E}$ and $\gamma \in \mathscr{B}$, define a new asymmetric information economy $\mathscr{E}(\gamma, f)$ which has the same characteristics as those of $\mathscr{E}$ except for the initial endowment for every agent $t \in T$ which is defined in each state $\omega \in \Omega$ as

$$a_\gamma(t,\omega) = \gamma(t)a(t,\omega) + (1 - \gamma(t))f(t,\omega).$$

Graziano and Meo [41] proved that $\mathscr{W}(\mathscr{E})$ coincides with the set of those feasible allocations that are not privately blocked by the grand coalition in the economy $\mathscr{E}(\gamma, f)$ for any $\gamma \in \mathscr{B}$. The next theorem not only illustrates the connection between Theorem 5.2.3 and this result, but also provides an answer to the question posed in [48] and extends the main result in [48] to a mixed economy as well.

Theorem 5.2.4. *Assume* $(\mathbf{A}_1)$-$(\mathbf{A}_4)$, $(\mathbf{A}_7)$ *and that* Y *is separable. If* $|\mathscr{A}| = 0$ *or* $|\mathscr{A}| \geq 2$, *then the following are equivalent for a feasible allocation* f *in* $\mathscr{E}$*:*

- (**i**) $f \in \mathscr{W}(\mathscr{E})$.
- (**ii**) f *is robustly efficient.*
- (**iii**) f *is not privately blocked by the grand coalition in the economy* $\mathscr{E}(\gamma, f)$ *for any* $\gamma \in \mathscr{B}$.
- (**iv**) f *is not privately blocked by the grand coalition in every economy* $\mathscr{E}(S, f, r)$ *with* $\mu(S) < \mu(T)$ *and* $0 \leq r \leq 1$.

Proof. (**i**) implies (**ii**) and (**i**) implies (**iii**) are similar, and they can be proved by simple arguments similar to that in the proof of Theorem 5.2.3. (**ii**) implies (**iv**) is straightforward. To show that (**iii**) implies (**iv**), for a coalition S of $\mathscr{E}$ and $0 \leq r \leq 1$, define a simple function $\gamma_{S,r} : T \to [0,1]$ by

$$\gamma_{S,r}(t) = \begin{cases} 1 - r, & \text{if } t \in S; \\ 1, & \text{if } \in T \setminus S. \end{cases}$$

Note that $\gamma_{S,r} \in \mathscr{B}$ if $\mu(S) < \mu(T)$. So for every coalition S of $\mathscr{E}$ with $\mu(S) < \mu(T)$ and $0 \leq r \leq 1$, one has $\mathscr{E}(S, f, r) = \mathscr{E}(\gamma_{S,r}, f)$ and thus (**iv**) holds. Finally, (**iv**) implies (**i**) follows directly from Theorem 5.2.3. □

5.3 Walrasian Allocations in a Discrete Economy

In this section, characterizations of Walrasian allocations in terms of the privately blocking power of the grand coalition are studied in economies with finitely many agents and Banach lattices as commodity spaces. These results are extensions of those in [36, 46, 47] and the proofs are similar to those in [47]. Throughout this section, let $T = N$ and Y be a Banach lattice

Definition 5.3.1. [47] A coalition $S \subseteq N$ *privately blocks an allocation* x of $\mathscr{E}$ in the sense of Aubin via $y = (y_i : i \in S)$ if for all $i \in S$, there is an element $\alpha_i \in (0, 1]$ such that $y_i \in P_i(x_i) \cap L_i$ and $\sum_{i \in S} \alpha_i y_i \leq \sum_{i \in S} \alpha_i a_i$. The *Aubin private core* of $\mathscr{E}$ is the set of all feasible allocations which cannot be privately blocked in the sense of Aubin by any coalition, and an allocation x of $\mathscr{E}$ is called *Aubin non-dominated* if x is not privately blocked in the sense of Aubin by the grand coalition. An allocation is called an *Aubin dominated* if it is not Aubin non-dominated.

Theorem 5.3.1. *Assume* ($\mathbf{B}_1$)-($\mathbf{B}_5$) *and* ($\mathbf{B}_7$), *and* $\mathrm{int} Y_+ \neq \emptyset$. *Then, a feasible allocation* $x \in \mathscr{W}(\mathscr{E})$ *if and only if* x *is Aubin non-dominated.*

Proof. Let $x \in \mathscr{W}(\mathscr{E})$. Suppose that x is Aubin dominated. Then, $f \notin \mathscr{PC}(\mathscr{E}_c)$, where $f = \Xi(x)$. Hence, by Corollary 3.2.3, Proposition 2.2.2 and Remark 2.2.1, $x \notin \mathscr{W}(\mathscr{E})$, which is a contradiction.

Conversely, let x be an Aubin non-dominated feasible allocation in $\mathscr{E}$. Suppose that $x \notin \mathscr{W}(\mathscr{E})$. By Remark 2.2.1, $f \notin \mathscr{W}(\mathscr{E}_c)$. Corollary 3.2.3 and Proposition 2.2.2 imply that $f \notin \mathscr{PC}(\mathscr{E}_c)$. By Theorem 4.1.2, there exist a coalition $S \subseteq I$ with $\hat{\mu}(S) > 1 - \frac{1}{n}$ and an assignment g such that $g(t, \cdot) \in L_t$ and $g(t, \cdot) \in P_t(f(t, \cdot))$ for almost all $t \in S$, and

$$\int_S g(\cdot, \omega) d\hat{\mu} \leq \int_S a(\cdot, \omega) d\hat{\mu}$$

for all $\omega \in \Omega$. Let $S_i = S \cap I_i$ and $\tilde{\alpha}_i = n\hat{\mu}(S_i)$ for all $i \in N$. Since

$$\hat{\mu}(S) > 1 - \frac{1}{n},$$

then $0 < \tilde{\alpha}_i \leq 1$ for all $i \in N$. Now,

$$y_i = \frac{1}{\hat{\mu}(S_i)} \int_{S_i} g(\cdot,\cdot) d\hat{\mu} \in L_i$$

for each $i \in N$, and $y = (y_1, ..., y_n)$ is an allocation in $\mathscr{E}$. By Lemma 2.2.3, $y_i \in P_i(x_i)$ for all $i \in N$. Since

$$\sum_{i \in N} \hat{\mu}(S_i) y_i \leq \sum_{i \in N} \hat{\mu}(S_i) a_i,$$

one has

$$\sum_{i \in N} \tilde{\alpha}_i y_i \leq \sum_{i \in N} \tilde{\alpha}_i a_i.$$

This contradicts with the fact that x is Aubin non-dominated. Thus, $x \in \mathscr{W}(\mathscr{E})$. $\square$

Remark 5.3.1. The participation of an agent i in the grand coalition of $\mathscr{E}$ is said to be *close to complete* if $\tilde{\alpha}_i > 1 - \delta$ for sufficiently small $\delta > 0$. Indeed, Theorem 5.3.1 shows that the participation of each agent can be chosen to be close to complete. To see this, for any given $0 < \delta < 1$, by Theorem 4.1.2, one can choose a privately blocking coalition S such that

$$\hat{\mu}(S) > 1 - \frac{\delta}{n}.$$

Hence, it follows from the proof that $\tilde{\alpha}_i = n\hat{\mu}(S_i) > 1 - \delta$ for all $i \in N$.

Remark 5.3.2. Conclusions of Theorem 5.3.1 and Remark 5.3.1 are also true under $(\mathbf{B}_1)$-$(\mathbf{B}_8)$, or $(\mathbf{B}_1)$-$(\mathbf{B}_3)$, $(\mathbf{B}_5)$-$(\mathbf{B}_7)$ and $(\mathbf{B}_9)$ respectively.

To see the next characterization theorem, let $f = x$ and $\gamma = r \in [0,1]^n$ in the definition of $\mathscr{E}(\gamma, f)$, where agent i's initial endowment is given by

$$a_i(r_i, x_i) = r_i a_i + (1 - r_i) x_i.$$

Theorem 5.3.2. *Assume* $(\mathbf{B}_1)$-$(\mathbf{B}_5)$ *and* $(\mathbf{B}_7)$, *and* $\text{int}Y_+ \neq \emptyset$. *Then, a feasible allocation $x \in \mathscr{W}(\mathscr{E})$ if and only if x is privately non-dominated in $\mathscr{E}(r,x)$ for any $r \in [0,1]^n$.*

Proof. Let $x \in \mathscr{W}(\mathscr{E})$. Applying an argument similar to that in Theorem 5.2.3 with $S = N$, one can show that x is privately non-dominated in $\mathscr{E}(r,x)$ for any $r \in [0,1]^n$.

