Towards Combining Dense Linear Order with Random Graph

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Abstract. In this paper we present our work in progress towards obtaining a Nelson-Oppen style combination for combining quantified theories, where each individual component theory admits quantifier elimination. We introduces the notion of *good model* for union theories, and show that for the good models of union theories, there exists a simple quantifier elimination scheme which uses the elimination procedures for individual component theories as black boxes. Using a priority argument, we show that good models exist for the union theory of dense linear order and random graph.

1 Introduction

In [1] in 1979, Nelson and Oppen proposed a framework for combining decision procedures on quantifier-free formulas: if theories T_1 and T_2 are over disjoint signatures and stably infinite, then one can obtain a decision procedure, on quantifier-free formulas, for the union theory $T_1 \cup T_2$, using the decision procedures for T_1 and T_2 as modules. Ever since the foundational work of Nelson and Oppen, researchers have been asking the general question: under what condition do we have a combination method for arbitrary first-order (not necessarily quantifier-free) formulas? Recently, a lot of progresses have been made to relax the conditions on component theories to be combined [2, 3].

In this paper we consider a restricted version of the question: providing that two theories T_1 and T_2 both admit quantifier elimination, does the union theory $T_1 \cup T_2$ also admit quantifier elimination? If it does, can we find an elimination procedure for $T_1 \cup T_2$, using the elimination procedures for T_1 and T_2 as modules?

Suppose for $i \in \{1, 2\}$, T_i is a L_i theory and φ_i is a conjunction of L_i-literals. It is not hard to see that we can obtain a quantifier elimination procedure for $T_1 \cup T_2$ by using individual elimination procedures for T_1 and T_2 as modules, if we have

$$T_1 \cup T_2 \models \forall \bar{x} \left[(\exists y\, \varphi_1(\bar{x}, y) \wedge \exists y\, \varphi_2(\bar{x}, y)) \leftrightarrow \exists y\, (\varphi_1(\bar{x}, y)) \wedge \varphi_2(\bar{x}, y)) \right]. \quad (1)$$

However, Condition (1) does not hold in general, as shown by the following example.

Example 1 (Incompatible Dense Linear Orders) *For $i \in \{1, 2\}$, let L_i be the signature $\{<_i\}$, and let T_i be the L_i-theory of dense linear orders. Consider $\mathcal{A} = \langle A, <_1, <_2 \rangle$ where A is the set of rational numbers, and $<_1$, $<_2$ are such that for any $u, v \in A$, $u <_1 v$ iff $v <_2 u$ iff $u < v$. Obviously $\mathcal{A}$ is a model of $T_1 \cup T_2$. However, for any $a \in A$, $\mathcal{A} \models \exists x\,(x <_1 a)$, $\mathcal{A} \models \exists x\,(x <_2 a)$, but $\mathcal{A} \not\models \exists x\,(x <_1 a \wedge x <_2 a)$.*

In this paper we call models of $T_1 \cup T_2$ that satisfy Condition (1) *good models*. Using a priority argument, we show that good models do exist for the union theory of dense linear order and random graph, and hence we obtain a decision procedure for the union theory restricted to those good models.

Paper Organization Section 2 provides basic notions and terminology in model theory, and introduces some notations in our presentation. Section 3 proves the existence of good models for the union theory of dense linear order and random graphs. Section 4 concludes with a discussion of future work.

2 Preliminary

In this section we introduce notions and terminology used in this paper. We assume the first-order syntactic notions of variables, parameters and quantifiers, and semantic notions of structures, satisfiability and validity as in [4].

Basic Notations. We use $\mathbb{N}$ to denote the set of natural numbers, and $\mathbb{Q}$ the set of rational numbers. We use $\overline{u}$ to denote the sequence $u_1, \ldots, u_n$ (for some $n > 0$). We abuse notation a bit by also using $\overline{u}$ to denote the set that consists of elements in the sequence. For example, by $\overline{u} \in S$ we mean that all elements in $\overline{u}$ are contained in S. The meaning should be clear from the context. Also by $(u_i)_{i<\omega}$ we mean an infinite enumeration of the form $u_0, u_1, \ldots$.

