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Cycle conditions for "Luce rationality"[☆]

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ABSTRACT

We extend and refine conditions for "Luce rationality" (i.e., the existence of a Luce – or logit – model) in the context of stochastic choice. When choice probabilities satisfy *positivity*, the *cyclical independence (CI)* condition of Ahumada and Ülkü (2018) and Echenique and Saito (2019) is necessary and sufficient for Luce rationality, even if choice is only observed for a restricted set of menus. We adapt results from the *cycles approach* (Rodrigues-Neto, 2009) to the common prior problem Harsanyi (1967–1968) to refine the CI condition, by reducing the number of cycle equations that need to be checked. A general algorithm is provided to identify a minimal sufficient set of equations. Three cases are discussed in detail: (i) when choice is only observed from binary menus, (ii) when all menus contain a common default; and (iii) when all menus contain an element from a common binary default set. Investigation of case (i) leads to a refinement of the famous product rule.

1. Introduction

The classical Luce – or multinomial logit – model (Luce, 1959) assumes *positivity* of choice probabilities: each available option is chosen with strictly positive probability. Given positivity, the Luce model is characterised by Luce's *independence of irrelevant alternatives (IIA)* axiom. This characterisation assumes that choice is observed from a "comprehensive" set of menus — all possible subsets of a given (finite) universe, X, of alternatives. This is a very restrictive assumption in practical applications. When the restriction is relaxed, the Luce model is characterised by a strengthening of the IIA axiom called *cyclical independence (CI)*. This fact is established in Section 2 as a straightforward corollary of an important result of Ahumada and Ülkü (2018) and Echenique and Saito (2019).

Cyclical independence is a restriction on the probability of choice cycles. A choice cycle is a sequence of choices where each successively chosen element is also available in the immediately preceding menu and the final chosen element is identical to the first; it is a cycle in "revealed (weak) preference". Cyclical independence requires (roughly speaking) that any choice cycle has the same probability as the reverse cycle (with which it is paired). This requirement generates a *cycle equation* associated with the cycle pair.

The CI condition may generate a very large number of cycle equations to check. However, we show that it is almost never necessary to check all of them — there exists a subset of cycle equations whose satisfaction guarantees that the remaining equations are also satisfied. We also provide a simple algorithm to identify a minimal sufficient set of cycle equations — a *cycle basis*.

Our analysis draws heavily on insights from the literature on Harsanyi's (1967-1968) *common prior* problem. It is well known that the existence of a common prior is mathematically analogous to the existence of a Luce model. It should therefore come as little surprise that an analogue of cyclical independence also makes an appearance in the common prior literature. It was introduced by Rodrigues-Neto (2009). Several papers have explored the so-called *cycles approach* to the common prior problem.[1] Despite the well-known similarity between the two problems, the stochastic choice literature seems to have overlooked the lessons to be learned from the cycles approach.

The amount of redundancy in the cycle equations depends on the structure of the menu set (i.e., the collection of subsets of X for which choice behaviour is specified). Two such results are already familiar. When the menu set comprises all binary menus, it suffices to satisfy the cycle equations for cycles of length three (Luce and Suppes, 1965, Theorem 48): this is equivalent to the well-known *product rule*.[2] When

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[1] See, for example, Rodrigues-Neto (2012, 2014), Hellwig (2013), Hellman Z. Samet (2012) and Fiorini and Rodrigues-Neto (2017).

[2] The product rule condition appears, unnamed, in (Luce, 1959, p.16). The *product rule* terminology seems to have been coined by (Estes, 1960, p.272).

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all possible menus are present – the "comprehensive" menus case – it suffices to check cycles of length two (Luce, 1959, Lemmas 2–3 and Theorem 3): this is precisely the IIA condition.

The central result of Rodrigues-Neto (2012) may be adapted to establish the cardinality of a cycle basis for any given menu set, which we do in Theorem 4. We use this result to further refine the product rule for binary menus (Theorem 5), and to show that a subset of the IIA conditions suffice to ensure cyclical independence when all menus share a common default (Theorems 2 and 6). We also consider menu sets where every menu contains at least one element from a *pair* of defaults. In this case, we show that it suffices to check (a subset of) the cycles of length four (Theorem 3).

Rodrigues-Neto (2012) did not provide an algorithm to identify a minimal sufficient set of cycle equations. However, it is straightforward to construct one using elementary results from graph theory. This is done in Section 3.3.

The tools of graph theory have been usefully applied to analyse the random utility model (Block and Marschak, 1959) since at least the work of Fiorini (2004). Recent contributions include Turansick (2022) and Chambers and Turansick (2024). The present paper shows that these tools are also valuable for analysing the Luce model. In particular, the notion of the *cyclomatic number* (Berge, 1962) of a graph plays a central role here, as it does in Rodrigues-Neto (2012) and in Chambers and Turansick (2024).

2. Cyclical independence

Let X be a non-empty, finite set — the universal domain of alternatives. Let $\mathcal{M}$ be a non-empty collection of non-empty subsets of X. An element of $\mathcal{M}$ is a menu from which a single alternative must be chosen.

A *random choice function (RCF)* describes the stochastic choice behaviour of some individual. An RCF is a function $p : X \times \mathcal{M} \to [0,1]$ satisfying $\sum_{x\in A} p(x, A) = 1$ for any $A \in \mathcal{M}$, and $p(x, A) = 0$ for any $A \in \mathcal{M}$ and any $x \in X \setminus A$. We interpret $p(x, A)$ as the probability that the individual chooses x when confronted with menu A. For notational convenience, we treat $p(x, y)$ as synonymous with $p(x, \{x, y\})$, and we define

$$p(B, A) = \sum_{x\in B} p(x, A)$$

for any $B \subseteq X$ and $A \in \mathcal{M}$.

Unless otherwise stated, we assume that $\mathcal{M}$ includes all singletons — the definition of a random choice function fixes its value on any singleton, so this assumption is without loss of generality. If $\mathcal{M}$ contains all non-empty subsets of X then we say that the menu set is *comprehensive*.

If p is an RCF we define $\Gamma_p : \mathcal{M} \to 2^X$ to be the support function for p:

$$\Gamma_p(A) = \{x \in A \mid p(x, A) > 0\}$$

for each $A \in \mathcal{M}$. Note that Γ_p satisfies the properties of a choice function (Arrow, 1959), since $\emptyset \neq \Gamma_p(A) \subseteq A$ for each $A \in \mathcal{M}$. If $\Gamma_p(A) = A$ for all $A \in \mathcal{M}$ we say that p satisfies *positivity*.

