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# A stricter canon: General Luce models for arbitrary menu sets[☆]

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ABSTRACT

Alós-Ferrer and Mihm (2025, Corollary 1) recently provided a characterisation the classical Luce model (Luce, 1959) when choices are observed for an arbitrarily restricted collection of menus, as is typical in experimental settings or when working with field data. They also characterise the general Luce model (*ibid.*, Theorem 1), which allows choice probabilities to be zero, for the same setting. The latter characterisation involves a single axiom – the *general product rule (GPR)*. An important special case of the general Luce model is obtained when the mapping from menus to the support of choice probabilities can be rationalised by a weak order. Cerreia-Vioglio et al. (2021) show that this special case is characterised by Luce's (1959) *choice axiom*, provided choice is observed for all possible (finite) menus. The choice axiom is thus a fundamental "canon of probabilistic rationality". We show that a natural – and surprisingly simple – strengthening of the GPR characterises the model of Cerreia-Vioglio et al. (2021) when the menu set is arbitrarily restricted. Our axiom implies the choice axiom, and is therefore a "stricter canon".

## 1. Introduction

A *random choice function (RCF)* specifies the probability, $p(x, E)$, that alternative $x$ is chosen when the decision-maker is confronted with menu $E$. Alternatives come from some universal domain, $X$, and menus are non-empty, *finite* subsets of $X$. The classical Luce model (Luce, 1959) generates choice probabilities from a utility function, $v : X \to \mathbb{R}_{++}$. The probability of choosing alternative $x \in E$ from menu $E \subseteq X$ is equal to the utility of $x$ as a proportion of the total utility of alternatives in $E$:

$$p(x, E) = \frac{v(x)}{\sum_{y \in E} v(y)} \tag{1}$$

The Luce model embodies an implicit assumption of *positivity*: any element of any menu is chosen with strictly positive probability. In order to characterise the Luce model – to identify the set of RCFs which possess such a model – Luce makes additional assumptions about the menu domain.[1] He provides separate characterisations for two such assumptions. First, when $p(\cdot, E)$ is defined for *any* non-empty, finite $E \subseteq X$, we say that menus are *unrestricted*.[2] In this case, the Luce model is characterised by the *choice axiom* or, equivalently (given positivity), by *independence of irrelevant alternatives (IIA)*. Second, if menus are restricted to the *binary* subsets of $X$, the model is characterised by the *product rule*.

The *general Luce model (GLM)* relaxes the positivity assumption.[3] The GLM requires that if $x$ is chosen with positive probability from menu $E$, then the probability of choosing $x$ from $E$ is equal to the utility of $x$ as a proportion of the total utility of alternatives in the *support of* $p(\cdot, E)$. That is:

$$p(x, E) = \frac{v(x)}{\sum_{y \in \Gamma_p(E)} v(y)} \tag{2}$$

where $\Gamma_p(E)$ denotes the support of $p(\cdot, E)$. Note the difference in the denominators in (1) and (2). The GLM was characterised by Ahumada and Ülkü (2018) and Echenique and Saito (2019) under the assumption

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[1] He also assumes that $X$ is finite, but this assumption is redundant, as we discuss below.

[2] That is, there is no restriction (beyond finiteness) on the menus for which choice behaviour is specified. Our terminology follows that in Alós-Ferrer and Mihm (2025).

[3] The GLM terminology is from Echenique and Saito (2019). Horan (2021) includes a thorough and insightful overview of the literature on the GLM.

[4] It is unnamed in Ahumada and Ülkü (2018).

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of unrestricted menus. Their characterisation involves a single axiom, which Echenique and Saito (2019) call *cyclical independence (CI)*.[4]

Various special cases of the GLM have also been studied, each imposing a constraint on the form of $\Gamma_p$. The leading special case requires $\Gamma_p$ to be rationalisable by a weak order in the sense of Arrow (1959).[5] We call this special case the *rationalisable GLM*. In a rationalisable GLM, a set of "acceptable" alternatives is selected according to a rational (deterministic) choice function, $\Gamma_p$, with "indifference" resolved in Luce fashion. Cerreia-Vioglio et al. (2021) show that, remarkably, the rationalisable GLM is characterised by Luce's choice axiom under the unrestricted menus assumption. The choice axiom therefore emerges as a fundamental "canon of probabilistic rationality".