Conversely, let x be a privately non-dominated feasible allocation in $\mathscr{E}(r, x)$ for each $r \in [0,1]^n$. Suppose that $x \notin \mathscr{W}(\mathscr{E})$. By Remark 2.2.1, $f \notin \mathscr{W}(\mathscr{E}_c)$. Corollary 3.2.3 and Proposition 2.2.2 imply that $f \notin \mathscr{P}\mathscr{C}(\mathscr{E}_c)$. By Theorem 4.1.2, one has a coalition $S \subseteq I$ with

$$\hat{\mu}(S) > 1 - \frac{1}{n}$$

and an assignment g such that $g(t, \cdot) \in L_t$ and $g(t, \cdot) \in P_t(f(t, \cdot))$ for almost all $t \in S$, and

$$\int_S g(\cdot, \omega) d\hat{\mu} \leq \int_S a(\cdot, \omega) d\hat{\mu}$$

for all $\omega \in \Omega$. Let $S_i = S \cap I_i$ and $\tilde{r}_i = n\hat{\mu}(S_i)$ for all $i \in N$. Put $\tilde{r} = (\tilde{r}_1, ..., \tilde{r}_n)$. Since

$$\hat{\mu}(S) > 1 - \frac{1}{n},$$

then $\tilde{r}_i > 0$ for all $i \in N$. Now,

$$y_i = \frac{1}{\hat{\mu}(S_i)} \int_{S_i} g(\cdot, \cdot) d\hat{\mu} \in L_i$$

for all $i \in N$. Hence, $y = (y_1, ..., y_n)$ is an allocation in $\mathscr{E}(\tilde{r}, x)$. By Lemma 2.2.3, $y_i \in P_i(x_i)$ for all $i \in N$. Further, since

$$\sum_{i \in N} \hat{\mu}(S_i) y_i \leq \sum_{i \in N} \hat{\mu}(S_i) a_i,$$

then

$$\sum_{i \in N} \tilde{r}_i y_i \leq \sum_{i \in N} \tilde{r}_i a_i.$$

If one put $z_i = \tilde{r}_i y_i + (1 - \tilde{r}_i) x_i$ for all $i \in N$, then $z_i \in L_i$ for all $i \in N$ and

$$\sum_{i \in N} z_i \leq \sum_{i \in N} \{\tilde{r}_i a_i + (1 - \tilde{r}_i) x_i\} = \sum_{i \in N} a_i(\tilde{r}_i, x_i).$$

Since $x_i \in \mathrm{cl} P_i(x_i)$, one obtains $z_i \in P_i(x_i)$ for all $i \in N$, which contradicts with the fact that x is privately non-dominated in $\mathscr{E}(r, x)$ for every $r \in [0,1]^n$. Hence, $x \in \mathscr{W}(\mathscr{E})$. $\square$

Remark 5.3.3. Note that for each agent $i \in N$, $\tilde{r}_i$ can be selected arbitrarily close to 1. To see this, for any given $0 < \delta < 1$, by Theorem 4.1.2, one can choose a privately blocking coalition S such that

$$\hat{\mu}(S) > 1 - \frac{\delta}{n}.$$

Hence, $\tilde{r}_i = n\hat{\mu}(S_i) > 1 - \delta$ for all $i \in N$.

Remark 5.3.4. Conclusion of Theorem 5.3.2 is also true under $(\mathbf{B}_1)$-$(\mathbf{B}_8)$, or $(\mathbf{B}_1)$-$(\mathbf{B}_3)$, $(\mathbf{B}_5)$-$(\mathbf{B}_7)$ and $(\mathbf{B}_9)$ respectively. The same is true for Remark 5.3.3 in these cases.

The following corollary can be obtained under the assumptions mentioned in Remark 5.3.2.

Corollary 5.3.3. *Assume that* $(\mathbf{B}_1)$-$(\mathbf{B}_5)$, $(\mathbf{B}_7)$ *and* $\text{int}Y_+ \neq \emptyset$. *The following are equivalent for any feasible allocation* x *in* $\mathscr{E}$.

(i) $x \in \mathscr{W}(\mathscr{E})$.

(ii) *x is in the Aubin private core of $\mathscr{E}$.*

(iii) *x is Aubin non-dominated in $\mathscr{E}$.*

(iv) *x is Aubin non-dominated in $\mathscr{E}$ with a participation of each agent as close to the complete participation as one wants.*

(v) *x is non-dominated in any $\mathscr{E}(r, x)$ with $r = (r_1, ..., r_n) \in [0, 1]^n$.*

(vi) *x is non-dominated in any $\mathscr{E}(r, x)$ with r_i as close to 1 as one wants.*

Chapter 6

Existence of a Maximin Rational Expectations Equilibrium

Recently, de Castro et al. [32] introduced the concept of a maximin rational expectations equilibrium and showed that such an equilibrium always exists in a pure exchange asymmetric information economy with finitely many agents, finitely many states of nature, and finitely many commodities. The main purpose of this chapter is to extend this result into several directions. In contrast to finitely many agents and finitely many states of nature in [32], the spaces of agents and states of nature in this chapter are taken as a finite measure space and a probability space respectively. The commodity space is the same as that in [32]. In Section 6.1, the aggregate preferred correspondence is introduced in the sense of Aumann in [17]. In addition to this, the non-empty valuedness, compactness, continuity and measurability of the aggregate preferred correspondence are studied in this section. Section 6.2 is the last section of this chapter and is devoted to establish the existence of a maximin rational expectations equilibrium under the assumption that the space of states of nature is complete. In this chapter, the economic model given in Subsection 2.2.3 is studied. The main results in this chapter are taken from [23].

6.1 The Aggregate Preferred Correspondence

In this section, the concept of aggregate preferred correspondence is introduced and then the non-empty valuedness, compactness, continuity and measurability of this correspondence are established. To see these, some concepts are presented below.

Let $\Delta = \text{int}\Im^\ell$. The *budget correspondence* $B : T \times \Omega \times \Delta \rightrightarrows \mathbb{R}^\ell_+$ is defined by

$$B(t, \omega, p) = \left\{ x \in \mathbb{R}^\ell_+ : \langle x, p \rangle \leq \langle a(t, \omega), p \rangle \right\}$$

for all $(t, \omega, p) \in T \times \Omega \times \Delta$. Obviously, B is non-empty closed-valued. For each $\omega \in \Omega$, by Theorem 2 in [51, p.151] and $(\mathbf{C}_1)$-$(\mathbf{C}_4)$, there are $p(\omega) \in \Delta$ and an allocation f such that $(f(\cdot, \omega), p(\omega))$ is a Walrasian equilibrium of the deterministic economy $\mathscr{E}(\omega)$, given by

$$\mathscr{E}(\omega) = \left\{ (T, \Sigma, \mu); \mathbb{R}^\ell_+; (U_t(\omega, \cdot), a(t, \omega)) : t \in T \right\}.$$

Define a function $\delta : \Delta \to \mathbb{R}_+$ by

$$\delta(p) = \min \left\{ p^h : h = 1, \cdots, \ell \right\},$$

where $p = \left(p^1, \cdots, p^\ell\right) \in \Delta$. For any $(t, \omega, p) \in T \times \Omega \times \Delta$, let

$$\gamma(t, \omega, p) = \frac{1}{\delta(p)} \sum_{h=1}^{\ell} a^h(t, \omega)$$

and

$$b(t, \omega, p) = (\gamma(t, \omega, p), \cdots, \gamma(t, \omega, p)).$$

Define $X : T \times \Omega \times \Delta \rightrightarrows \mathbb{R}^\ell_+$ by

$$X(t, \omega, p) = \left\{ x \in \mathbb{R}^\ell_+ : x \leq b(t, \omega, p) \right\}$$

for all $(t, \omega, p) \in T \times \Omega \times \Delta$. Note that X is non-empty compact- and convex-valued such that $B(t, \omega, p) \subseteq X(t, \omega, p)$ for all $(t, \omega, p) \in T \times \Omega \times \Delta$. It can be readily verified that for every $(t, \omega) \in T \times \Omega$, the correspondence $X(t, \omega, \cdot) : \Delta \rightrightarrows \mathbb{R}^\ell_+$ is Hausdorff continuous. Define two correspondences $C, C^X : T \times \Omega \times \Delta \rightrightarrows \mathbb{R}^\ell_+$ by

$$C(t, \omega, p) = \left\{ y \in \mathbb{R}^\ell_+ : U_t(\omega, y) \geq U_t(\omega, x) \text{ for all } x \in B(t, \omega, p) \right\}$$

and

$$C^X(t, \omega, p) = C(t, \omega, p) \cap X(t, \omega, p).$$

Obviously,

$$B(t, \omega, p) \cap C(t, \omega, p) = B(t, \omega, p) \cap C^X(t, \omega, p)$$

holds for all $(t, \omega, p) \in T \times \Omega \times \Delta$. Note that under $(\mathbf{C}_2)$, $U_t(\omega, \cdot)$ is continuous on the non-empty compact set $B(t, \omega, p)$. Thus, one has

$$B(t, \omega, p) \cap C(t, \omega, p) \neq \emptyset$$

for all $(t, \omega, p) \in T \times \Omega \times \Delta$.

