Bending waves in composite structures with random parameters

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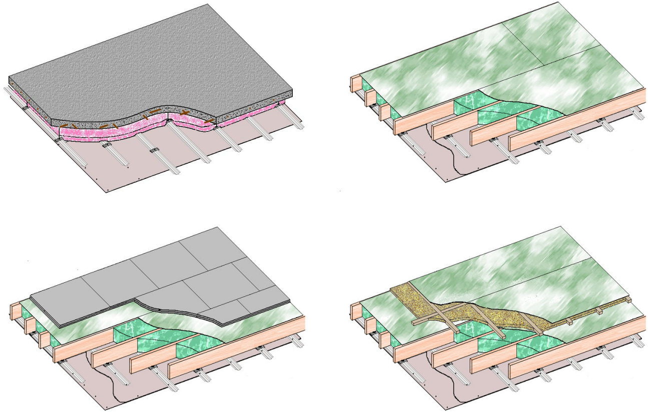
Outline

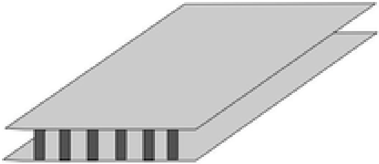
Modelling Composite structures

- Double-leaf walls
- Lightweight floor/ceiling systems
- Consist of many components

Examples: lightweight floor/ceiling systems



Elastic double-leaf plate

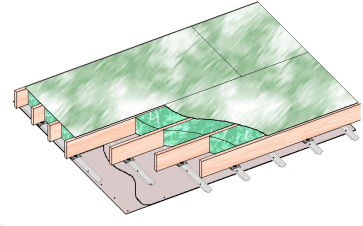


Components and elements of modelling

- Two elastic plates
- Re-enforcement beams
- Coupling between beams and plates

Example

Examples - Timber-framed floor/ceiling

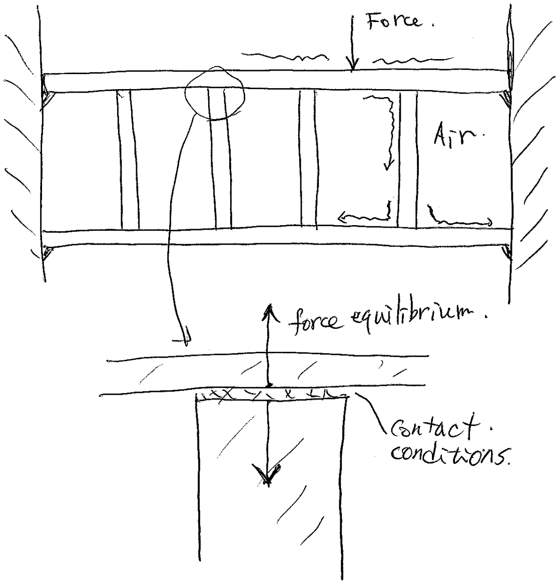


Questions

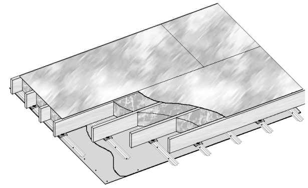
- How each components affects the whole structure?
- How each junction affects the whole structure?
- How to simulate the random inhomogeneities in components and junctions?