The following notion was introduced by Ahumada and Ülkü (2018) and Echenique and Saito (2019)[3]:

Definition 1. Let $p : X \times \mathcal{M} \to [0,1]$ be an RCF. Then p has a **general Luce model (GLM)** if there exists a (utility) function $v : X \to \mathbb{R}_{++}$ such that

$$p(x, A) = \frac{v(x)}{\sum_{y\in\Gamma_p(A)} v(y)}$$

whenever $A \in \mathcal{M}$ and $x \in \Gamma_p(A)$.

[3] We follow the terminology of Echenique and Saito. Ahumada and Ülkü use the term *Luce rule with limited consideration* rather than general Luce model, and provide a different interpretation.

If (and only if) p satisfies positivity, the GLM specialises to the classical Luce model (Luce, 1959).

Ahumada and Ülkü (2018) and Echenique and Saito (2019) characterise the set of RCFs which possess a GLM. To state their result we require some additional notation and terminology, which are mostly adopted from Echenique and Saito (2019).[4] A sequence $\{(x_i, x_{i+1}, E_i)\}_{i=1}^m$ with $m \in \{1, 2, \ldots\}$ and $\{x_i, x_{i+1}\} \subseteq E_i \in \mathcal{M}$ for each i (and repetition allowed)[5] is called a *connected sequence* of *length* m. It is a *cycle* if $x_1 = x_{m+1}$. A cycle of length m will be called an m-cycle. A connected sequence $\{(x_i, x_{i+1}, E_i)\}_{i=1}^m$ will be abbreviated as

$$x_1 E_1 x_2 E_2 \cdots E_{m-1} x_m E_m x_{m+1}$$

when notationally convenient.

A connected sequence $\{(x_i, x_{i+1}, E_i)\}_{i=1}^m$ is *positive* (for a given RCF, p) if

$$p(x_i, E_i) p(x_{i+1}, E_i) > 0$$

for each i. We then say that there is a positive connected sequence from x_1 to x_{m+1} and denote this fact by $x_1 \to_p x_{m+1}$. Of course, all connected sequences are positive when p satisfies positivity. In general, $\to_p$ is a reflexive,[6] symmetric and transitive binary relation on X. Hence, the equivalence classes of $\to_p$ partition X. The symmetry of $\to_p$ reflects the fact that positive connected sequences come in natural pairs. If $\{(x_i, x_{i+1}, E_i)\}_{i=1}^m$ is a positive connected sequence (hence $x_1 \to_p x_{m+1}$), then so is the "reverse" sequence

$$\{(x_{m+2-i}, x_{m+1-i}, E_{m+1-i})\}_{i=1}^m$$

(which shows that $x_{m+1} \to_p x_1$).

Definition 2. An RCF, $p : X \times \mathcal{M} \to [0,1]$, satisfies **cyclical independence (CI)** if

$$\prod_{i=1}^m p(x_i, E_i) = \prod_{i=1}^m p(x_{i+1}, E_i) \qquad (*)$$

for any positive cycle $\{(x_i, x_{i+1}, E_i)\}_{i=1}^m$.

We refer to $(*)$ as the *cycle equation* for the associated cycle (pair).[7] We say that a cycle is *consistent* if it satisfies $(*)$.

The cyclical independence condition characterises the GLM:[8]

Theorem 1. *An RCF possesses a general Luce model if and only if it satisfies CI.*

Ahumada and Ülkü (2018) and Echenique and Saito (2019) prove Theorem 1 under the additional assumption that $\mathcal{M}$ is comprehensive, but this assumption is superfluous. To see why, let $\mathcal{X}$ denote the set of all non-empty subsets of X. If p is an RCF with $\mathcal{M} \neq \mathcal{X}$, let $\hat{p}$ be an extension of p to $\mathcal{X}$ such that $\Gamma_{\hat{p}}(A)$ is a singleton for every $A \in \mathcal{X} \setminus \mathcal{M}$. Then $\hat{p}$ has a GLM if and only if p has a GLM. Moreover, terms involving menus with singleton supports cancel from the cycle equation $(*)$, so p satisfies CI if and only if $\hat{p}$ does.

We may immediately deduce:

Corollary 1. *An RCF that satisfies positivity possesses a Luce model if and only if it satisfies CI.*

Hence, when p satisfies positivity, CI is equivalent to the existence of a Luce model. Importantly, $\mathcal{M}$ is arbitrary in Corollary 1. This result will be the foundation for our subsequent analysis.

[4] In particular, the cyclical independence terminology is theirs. Ahumada and Ülkü do not name this property but it appears as their Axiom 1.

[5] A connected sequence is an indexed family, not a set.

[6] Recall that $\mathcal{M}$ includes all singletons.

[7] The cycle equation for the reverse cycle is identical.

[8] After completing the first draft of this paper, we discovered that the same observation appears in Alós-Ferrer and Mihm (2024).

3. Independent cycle equations

Theorem 1 establishes that cyclical independence is a fundamental principle of "stochastic rationality" in the Luce (1959) sense. As we have argued elsewhere,[9] it is a stochastic analogue of Richter's (1966) *congruence* condition on deterministic choice. But in many practical applications, the CI condition may require us to check an inconveniently large number of cycle equations.

Fortunately, it is almost never necessary to check all of them. As noted by Alós-Ferrer and Mihm (2024), any positive cycle with $E_i = E_j$ for all $i, j \in \{1, 2, \ldots, m\}$ evidently satisfies ($*$). Moreover, as also noted by the aforementioned authors, if two positive cycles each start and end at $x \in X$, and each satisfies ($*$), then their concatenation is another positive cycle that also satisfies ($*$). Alós-Ferrer and Mihm (2024) call a cycle *elementary* if it is not contained in a single menu and is not the concatenation of two sub-cycles. It therefore suffices to check elementary cycles.

If satisfaction of the cycle equations for a given subset of positive cycles ensures that the remaining cycle equations are also satisfied, we say that the subset "spans" the excluded cycles.[10] The set of elementary cycles is such a spanning subset. However, depending on the structure of $\mathcal{M}$, there typically exist spanning subsets that are smaller than this, and which may be identified in a more straightforwardly mechanical fashion.

3.1. Bounding the length of cycles

In some situations, we only need to check the cycle equations for positive cycles of a given length.[11] Two situations of this sort are well-known. In both cases we assume p satisfies positivity.

The first is when $\mathcal{M}$ is comprehensive. In this case, a sufficient condition for the existence of a Luce model is that p satisfy the IIA axiom (Luce, 1959). The IIA condition is equivalent to requiring that all 2-cycles are consistent:

$$p(x, A)\, p(y, B) = p(x, B)\, p(y, A) \tag{IIA}$$

whenever $A, B \in \mathcal{M}$ and $\{x, y\} \subseteq A \cap B$.