By default we use calligraphic letters $\mathcal{A}, \mathcal{B}, \mathcal{C}, \ldots$, to denote structures and the capital letters $A, B, C, \ldots$, to denote the corresponding domains. For example, a model of graph is denoted by $\mathcal{G} = \langle G, E^{\mathcal{G}} \rangle$. When there is no confusion, we drop superscripts on function symbols and predicate symbols.

We use $\mathcal{A} \cong \mathcal{B}$ to mean that $\mathcal{A}$ and $\mathcal{B}$ are isomorphic. We use $\mathcal{A} \subset \mathcal{B}$ to mean that $\mathcal{A}$ properly embeds into $\mathcal{B}$, i.e., $\mathcal{A}$ is isomorphic to a proper substructure of $\mathcal{B}$. For a structure $\mathcal{A}$ and a tuple $\overline{a} \in A$, whenever we use $\overline{a}$ in variable substitution, it should be understood that the underlying language is extended with constants $\overline{a}$, each of which names itself in the extended structure $\mathcal{A}' = (\mathcal{A}, \overline{a})$.

Dense Linear Order. A *dense linear order* (*DLO*) is a linear order $\mathcal{D} = \langle D, < \rangle$ such that $\mathcal{D}$ has no endpoints and

$$\forall x, y \in D\ (x < y \rightarrow \exists z\,(x < z \wedge z < y))\,. \tag{2}$$

Let $L_{\mathcal{D}}$ denote the language of $\mathcal{D}$ and $T_{\mathcal{D}}$ the theory of $\mathcal{D}$. It is well-known that $T_{\mathcal{D}}$ is ω-categorical, complete and decidable, and it admits quantifier elimination.

In particular, the linear order on rational numbers, denoted by $\mathcal{Q} = \langle \mathbb{Q}, <^{\mathcal{Q}} \rangle$, is the unique countable model of $T_{\mathcal{D}}$ up to isomorphism. In the paper we identify $\mathcal{Q}$ with $\mathcal{D}$.

Lemma 1. *For any conjunction of* positive *$L_{\mathcal{Q}}$-literals $\Phi(\overline{x}, y)$, where y does not appear in equalities, for any $\overline{a} \in Q$, if $\mathcal{Q} \models \exists y \Phi(\overline{a}, y)$, then there are infinitely many $b \in Q$ such that $\mathcal{Q} \models \Phi(\overline{a}, b)$.*

Proof. Let $\Phi(\overline{x}, y)$ be a $L_{\mathcal{Q}}$conjunction of *positive* $L_{\mathcal{Q}}$-literals, where y does not appear in equalities, and let $\overline{a}$ be any tuple in $\mathcal{Q}$. It is easily seen that $\Phi(\overline{a}, y)$ states that y is contained in the intersection of finitely many open intervals whose boundaries are elements in $\overline{a}$. Since the intersection of finitely many open intervals is an open interval, if there is a solution of $\Phi(\overline{a}, y)$, then by the denseness property of $\mathcal{Q}$, there exist infinitely many such solutions. $\square$

Random Graph. A *Random Graph* (*RG*) is a countable graph $\mathcal{G} = \langle G, E \rangle$ such that for any $n, m > 0$,

$$\forall x_1 \ldots \forall x_n \forall y_1 \ldots \forall y_m \left(\bigwedge_{i=1}^{n} \bigwedge_{j=1}^{m} x_i \neq y_j \rightarrow \exists z \left(\bigwedge_{i=1}^{n} E(x_i, z) \wedge \bigwedge_{j=1}^{m} \neg E(y_j, z) \right) \right) . \quad (3)$$

Let $L_{\mathcal{G}}$ denote the language of $\mathcal{G}$ and $T_{\mathcal{G}}$ the theory of random graph. Like $T_{\mathcal{D}}$, $T_{\mathcal{G}}$ is ω-categorical, complete and decidable, and it admits quantifier elimination.