The literature has paid comparatively scant attention to relaxing constraints on the menu domain. Given that most laboratory or field data are only available for restricted sets of menus, it is important to understand the empirical implications of the models for these settings. In a recent contribution, Alós-Ferrer and Mihm (2025) characterise the classical Luce model and the general Luce model for an arbitrary collection of menus. The GLM is characterised by the *cyclical independence* condition:[6] in other words, the "unrestricted menus" assumption is redundant to the result of Ahumada and Ülkü (2018) and Echenique and Saito (2019). The classical Luce model is characterised by CI together with positivity.

In this paper we characterise the *rationalisable* GLM when the set of menus is arbitrarily restricted. The choice axiom is too weak to characterise the model in this general setting: in other words, the "unrestricted menus" assumption is not redundant to the Cerreia-Vioglio et al. (2021) result. We prove (Theorem 5) that a mild strengthening of CI, which we call *strong cyclical independence*, does the needful. Since strong cyclical independence implies the choice axiom (Theorem 6) it constitutes a "stricter canon", one that is robust to restrictions on the menu set for which choice behaviour is defined.

The next section reviews characterisations of the classical and general Luce models, and the rationalisable GLM. Section 3 contains the definition of strong cyclical independence and our main results. Section 4 concludes.

## 2. The Luce model and its generalisations

### 2.1. The classical Luce model

Let $X$ be a non-empty set, interpreted as the universal domain of alternatives. Define $\mathcal{X}$ to be the set of all non-empty, finite subsets of $X$. Let $\mathcal{M}$ be the set of menus for which choice behaviour is defined. We assume throughout that $\emptyset \neq \mathcal{M} \subseteq \mathcal{X}$. If $\mathcal{M} = \mathcal{X}$ we say that the menu set is *unrestricted*.

When confronting a given menu, $E \in \mathcal{M}$, the decision-maker must choose a single alternative from $E$ – abstention is not allowed. A *random choice function (RCF)* describes the stochastic choice behaviour of some individual. An RCF specifies a probability function on each menu in $\mathcal{M}$; it is a mapping $p : X \times \mathcal{M} \to [0,1]$ satisfying $\sum_{x\in A} p(x,A) = 1$ for any $A \in \mathcal{M}$ and $p(x,A) = 0$ for any $A \in \mathcal{M}$ and any $x \in X \setminus A$. We interpret $p(x,A)$ as the probability that the individual chooses $x$ when confronted with menu $A$. For notational convenience, define

$$p(B,A) = \sum_{x\in B} p(x,A)$$

for any $B \subseteq X$ and $A \in \mathcal{M}$.

It is without loss of generality, to assume that $\mathcal{M}$ includes all singletons, since the definition of a random choice function fixes its value on any singleton menu. We therefore maintain this assumption throughout.

[5] This case has been analysed by, *inter alia*, Ahumada and Ülkü (2018), Cerreia-Vioglio et al. (2021), Doğan and Yıldız (2021), and Horan (2021).

[6] Alós-Ferrer and Mihm (2025) refer to CI as the *general product rule*.

If $p$ is an RCF we define $\Gamma_p : \mathcal{M} \to \mathcal{X}$ to be the support function for $p$:

$$\Gamma_p(A) \;=\; \{\, x \in A \mid p(x,A) > 0 \,\}$$

for each $A \in \mathcal{M}$. Note that $\Gamma_p$ satisfies the properties of a choice function: $\emptyset \neq \Gamma_p(A) \subseteq A$ for each $A \in \mathcal{M}$. We say that $\Gamma_p$ is *rationalisable* if there exists a weak order $\succsim \subseteq X \times X$ such that

$$\Gamma_p(A) \;=\; \{\, x \in A \mid x \succsim y \;\text{ for all } y \in A \,\}$$

for each $A \in \mathcal{M}$.