Second, if $\mathcal{M}$ comprises all menus of cardinality (up to) two – the *binary menus* case – then CI holds if and only if the *product rule* is satisfied (Luce and Suppes, 1965, Theorem 48). The product rule requires that all 3-cycles are consistent:

$$p(x, y)\, p(z, x)\, p(y, z) \;=\; p(y, x)\, p(x, z)\, p(z, y) \tag{PR}$$

for any $\{x, y, z\} \subseteq X$. Indeed, we may leverage the latter result into a stronger one. If $\mathcal{M}$ *includes* all binary menus, then it is straightforward to show that a Luce model exists (hence CI is satisfied) if and only if p satisfies the product rule and IIA.[12] Note that the product rule only makes reference to binary menus, while the IIA condition applies to all menus. It follows that when $\mathcal{M}$ includes all binary menus, CI holds if and only if all 3-cycles are consistent.

Another important context in which economies are possible is when every non-singleton menu in $\mathcal{M}$ includes a common "default" option. When $\mathcal{M}$ has this structure, it suffices to check cycles of length two.[13]

[9] See Rodrigues-Neto et al. (2024).

[10] The linear algebra terminology will be motivated shortly.

[11] Note that if every (positive) m-cycle is consistent, then so is every (positive) k-cycle for any $k < m$: a (positive) k-cycle $\{(x_i, x_{i+1}, E_i)\}_{i=1}^{k}$ can be extended to a (positive) m-cycle by appending $m - k$ terms of the form $(x, x, \{x\})$, and the extended cycle generates the same cycle equation as the original k-cycle.

[12] Kovach and Tserenjigmid (2022, Theorem 1) extend this idea to characterise a generalisation of the LM which they call the *focal Luce model.*

[13] All proofs are in Appendix A.

Theorem 2. *Suppose there is some fixed alternative $d \in X$ such that $d \in E$ for all non-singleton $E \in \mathcal{M}$. Let $p : X \times \mathcal{M} \to [0, 1]$ be an RCF that satisfies positivity. Then p satisfies CI if and only if it satisfies IIA.*

Combining Theorem 2 with Theorem 1 we have:[14]

Corollary 2. *Suppose there is some fixed alternative $d \in X$ such that $d \in E$ for all non-singleton $E \in \mathcal{M}$. Let $p : X \times \mathcal{M} \to [0, 1]$ be an RCF that satisfies positivity. Then p has a Luce model if and only if it satisfies IIA.*

Defaults play an important role in empirical applications of the logit model. Discrete choice models typically include an outside option that is available in every choice set. In this situation, IIA suffices to justify the logit model no matter how rich (or otherwise) is the set of menus that are observed in the data. Note that Corollary 2 does not impose any other restriction on $\mathcal{M}$ other than the presence of a common default. This is reassuring given that standard logit specification tests, such as the Hausman–McFadden test (Hausman and McFadden, 1984), are effectively tests of the IIA condition.

We also have an analogue of Corollary 2 for the case of two potential default options. We say that $\mathcal{M}$ has a *default pair*, $\{a, b\} \subseteq X$, if $a \neq b$ and for each non-singleton $E \in \mathcal{M}$, either $a \in E$ or $b \in E$.

Theorem 3. *Let $p : X \times \mathcal{M} \to [0, 1]$ be an RCF that satisfies positivity and let $\mathcal{M}$ have the default pair, $\{a, b\} \subseteq X$. Then:*

(i) p has a Luce model if and only if all 4-cycles are consistent.
(ii) If $\{a, b\} \subseteq A$ for some $A \in \mathcal{M}$, then p has a Luce model if and only if all 3-cycles are consistent.

3.2. Bounding the number of cycles

The following example shows that the (PR) condition contains some residual redundancy in the binary menus case.

Example 1. Let $X = \{a, b, c, d\}$ and let $\mathcal{M}$ consist of all non-empty subsets of X with cardinality no greater than two. Suppose p satisfies positivity and (PR) holds for

$$(x, y, z) \in \{(a, b, c), (a, b, d), (b, c, d)\}.$$

Then (PR) also holds when $(x, y, z) = (a, c, d)$:

$$\begin{aligned}
\frac{p(a,c)}{p(c,a)}\frac{p(d,a)}{p(a,d)}\frac{p(c,d)}{p(d,c)} &= \frac{p(a,c)}{p(c,a)}\frac{p(d,a)}{p(a,d)}\left(\frac{p(c,b)}{p(b,c)}\frac{p(b,d)}{p(d,b)}\right)\\
&= \left(\frac{p(a,c)}{p(c,a)}\frac{p(c,b)}{p(b,c)}\right)\frac{p(d,a)}{p(a,d)}\frac{p(b,d)}{p(d,b)}\\
&= \frac{p(a,b)}{p(b,a)}\frac{p(d,a)}{p(a,d)}\frac{p(b,d)}{p(d,b)}\\
&= 1
\end{aligned}$$

where the first equality uses (PR) for $(x, y, z) = (b, c, d)$; the third uses (PR) for $(x, y, z) = (a, b, c)$ and the final equality uses (PR) for $(x, y, z) = (a, b, d)$.

Similar redundancies exist in the conditions asserted in Theorems 2 and 3.

Ideally, we would like an algorithm that identifies, for any given $\mathcal{M}$, a "minimal" subset of cycles – a *cycle basis* – whose cycle equations span the rest. We will adapt ideas from the literature on Harsanyi's (1967–1968) *common prior* problem to develop such an algorithm.

The common prior problem concerns the existence of a prior distribution that rationalises a given collection of posteriors. It is a close mathematical cousin of the Luce model existence problem. If we reinterpret X as a state space, $\mathcal{M}$ as a collection of conditioning events and $p(\cdot, A)$ as a conditional probability on A, then a suitably normalised Luce model is probability mass function on X that rationalises the set of conditional probabilities represented by p.

An analogue of cyclical independence plays a central role in what has become known as the *cycles approach* to the common prior problem.

[14] A similar result, though in a different context, appears in Hellwig (2013).

In fact, it is not hard to show that Theorem 1 is implied by Proposition 1 of Rodrigues-Neto (2009).[15]

Rodrigues-Neto (2012, Corollary 2) uses graph-theoretic techniques to identify an upper bound on the number of cycle equations that need to be checked in order to verify the (common prior analogue of the) CI condition. This result is readily translated into the stochastic choice context. The bound depends only the structure of the menu set; it uses no other information about p.