Lemma 2. *For any conjunction of $L_{\mathcal{G}}$-literals $\Phi(\overline{x}, y)$, where y does not appear in equalities, for any $\overline{a} \in G$, if $\mathcal{G} \models \exists y \Phi(\overline{a}, y)$, then there are infinitely many $b \in G$ such that $\mathcal{G} \models \Phi(\overline{a}, b)$.*

Proof. Let $\Phi(\overline{x}, y)$ be a conjunction of $L_{\mathcal{G}}$-literals, where y does not appear in equalities, and let $\overline{a} = a_1, \ldots, a_n$ be any tuple in G. It is not hard to see that $\Phi(\overline{a}, y)$ is of the form

$$\bigwedge_{i=1}^{s} E(a_i, y) \wedge \bigwedge_{j=s+1}^{n} \neg E(a_j, y) \wedge \bigwedge_{b \in P} y \neq b \wedge \Phi'(\overline{a}) \ , \quad (4)$$

where $P \subseteq \overline{a}$, y does not appear in Φ' and $s \leq n$. Since $\mathcal{G} \models \exists y \Phi(\overline{a}, y)$, we have $\bigwedge_{i=1}^{s} \bigwedge_{j=s+1}^{n} a_i \neq a_j$ and $\mathcal{G} \models \Phi'(\overline{a})$. Now take a finite set $S \subseteq G$ such that $S \cap \overline{a} = \emptyset$. Then by (3) we have, for any $S' \subseteq S$,

$$\mathcal{G} \models \exists y \left(\bigwedge_{i=1}^{s} (E(a_i, y) \wedge \bigwedge_{b \in S'} (E(b, y) \wedge \bigwedge_{j=s+1}^{n} \neg E(a_j, y) \wedge \bigwedge_{b \in S \setminus S'} \neg E(b, y) \right) \ , \quad (5)$$

which tells us that there are at least $2^{|S|}$ distinct witnesses to $\bigwedge_{i=1}^{s} E(a_i, y) \wedge \bigwedge_{j=s+1}^{n} \neg E(a_j, y)$, and hence at least $2^{|S|} - |P|$ solutions to (4). In fact there must be infinitely many solutions to (4) because S can be arbitrarily large. $\square$

3 Combining Dense Linear Order with Random Graph

In this section we present a model of $T_{\mathcal{D}} \cup T_{\mathcal{G}}$ which admits quantifier elimination.

Lemma 3. *There exists a model $\mathcal{A} = \langle A, <^{\mathcal{A}}, E^{\mathcal{A}} \rangle$ of $T_{\mathcal{D}} \cup T_{\mathcal{G}}$ such that for any conjunction of* positive *$L_{\mathcal{D}}$-literals $\Phi(\overline{x}, y)$, and for any conjunction of $L_{\mathcal{G}}$-literals $\Psi(\overline{x}, y)$, if y does not occur in equality in either Φ or Ψ, then*

$$\mathcal{A} \models \forall \overline{x} \left[(\exists y \Phi(\overline{x}, y) \wedge \exists y \Psi(\overline{x}, y)) \leftrightarrow \exists y \left(\Phi(\overline{x}, y) \wedge \Psi(\overline{x}, y) \right) \right] \ . \tag{6}$$

Proof. We first outline our construction idea for $\mathcal{A}$. Then we present the detailed construction. Finally we prove that $\mathcal{A}$ is our desired model.

Construction Plan. The direction "$\leftarrow$" is obvious as it holds for any models. The other direction is considerably involved. We construct an infinite ascending chain of finite structures, $\mathcal{A}_0 \subset \mathcal{A}_1 \subset \ldots$, whose limit is our desired $\mathcal{A}$, i.e., $\mathcal{A} = \bigcup_{i \in \mathbb{N}} \mathcal{A}_i$. The domain A of $\mathcal{A}$ consists of tuples of the form (u, v) where $u \in \mathbb{Q}$ and $v \in G$. Moreover, every $u \in \mathbb{Q}$ and every $v \in G$ appear in exactly one tuple in A. Essentially we construct an infinite ascending chain of functions $f_0 \subset f_1 \subset \ldots$, where each f_i is a $1-1$ partial function from $\mathbb{Q}$ to G with a finite graph. Let $dom(f_i)$ denote the effective domain of f_i. For each $i \in \mathbb{N}$, f_i induces $\mathcal{A}_i = \langle A_i, <^{\mathcal{A}_i}, E^{\mathcal{A}_i} \rangle$ as follows.