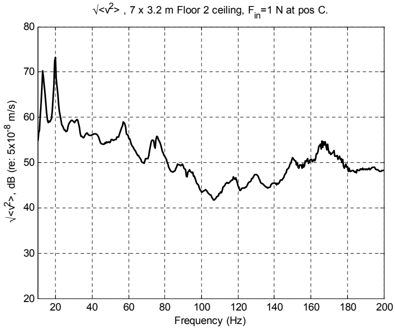
Modelling in the mid-frequency range

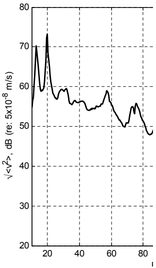
Low-frequency deterministic model

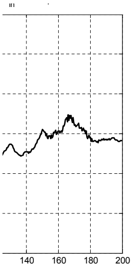


Prediction of vibrations at low-frequencies



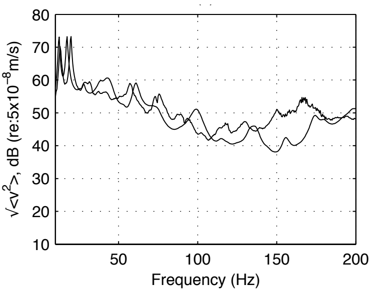






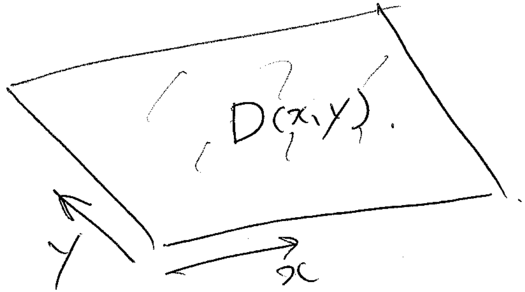
Modelling regimes for frequency ranges

- Low-frequency: clear fundamental resonances - peaks and troughs
- Mid-frequency: smeared and leveled out responses



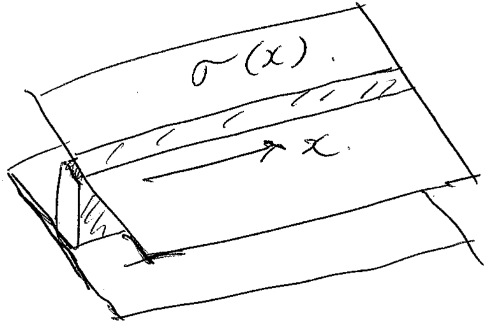
Random parameters

Stiffness of plates



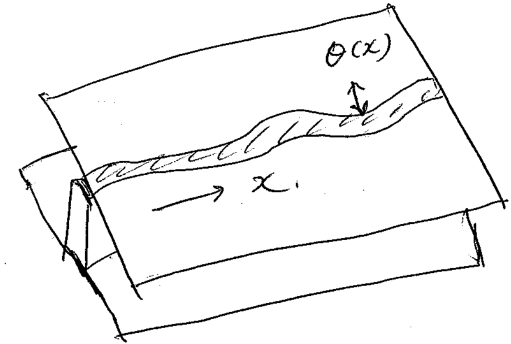
Random parameters

Slippage along the beam



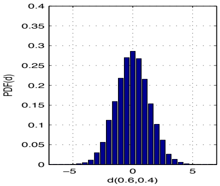
Random parameters

Twist of the beam

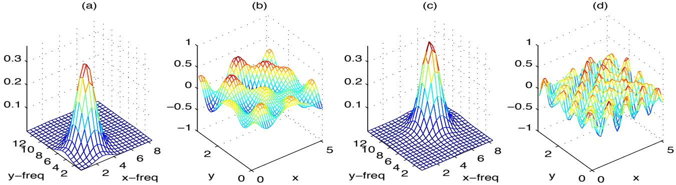


Simulating the parameters

1. Continuous and smooth in 1D or 2D
   - Superposition of sinusoidal functions
2. Predetermined probability density for the amplitude at any position



   - Gaussian amplitude
3. Predetermined average power spectral density
   - Smooth single peak



Mathematical formulations

Variational formulation

$$\mathcal{L}(\mathbf{w}) = \int_0^T \{\mathcal{P}(t) + \mathcal{K}(t) - \mathcal{F}(t)\}\, dt$$

- $\mathbf{w} = \mathbf{w}(x, y, z, t)$ deformation of the structure at (x,y,z)
- $\mathcal{P}$: potential energy
- $\mathcal{K}$: kinetic energy
- $\mathcal{F}$: the work done to the object

The classical Kirchhoff (thin elastic) plate model

$$\mathcal{P}_1 = \frac{1}{2}\int_0^A \int_0^B D_1(x,y)\left|\nabla^2 w_1\right|^2 \, dxdy,$$

$$\mathcal{K}_1 = \frac{\rho_1 h_1 \omega^2}{2}\int_0^A \int_0^B |w_1(x,y)|^2 \, dxdy,$$

$$D_1(x,y) = E_1(x,y)h_1^3 / \left(12\left(1-\nu^2\right)\right) \quad \text{is the flexural rigidity}$$