Proposition 6.1.1. *Let $(t, \omega, p) \in T \times \Omega \times \Delta$. Under $(\mathbf{C}_3)$, $\langle x, p\rangle \geq \langle a(t,\omega), p\rangle$ for every point $x \in C^X(t, \omega, p)$.*

Proof. Assume that $\langle x_0, p\rangle < \langle a(t,\omega), p\rangle$ for some point $x_0 \in C^X(t, \omega, p)$. Then, one can choose some $y \in \mathbb{R}^\ell_+$ such that $y > 0$ and $\langle p, x_0 + y\rangle < \langle p, a(t,\omega)\rangle$. Thus, $x_0 + y \in B(t, \omega, p)$. Since $x_0 \in C^X(t, \omega, p)$, one has $U_t(\omega, x_0) > U_t(\omega, x_0 + y)$. However, $(\mathbf{C}_3)$ implies $U_t(\omega, x_0+y) > U_t(\omega, x_0)$. This is a contradiction, which completes the proof. □

Following [17], $C^X(t, \omega, p)$ is called the *preferred set* of agent t at the price p and state ω, and $\int_T C^X(\cdot, \omega, p) d\mu$ is called the *aggregate preferred set* at the price p and state ω. Moreover,

$$\int_T C^X(\cdot, \cdot, \cdot) d\mu : \Omega \times \Delta \rightrightarrows \mathbb{R}^\ell_+$$

is termed as the *aggregate preferred correspondence*. The first important result in this section is the non-emptiness and the compactness of the aggregate preferred correspondence. For this end, the following lower measurability of $B(\cdot, \omega, p)$ and $X(\cdot, \omega, p)$ are crucial for all $(\omega, p) \in \Omega \times \Delta$.

Proposition 6.1.2. *Under $(\mathbf{C}_1)$, for every $(\omega, p) \in \Omega \times \Delta$, $B(\cdot, \omega, p) : T \rightrightarrows \mathbb{R}^\ell_+$ and $X(\cdot, \omega, p) : T \rightrightarrows \mathbb{R}^\ell_+$ are lower measurable.*

Proof. Here, only the proof of lower measurability of $B(\cdot, \omega, p)$ is provided. The other case can be done analogously. Fix $(\omega, p) \in \Omega \times \Delta$. Define a function $h : T \times \mathbb{R}^\ell_+ \to \mathbb{R}$ by letting

$$h(t, x) = \langle x, p\rangle - \langle a(t, \omega), p\rangle$$

for all $(t, x) \in T \times \mathbb{R}^\ell_+$. Then, $h(\cdot, x)$ is measurable for all $x \in \mathbb{R}^\ell_+$. Note that

$$B(t, \omega, p) = h(t, \cdot)^{-1}((-\infty, 0]).$$

Let V be a non-empty open subset of $\mathbb{R}^\ell$, and put $V \cap \mathbb{Q}^\ell_+ = \{x_k : k \geq 1\}$. It is worth to point out that if $x \in B(t, \omega, p) \cap V$, then $x_k \in B(t, \omega, p)$ for some $k \geq 1$. Since $h(\cdot, x_k)$ is measurable,

$$\{t \in T : h(t, x_k) \in (-\infty, 0]\} \in \Sigma$$

for all $k \geq 1$. Thus,

$$\begin{aligned} B(\cdot, \omega, p)^{-1}(V) &= \bigcup_{k \geq 1} \{t \in T : x_k \in B(t, \omega, p)\} \\ &= \bigcup_{k \geq 1} \{t \in T : h(t, x_k) \in (-\infty, 0]\} \end{aligned}$$

belongs to Σ. It follows that $B(\cdot, \omega, p)$ is lower measurable. □

Proposition 6.1.3. *Under* ($\mathbf{C}_1$)-($\mathbf{C}_3$), $\int_T C^X(\cdot, \cdot, \cdot) d\mu$ *is non-empty compact-valued.*

Proof. Fix $(\omega, p) \in \Omega \times \Delta$. By ($\mathbf{C}_2$), $C^X(t, \omega, p)$ is non-empty closed for all $t \in T$. By the lower measurability of $B(\cdot, \omega, p)$, there exists a sequence $\{f_n : n \geq 1\}$ of measurable functions from (T, Σ, μ) to $\mathbb{R}_+^\ell$ such that

$$B(t, \omega, p) = \text{cl}\{f_n(t) : n \geq 1\}$$

for all $t \in T$. For each $n \geq 1$, define a correspondence $C_n : T \rightrightarrows \mathbb{R}_+^\ell$ by letting

$$C_n(t) = \left\{ x \in \mathbb{R}_+^\ell : U_t(\omega, x) \geq U_t(\omega, f_n(t)) \right\}$$

for all $t \in T$. Obviously,

$$C(t, \omega, p) \subseteq \bigcap_{n \geq 1} C_n(t).$$

If $x \in \mathbb{R}_+^\ell \setminus C(t, \omega, p)$ for some $t \in T$, there is an element $y \in B(t, \omega, p)$ such that $U_t(\omega, y) > U_t(\omega, x)$. By ($\mathbf{C}_2$), there is some $n_0 \geq 1$ such that $U_t(\omega, f_{n_0}(t)) > U_t(\omega, x)$. This implies that $x \notin C_{n_0}(t)$. Thus,

$$C(t, \omega, p) = \bigcap_{n \geq 1} C_n(t)$$

for all $t \in T$. Fix $n \geq 1$, and define a function $h : (T, \Sigma, \mu) \times \mathbb{R}_+^\ell \to \mathbb{R}$ by

$$h(t, x) = U_t(\omega, f_n(t)) - U_t(\omega, x).$$

Clearly, h is Carathéodory. Similar to Proposition 6.1.2, one can show that C_n is lower measurable. Since $X(\cdot, \omega, p)$ is compact-valued, each C_n is closed-valued and

$$C^X(\cdot, \omega, p) = \bigcap_{n \geq 1} C_n(\cdot) \cap X(\cdot, \omega, p),$$

then $C^X(\cdot,\omega,p) : (T,\Sigma,\mu) \rightrightarrows \mathbb{R}^\ell_+$ is lower measurable. By the Kuratowski-Ryll-Nardzewski measurable selection theorem, $C^X(\cdot,\omega,p)$ has a measurable selection which is also integrable, as $b(\cdot,\omega,p)$ is so. Since $C^X(\cdot,\omega,p)$ is closed-valued and integrably bounded, $\int_T C^X(\cdot,\omega,p)d\mu$ is compact. □

Next, the Hausdorff continuity of the aggregate preferred correspondence with respect to the variable $p \in \Delta$ is established.

Theorem 6.1.4. *Assume* $(\mathbf{C}_1)$-$(\mathbf{C}_3)$. *For each* $\omega \in \Omega$, $\int_T C^X(\cdot,\omega,\cdot)d\mu : \Delta \rightrightarrows \mathbb{R}^\ell_+$ *is Hausdorff continuous.*

Proof. Fix $\omega \in \Omega$. Let $\{p_n : n \geq 1\} \subseteq \Delta$ converge to $p \in \Delta$. Choose $\varepsilon > 0$ and $N \geq 1$ such that $\varepsilon < \delta(p)$ and $\varepsilon < \delta(p_n)$ for all $n \geq N$. Let

$$c = \min\left\{\delta(p_n), \varepsilon : n = 1, 2, \cdots, N-1\right\},$$

$$d(t,\omega) = \frac{1}{c}\sum_{h=1}^{\ell} a^h(t,\omega), \text{ and } \xi(t,\omega) = (d(t,\omega), \cdots, d(t,\omega)).$$

Define $M(\omega)$ by

$$M(\omega) = \left\{x \in \mathbb{R}^\ell_+ : x \leq \int_T \xi(\cdot,\omega)d\mu\right\}.$$