To determine this bound we first construct a *multigraph* from $(X, \mathcal{M})$.[16] The vertex set is X and the edge set is $\{[xy;E] \mid \{x,y\} \subseteq E \in \mathcal{M}\}$. The edge $[xy;E]$ joins vertices x and y.[17] Hence, there are as many edges joining x to y as there are menus containing both x and y. We say that edge $[xy;E]$ joins x to y *within* E. Let $G(X,\mathcal{M})$ denote this (undirected) multigraph. Each connected sequence determines a *walk* in the multigraph $G(X,\mathcal{M})$, and *vice versa*. The connected sequence $\left\{\left(x_i, x_{i+1}, E_i\right)\right\}_{i=1}^{m}$ is associated with the walk:

$$\left(x_1, \left[x_1 x_2; E_1\right], x_2, \left[x_2 x_3; E_2\right], x_3, \ldots, x_m, \left[x_m x_{m+1}; E_m\right], x_{m+1}\right) .$$

A *simple walk* is a walk with no repeated edges; and a simple walk connecting two distinct vertices is a *path* if no vertex is encountered more than once along the walk. If each edge in a walk (or path) is within $E \in \mathcal{M}$ then we say that the walk (or path) itself is within E. A *connected component* of $G(X,\mathcal{M})$ is a set of vertices with the property that every distinct pair of vertices in the set is connected by a path, but there is no path from any element of the set to any vertex outside the set.

Rodrigues-Neto (2012) defines a *version* of $G(X,\mathcal{M})$ to be any subgraph obtained by deleting just enough edges to satisfy the following requirement: for every $E \in \mathcal{M}$ and every distinct $x, y \in E$ there is a unique path from x to y within E. Note that any version of $G(X,\mathcal{M})$ has the same connected components as $G(X,\mathcal{M})$ itself. The result of Rodrigues-Neto (2012), adapted to our setting, says the following: to ensure CI it suffices to check a number of cycle equations equal to the *cyclomatic number* (Berge, 1962) of any version of $G(X,\mathcal{M})$. The cyclomatic number is the same for all versions and is equal to:

$$\left(\sum_{A \in \mathcal{M}} |A|\right) - |\mathcal{M}| - |X| + \kappa$$

where κ is the number of connected components in $G(X,\mathcal{M})$. Let us denote this quantity by $C(X,\mathcal{M})$. We therefore have:

Theorem 4 (*Cf.*, *Rodrigues-Neto, 2012*)**.** *Let $p : X \times \mathcal{M} \to [0,1]$ be a random choice function satisfying positivity. There exists a set of cycle equations, with cardinality $C(X,\mathcal{M})$, such that p has a Luce model if and only if these cycle equations are satisfied.*

For completeness, we give a proof of Theorem 4 in Appendix A. It is a direct adaptation of the proof of Corollary 2 in Rodrigues-Neto (2012).

The following example computes $C(X,\mathcal{M})$ for the case of binary menus.

Example 2. Suppose $|X| = n$ and $\mathcal{M}$ consists of all binary menus.[18] The cyclomatic number for this case is:

$$\begin{aligned}\left(\sum_{A \in \mathcal{M}} |A|\right) - |\mathcal{M}| - |X| + \kappa &= 2|\mathcal{M}| - |\mathcal{M}| - |X| + \kappa \\ &= |\mathcal{M}| - n + 1 \\ &= \frac{n(n-1)}{2} - (n-1) \\ &= \frac{(n-1)(n-2)}{2}\end{aligned}$$

Example 2 reinforces our earlier observation about redundancy in the product rule conditions. The product rule specifies one equation for every three-element subset of X. The number of such subsets is

$$\frac{n(n-1)(n-2)}{6}$$

which exceeds $C(X,\mathcal{M})$ when $n > 3$. In particular:

$$\frac{(n-1)(n-2)}{2} = \frac{n(n-1)(n-2)}{6} - \frac{(n-1)(n-2)(n-3)}{6}$$

so the product rule contains

$$\frac{(n-1)(n-2)(n-3)}{6}$$

redundant equations. This represents a fraction $(n-3)/n$ of the total. For example, 50% of the product rule equations are redundant when $n = 6$.

It is not difficult to identify a subset of the product rule conditions that achieves the bound in Example 2 and suffices for CI. The following may be proved using the algebra in Example 1.[19]

Theorem 5. *Let $X = \{x_1, x_2, \ldots, x_n\}$ and let $\mathcal{M}$ be the set of all binary subsets of X. A random choice function satisfying positivity has a Luce model if and only if it satisfies (PR) for all $\{x,y,z\} \subseteq X$ with $|\{x,y,z\}| = 3$ and $x_1 \in \{x,y,z\}$*

The CI condition is therefore guaranteed provided all instances of the product rule that include x_1 are satisfied. Of course, x_1 may be fixed arbitrarily. The number of three-element subsets of X that include x_1 is

$$\frac{(n-1)(n-2)}{2}$$

which is equal to the value of $C(X,\mathcal{M})$ computed in Example 2.

It is also interesting to revisit Theorem 2 and its corollary (Corollary 2) in the light of Theorem 4. When all menus contain a common default, Theorem 2 tells us that it suffices to check the cycle equations for 2-cycles. However, the proof of Theorem 2 makes it clear that we need only check a subset of these: just those 2-cycles which have the default option in the "middle" position and which are not contained within a single menu. The associated cycle equations embody a very intuitive condition: for any non-default option that is available in more than one menu, the likelihood of this option being chosen, relative to the likelihood of the default being chosen, should be the same across all menus in which the option appears. This condition is clearly necessary for the existence of a Luce model, and it takes but a little thought to convince oneself that it is also sufficient. This subset of the 2-cycles contains one cycle for every common element (other than d) of every pair of distinct menus in $\mathcal{M}$, so its cardinality is equal to that of the set

$$\{(x, \{E, F\}) \mid E, F \in \mathcal{M},\ x \in (E \cap F) \setminus \{d\} \text{ and } E \neq F\} \qquad (\spadesuit)$$

Nevertheless, Theorem 4 implies that even this reduced set of 2-cycles is excessive to the purpose of verifying CI. Unless $|\mathcal{M}| < 3$, the cardinality of ($\spadesuit$) may exceed $C(X,\mathcal{M})$.

Theorem 6. *Suppose there is some $d \in X$ such that $d \in E$ for all $E \in \mathcal{M}$.*[20] *Then the cardinality of ($\spadesuit$) weakly exceeds $C(X,\mathcal{M})$. They are*

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[15] In the set-up of Rodrigues-Neto (2009), $\mathcal{M}$ consists of a union of partitions of X, but that is without loss of generality for our purposes. Since our $\mathcal{M}$ is assumed to contain all singletons, we may associate each non-singleton $A \in \mathcal{M}$ with the partition comprising A together with all singletons not contained in A. Then $\mathcal{M}$ is the union of these partitions.