$$A_i = \{ (u, f_i(u)) \in \mathbb{Q} \times G \mid u \in dom(f_i) \} \tag{7}$$

$$<^{\mathcal{A}_i} = \{ ((u, v), (u', v')) \in A_i \times A_i \mid u <^{\mathbb{Q}} u' \} \tag{8}$$

$$E^{\mathcal{A}_i} = \{ ((u, v), (u', v')) \in A_i \times A_i \mid E^{\mathcal{G}}(v, v') \} \tag{9}$$

Note that the limit of this chain is a bijective function $f : \mathbb{Q} \to G$ which induces $\mathcal{A} = \langle A, <^{\mathcal{A}}, E^{\mathcal{A}} \rangle$ in the same way as defined above.

The essential construction from stage i to stage $i+1$ is to, for each tuple $\overline{a} \in \mathcal{A}_i$, find witnesses for formulas of the form $\exists y (\Phi(\overline{a}, y) \wedge \Psi(\overline{a}, y))$, providing that both $\exists y (\Phi(\overline{a}, y)$ and $\exists y (\Psi(\overline{a}, y)$ hold separately in $\mathcal{A}_i$. Obviously, at a single stage we might not find witnesses for all pairs of formulas of the form $(\Phi(\overline{x}, y), \Psi(\overline{x}, y))$ as there could be infinitely many such pairs. However, by a standard encoding technique we make sure that witnesses for every such pair will eventually be discovered at certain stage. We present the detailed construction as follows, which is essentially a priority argument.

Construction. Let $(\Phi_i)_{i<\omega}$ be an enumeration of all finite conjunctions of *positive* $L_{\mathcal{D}}$-literals of the form $\varphi(\overline{x}, y)$ where y does not appear in equalities, and $(\Psi_i)_{i<\omega}$ an enumeration of all finite conjunctions of $L_{\mathcal{G}}$-literals of the form $\psi(\overline{x}, y)$ where y does not appear in equalities. Note that such enumerations exist since both $L_{\mathcal{D}}$ and $L_{\mathcal{G}}$ are countable languages. Let $\langle \cdot, \cdot \rangle : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a pairing function (i.e., a bijection from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$), and $l : \mathbb{N} \to \mathbb{N}$, $r : \mathbb{N} \to \mathbb{N}$ be the corresponding projection functions such that for any $n \in \mathbb{N}$, $\langle l(n), r(n) \rangle = n$. This pairing function is used to enumerate $\{(\Phi_i, \Psi_j) \mid i, j \in \mathbb{N}\}$. Also let $(u_i)_{i<\omega}$ be an enumeration of $\mathbb{Q}$, and $(v_i)_{i<\omega}$ an enumeration of G.

Let $f_0 = \emptyset$ and hence $\mathcal{A}_0$ be an empty structure. Suppose f_i and $\mathcal{A}_i$ have been obtained. We run Algorithm 1 to obtain f_{i+1} and $\mathcal{A}_{i+1}$.

Algorithm 1 Construction of $\mathcal{A}_{i+1}$.

1: Set $f_{i+1} = f_i$.
2: Find the first *unused* element $u \in (u_i)_{i<\omega}$ and the first *unused* element $v \in (v_i)_{i<\omega}$. Mark u, v as *used*. Set $f_{i+1} = f_{i+1} \cup (u, v)$.
3: **for all** $\bar{a} \in \mathcal{A}_i$ and $j < i$ **do**
4:     **if** $\mathcal{A}_i \models \exists y \Phi_{l(j)}(\bar{a}, y) \wedge \mathcal{A}_i \models \exists y \Psi_{r(j)}(\bar{a}, y)$ **then**
5:         Find the first *unused* element $u \in (u_i)_{i<\omega}$ such that $\mathcal{Q} \models \Phi_{l(j)}(\bar{a}, u)$ and the first *unused* element $v \in (v_i)_{i<\omega}$ such that $\mathcal{G} \models \Psi_{r(j)}(\bar{a}, v)$. Mark u, v as *used*. Set $f_{i+1} = f_{i+1} \cup (u, v)$.
6:     **end if**
7: **end for**