Next, we recall some properties of RCFs and a classical result:[7]

**Definition 1.** Given an RCF, $p : X \times \mathcal{M} \to [0,1]$, we say that $p$ satisfies:

(i) **positivity** if $p(x,A) > 0$ when $x \in A \in \mathcal{M}$.
(ii) the **choice axiom (CA)** if

$$p(x,A) \;=\; p(x,B)\, p(B,A)$$

whenever $A, B \in \mathcal{M}$ and $x \in B \subseteq A$.
(iii) **independence of irrelevant alternatives (IIA)** if

$$p(x,A)\, p(y,B) = p(x,B)\, p(y,A)$$

whenever $A, B \in \mathcal{M}$ and $\{x,y\} \subseteq A \cap B$.

**Definition 2.** An RCF, $p : X \times \mathcal{M} \to [0,1]$, has a **Luce model (LM)** if there exists some (utility) function $v : X \to \mathbb{R}_{++}$ such that

$$p(x,A) \;=\; \frac{v(x)}{\sum_{y\in A} v(y)}$$

whenever $x \in A \in \mathcal{M}$. We say that $v$ is a Luce model for $p$.

**Theorem 1** (*Luce, 1959*). *Let $X$ be finite and let $p : X \times \mathcal{M} \to [0,1]$ be an RCF. Suppose $\mathcal{M}$ is unrestricted and $p$ satisfies positivity. Under these assumptions, the following are equivalent:*

*(i) $p$ satisfies CA.*
*(ii) $p$ satisfies IIA.*
*(iii) $p$ has a Luce model*

The assumption of finite $X$ is actually redundant to this classical result. Given positivity, standard arguments show that (i) is equivalent to (ii) for arbitary $X$. Theorem 1 in Rodrigues-Neto et al. (2025) – henceforth, RRT25 – establishes that (ii) is also equivalent to (iii) for arbitrary $X$ (again, given positivity).

### 2.2. The general Luce model

Positivity is obviously a necessary condition for an RCF to possess a Luce model. Ahumada and Ülkü (2018) and Echenique and Saito (2019) relax the positivity assumption and consider the following generalisation of the LM:

**Definition 3.** Let $p : X \times \mathcal{M} \to [0,1]$ be an RCF. Then $p$ has a **general Luce model (GLM)** if there exists a (utility) function $v : X \to \mathbb{R}_{++}$ such that

$$p(x,A) \;=\; \frac{v(x)}{\sum_{y\in \Gamma_p(A)} v(y)}$$

whenever $A \in \mathcal{M}$ and $x \in \Gamma_p(A)$. We say that $v$ is a GLM for $p$.

Note that any LM is a GLM. If $v$ is a GLM for $p$ and $p$ satisfies positivity, then $v$ is a Luce model for $p$.

One may think of $\Gamma_p(A)$ as the "acceptable" choices from menu $A$, with $v$ used to randomly resolve "indifference" in Luce fashion. Alternatively – the interpretation favoured by Ahumada and Ülkü (2018)

[7] The IIA condition is usually expressed in ratio form. The version below is equivalent under the usual Luce assumption of positivity.

– we may think of $\Gamma_p(A)$ as a *consideration set*, to which the decision-maker restricts attention for the purpose of choice. In this paper, we remain agnostic as to interpretation of the model; our concern is with its empirical signature when $\mathcal{M}$ is an arbitrary non-empty subset of $\mathcal{X}$.