[16] The basic concepts and results from graph theory needed for our analysis may be found in any introductory text (e.g., Diestel, 2010; Wallis, 2000).

[17] The expression "$[xy;E]$" is synonymous with "$[yx;E]$" — the multigraph is undirected.

[18] The presence of singletons is inconvenient for present purposes so we exclude them from $\mathcal{M}$. The reader will easily verify that including them leads to the same answer by a slightly longer route.

[19] Simply identify b with x_1 and $\{a,c,d\}$ with any three-element subset of X that excludes x_1.

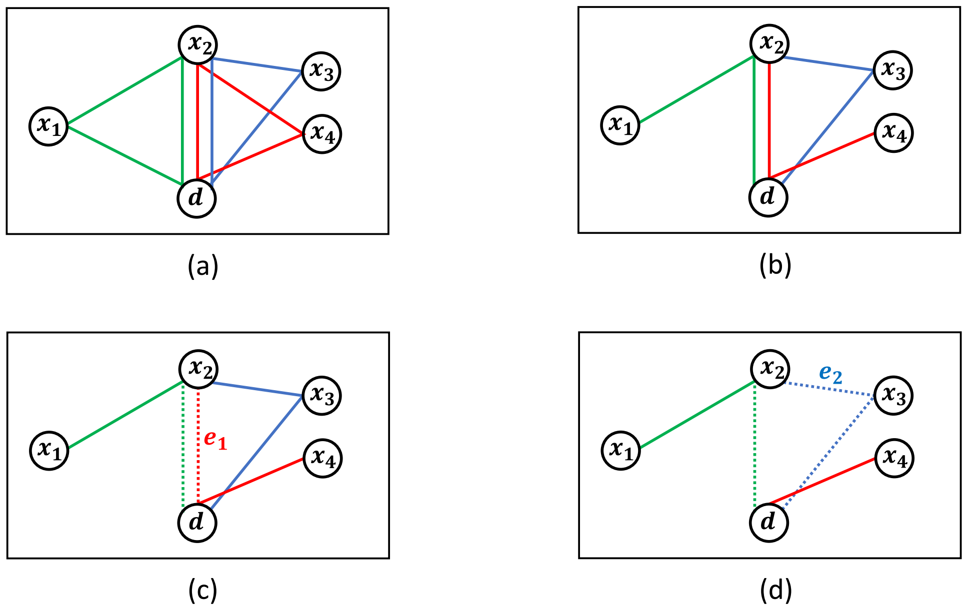

Fig. 1. Graphs of Example 4.

equal if $|\mathcal{M}| < 3$.

It is not difficult to see why there may be redundancy in (♠) when $|\mathcal{M}| \geq 3$. As the proof of Theorem 6 makes clear, there are redundant elements in (♠) whenever there exists some $x \in X\backslash\{d\}$ that is contained in at least three distinct menus. The following example illustrates.

Example 3. Suppose $X = \{x_1, x_2, x_3, y, d\}$ and $\mathcal{M} = \{E_1, E_2, E_3\}$ with $E_i = \{x_i, y, d\}$. Let p be an RCF satisfying positivity and define

$$p_i(a, b) = \frac{p(a, E_i)}{p(b, E_i)}$$

for each $i \in \{1, 2, 3\}$ and each $\{a, b\} \subseteq E_i$. The set (♠) is

$$\{(y, E_1, E_2),\ (y, E_2, E_3),\ (y, E_1, E_3)\}.$$

and element (y, E_i, E_j) has associated cycle equation

$$p_i(y, d) = p_j(y, d).$$

If $p_1(y, d) = p_2(y, d)$ and $p_2(y, d) = p_3(y, d)$ then $p_1(y, d) = p_3(y, d)$, so one of the three cycle equations is redundant.

3.3. Finding a "cycle basis"

To apply Theorem 4, we require a systematic way of identifying a set of $C(X, \mathcal{M})$ cycles such that CI holds if and only if the cycle equations for these $C(X, \mathcal{M})$ cycles are satisfied. As explained in the proof of Theorem 4, this amounts to finding a basis for a particular vector subspace. Appendix B describes a simple graph-theoretic algorithm for finding such a basis. The algorithm fixes some version, $G^*(X, \mathcal{M})$, of $G(X, \mathcal{M})$, then proceeds as follows:

Initialisation: Let $G_1 = G^*(X, \mathcal{M})$ and $k = 1$

Step k Do while G_k has a cycle:

Step k-1 Choose a cycle of G_k and denote it by C_k

Step k-2 Choose an edge in C_k and denote it by e_k

Step k-3 Let G_{k+1} be the subgraph $G_k - e_k$

Step k-4 $k + 1 \leftarrow k$

This algorithm will terminate in finitely many steps, say K of them. It can be shown (see Appendix B) that $K = C(X, \mathcal{M})$. The set $\{C_1, C_2, \ldots, C_K\}$ of cycles is called a *cycle basis* for $G^*(X, \mathcal{M})$. Fixing any cycle basis for any version of $G(X, \mathcal{M})$, satisfaction of the cycle equations associated with the cycles in this basis will suffice for CI. Again, see Appendix B for details.

We illustrate the algorithm with the following example.

Example 4. Let $X = \{d, x_1, x_2, x_3, x_4\}$ and

$$\mathcal{M} = \left\{\{d, x_1, x_2\}, \{d, x_2, x_3\}, \{d, x_2, x_4\}\right\}.$$

Note that every menu contains the "default" option, d, and

$$C(X, \mathcal{M}) = 9 - 3 - 5 + 1 = 2 < |(♠)| = 3.$$

Fig. 1(a) depicts the multigraph, $G(X, \mathcal{M})$, and Fig. 1(b) depicts one version of $G(X, \mathcal{M})$. Edges of the same colour belong to the same menu.[21] Fig. 1(c) indicates a possible choice of C_1 (dashed) and e_1. Fig. 1(d) exhibits a subsequent possible choice of C_2 and e_2. The algorithm terminates at this point, so $\{C_1, C_2\}$ is a cycle basis.