Equation of a thin plate

$w_1(x,y)$ is the solution of the thin plate equation,

$$\nabla^2\left(D_1(x,y)\nabla^2 w_1(x,y)\right) - \omega^2 \rho_1 h_1 w_1(x,y) = p(x,y)$$

Euler beam model

The strain and kinetic energies for the beams are given by

$$\mathcal{P}_2 = \frac{1}{2}\sum_{j=1}^{S}\int_0^A E_2 I\,|w_{2xx}(x,j)|^2\,dx, \quad \mathcal{K}_2 = \frac{\rho_2 h_2 \omega^2}{2}\sum_{j=1}^{S}\int_0^A |w_2(x,j)|^2\,dx,$$

where

- E_2 : the Young's modulus of the beam
- I : the moment of inertia of the beam
- ρ_2 and h_2 : mass density per unit length and thickness of the beam

Energy from the cavity air

- Ψ be the velocity potential for air motion, so that
- $p = \rho\dot{\Psi}$ is the pressure and
- $\mathbf{u} = -\nabla\Psi$ is the velocity. Here
- $\dot{\Psi}$ denotes the time derivative of Ψ and
- ρ is the density of air.

Energy

$$\mathcal{K} - \mathcal{P} = \frac{1}{2}\int_{V_{\text{cavity}}} \rho \left[\|\nabla\Psi\|^2 - \frac{1}{c^2}\left(\dot{\Psi}\right)^2 \right] dv$$

Contribution on the cavity surfaces

The Lagrangian for the air cavity then reduces to the contribution over the surface of the cavity

$$\mathcal{L}_{1,3} = \frac{1}{2}\rho \int_0^T \int_{\Gamma_{1,3}} \frac{\partial \Psi(x)}{\partial n} \Psi(x)\, dx\, dt$$

$\mathcal{L}_{1,3}$: Cavity suface.

- The motion of the cavity surfaces determine the normal velocity, $\dfrac{\partial \Psi}{\partial n}$,
- Use the boundary-integral equation to write the term $\Psi(x)$, for $x \in \Gamma_{1,3}$ using $\dfrac{\partial \Psi(x)}{\partial n}$, for $x \in \Gamma_{1,3}$.

Expressing the potential using Normal velocity and Green's function

$$\Psi(x) = -2\int_{\Gamma_{1,3}} \frac{\partial \Psi(\xi)}{\partial n} h(x|\xi)\, ds(\xi)$$

$h(x|\xi)$ is the Neumann Green's function, satisfying for each ξ

$$\left(\nabla^2 + \frac{\omega^2}{c^2}\right) h(x|\xi) = \delta(x-\xi)$$

$$\left.\frac{\partial h}{\partial n}\right|_{\xi\in\Gamma_{1,3}}$$

Boundary integral expression of the cavity energy using surface velocity

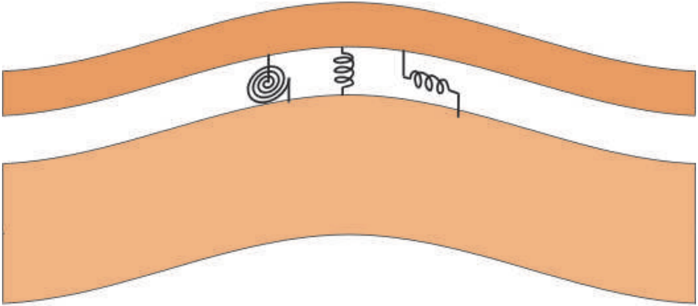
Then the time-averaged Lagrangian for the cavity takes the form

$$\begin{aligned}\mathcal{L}_{1,3} &= \rho \,\mathrm{Re} \int_{\Gamma_{1,3}} \int_{\Gamma_{1,3}} \left(\frac{\partial \Psi(x)}{\partial n} h^* (x|\xi) \frac{\partial \Psi^* (\xi)}{\partial n} \right) dx d\xi \\ &= \rho \int_{\Gamma_{1,3}} \int_{\Gamma_{1,3}} w_{1,3}(x) \,\mathrm{Re}\,(h(\xi|x))\, w_{1,3}^*(\xi) \; dx\, d\xi\end{aligned}$$

$w_{1,3}(x) = \dfrac{\partial \Psi(x)}{\partial n}$: the velocity normal to the surface $\Gamma_{1,3}$.