Ahumada and Ülkü (2018, Theorem 1) and Echenique and Saito (2019, Theorem 1) prove that, when $X$ is finite and $\mathcal{M}$ is unrestricted (i.e., $\mathcal{M} = \mathcal{X}$), an RCF has a general Luce model if and only if it satisfies a condition known as *cyclical independence*. Theorem 1 in Alós-Ferrer and Mihm (2025) shows that the unrestricted menus assumption is redundant to this result. Theorem 3 in RRT25 shows that the finiteness restriction on $X$ is likewise redundant.[8]

To define cyclical independence we require some additional notation and terminology, mostly adapted from Echenique and Saito (2019). A *connected sequence* is any sequence of the form $\{(x_i, x_{i+1}, E_i)\}_{i=1}^m$ with $m \in \{1, 2, \ldots\}$ and $\{x_i, x_{i+1}\} \subseteq E_i \in \mathcal{M}$ for each $i$ (and repetition allowed). A *cycle* of *length* $m$ is a connected sequence $\{(x_i, x_{i+1}, E_i)\}_{i=1}^m$ with $x_1 = x_{m+1}$. Given an RCF, $p : X \times \mathcal{M} \to [0,1]$, the connected sequence $\{(x_i, x_{i+1}, E_i)\}_{i=1}^m$ is *positive* if $p(x_i, E_i)p(x_{i+1}, E_i) > 0$ for each $i$. Of course, all connected sequences are positive when $p$ satisfies positivity. A positive connected sequence that is also a cycle is called a *positive cycle*.

**Definition 4.** The RCF, $p : X \times \mathcal{M} \to [0,1]$, satisfies **cyclical independence (CI)** if

$$\prod_{i=1}^m p(x_i, E_i) = \prod_{i=1}^m p(x_{i+1}, E_i) \tag{3}$$

for any positive cycle $\{(x_i, x_{i+1}, E_i)\}_{i=1}^m$.

It is evident that CI implies IIA when positivity is assumed. Indeed, CI is equivalent to IIA when $\mathcal{M}$ is unrestricted and $p$ satisfies positivity: see the proof of Theorem 1 in RRT25. When $\mathcal{M}$ comprises all (singleton and) binary menus, the *product rule* requires (3) to hold for cycles with $m \leq 3$. Horan (2021) therefore calls CI the *strong product rule*, while Alós-Ferrer and Mihm (2025) call it the *general product rule*.

**Theorem 2** (*Alós-Ferrer and Mihm (2025), RRT25*)**.** *Suppose $p : X \times \mathcal{M} \to [0,1]$ is an RCF. Then $p$ possesses a GLM if and only if it satisfies CI.*

As noted above, Ahumada and Ülkü (2018, Theorem 1) and Echenique and Saito (2019, Theorem 1) proved the special case of this result for finite $X$ and unrestricted $\mathcal{M}$. The proof of Theorem 2 requires only minor modifications to their arguments.

Since the Luce model and general Luce model are equivalent when $p$ satisfies positivity we have:

**Corollary 1.** *Suppose $p : X \times \mathcal{M} \to [0,1]$ is an RCF satisfying positivity. Then $p$ possesses a Luce model if and only if it satisfies CI.*

Theorem 2 and Corollary 1 characterise the Luce model and general Luce model for arbitrary $X$ and an arbitrary $\mathcal{M} \subseteq \mathcal{X}$ with $\mathcal{M} \neq \emptyset$. The CI condition is common, with positivity added in the Luce model characterisation.

### 2.3. The rationalisable GLM

An important special case of the GLM is obtained if $\Gamma_p$ is rationalisable. This special case has been studied by various authors under a range of names: Ahumada and Ülkü (2018) call it a *Luce rule with rationalisable consideration* while Doğan and Yıldız (2021) call it a *preference oriented Luce rule*. Within the present paper we shall refer to it as a *rationalisable GLM*.

[8] We completed RRT25 before becoming aware of the work of Alós-Ferrer and Mihm. The latter authors illustrate several implications of their result for the design of choice experiments.

**Definition 5.** An RCF, $p : X \times \mathcal{M} \to [0,1]$, has a **rationalisable GLM** if it has a GLM and $\Gamma_p$ is rationalisable.