3.4. Independent cycles and uniqueness of Luce models

There is a close link between Theorem 4 and the uniqueness properties of Luce models. Fixing $(X, \mathcal{M})$, note that any (graph-theoretic) cycle in the multigraph, $G(X, \mathcal{M})$, must be contained within a connected component. These connected components determine, in the obvious fashion, partitions $\{X_k\}_{k=1}^{K}$ and $\{\mathcal{M}_k\}_{k=1}^{K}$ of X and $\mathcal{M}$ respectively. If $p : X \times \mathcal{M} \to [0, 1]$ is an RCF, let p_k denote the restriction of p to $X_k \times \mathcal{M}_k$. It is clear that p is uniquely determined by $\{p_k\}_{k=1}^{K}$. It is equally clear that p has a Luce model if and only if there is a Luce model for each p_k. The following result establishes that a Luce model for p_k is unique up to multiplication by a strictly positive scalar.

[20] Here it is convenient, but inessential, to exclude singleton menus (other than $\{d\}$) from $\mathcal{M}$.

[21] The colours only matter for identifying versions. They are superfluous to the rest of the algorithm.

Theorem 7. *Let $p_k : X_k \times \mathcal{M}_k \to [0,1]$ be an RCF satisfying positivity. Suppose there is a connected sequence joining any two distinct elements of X_k. Let v be a Luce model for p_k. Then u is a Luce model for p_k if and only if $u = \lambda v$ for some $\lambda > 0$.*

Thus, if p has a Luce model, then this model is unique up to one strictly positive multiplicative constant per connected component of $G(X, \mathcal{M})$. If p is decomposed as above, we may normalise any Luce model so that it sums to 1 on each X_k. This leaves $|X| - \kappa$ free parameters to specify. By Theorem 4:

$$|X| - \kappa = \left[\left(\sum_{A \in \mathcal{M}} |A|\right) - |\mathcal{M}|\right] - C(X, \mathcal{M})$$

The square-bracketed term is the number of free parameters in the specification of an RCF: for each $A \in \mathcal{M}$ we need to specify $|A| - 1$ values. The quantity $C(X, \mathcal{M})$ is precisely the difference between the degrees of freedom in the specification of an RCF and the degrees of freedom in the specification of a Luce model for p. Given positivity, we may specify p using log-probabilities, in which case the cycle equations are *linear* restrictions on the log-probabilities (see the proof of Theorem 4). A Luce model imposes no more than $C(X, \mathcal{M})$ independent restrictions on this representation of an RCF.

4. Concluding remarks

Given positivity, CI is the empirical signature of Luce rationality (i.e., existence of a Luce model) when choice is observed from an arbitrarily fixed set of menus. Essentially the same insight underpins the so-called “cycles approach” to the common prior problem. The literature on the latter problem provides graph-theoretic tools to reduce the number of cycle equations that need to be checked in order to verify CI. Elementary graph theory also furnishes a simple algorithm for identifying a suitable “cycle basis”.

This theoretical apparatus reduces the empirical signature of Luce rationality to a small set of *linear* restrictions on the *logs* of choice probabilities (i.e., the log-transformed cycle equations). It remains to develop a suitable empirical methodology for testing these conditions. This is the subject of on-going research.

CRediT authorship contribution statement

José A. Rodrigues-Neto: Writing – review & editing, Writing – original draft, Project administration, Methodology, Investigation, Formal analysis, Conceptualization. **Matthew Ryan:** Writing – review & editing, Writing – original draft, Project administration, Methodology, Investigation, Formal analysis, Conceptualization. **James Taylor:** Writing – review & editing, Writing – original draft, Project administration, Methodology, Investigation, Formal analysis, Conceptualization.

Declaration of competing interest

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Appendix A. Proofs

A.1. Proof of Theorem 2

Recall that IIA is equivalent to all 2-cycles being consistent, which is equivalent to all positive 2-cycles being consistent when p satisfies positivity. Before proving Theorem 2, we first establish a useful Lemma.

Lemma 1. *Let $p : X \times \mathcal{M} \to [0,1]$ be an RCF that satisfies positivity. Let*

$$c = \left\{\left(x_i, x_{i+1}, E_i\right)\right\}_{i=1}^m$$

be a cycle with $x_k \in E_1 \cap \cdots \cap E_m$ for some $k \in \{1, 2, \dots, m\}$. If p satisfies IIA, then the cycle equation for c is satisfied.

Proof. First, suppose $x_1 \in E_1 \cap \cdots \cap E_m$. Recall that $x_1 = x_{m+1}$. For ease of exposition, let $y = x_1 = x_{m+1}$. In this case, we must show that

$$\frac{p\left(x_2, E_2\right)}{p\left(x_2, E_1\right)} \frac{p\left(x_3, E_3\right)}{p\left(x_3, E_2\right)} \cdots \frac{p\left(x_m, E_m\right)}{p\left(x_m, E_{m-1}\right)} \frac{p\left(y, E_1\right)}{p\left(y, E_m\right)} = 1$$

or, equivalently,

$$\left[\frac{p\left(x_2, E_2\right) p\left(y, E_1\right)}{p\left(x_2, E_1\right) p\left(y, E_2\right)}\right]\left[\frac{p\left(x_3, E_3\right) p\left(y, E_2\right)}{p\left(x_3, E_2\right) p\left(y, E_3\right)}\right] \cdots \left[\frac{p\left(x_m, E_m\right) p\left(y, E_{m-1}\right)}{p\left(x_m, E_{m-1}\right) p\left(y, E_m\right)}\right] = 1.$$

But this equation holds since each square-bracketed term is equal to 1 by IIA.

Next, suppose $m > 1$ and $x_k \in E_1 \cap \cdots \cap E_m$ for some $k \in \{2, 3, \dots, m\}$. Let $y = x_k$. From the previous case, we know that the cycles

$$\left\{\left(y, x_1, E_1\right), \left(x_1, x_2, E_1\right) \dots, \left(x_{k-1}, y, E_{k-1}\right)\right\}$$

and

$$\left\{\left(y, x_{k+1}, E_k\right), \left(x_{k+1}, x_{k+2}, E_{k+1}\right), \dots, \left(x_m, x_{m+1}, E_m\right) \left(x_1, y, E_1\right)\right\}$$

are consistent. The cycle equations for these two cycles are, respectively:

$$\frac{p(y, E_1)}{p(x_1, E_1)} \prod_{i=1}^{k-1} \frac{p\left(x_i, E_i\right)}{p\left(x_{i+1}, E_i\right)} = 1$$

and

$$\frac{p(x_1, E_1)}{p(y, E_1)} \prod_{i=k}^{m} \frac{p\left(x_i, E_i\right)}{p\left(x_{i+1}, E_i\right)} = 1.$$

Multiplying these together gives (∗). □

Since p satisfies positivity, all cycles are positive. The “only if” part of Theorem 2 is therefore immediate. We show the “if” part.

Suppose p satisfies IIA.