Since all $X(\cdot,\omega,p_n)$ and $X(\cdot,\omega,p)$ are upper bounded by $\xi(\cdot,\omega)$, then $\int_T C^X(\cdot,\omega,p_n)d\mu$ and $\int_T C^X(\cdot,\omega,p)d\mu$ are contained in the compact subset $M(\omega)$ of $\mathbb{R}^\ell_+$. Thus, one only needs to show that $\left\{\int_T C^X(\cdot,\omega,p_n)d\mu : n \geq 1\right\}$ converges to $\int_T C^X(\cdot,\omega,p)d\mu$ in the Hausdorff metric topology on $\mathscr{K}_0(M(\omega))$, which is equivalent to

$$\text{Li}\int_T C^X(\cdot,\omega,p_n)d\mu = \text{Ls}\int_T C^X(\cdot,\omega,p_n)d\mu = \int_T C^X(\cdot,\omega,p)d\mu.$$

The verification of the above equation can be split into two steps. First, one verifies

$$\text{Ls}\int_T C^X(\cdot,\omega,p_n)d\mu \subseteq \int_T C^X(\cdot,\omega,p)d\mu.$$

To do this, it is enough to verify that for any $t \in T$,

$$\text{Ls}C^X(t,\omega,p_n) \subseteq C^X(t,\omega,p).$$

Pick $t \in T$ and $x \in \text{Ls}C^X(t,\omega,p_n)$. Then, there exist positive integers $n_1 < n_2 < n_3 < \cdots$ and for each k a point $x_k \in C^X(t,\omega,p_{n_k})$ such that $\{x_k : k \geq 1\}$ converges to x.

It is obvious that $x \in X(t, \omega, p)$. If $x \notin C^X(t, \omega, p)$, by the continuity of $U_t(\omega, \cdot)$, one can choose some $y \in \mathbb{R}^\ell_+$ such that $\langle y, p\rangle < \langle a(t, \omega), p\rangle$ and $U_t(\omega, y) > U_t(\omega, x)$. By the Hausdorff continuity of $X(t, \omega, \cdot)$, $\{X(t, \omega, p_{n_k}) : k \geq 1\}$ converges to $X(t, \omega, p)$ in the Hausdorff metric topology. Since $y \in X(t, \omega, p)$, there exists a sequence $\{y_k : k \geq 1\}$ such that $y_k \in X(t, \omega, p_{n_k})$ for all $k \geq 1$ and $\{y_k : k \geq 1\}$ converges to y. It follows that $U_t(\omega, y_k) > U_t(\omega, x_k)$ and $\langle y_k, p_{n_k}\rangle < \langle a(t, \omega), p_{n_k}\rangle$ for all sufficiently large k, which is a contradiction. Therefore, one must have $x \in C^X(t, \omega, p)$. Secondly, one needs to verify

$$\int_T C^X(\cdot, \omega, p)d\mu \subseteq \mathrm{Li} \int_T C^X(\cdot, \omega, p_n)d\mu.$$

It is enough to verify that for all $t \in T$,

$$C^X(t, \omega, p) \subseteq \mathrm{Li}C^X(t, \omega, p_n).$$

Fix $t \in T$ and pick $d \in C^X(t, \omega, p)$. If $d = b(t, \omega, p)$, $b(t, \omega, p_n) \in C^X(t, \omega, p_n)$ and $\{b(t, \omega, p_n) : n \geq 1\}$ converges to d. Assume $d < b(t, \omega, p)$. Select $\delta > 0$ such that

$$d + (0, \cdots, \delta, \cdots, 0) \leq b(t, \omega, p).$$

Further, choose a sequence $\{\delta_i : i \geq 1\}$ in $(0, \delta]$ converging to 0. For each $i \geq 1$, let

$$d^i = d + (0, \cdots, \delta_i, \cdots, 0),$$

and choose a sequence $\{d^i_n : n \geq 1\}$ such that $d^i_n \in X(t, \omega, p_n)$ for each $n \geq 1$ and $\{d^i_n : n \geq 1\}$ converges to d^i. It is claimed that for each $i \geq 1$, $d^i_n \in C^X(t, \omega, p_n)$ for sufficiently large n. Otherwise, there must exist an $i_0 \geq 1$ and a subsequence $\{d^{i_0}_{n_k} : k \geq 1\}$ of $\{d^{i_0}_n : n \geq 1\}$ such that $d^{i_0}_{n_k} \notin C^X(t, \omega, p_{n_k})$. Let $b_k \in B(t, \omega, p_{n_k})$ and $U_t(\omega, b_k) > U_t(\omega, d^{i_0}_{n_k})$ for all $k \geq 1$. Then $\{b_k : k \geq 1\}$ has a subsequence converging to $b \in B(t, \omega, p)$. By $(\mathbf{C}_2)$ and $(\mathbf{C}_3)$, one obtains

$$U_t(\omega, b) \geq U_t(\omega, d^{i_0}) > U_t(\omega, d),$$

which contradicts with the fact that $d \in C^X(t, \omega, p)$. To complete the proof, note that the previous claim implies that for each i, $\left\{\mathrm{dist}(d^i, C^X(t, \omega, p_n)) : n \geq 1\right\}$ converges to 0. Since $\{d^i : i \geq 1\}$ converges to d, one concludes that $\{\mathrm{dist}(d, C^X(t, \omega, p_n)) : n \geq 1\}$ also converges to 0. This means that $d \in \mathrm{Li}C^X(t, \omega, p_n)$. $\square$

The next result is crucial for the existence theorem in Section 6.2.

Theorem 6.1.5. *Assume* ($\mathbf{C}_1$)-($\mathbf{C}_3$). *For each* $p \in \Delta$, $\int_T C^X(\cdot,\cdot,p)d\mu : \Omega \rightrightarrows \mathbb{R}_+^\ell$ *is lower measurable.*

Proof. Fix $p \in \Delta$. Since $a(\cdot,\cdot)$ and $U_\cdot(\cdot,\cdot)$ are $\Sigma\otimes\mathscr{F}$-measurable and $\Sigma\otimes\mathscr{F}\otimes\mathscr{B}(\mathbb{R}_+^\ell)$-measurable respectively, by Proposition 2.1.3, there exist two sequences $\{a_n : n \geq 1\}$ and $\{\psi_n : n \geq 1\}$ of $\Sigma\otimes\mathscr{F}$-measurable and $\Sigma\otimes\mathscr{F}\otimes\mathscr{B}(\mathbb{R}_+^\ell)$-measurable functions respectively such that $\{a_n : n \geq 1\}$ uniformly converges to $a(\cdot,\cdot)$ on $T\times\Omega$ and $\{\psi_n : n \geq 1\}$ uniformly converges to $U_\cdot(\cdot,\cdot)$ on $T\times\Omega\times\mathbb{R}_+^\ell$. For each $n \geq 1$, a_n and ψ_n are written as

$$a_n = \sum_{i\geq 1} e_i \chi_{T_i^n\times\Omega_i^n} \text{ and } \psi_n = \sum_{i\geq 1} v_i \chi_{T_i^n\times\Omega_i^n\times B_i^n},$$

where $e_i \in \mathbb{R}_+^\ell$, $v_i \in \mathbb{R}$, and $\{T_i^n\times\Omega_i^n\times B_i^n : i \geq 1\}$ is a partition of $T\times\Omega\times\mathbb{R}_+^\ell$ for all $n \geq 1$. Choose some $N \geq 1$ such that $\|a_n - a\|_\infty < 1$ for all $n \geq N$. By the measurability of $a_n(\cdot,\omega)$, one has $a_n(\cdot,\omega) \in L_1\left(\mu,\mathbb{R}^\ell\right)$ for all $\omega \in \Omega$ and $n \geq 1$ (replacing a_n for all $1 \leq n < N$ by some constant functions, if necessary). Let

$$\gamma_n(t,\omega) = \frac{1}{\delta(p)}\sum_{h=1}^{\ell} a_n^h(t,\omega) \text{ and } b_n(t,\omega) = (\gamma_n(t,\omega),\cdots,\gamma_n(t,\omega)).$$

Define $X_n, B_n, C_n : T\times\Omega \rightrightarrows \mathbb{R}_+^\ell$ such that

$$X_n(t,\omega) = \left\{x \in \mathbb{R}_+^\ell : x \leq b_n(t,\omega)\right\},$$

$$B_n(t,\omega) = \left\{x \in \mathbb{R}_+^\ell : \langle x,p\rangle \leq \langle a_n(t,\omega),p\rangle\right\},$$

and

$$C_n(t,\omega) = \left\{y \in \mathbb{R}_+^\ell : \psi_n(t,\omega,y) \geq \psi_n(t,\omega,x) \text{ for all } x \in B_n(t,\omega)\right\}.$$

In addition, define $C_n^X : T\times\Omega \rightrightarrows \mathbb{R}_+^\ell$ such that for all $(t,\omega) \in T\times\Omega$,

$$C_n^X(t,\omega) = (C_n(t,\omega)\cup\{b_n(t,\omega)\})\cap X_n(t,\omega).$$

For every $n \geq 1$, define the correspondence $H_n : (\Omega,\mathscr{F},\nu) \rightrightarrows L_1\left(\mu,\mathbb{R}^\ell\right)$ by letting

$$H_n(\omega) = \left\{f \in L_1(\mu,\mathbb{R}_+^\ell) : f(t) \in C_n^X(t,\omega) \text{ for almost all } t \in T\right\}.$$

Obviously, $H_n(\omega) \neq \emptyset$ for all $\omega \in \Omega$.