Coupling conditions

Modelling by springs connecting the beams and the plates



Twisting, stretching and slipping

Coupling conditions

Energy contribution from the coupling

- Separation

$$\mathcal{P}_{1,2}^{\text{sep}} = \frac{1}{2} \sum_{j=1}^{S} \int_0^A \sigma_{\text{sep}}(x,j) \left| w_1(x,y_j) - w_2(x,j) \right|^2 \; dx$$

- Slippage

$$\mathcal{P}_{1,2}^{\text{slip}} = \frac{1}{2} \sum_{j=1}^{S} \int_0^A \sigma_{\text{slip}}(x,j) \left| h_1 w_1'(x,y_j) + h_2 w_2'(x,j) \right|^2 \; dx$$

- Rotation

$$\mathcal{P}_{1,2}^{\text{rot}} = \frac{1}{2} \sum_{j=1}^{S} \int_0^A \sigma_{\text{rot}}(x,j) \left| w_1'(x,y_j) - w_2'(x,j) \right|^2 \; dx$$

Total energy

$$\text{The total potential energy} = \text{potential energies from components} + \text{and coupling between components}$$

$$\mathcal{P} = \sum_{l=1}^{3} \mathcal{P}_l + \sum_{(l,j)=(1,2),(3,2)} \left\{ \mathcal{P}_{l,j}^{\text{sep}} + P_{l,j}^{\text{slip}} + P_{l,j}^{\text{rot}} \right\},$$

The number of components and coupling methods are not limited.

Method of solution

Fourier series

The plates

$$w_l(x,y) = \sum_{m,n=1}^{N} C_{mn}^{(l)} \phi_m(x)\psi_n(y), \quad l = 1, 3,$$

and the beams by

$$w_2(x,j) = \sum_{m=1}^{N} C_{mj}^{(2)} \phi_m(x), \quad j = 1, 2, ..., S,$$

where the basis functions are

$$\phi_m(x) = \sqrt{2/A}\sin k_m x, \quad \psi_n(y) = \sqrt{2/B}\sin \kappa_n y.$$

Solution

Solve for the coefficients $\{C_{mn}^{(1)}, C_{mj}^{(2)}, C_{mn}^{(3)}\}$.

Vectorization: Plate, beams, plate

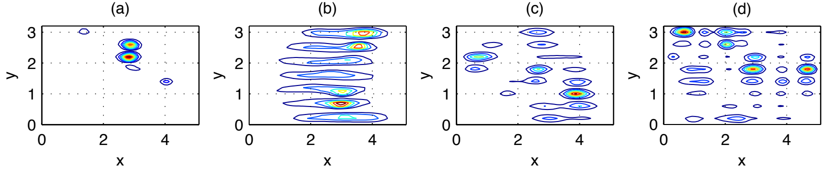
$$\mathbf{c}_1 = \begin{pmatrix} C_{11}^{(1)} \\ C_{21}^{(1)} \\ \vdots \\ C_{NN}^{(1)} \end{pmatrix}, \mathbf{c}_2 = \begin{pmatrix} C_{11}^{(2)} \\ C_{21}^{(2)} \\ \vdots \\ C_{NS}^{(2)} \end{pmatrix}, \mathbf{c}_3 = \begin{pmatrix} C_{11}^{(3)} \\ C_{21}^{(3)} \\ \vdots \\ C_{NN}^{(3)} \end{pmatrix}$$

The variational formulation with vectors and matrices

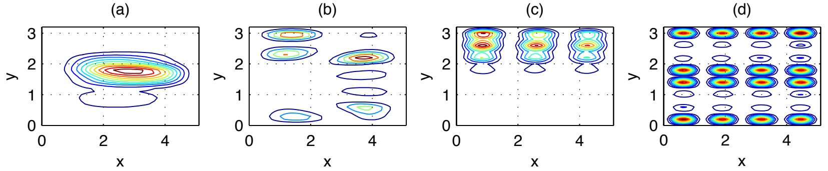
$$\frac{1}{2}\begin{bmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \\ \mathbf{c}_3 \end{bmatrix}^{t} \mathbf{L} \begin{bmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \\ \mathbf{c}_3 \end{bmatrix} = \mathbf{f}^{t} \begin{bmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \\ \mathbf{c}_3 \end{bmatrix} \implies \mathbf{L} \begin{bmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \\ \mathbf{c}_3 \end{bmatrix} = \mathbf{f}$$