First, consider a cycle $\left\{\left(x_i, x_{i+1}, E_i\right)\right\}_{i=1}^m$ with $m > 2$ and $d \in \{x_1, x_2, \dots, x_m\}$. Since d is in every menu, this cycle is consistent by Lemma 1.

Next, consider a cycle $\left\{\left(x_i, x_{i+1}, E_i\right)\right\}_{i=1}^m$ with $m > 2$ and suppose $d \notin \{x_1, x_2, \dots, x_m\}$. Since $x_1 = x_{m+1}$ we must show that

$$\frac{p\left(x_2, E_2\right)}{p\left(x_2, E_1\right)} \frac{p\left(x_3, E_3\right)}{p\left(x_3, E_2\right)} \cdots \frac{p\left(x_m, E_m\right)}{p\left(x_m, E_{m-1}\right)} \frac{p\left(x_1, E_1\right)}{p\left(x_1, E_m\right)} = 1$$

which is equivalent to

$$\left[\frac{p\left(x_2, E_2\right) p\left(d, E_1\right)}{p\left(x_2, E_1\right) p\left(d, E_2\right)}\right]\left[\frac{p\left(x_3, E_3\right) p\left(d, E_2\right)}{p\left(x_3, E_2\right) p\left(d, E_3\right)}\right] \cdots \left[\frac{p\left(x_1, E_m\right) p\left(d, E_m\right)}{p\left(x_1, E_m\right) p\left(d, E_1\right)}\right] = 1.$$

This equation holds since each square-bracketed term is equal to 1 by IIA. Hence, the cycle is consistent.

A.2. Proof of Theorem 3

We begin with a useful lemma:

Lemma 2. *Let $p : X \times \mathcal{M} \to [0,1]$ be an RCF which satisfies positivity, and let $\mathcal{M}$ contain a default pair, $\{a, b\} \subseteq X$. Then p has a Luce model if* *and only if all cycles in the following categories are consistent: all 3-cycles, and all 4-cycles of the form*

$$xE_1aE_2yE_3bE_4x \quad (\clubsuit)$$

Proof. The "only if" part is immediate from Corollary 1. We prove the "if" part.

We begin by defining some useful notation. If $q = \{(x_i, x_i', E_i)\}_{i=1}^m$ is a sequence with $\{x_i, x_i'\} \subseteq E_i \in \mathcal{M}$ for all $i \in \{1,2,\dots,m\}$ (not necessarily a connected sequence) we call the elements of the sequence *edges* and we define

$$v(q) = \prod_{i=1}^m \frac{p(x_i', E_i)}{p(x_i, E_i)}.$$

If q_1 and q_2 are two such sequences, let $q_1 \circ q_2$ denote their concatenation. Observe that $v(q_1 \circ q_2) = v(q_1)v(q_2)$.

Consider a cycle $c = x_1E_1x_2E_2 \cdots E_{m-1}x_mE_mx_1$. Since p satisfies positivity, c is a positive cycle so we must show that $v(q) = 1$ (i.e., c is consistent). We start by decomposing c as follows:

$$c = q_1 \circ q_2 \circ \cdots \circ q_k$$

where the connected subsequences $\{q_j\}_{j=1}^k$ are associated with an alternating sequence of defaults: there exist $d_j \in \{a,b\}$ for each $j \in \{1,2,\dots,k\}$ such that each E_i appearing in q_j contains d_j, and $d_j \neq d_{j+1}$ for each $j \in \{1,2,\dots,k-1\}$. Of course, many such decompositions may be possible; just fix one. Let $\mathcal{J}_a \subseteq \{1,2,\dots,k\}$ be defined by $j \in \mathcal{J}_a$ if and only if $d_j = a$, and let $\mathcal{J}_b$ be the complementary set of indices.

Define $E^j \in \{E_1,\dots,E_m\}$ to be the first menu appearing in q_j and $F^j \in \{E_1,\dots,E_m\}$ to be the last. Likewise, let $y^j \in \{1,2,\dots,x_m\}$ be the first, and $z^j \in \{1,2,\dots,x_m\}$ the last, alternative appearing in q_j. Hence $q_j = y^jE^j \cdots F^jz^j$. For each $j \in \{1,2,\dots,k\}$ define edges $e_j^1 = (z^j, d_j, F^j)$ and $e_j^2 = (d_j, y^j, E^j)$, and associated connected sequence $\pi_j = e_j^1 \circ e_j^2 = z^jF^jd_jE^jy^j$. Finally, let $\hat{q}_j = q_j \circ \pi_j$ and note that $\hat{q}_j$ is a positive cycle. This cycle follows q_j then loops back to the start of q_j by going through d_j (recall that each menu in q_j contains d_j). Since p satisfies IIA, Lemma 1 implies that $v(\hat{q}_j) = 1$ for each j. Hence

$$v(\hat{q}_1 \circ \hat{q}_2 \circ \cdots \circ \hat{q}_k) = 1 \quad (1)$$

We will use this fact to show that $v(c) = 1$.

The value of $v(\hat{q}_1 \circ \hat{q}_2 \circ \cdots \circ \hat{q}_k)$ is unaffected by re-arranging the edges in $\hat{q}_1 \circ \hat{q}_2 \circ \cdots \circ \hat{q}_k$. For each $j \in \mathcal{J}_a$ define a new connected sequence

$$\rho_j = \pi_j \circ e_{j\ominus 1}^2 \circ e_{j\oplus 1}^1$$

where

$$j \ominus 1 = \begin{cases} j-1 & \text{if } j > 1 \\ k & \text{if } j = 1 \end{cases}$$

and

$$j \oplus 1 = \begin{cases} j+1 & \text{if } j < k \\ 1 & \text{if } j = k \end{cases}$$

Since $y^j = z^{j\ominus 1}$ and $z^j = y^{j\oplus 1}$, we see that

$$\rho_j = z^jF^jaE^jy^jF^{j\ominus 1}bE^{j\oplus 1}z^j$$

is a cycle of the form (♣). Hence $v(\rho_j) = 1$ for each $j \in \mathcal{J}_a$. Moreover, the sequence $\rho_1 \circ \rho_2 \circ \cdots \circ \rho_k$ is obtained from $\pi_1 \circ \pi_2 \circ \cdots \circ \pi_k$ by rearranging edges. We therefore have:

$$\begin{aligned} v(\hat{q}_1 \circ \hat{q}_2 \circ \cdots \circ \hat{q}_k) &= v(q_1 \circ \cdots \circ q_k) \prod_{j \in \mathcal{J}_a} v(\rho_j) \\ &= v(c) \end{aligned}$$

so $v(c) = 1$ from (1). □

We may now establish Theorem 3 as follows. (i) If p has a Luce model then p satisfies Corollary 1 so all cycles (hence all 4-cycles) are consistent. The converse holds by virtue of Lemma 2.