Proof Continued. We show by induction that for each $i \in \mathbb{N}$, Algorithm 1 is sound and terminates, and for each $i \in \mathbb{N}$, $\mathcal{A}_i \subset \mathcal{A}_{i+1}$, $\mathcal{A}_i^{L_\mathcal{D}} \subset \mathcal{Q}$, $\mathcal{A}_i^{L_\mathcal{G}} \subset \mathcal{G}$, and

$$\forall j < i \, \forall \bar{a} \in A_i \Big[\Big(\mathcal{A}_i \models \exists y \Phi_{l(j)}(\bar{a}, y) \ \wedge \ \mathcal{A}_i \models \exists y \Psi_{r(j)}(\bar{a}, y) \Big) \Rightarrow \mathcal{A}_{i+1} \models \exists z \left(\Phi_{l(j)}(\bar{a}, z) \ \wedge \ \Psi_{r(j)}(\bar{a}, z) \right) \Big]. \tag{10}$$

The case $i = 0$ is trivial. By Step (2), $f_i \subset f_{i+1}$ and hence $A_i \subset A_{i+1}$. By (7)-(9), we have $\mathcal{A}_i^{L_\mathcal{D}} \subset \mathcal{Q}$ and $\mathcal{A}_i^{L_\mathcal{G}} \subset \mathcal{G}$. Now $\mathcal{A}_i \models \exists y \Phi_{l(j)}(\bar{a}, y)$ implies $\mathcal{A}_i^{\mathcal{D}} \models \exists y \Phi_{l(j)}(\bar{a}, y)$, which in turn implies $\mathcal{Q} \models \exists y \Phi_{l(j)}(\bar{a}, y)$. Similarly, we have $\mathcal{A}_i^{\mathcal{G}} \models \exists y \Psi_{r(j)}(\bar{a}, y)$ implies $\mathcal{G} \models \exists y \Psi_{r(j)}(\bar{a}, y)$. Therefore, Step (5) can be realized due to Lemma 1 and Lemma 2. The termination of Algorithm 1 follows because there are only finitely many $\bar{a} \in A_i$ and $j < i$. Property (10) holds obviously thanks to Step (5).

Since Step (2) pairs elements in Q with elements in G according to the enumerations $(u_i)_{i<\omega}$ and $(v_i)_{i<\omega}$, eventually every element in $\mathbb{Q}$ is paired with one element in G, and vice versa. Therefore, we have $\mathcal{A}^{L_\mathcal{D}} \cong \mathcal{Q}$ and $\mathcal{A}^{L_\mathcal{G}} \cong \mathcal{G}$, and hence $\mathcal{A}$ is a model of $T_\mathcal{D} \cup T_\mathcal{G}$.

Let $\Phi \equiv \Phi_i$ and $\Psi \equiv \Psi_j$ for some $i, j \in \mathbb{N}$, and $\bar{a}$ be an arbitrary tuple in A. Suppose that $\mathcal{A} \models \Phi(\bar{a}, u) \wedge \Psi(\bar{a}, v)$ for some $u, v \in A$. Take $k \in \mathbb{N}$ such that $k > \langle i, j \rangle$, and $\bar{a}, u, v \in A_k$. We have

$$\begin{aligned} \mathcal{A} \models \Phi(\bar{a}, u) \ \wedge \ \Psi(\bar{a}, v) \quad & \Rightarrow \quad \mathcal{A}_k \models \Phi(\bar{a}, u) \ \wedge \ \Psi(\bar{a}, v) \\ & \Rightarrow \quad \mathcal{A}_k \models \exists y \Phi(\bar{a}, y) \ \wedge \ \exists y \Psi(\bar{a}, y) \\ & \Rightarrow \quad \mathcal{A}_{k+1} \models \exists y \, (\Phi(\bar{a}, y) \ \wedge \ \Psi(\bar{a}, y)) \\ & \Rightarrow \quad \mathcal{A} \models \exists y \, (\Phi(\bar{a}, y) \ \wedge \ \Psi(\bar{a}, y)) \end{aligned}$$