Claim 1. For each fixed $n \geq 1$, H_n *is lower measurable.* For convenience, define a function $\Theta : L_1\left(\mu, \mathbb{R}^\ell\right) \times \Omega \to \mathbb{R}_+$ such that

$$\Theta(g, \omega) = \operatorname{dist}(g, H_n(\omega))$$

for all $g \in L_1\left(\mu, \mathbb{R}^\ell\right)$ and $\omega \in \Omega$. To verify the claim, one needs to verify that for each $g \in L_1\left(\mu, \mathbb{R}^\ell\right)$, $\Theta(g, \cdot)$ is measurable. Since $\Theta(\cdot, \omega) : L_1\left(\mu, \mathbb{R}^\ell\right) \to \mathbb{R}_+$ is norm-continuous, it suffices to show that $\Theta(g, \cdot) : (\Omega, \mathscr{F}, \nu) \to \mathbb{R}_+$ is measurable for every simple function $g = \sum_{j=1}^{r} x_j \chi_{T_j}$, where $x_j \in \mathbb{R}^\ell$. To this end, consider the function $\Gamma : (T, \Sigma, \mu) \times (\Omega, \mathscr{F}, \nu) \to \mathbb{R}_+$ such that

$$\Gamma(t, \omega) = \operatorname{dist}\left(g(t), C_n^X(t, \omega)\right)$$

for all $(t, \omega) \in T \times \Omega$. Since Γ is constant on each $(T_i^n \cap T_j) \times \Omega_i^n$, it is jointly measurable. Note that

$$\Gamma(t, \omega) \leq \|g(t) - b_n(t, \omega)\|$$

for all $(t, \omega) \in T \times \Omega$. This implies for all $\omega \in \Omega$, $\Gamma(\cdot, \omega)$ is integrable. Thus, $\Theta(g, \cdot)$ is measurable and the claim is verified if one shows for all $\omega \in \Omega$,

$$\int_T \Gamma(\cdot, \omega) d\mu = \Theta(g, \omega).$$

Assume that

$$\int_T \Gamma(\cdot, \omega_0) d\mu < \Theta(g, \omega_0)$$

for some $\omega_0 \in \Omega$. Pick $\varepsilon > 0$ such that

$$\int_T \Gamma(\cdot, \omega_0) d\mu + \varepsilon\mu(T) < \Theta(g, \omega_0).$$

Further, pick $t \in T_i^n \cap T_j$ and $y_{(i,j)} \in C_n^X(t, \omega_0)$ such that

$$\|x_j - y_{(i,j)}\| < \Gamma(t, \omega_0) + \varepsilon.$$

Define $\zeta : T \to \mathbb{R}_+^\ell$ by $\zeta(t) = y_{(i,j)}$ for all $t \in T_i^n \cap T_j$. Then, $\zeta \in H_n(\omega_0)$ and

$$\|g - \zeta\|_1 < \int_T \Gamma(\cdot, \omega_0) d\mu + \varepsilon\mu(T),$$

which is a contradiction.

Claim 2. The correspondence $\int_T C_n^X(\cdot, \cdot) d\mu : (\Omega, \mathscr{F}, \nu) \rightrightarrows \mathbb{R}_+^\ell$ is lower measurable.

To see this, consider the function $\xi : L_1\left(\mu, \mathbb{R}^\ell\right) \to \mathbb{R}^\ell$ defined by $\xi(f) = \int_T f d\mu$ for all $f \in L_1\left(\mu, \mathbb{R}^\ell\right)$. Let V be an open subset of $\mathbb{R}^\ell$. Note that

$$\xi \circ H_n(\omega) = \int_T C_n^X(\cdot, \omega) d\mu$$

for all $\omega \in \Omega$, and

$$(\xi \circ H_n)^{-1}(V) = \{\omega \in \Omega : H_n(\omega) \cap \xi^{-1}(V) \neq \emptyset\}.$$

Since ξ is norm-continuous, by Claim 1, $(\xi \circ H_n)^{-1}(V) \in \mathscr{F}$. This verifies the claim.

Claim 3.

$$\mathrm{Li} \int_T C_n^X(\cdot, \omega) d\mu = \mathrm{Ls} \int_T C_n^X(\cdot, \omega) d\mu = \int_T C^X(\cdot, \omega, p) d\mu.$$

To see this, for each $\omega \in \Omega$, put

$$\alpha(\cdot, \omega) = \sup \left\{ b_1(\cdot, \omega), \cdots, b_{N-1}(\cdot, \omega), b(\cdot, \omega, p) + \left(\frac{\ell}{\delta(p)}, \cdots, \frac{\ell}{\delta(p)} \right) \right\}.$$

Then, $C^X(\cdot, \omega, p)$ and all $C_n^X(\cdot, \omega)$ are upper bounded by $\alpha(\cdot, \omega)$. Now, it suffices to verify that for all $t \in T$,

$$\mathrm{Ls} C_n^X(t, \omega) \subseteq C^X(t, \omega, p) \text{ and } C^X(t, \omega, p) \subseteq \mathrm{Li} C_n^X(t, \omega).$$

First, let $x \in \mathrm{Ls} C_n^X(t, \omega)$. If $x = b(t, \omega, p)$, then $\{b_n(t, \omega) : n \geq 1\}$ converges to x and $b_n(t, \omega) \in C_n^X(t, \omega)$ for all $n \geq 1$. Assume now that $x \neq b(t, \omega, p)$. Then, there exist positive integers $n_1 < n_2 < n_3 < \cdots$ and for each k a point $x_k \in C_{n_k}^X(t, \omega)$ such that $\{x_k : k \geq 1\}$ converges to x and $x_k \neq b_{n_k}(t, \omega)$ for all sufficiently large k. Obviously, $x \in X(t, \omega, p)$. If $x \notin C^X(t, \omega, p)$, there is some $y \in \mathbb{R}_+^\ell$ such that

$$\langle y, p \rangle < \langle a(t, \omega), p \rangle \text{ and } U_t(\omega, y) > U_t(\omega, x).$$

Since $\{X_{n_k}(t, \omega) : k \geq 1\}$ converges to $X(t, \omega, p)$ in the Hausdorff metric topology, there is a sequence $\{y_k : k \geq 1\}$ such that $y_k \in X_{n_k}(t, \omega)$ for all $k \geq 1$ and $\{y_k : k \geq 1\}$ converges to y. By the inequality

$$\begin{aligned} |U_t(\omega, x) - \psi_{n_k}(t, \omega, x_k)| &< |U_t(\omega, x) - U_t(\omega, x_k)| \\ &+ |U_t(\omega, x_k) - \psi_{n_k}(t, \omega, x_k)|, \end{aligned}$$

the continuity of $U_t(\omega, \cdot)$ and the uniform convergence of $\psi_{n_k}(t, \omega, \cdot)$ to $U_t(\omega, \cdot)$, one concludes that

$$\psi_{n_k}(t, \omega, y_k) > \psi_{n_k}(t, \omega, x_k) \text{ and } \langle y_k, p\rangle < \langle a_{n_k}(t, \omega), p\rangle$$

for sufficiently large k, which contradicts with $x_k \in C^X_{n_k}(t, \omega)$ for all $k \geq 1$. Hence, $x \in C^X(t, \omega, p)$. Now, let $d \in C^X(t, \omega, p)$. If $d = b(t, \omega, p)$, then $\{b_n(t, \omega) : n \geq 1\}$ converges to d and $b_n(t, \omega) \in C^X_n(t, \omega)$ for all $n \geq 1$. Thus, $d \in \mathrm{Li}C^X_n(t, \omega)$. Suppose that $d < b(t, \omega, p)$. Similar to that in the proof of Theorem 6.1.4, one can show that $d \in \mathrm{Li}C^X_n(t, \omega)$.