Cerreia-Vioglio et al. (2021) provide an important characterisation of the rationalisable GLM:[9]

**Theorem 3** (*Cerreia-Vioglio et al., 2021; Theorem 2*)**.** *Suppose $\mathcal{M}$ is unrestricted and $p : X \times \mathcal{M} \to [0,1]$ is an RCF. Then $p$ has a rationalisable GLM if and only if it satisfies CA.*

The unrestricted menus assumption is not redundant in Theorem 3. The choice axiom loses too much bite if the menu set is not sufficiently rich. For example, if $\mathcal{M}$ contains no (non-singleton) menu that is properly contained in another menu, then CA has no bite whatsoever.

## 3. The main result

Our main result (Theorem 5) provides a necessary and sufficient condition for the existence of a rationalisable GLM for an arbitrary non-empty $\mathcal{M} \subseteq \mathcal{X}$.

Consider the following strengthening of the CI condition:

**Definition 6.** An RCF, $p : X \times \mathcal{M} \to [0,1]$, satisfies **strong cyclical independence (SCI)** if (3) holds for any cycle $\{(x_i, x_{i+1}, E_i)\}_{i=1}^m$.

Strong cyclical independence requires that the *cycle equation* (3) holds for every cycle, not just the positive ones. There is obviously no difference between CI and SCI when $p$ satisfies positivity. It turns out that SCI is precisely the condition we seek.

Before stating our main result, we make some preliminary observations on rationalisability of $\Gamma_p$. When $\mathcal{M}$ is unrestricted, rationalisability of $\Gamma_p$ is equivalent to $\Gamma_p$ satisfying the weak axiom of revealed preference (WARP) – a property that is underwritten by the choice axiom (Cerreia-Vioglio et al., 2021). When $\mathcal{M}$ is an arbitrary collection of menus, a stronger condition is required. To state it, we first define a pair of revealed preference relations: for any $x, y \in X$

$$x \succsim^p y \quad \Leftrightarrow \quad x \in \Gamma_p(E) \text{ and } \{x, y\} \subseteq E \text{ for some } E \in \mathcal{M}$$

$$x \succ^p y \quad \Leftrightarrow \quad \{x, y\} \cap \Gamma_p(E) = \{x\} \text{ and } \{x, y\} \subseteq E \text{ for some } E \in \mathcal{M}$$

Note that $\succ^p \subseteq \succsim^p$ but $\succ^p$ need not be (asymmetric or) the asymmetric part of $\succsim^p$. The following appropriates a familiar property of choice functions, due to Richter (1966), to our context.

**Definition 7.** The support function $\Gamma_p : \mathcal{M} \to \mathcal{X}$ satisfies **congruence** if there does not exist any sequence $\{x_i\}_{i=1}^m \subseteq X$ with $x_i \succsim^p x_{i+1}$ for each $i \in \{1, \ldots, m-1\}$ and $x_m \succ^p x_1$.

Restricting the congruence condition to sequences with $m = 2$ gives WARP. The following is well-known (e.g., Chambers and Echenique, 2016, Theorem 2.6):[10]

**Theorem 4.** *The support function $\Gamma_p : \mathcal{M} \to \mathcal{X}$ is rationalisable if and only if it satisfies congruence.*

We can now state our main result:

[9] Alternative characterisations for the special case of finite $X$ are obtained by Ahumada and Ülkü (2018, Corollary 2), Horan (2021), Theorem 4* of the online Appendix) and, in strikingly novel fashion, by Doğan and Yıldız (2021, Theorem 1).

[10] The SCI condition ensures that $\succ^p$ is the asymmetric part of $\succsim^p$, as one may easily verify. When $\succ^p$ is the asymmetric part of $\succsim^p$, congruence is *Suzumura consistency*. This is a necessary and sufficient condition for $\succsim^p$ to have a weak order extension (Suzumura, 1976) – an important generalisation of Szpilrajn's Extension Theorem.