Numerical simulations: Variance distribution

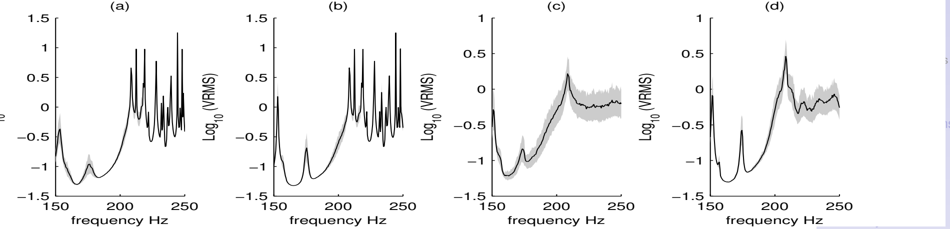
Random stiffness: 100Hz, 150Hz, 200Hz, 250Hz



Random slippage: 100Hz, 150Hz, 200Hz, 250Hz



Root-Mean-Square-velocity 150Hz - 250Hz



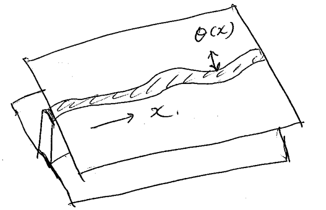
- (a,b) : slippage
- (c,d) : stiffness

Twisted beams

Fourier expansion of twisted beam

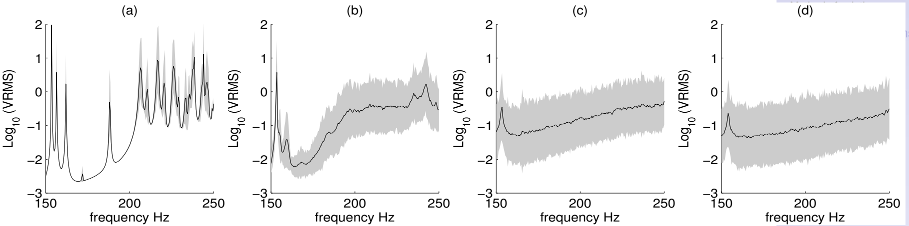
- The shape of j'th beam is given by $y_j + \theta_j(x)$,
- Modified Fourier series where the plate and the beam meet

$$w(x) = \sum_{m,n=1}^{N} \{\psi_n(y_j) + \kappa_n \theta_j(x) \chi_n(y_j)\} \, C_{mn}^{(1)} \phi_m(x).$$



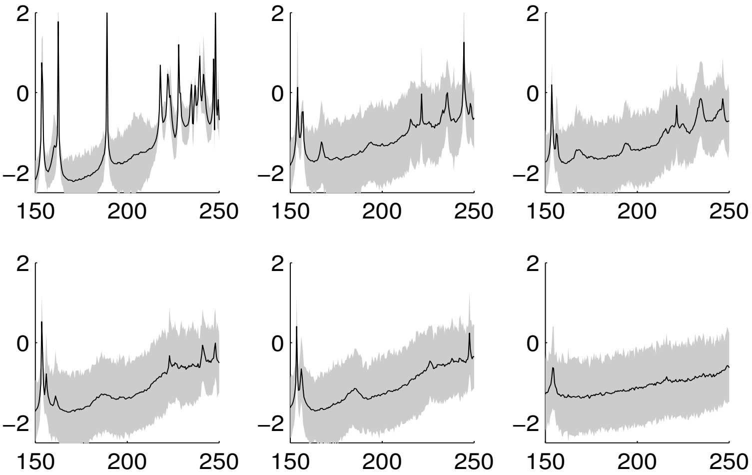
Twisted beams

Twist amplitude standard deviation, 1cm



Sensitivity to the straightness

Twist amplitude standard deviation, 1cm



Summary

- Modelling a double-leaf plate using the variational formulation
- Including the non-homogeneous parameters as the Fourier components
- Monte-Carlo simulation from the given PSD and PDF of the parameters
- Sensitivity to the *straightness* of the beams