To complete the proof, for each $\omega \in \Omega$, put

$$M(\omega) = \left\{ x \in \mathbb{R}^\ell_+ : x \leq \int_T \alpha(\cdot, \omega) d\mu \right\}.$$

Clearly, $\mathrm{cl} \int_T C^X_n(\cdot, \omega) d\mu$ and $\int_T C^X(\cdot, \omega, p) d\mu$ are contained in the compact set $M(\omega)$. It follows from Claim 3 that $\left\{\mathrm{cl} \int_T C^X_n(\cdot, \omega) d\mu : n \geq 1\right\}$ converges to $\int_T C^X(\cdot, \omega, p) d\mu$ in $M(\omega)$ under the Hausdorff metric topology. By Claim 2, $\mathrm{cl} \int_T C^X_n(\cdot, \cdot) d\mu : (\Omega, \mathscr{F}, \nu) \to \mathscr{K}_0(\mathbb{R}^\ell_+)$ is measurable, and thus, $\int_T C^X(\cdot, \cdot, p) d\mu$ is lower measurable. $\square$

Corollary 6.1.6. *Assume* $(\mathbf{C}_1)$-$(\mathbf{C}_3)$. *Then* $\int_T C^X(\cdot, \cdot, \cdot) d\mu : (\Omega, \mathscr{F}, \nu) \times (\Delta, \mathscr{B}(\Delta)) \to (\mathscr{K}_0(\mathbb{R}^\ell_+), \mathscr{T}_H)$ *is a jointly measurable function.*

Proof. By Theorem 6.1.4, for every $\omega \in \Omega$, $\int_T C^X(\cdot, \omega, \cdot) d\mu : \Delta \to \mathscr{K}_0\left(\mathbb{R}^\ell_+\right)$ is continuous. Furthermore, by Theorem 6.1.5, for every $p \in \Delta$, $\int_T C^X(\cdot, \cdot, p) d\mu : \Omega \rightrightarrows \mathbb{R}^\ell_+$ is lower measurable. Thus, for every $p \in \Delta$, $\int_T C^X(\cdot, \cdot, p) d\mu : \Omega \to \mathscr{K}_0(\mathbb{R}^\ell_+)$ is a measurable function. This means that $\int_T C^X(\cdot, \cdot, \cdot) d\mu : (\Omega, \mathscr{F}, \nu) \times (\Delta, \mathscr{B}(\Delta)) \to \mathscr{K}_0(\mathbb{R}^\ell_+)$ is Carathéodory, and thus is jointly measurable. $\square$

6.2 Existence of a Maximin REE

A *price system* of $\mathscr{E}$ is a measurable function $\pi : (\Omega, \mathscr{F}, \nu) \to (\Delta, \mathscr{B}(\Delta))$. Let $\mathscr{G}_t = \mathscr{F}_t \vee \sigma(\pi)$. For each $\omega \in \Omega$, let $\mathscr{G}_t(\omega)$ denote the smallest element of $\mathscr{G}_t$ containing ω. Given $t \in T$, $\omega \in \Omega$ and a price system π, let $B^{REE}(t, \omega, \pi)$ be defined by

$$B^{REE}(t, \omega, \pi) = \left\{ x \in (\mathbb{R}^\ell_+)^\Omega : x(\omega') \in B(t, \omega', \pi(\omega')) \text{ for all } \omega' \in \mathscr{G}_t(\omega) \right\}.$$

The *maximin utility* of each agent $t \in T$ with respect to $\mathscr{G}_t$ at an allocation $f : T \times \Omega \to \mathbb{R}^\ell_+$ in state ω, denoted by $\underline{U}_t^{REE}(\omega, f(t, \cdot))$, is defined by

$$\underline{U}_t^{REE}(\omega, f(t, \cdot)) = \inf_{\omega' \in \mathscr{G}_t(\omega)} U_t(\omega', f(t, \omega')).$$

Comparing with $\underline{U}_t^{REE}$, the function U_t is sometime called the *ex post* utility of agent t.

Remark 6.2.1. The maximin utility formation in the sense of REE was introduced by de Castro et al. in [31], where Ω is finite. In this case, for each $t \in T$, Π_t is a partition of Ω consisting of only finitely many elements and $\sigma(\pi)$ is generated by a partition Π_π also consisting of only finitely many elements. Thus, the σ-algebra $\mathscr{G}_t = \mathscr{F}_t \vee \sigma(\pi)$ is generated by the partition $\Pi_t \vee \Pi_\pi$. For each $\omega \in \Omega$, there exists a unique element $\mathscr{G}_t(\omega)$ in $\Pi_t \vee \Pi_\pi$ containing ω. It is clear that $\mathscr{G}_t(\omega)$ is the smallest element of $\mathscr{G}_t$ containing ω. Moreover, since $\mathscr{G}_t(\omega)$ is a finite set, $\underline{U}_t^{REE}(\omega, f(t, \cdot))$ is well-defined.

In our case, Ω is fairly general, particularly, can be infinite. The structure of $\sigma(\pi)$ can be complicated. If Ω is infinite, $\sigma(\pi)$ may not be generated by a partition. But, for each $\omega \in \Omega$, there always exists a (unique) smallest element in $\sigma(\pi)$ containing ω. This means that there also exists a (unique) smallest element $\mathscr{G}_t(\omega)$ in $\mathscr{G}_t$ containing ω. Since $\mathscr{G}_t(\omega)$ can be infinite, $\underline{U}_t^{REE}(\omega, f(t, \cdot))$ is allowed to take the value $-\infty$ if the above infimum does not exist. This adaptation will not affect the proof of Theorem 6.2.1.

Definition 6.2.1. Given a feasible allocation f and a price system π, the pair (f, π) is called a *maximin rational expectations equilibrium* (abbreviated as maximin REE) of $\mathscr{E}$ if $f(t, \omega) \in B(t, \omega, \pi(\omega))$ and $f(t, \cdot)$ maximizes $\underline{U}_t^{REE}(\omega, \cdot)$ on $B^{REE}(t, \omega, \pi)$ for almost all $(t, \omega) \in T \times \Omega$. In this case, f is called a *maximin rational expectations allocation*, and the set of such allocations is denoted by $MREE(\mathscr{E})$.

Definition 6.2.1 indicates that at a maximin rational expectations allocation, except for some negligible sets of agents and states of nature, each agent maximizes his maximin utility conditioned on his private information and the information generated by the equilibrium prices, subject to the budget constraint. Recently, de Castro et al. [32] showed that $MREE(\mathscr{E}) \neq \emptyset$ when Ω and T are finite. The next theorem extends their result to a more general case. To do this, suppose that $(\Omega, \mathscr{F}, \nu)$ is complete.

Theorem 6.2.1. *Under* $(\mathbf{C}_1)$-$(\mathbf{C}_4)$*,* $MREE(\mathscr{E}) \neq \emptyset$*.*

Proof. Consider the correspondence $Z : (\Omega, \mathscr{F}, \nu) \times (\Delta, \mathscr{B}(\Delta)) \rightrightarrows \mathbb{R}^\ell$ defined by

$$Z(\omega, p) = \int_T C^X(\cdot, \omega, p) d\mu - \int_T a(\cdot, \omega) d\mu.$$

By Proposition 6.1.3, Z is non-empty compact-valued. In addition, by Corollary 6.1.6 and $(\mathbf{C}_1)$, $Z : (\Omega, \mathscr{F}, \nu) \times (\Delta, \mathscr{B}(\Delta)) \to \mathscr{K}_0(\mathbb{R}^\ell)$ is jointly measurable. Define another correspondence $F : (\Omega, \mathscr{F}, \nu) \rightrightarrows (\Delta, \mathscr{B}(\Delta))$ such that

$$F(\omega) = \{p \in \Delta : Z(\omega, p) \cap \{0\} \neq \emptyset\}.$$

Since $\mathscr{E}(\omega)$ has a Walrasian equilibrium, F is non-empty valued. Since $\mathrm{Gr}_F = Z^-(\{0\})$, one obtains $\mathrm{Gr}_F \in \mathscr{F} \otimes \mathscr{B}(\Delta)$, and thus, F is lower measurable. It follows from Theorem 6.1.4 that $F(\omega)$ is closed for all $\omega \in \Omega$. By the Kuratowski-Ryll-Nardzewski measurable selection theorem, there is a measurable function $\hat{\pi} : (\Omega, \mathscr{F}, \nu) \to (\Delta, \mathscr{B}(\Delta))$ such that $\hat{\pi}(\omega) \in F(\omega)$ for all $\omega \in \Omega$. By the definition of Z, there exists an allocation f such that $f(t, \omega) \in C^X(t, \omega, \hat{\pi}(\omega))$ and

$$\int_T f(\cdot, \omega) d\mu = \int_T a(\cdot, \omega) d\mu$$

for almost all $t \in T$ and all $\omega \in \Omega$. By Proposition 6.1.1, one has

$$\langle f(t, \omega), \hat{\pi}(\omega) \rangle \geq \langle a(t, \omega), \hat{\pi}(\omega) \rangle$$

for almost all $t \in T$ and all $\omega \in \Omega$. It follows that

$$\langle f(t, \omega), \hat{\pi}(\omega) \rangle = \langle a(t, \omega), \hat{\pi}(\omega) \rangle$$

for almost all $t \in T$ and all $\omega \in \Omega$. Thus, $f(t, \omega) \in B(t, \omega, \hat{\pi}(\omega))$ for almost all $t \in T$ and all $\omega \in \Omega$. For all $\omega \in \Omega$, define $T_\omega \subseteq T$ by

$$T_\omega = \{t \in T : f(t, \omega) \in B(t, \omega, \hat{\pi}(\omega)) \cap C(t, \omega, \hat{\pi}(\omega))\}.$$