**Theorem 5.** *Suppose* $p : X \times \mathcal{M} \to [0,1]$ *is an RCF. Then* $p$ *possesses a rationalisable GLM if and only if it satisfies strong cyclical independence. In particular, if* $p$ *satisfies SCI then* $\Gamma_p$ *satisfies congruence.*

**Proof.** We start with the "if" part of the claim. Suppose $p$ satisfies SCI. Let

$$\mathcal{M}^* = \left\{ \Gamma_p(A) \mid A \in \mathcal{M} \right\}$$

be the collection of support sets. Since $\mathcal{M}$ contains all singletons, so does $\mathcal{M}^*$ and it follows that $X = \bigcup \mathcal{M}^*$. Define $p^*$ to be the RCF on $X \times \mathcal{M}^*$ which is obtained by setting $p^*\left(x, \Gamma_p(A)\right) = p(x, A)$ for each $x \in X$ and each $A \in \mathcal{M}$. We show that $p^*$ is well-defined by strong cyclical independence: if $\Gamma_p(A) = \Gamma_p(B)$ then $p(\cdot, A) \equiv p(\cdot, B)$. To see why, suppose $\Gamma_p(A) = \Gamma_p(B) = E$ and $\{x, y\} \subseteq E$. Then

$$\{(x, y, A), (y, x, B)\}$$

is a positive cycle so

$$p(y, A)\, p(x, B) = p(y, B)\, p(x, A) \quad \Leftrightarrow \quad \frac{p(y, A)}{p(x, A)} = \frac{p(y, B)}{p(x, B)}.$$

Since $p(\cdot, A)$ and $p(\cdot, B)$ both sum to 1 on $E$, and $E$ is finite, it follows that $p(\cdot, A)$ and $p(\cdot, B)$ coincide on $E$ (hence on $X$).

Now observe that $p^*$ satisfies positivity and inherits (strong) cyclical independence from $p$. Therefore, by Corollary 1 there is a Luce model $v : X \to \mathbb{R}_{++}$ for $p^*$. It follows that if $x \in \Gamma_p(A)$ then

$$p(x, A) = p^*\left(x, \Gamma_p(A)\right) = \frac{v(x)}{\sum_{y \in \Gamma_p(A)} v(y)}.$$

It remains to show that $\Gamma_p$ satisfies congruence. Suppose, by way of contradiction, that there exists a sequence $\{x_i\}_{i=1}^m \subseteq X$ with $x_i \succsim^p x_{i+1}$ for each $i \in \{1, \dots, m-1\}$ and $x_m \succ^p x_1$. Then there exists $\{E_i\}_{i=1}^m \subseteq \mathcal{M}$ such that $\{(x_i, x_{i+1}, E_i)\}_{i=1}^m$ is a cycle (hence $x_{m+1} = x_1$), with $x_i \in \Gamma_p(E_i)$ for each $i \in \{1, \dots, m\}$ and $x_{m+1} \notin \Gamma_p(E_m)$. It follows that we have a violation of SCI, since

$$\prod_{i=1}^m p\left(x_i, E_i\right) > 0$$

while

$$\prod_{i=1}^m p\left(x_{i+1}, E_i\right) = 0.$$

This proves the "if" part of Theorem 5.

Next, we prove the "only if" part. Suppose $v$ is a GLM for $p$ and $\Gamma_p$ is rationalisable. Let $\left\{\left(x_i, x_{i+1}, E_i\right)\right\}_{i=1}^m$ be a connected sequence with $x_1 = x_{m+1}$ (i.e., a cycle). We show that (3) is satisfied by considering three exhaustive cases.

First, if $\left\{x_i, x_{i+1}\right\} \subseteq \Gamma_p(E_i)$ for each $i \in \{1, 2, \dots, m\}$ then the cycle is positive and we have:

$$\prod_{i=1}^m \frac{p\left(x_i, E_i\right)}{p\left(x_{i+1}, E_i\right)} = \frac{v(x_1)}{v(x_2)} \cdots \frac{v(x_{m-1})}{v(x_m)} \frac{v(x_m)}{v(x_1)} = 1$$

so (3) is satisfied.