□

We call the models that satisfy Lemma 3 **good models** of $T_{\mathcal{D}} \cup T_{\mathcal{G}}$. Let $(T_{\mathcal{D}} \cup T_{\mathcal{G}})_{GOOD}$ be the theory of all good models of $T_{\mathcal{D}} \cup T_{\mathcal{G}}$.

Theorem 2. *$(T_{\mathcal{D}} \cup T_{\mathcal{G}})_{GOOD}$ admits quantifier elimination.*

Proof. It suffices to show that one can eliminate $\exists y$ from formulas of the form $\exists y \varphi(\overline{x}, y)$ where $\varphi(\overline{x}, y)$ is a conjunction of literals. Since $L_{\mathcal{D}} \cup L_{\mathcal{G}}$ contains no function symbols, any such $\exists y \varphi(\overline{x}, y)$ can be rewritten as

$$\exists y \ (\Phi(\overline{x}, y) \ \wedge \ \Psi(\overline{x}, y)) \tag{11}$$

where $\Phi(\overline{x}, y)$ is conjunction of $L_{\mathcal{D}}$-literals, $\Psi(\overline{x}, y)$ is a conjunction of $L_{\mathcal{G}}$-literals. We further assume that $\Phi(\overline{x}, y)$ contains only positive literals as $\neg(x < y)$ can be replaced by $x = y \vee x > y$. We also assume that y does not appear in equalities (otherwise the elimination of $\exists y$ is trivial). Now $\Phi(\overline{x}, y)$ and $\Psi(\overline{x}, y)$ satisfy the requirements in Lemma 3. So (11) can be rewritten as

$$\exists y \ \Phi(\overline{x}, y) \ \wedge \ \exists y \Psi(\overline{x}, y) \tag{12}$$

Now $\exists y \Phi(\overline{x}, y)$ is a pure $L_{\mathcal{D}}$-formula and $\exists y \Psi(\overline{x}, y)$ is a pure $L_{\mathcal{G}}$-formula. We can carry out the elimination using the elimination procedure for $\mathcal{Q}$ and the elimination procedure for $\mathcal{G}$.

Corollary 1. *The decision problem for$(L_{\mathcal{D}} \cup L_{\mathcal{G}})$-formulas in good models of $T_{\mathcal{D}} \cup T_{\mathcal{G}}$ is decidable .*

Proof. Using the quantifier elimination described in Theorem 2, one can transform an arbitrary closed first-order $(L_{\mathcal{D}} \cup L_{\mathcal{G}})$-formula into an equivalent quantifier-free formula, which must be either $false$ or *true* as $(L_{\mathcal{D}} \cup L_{\mathcal{G}})$ has no constants.

4 Conclusion and Future Work

In this paper we introduced the notion of *good model* and showed a simple quantifier elimination scheme for good models of union theories. Using a priority argument we showed that $T_{\mathcal{D}} \cup T_{\mathcal{G}}$ has good models and hence admits quantifier elimination with respect to those good models.

This is a work in progress towards generalizing Nelson-Oppen combination for combining quantified theories, providing that each individual component theory admits quantifier elimination. Although our current result is only limited to good models, we think it is a good starting point for investigating more general schemes for combining quantifier elimination procedures. Note that our proof of the existence of good models relies on the "denseness" property of individual theories, that is, there are infinitely many witnesses to existential formulas (Lemmas 1 and 2). However, this property does not hold for many important theories in computer science, such as Presburger arithmetic and discrete orders. Therefore, we first plan to investigate the necessary conditions for the existence of good models and hope this would give us more insights on quantifier elimination schemes for the general models of union theories.

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