Then, $\mu(T_\omega) = \mu(T)$ for all $\omega \in \Omega$. Next, for every $\omega \in \Omega$ and every $t \in T \setminus T_\omega$, as $B(t, \omega, \hat{\pi}(\omega)) \cap C(t, \omega, \hat{\pi}(\omega)) \neq \emptyset$, one can pick a point

$$h(t, \omega) \in B(t, \omega, \hat{\pi}(\omega)) \cap C(t, \omega, \hat{\pi}(\omega)),$$

and then define a function $\hat{f} : T \times \Omega \to \mathbb{R}^\ell_+$ such that

$$\hat{f}(t, \omega) = \begin{cases} f(t, \omega), & \text{if } t \in T_\omega; \\ h(t, \omega), & \text{if } t \in T \setminus T_\omega. \end{cases}$$

It is obvious that $\hat{f}(t,\omega) \in B(t,\omega,\hat{\pi}(\omega)) \cap C(t,\omega,\hat{\pi}(\omega))$ for all $(t,\omega) \in T \times \Omega$. Assume that there are an agent $t_0 \in T$, a state of nature $\omega_{t_0} \in \Omega$ and an element $y(t_0,\cdot) \in B^{REE}(t_0,\omega_{t_0},\hat{\pi})$ such that

$$\underline{U}_{t_0}^{REE}(\omega_{t_0}, y(t_0,\cdot)) > \underline{U}_{t_0}^{REE}(\omega_{t_0}, \hat{f}(t_0,\cdot)).$$

Then, one obtains

$$U_{t_0}\left(\omega'_{t_0}, y(t_0,\omega'_{t_0})\right) > U_{t_0}\left(\omega'_{t_0}, \hat{f}(t_0,\omega'_{t_0})\right)$$

for some $\omega'_{t_0} \in \mathscr{G}_{t_0}(\omega_{t_0})$, which contradicts with $\hat{f}(t_0,\omega'_{t_0}) \in C\left(t_0,\omega'_{t_0},\hat{\pi}(\omega'_{t_0})\right)$. This verifies that $(\hat{f},\hat{\pi})$ is a maximin rational expectations equilibrium of $\mathscr{E}$. $\square$

Chapter 7

Conclusions and Future Work

The main goal of this thesis is to build up a general framework, which can be used to analyze some unsolved key problems on the blocking efficiency of the core allocations, characterizations and existence of Walrasian allocations in economies with asymmetric information. To do this, several mathematical concepts and techniques are employed. In the following sections, comparison of the results in this thesis with the others and some open problems are discussed. All of those open problems are left for future work. More precisely, Section 7.1 is devoted to discuss the blocking power of coalitions for allocations not belonging to the (strong) fine core and the private core of an asymmetric information economy. All results concerning the characterizations of Walrasian allocations established in the preceding chapters are summarized in Section 7.2. Section 7.3 is the last section of this chapter and deals with the existence of Walrasian allocations and maximin REE.

7.1 Blocking of Non-core Allocations

In Chapter 4, sharper characterizations of the (strong) fine core and the private core are provided in an asymmetric information economy with finitely many states and an infinite dimensional commodity space. It would be interesting to extend these results in the direction of infinitely many states. In Theorem 4.2.2 and Theorem 4.3.2, the positive cone of the commodity space has an interior point. At this stage, it is unclear that whether the conclusions of these theorems still hold when the positive cone of the commodity space has no interior points. The concept of the strong private core was used in [72], where a Vind-type theorem was also established in an asymmetric information economy with finitely many commodities. However, it is unclear whether this result can be extended to an asymmetric information economy with infinitely many commodities.

Definition 7.1.1. An allocation f in $\mathscr{E}$ is *coarsely blocked by a coalition* S if there is an assignment g such that $g(t,\cdot)$ is $\bigwedge \mathfrak{P}_S$-measurable and $V_t(g(t,\cdot)) > V_t(f(t,\cdot))$ for almost all $t \in S$, and $\int_S g(\cdot,\omega)d\mu = \int_S a(\cdot,\omega)d\mu$ for all $\omega \in \Omega$. The *coarse core* of $\mathscr{E}$, denoted by $\mathscr{C}\mathscr{C}(\mathscr{E})$, is the set of all feasible allocations which are not coarsely blocked by any coalition of $\mathscr{E}$.

Moreover, analogous to Definition 4.3.2 and Definition 7.1.1, the concept of the strong coarse core can also be introduced. In the light of Theorem 4.2.2 and Theorem 4.3.2, one may ask whether similar theorems can be established for the coarse core and the strong coarse core. These are left as open questions. In the rest of this section, several applications of Theorem 4.2.2 are given.

Remark 7.1.1. The first application of Theorem 4.2.2 leads to an affirmative solution to the question mentioned in Remark 1 of [72], when one considers $Y = \mathbb{R}^\ell$. This is just a special case of Theorem 4.2.2. Suppose that for an agent t, $U_t : \Omega \times Y_+ \to \mathbb{R}$ is the random utility function and q_t is the prior belief. Thus, agent t's ex ante expected utility is given by $V_t(x) = \sum_{\omega\in\Omega} U_t(\omega,x)q_t(\omega)$. By Remark 6 in [36], the measurability of the function $q_\cdot(\omega)$ and $U_\cdot(\omega,x)$ for all $x \in Y_+$ and the continuity of $U_t(\omega,\cdot)$ for all $t \in T$ imply $(\mathbf{A}_1)$. Similarly, $(\mathbf{A}_2)$ and $(\mathbf{A}_3)$ follow from continuity and monotonicity of $U_t(\omega,\cdot)$. Note that in the proof of Lemma 3 of [72], only the assumption that $U_t(\omega,\cdot)$ is concave for all $(t,\omega) \in T \times \Omega$ is needed. Thus, one may think that the same assumption is needed to solve Pesce's question. But here, instead of the concavity of $U_t(\omega,\cdot)$, measurability, continuity and monotonicity assumptions stated above are essential.

Remark 7.1.2. An equivalence theorem for the private core and the set of Walrasian expectations allocations without free disposal was given in [9]. Theorem 4.2.2 strengthens this result with a sharper interpretation. Under free disposal, it was shown in [22, 36, 46, 47] that Walrasian allocations can be characterized by privately non-dominated allocations and Aubin non-dominated allocations. The proofs of these results depend on some versions of the equivalence theorem and Vind's theorem. Applying Theorem 4.2.2 and the equivalence theorem in [9], one can easily extend those results to an asymmetric information economy with a finite dimensional commodity space and without free disposal. There is some potential to extend those results to an economy with an infinite dimensional commodity space, depending on whether the equivalence theorem in [9] can be extended to an asymmetric information economy with infinitely many commodities.

Remark 7.1.3. Recently, some work has been done in the formulation of maximin preferences, refer to [24, 31, 65]. With a little bit of extra efforts and modifications, results similar to Theorem 4.2.2 in the framework of maximin preferences could be established. In addition, based on non-linear price systems, the notion of a personalized equilibrium was introduced in [3]. As a potential future research direction, one may wonder if Theorem 4.2.2 can be applied to give some characterizations of personalized equilibria.