Second, if there exists $j \in \{1, \dots, m\}$ with $\{x_j, x_{j+1}\} \cap \Gamma_p(E_j) = \emptyset$ then both sides of (3) are zero.

Finally, we have the case in which there exists $j \in \{1, \dots, m\}$ with

$$\emptyset \neq \{x_j, x_{j+1}\} \cap \Gamma_p(E_j) \neq \{x_j, x_{j+1}\}.$$

Let $\mathcal{I}$ be the family of indifference classes for the weak order that rationalises $\Gamma_p$ and let $\succ$ be the induced strict linear order on $\mathcal{I}$. It follows that $\mathcal{I}$ is a partition of $X$, so each $x_i$ is contained in some $S_i \in \mathcal{I}$. Moreover, $S_j \neq S_{j+1}$. It follows that there exist $k, \ell \in \{1, 2, \dots, m\}$ with $S_{k+1} \succ S_k$ and $S_\ell \succ S_{\ell+1}$: otherwise we deduce that $S_1 \succ S_1$ by the transitivity of $\succ$. Therefore, $p\left(x_k, E_k\right) = 0$ and $p\left(x_{\ell+1}, E_\ell\right) = 0$. It follows that both sides of (3) must be zero. Thus, in all three cases, (3) holds. This completes the proof of Theorem 5. □

Combining Theorems 3 and 5, it follows that CA is equivalent to SCI when $\mathcal{M}$ is unrestricted. In general, SCI is stronger – a "stricter canon".[11]

**Theorem 6.** *Suppose* $p : X \times \mathcal{M} \to [0,1]$ *is an RCF that satisfies strong cyclical independence. Then* $p$ *satisfies the choice axiom.*

**Proof.** Suppose $A, B \in \mathcal{M}$ with $A \subseteq B$ and $x \in A$. If $\{x\} = A$ the result is trivial, so suppose $|A| \geq 2$. Let $y \in A \setminus \{x\}$. By considering the cycle $\{(x, y, A), (y, x, B)\}$ we have

$$p(x, A)\, p(y, B) = p(x, B)\, p(y, A)$$

Thus:

$$\sum_{y \in A \setminus \{x\}} p(x, A)\, p(y, B) = \sum_{y \in A \setminus \{x\}} p(x, B)\, p(y, A)$$

$$\Leftrightarrow \quad p(x, A)\, p(A \setminus \{x\}, B) = p(x, B)\,[1 - p(x, A)]$$

$$\Leftrightarrow \quad p(x, B) = p(x, A)\, p(A, B) \quad \square$$

## 4. Concluding remarks

Cerreia-Vioglio et al. (2021) showed that the choice axiom characterises the rationalisable GLM when the menu set is unrestricted. Our main result establishes that strong cyclical independence characterises the same model for an arbitrary menu set. The SCI condition embodies both cyclical independence, which characterises the GLM, and the congruence condition on the support function, $\Gamma_p$. Our result complements the characterisations of the Luce and general Luce models on arbitrary menu domains by Alós-Ferrer and Mihm (2025), highlighting the foundational role of cyclical independence properties.

## CRediT authorship contribution statement

**José A. Rodrigues-Neto:** Writing – review & editing, Investigation, Formal analysis, Conceptualization. **Matthew Ryan:** Writing – review & editing, Writing – original draft, Investigation, Formal analysis, Conceptualization. **James Taylor:** Writing – review & editing, Investigation, Formal analysis, Conceptualization.

## Data availability

No data was used for the research described in the article.

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[11] The following argument essentially rehearses the well-known proof that IIA implies the choice axiom, for which the auxiliary assumptions of positivity and unrestricted menus are redundant. Theorem 6 is also implied by the (iii)⇒(i) part of Lemma 8 in Cerreia-Vioglio et al. (2021), since their proof does not rely on menus being unrestricted. Their result shows that CA is necessary for a rationalisable GLM and hence implies our Theorem 6.