7.2 Characterizations of Walrasian Allocations

Since the work of Aumann in [15], different characterizations of Walrasian allocations by the co-operative solution concepts have been obtained in economic theory. In this thesis, two types of characterizations are considered. One deals with the veto power of infinitely many coalitions, and other claims that an allocation is a Walarasian allocation if and only if it is not privately blocked by the grand coalition, by considering perturbations of the original initial endowments in precise directions. All of these results presented in Chapter 3 and Chapter 5 are obtained in economies with finitely many states of nature. Possible extensions of these results in asymmetric information economies with infinitely many states of nature are still open questions. In addition, it is natural to ask whether the characterization of Walrasian allocations in terms of robustly efficient allocations can be extended to a framework with the commodity space has no interior points in its positive cone. Moreover, the major limitation of this theorem is that it does not allow the situation when either there is exactly one large agent, or at least two large agents have the different characteristics. Thus the following question arises: Does the conclusion of Theorem 5.2.4 hold in a mixed economy $\mathscr{E}$ with either $|\mathscr{A}| = 1$, or $|\mathscr{A}| \geq 2$ but whose large agents have different types of characteristics? In Section 5.3, some characterizations of Walrasian allocations are given in a discrete asymmetric information economy whose commodity space is a Banach lattice. At this stage, it is unclear whether similar characterizations hold in atomless (possibly mixed) economies with asymmetric information and Banach lattices as commodity spaces. This section concludes with several remarks.

Remark 7.2.1. The robust efficiency theorem in [48] is a particular case of Theorem 5.2.4. By Lemma 5.1.1 and Theorem 5.1.3, the coalition S and the number r in Theorem 5.2.4 can be chosen arbitrarily small in an atomless economy. As a result, slight perturbations of initial endowments of agents in small coalitions are enough to characterize

Walrasian allocations. Moreover, similar to Remark 3.2 in [48], as a particular case of Theorem 5.2.4, one obtains the welfare theorems in mixed economies with asymmetric information.

Remark 7.2.2. In the end of [48], the following question was mentioned: *More work is needed to verify whether the result still holds for other commodity spaces. Of particular interest is the spaces of measures, where the Walrasian allocations are known to resist manipulation by arbitrarily small coalitions and may, therefore, be regarded as true perfectly compatitive outcomes.* Our Theorem 5.2.4 provides an answer to the first part of the question. Under the continuum hypothesis, Podczeck [68], Tourky and Yannelis [81] constructed a continuum economy with a non-separable ordered Banach space having an interior point in its positive cone as the commodity space such that Aumann's core-Walras equivalence theorem fails. In the light of this construction, it is conjectured that the answer to the second part of the question is negative.

It is worth to point out that the argument in the proof appeared in [48] cannot be applied to extend the robust efficiency theorem to an asymmetric information economy with an atomless measure space of agents, finitely many commodities and without free disposal. In fact, to keep the measurability of $h(t, \cdot)$ for almost all $t \in S$ in their Lemma 3.1, the authors choose $\delta = \min\{\delta(\omega) : \omega \in \Omega\}$. So the equality sign in their Lemma 3.1 is replaced by "$\leq$" and hence also at the end of the proof of their Theorem 3.1. Thus, this approach does not provide an answer to the case without free disposal in an asymmetric information economy. However, one can apply Theorem 5.1.3 of this thesis to obtain a positive result. A similar result for a mixed economy requires some further modification of Theorem 5.2.2, which is left as an open problem. In this regard, Theorem 4.2.2 may be helpful to obtain a positive result.

Remark 7.2.3. Now, Theorem 5.2.4 of this thesis and Theorem 5.2 in [41] are compared. First, different assumptions on the concavity of utility functions and the convexity of preference relations are used in the proofs of these two results. Recall that if a preference relation is represented by a utility function, then the concavity of utility implies the convexity of preference relation. Note that in the proof of Theorem 5.2.4, the partial convexity of P_t for all $t \in T_1$ is used. When the economy $\mathscr{E}$ is atomless, no convexity is needed at all. But, in the proof of Theorem 5.2 in [41], the concavity of $U_t(\omega, \cdot)$ for all $(t, \omega) \in T \times \Omega$ is used. Second, Theorem 5.2.4 holds under assumptions that $\mathscr{E}$ either is atomless or contains at least two large agents, and all large agents have the same characteristics. But, Theorem 5.2 in [41] does not need these assumptions. Third, Theorem 5.2 in [41] gives a characterization of a Walrasian expectations allocation f of $\mathscr{E}$ in terms of the veto power of the grand coalition in every economy $\mathscr{E}(\gamma, f)$ for all

$\gamma \in \mathscr{B}$. Since $\mathscr{E}(S, f, r) = \mathscr{E}(\gamma_{S,r}, f)$ for all $S \in \Sigma$, $\mu(S) < \mu(T)$ and $0 \leq r \leq 1$, and

$$\mathscr{B}_0 = \{\gamma_{S,r} : S \in \Sigma,\ \mu(S) < \mu(T) \text{ and } 0 \leq r \leq 1\}$$

is a proper subfamily of $\mathscr{B}$, Theorem 5.2.4 of this thesis provides a sharper characterization of a Walrasian expectations allocation f in $\mathscr{E}$.

Remark 7.2.4. Applying Theorem 5.2.4 to a continuum economy with n different types of agents, one obtains Theorem 4.1 in [46, 47] and a similar result in [36] as particular cases.

Remark 7.2.5. The work presented in this thesis is within the Arrow-Debreu formulation of the Walrasian equilibrium model. Since many intertemporal general equilibrium models are derived from this microeconomic formulation, characterization theorems may have potential applications to some of intertemporal general equilibrium models. For instance, one may consider if some modifications of the approach in this paper can provide characterizations of equilibria in the economic model of asset markets without short-selling in [26] or the models mentioned in [55].

7.3 Non-emptiness of $\mathscr{W}(\mathscr{E})$ and $MREE(\mathscr{E})$

Several characterizations of Walrasian allocations are presented in Chapter 3 and Chapter 5. One may wonder whether a Walrasian equilibrium exists in each of those frameworks. The existence of a Walrasian equilibrium in a discrete economy can be established by using an argument similar to that in [71]. It is well known that a Walrasian equilibrium may not exist in a continuum economy with infinitely many agents and infinitely many commodities. In fact, it was shown in [90] that without any upper bound on the consumption sets, an equilibrium may not exist in an economy with a continuum of agents and infinitely many commodities. However, Khan and Yannelis [57] obtained a positive result by employing an additional assumption on the consumption set of each agent. To obtain a positive result for a mixed asymmetric information economy whose commodity space is an ordered Banach space having an interior point in its positive cone, the following assumption is posed: *The consumption correspondence $X : T \times \Omega \to Y_+$ is integrably bounded, norm-closed, convex, nonempty, weakly compact-valued. In addition, $X(\cdot, \omega)$ has a measurable graph for all $\omega \in \Omega$ and $X(t, \cdot)$ is $\mathscr{F}_t$-measurable for all $t \in T$.* Under this assumption, define a correspondence $\widehat{X} : T \to Y_+^{\Omega}$ by $\widehat{X}(t) = X(t, \cdot) \cap L_t$. It can be easily checked that the assumption (3.2) in [57] is satisfied. Thus, applying the main result in [57] with P_t as a prefernce on

$\widehat{X}(t)$ for $t \in T$, one can show the existence of a Walrasian equilibrium.

In Chapter 6, an existence theorem on a maximin rational expectations equilibrium (maximin REE) for an exchange asymmetric information economy is proved. Comparing with the existence result on maximin REE in [31], the existence theorem in Chapter 6 applies to a more general economic model with an arbitrary finite measure space of agents and an arbitrary complete probability measure space as the space of states of nature, while the later applies only to an economic model which has finitely many agents and finitely many states of nature. Assumptions in Theorem 6.2.1 are similar to those in [32], except the joint measurability and continuity of utility functions, and the joint measurability of the initial endowment function. The proof techniques in Theorem 6.2.1 are quite different from those in [32]. Since there are only finitely many agents and states of nature in the model considered in [32], neither measurability nor continuity of utility functions and the initial endowment function plays any role in the proof of the existence of a maximin REE. Instead, the existence of a competitive equilibrium for complete information economies is applied. In contrast, both measurability and continuity of utility functions and the initial endowment function play key roles in Theorem 6.2.1. To establish the existence theorem, techniques in [17] are adopted, the measurability and continuity of the aggregate preferred correspondence are investigated. However, for special cases, the techniques can be simplified. For instance, if there are finitely many states of nature, one can still apply the approach employed in [32] and obtains an existence theorem. On the other hand, if there are finitely many agents, then one can show that the demand of each agent is $\mathscr{F} \otimes \mathscr{B}(\Delta)$-measurable and so is the aggregate demand. Then, an approach similar to that in the proof of Theorem 6.2.1 can be applied to establish the existence theorem. Further, since the space of states of nature in our model is an abstract probability space, the existence theorem in Chapter 6 does not depend on the dimension of the space of states of nature.

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