(ii) Let $F \in \mathcal{M}$ be such that $a, b \in F$.

The "only if" part follows from Corollary 1. Conversely, suppose all 3-cycles are consistent. To show that p has a Luce model we need to show that all cycles of the form (♣) are also consistent. To that end, let $c = xE_1aE_2yE_3bE_4x$. Now consider the following 3-cycles: xE_1aFbE_4x and aE_2yE_3bFa. Both are consistent so we have:

$$p(x,E_1)p(a,F)p(b,E_4) = p(a,E_1)p(b,F)p(x,E_4)$$
$$p(a,E_2)p(y,E_3)p(b,F) = p(y,E_2)p(b,E_3)p(a,F).$$

Multiplying, cancelling like terms, and re-arranging gives:

$$p(x,E_1)p(a,E_2)p(y,E_3)p(b,E_4) = p(a,E_1)p(y,E_2)p(b,E_3)p(x,E_4).$$

Hence c is consistent.

A.3. Proof of Theorem 4

Cyclical independence requires that the following *cycle equation* be satisfied for any *positive* connected sequence $\{(x_i, x_{i+1}, E_i)\}_{i=1}^m$ with $x_1 = x_{m+1}$:

$$\prod_{i=1}^m \frac{p(x_i, E_i)}{p(x_{i+1}, E_i)} = 1 \quad (\dagger)$$

It suffices to consider connected sequences that also satisfy $|\{x_1, x_2, \dots, x_m\}| = m$. If the same alternative appears twice along the sequence then we can split it into two cyclical subsequences; provided the cycle equation for each subsequence is satisfied then the original cycle equation will be satisfied.

Recall that $G(X,\mathcal{M})$ is the (multi)graph with vertex set X and edge set

$$\mathcal{E} = \{[xy;E] \mid \{x,y\} \subseteq E \in \mathcal{M}\}.$$

We therefore need to check that (†) holds along any *cycle* in $G(X,\mathcal{M})$. To be clear, graph $G(X,\mathcal{M})$ is undirected while equation (†) assumes a particular direction of travel around the cycle associated with the connected sequence. However, (†) says precisely that the product on the left is independent of the direction of travel (or the vertex at which the walk starts and ends), so we can unambiguously talk about equation (†) holding, or being satisfied, along a cycle in $G(X,\mathcal{M})$.

In fact, we can restrict attention to cycles within the subgraph associated with some *version* of $G(X,\mathcal{M})$. If the subsequence from $x_j = x'$ to $x_k = x''$ (for $1 \le j < k \le m+1$) is *within* some menu (i.e., $E_j = E_{j+1} = \cdots = E_k$), then it may be replaced by any other subsequence from x' to x'' within the same menu without affecting the cycle equation, as is easily verified. Restricting attention to a version of $G(X,\mathcal{M})$ eliminates these redundancies in the cycle equations: for any menu E and any two distinct elements of E, a version contains exactly one path within E that joins these two elements. In particular, a version contains no cycle within any menu.

Let $G^* = (X, \mathcal{E}^*)$ be the subgraph generated by a given version of $G(X,\mathcal{M})$. Assuming positivity of p, cyclical independence is equivalent to satisfaction of the cycle equation (†) associated with any cycle in G^*.

The proof of Theorem 4 now proceeds as follows. We first map each cycle of G^* to a vector in a suitable vector space, and consider the subspace spanned (in the linear sense) by these vectors. We then use a standard result from graph theory to establish the $C(X,\mathcal{M})$ is the dimension of this subspace. Finally, we show that satisfaction of the cycle equations associated with the cycles in a basis for this subspace suffices for cyclical independence.

To construct a suitable vector space, let

$$\imath : \mathcal{E}^* \to \{1,2,\dots,|\mathcal{E}^*|\}$$

be an enumeration of the edges. Define $\mathcal{V}$ to be the vector space $\mathbb{R}^{|\mathcal{E}^*|}$ with edge $[xy;E]$ associated to coordinate $\imath([xy;E])$. We next map each cycle in G^* to a vector in $\mathcal{V}$.[22] To do so, let $\vec{G}^*$ be a *directed* multigraph that adds an orientation to each edge of G^* (so "edges" become "arcs"). Following Berge (1962), each cycle in G^* is mapped to a vector in $\mathcal{V}$ by setting the value of the $\imath([xy;E])$ coordinate equal to: 1 if the cycle traverses edge $[xy;E]$ in the direction of the corresponding arc in $\vec{G}^*$; -1 if the cycle traverses edge $[xy;E]$ in the opposite direction; and 0 if the cycle does not traverse $[xy;E]$. We call this the *cycle vector* for the given cycle in G^*. Let C denote the subspace of $\mathcal{V}$ spanned by the zero vector together with the cycle vectors for all cycles in G^*. This subspace is called the *cycle space* of $\vec{G}^*$.

By definition, the dimension of C is the *cyclomatic number* of G^*. This dimension is independent of the particular $\vec{G}^*$ used to fix the orientations of edges.[23] A classical result in graph theory (see, for example, (Berge, 1962), Chapter 4, Theorem 2)[24] shows that the cyclomatic number of G^* is equal to $|\mathcal{E}^*| - |X| + \kappa$ (recall that κ is the number of connected components of $G(X,\mathcal{M})$, and hence also of G^*). In our case,

$$|\mathcal{E}^*| \;=\; \sum_{A\in\mathcal{M}} (|A|-1) \;=\left(\sum_{A\in\mathcal{M}} |A|\right) - |\mathcal{M}|$$

so the cyclomatic number of G^* is $C(X,\mathcal{M})$. Since κ is the same for any version of $G(X,\mathcal{M})$, *all versions have the same cyclomatic number*, so we call this common value the cyclomatic number of $G(X,\mathcal{M})$.

Next, we re-write the cycle equation (†) in the following form:

$$\sum_{i=1}^{m} \left(\ln\left[p\left(x_i,E_i\right)\right] - \ln\left[p\left(x_{i+1},E_i\right)\right]\right) \;=\; 0 \qquad (\dagger\dagger)$$

Let $v \in \mathcal{V}$ be the vector representation for the associated cycle in $\vec{G}^*$. Then equation (††) may be expressed $\alpha \cdot v = 0$, where $\alpha \in \mathcal{V}$ is defined as follows:

$$\alpha_{\imath([xy;E])} \;=\; \ln\left(p\left(\underline{z},E\right)\right) - \ln\left(p\left(\overline{z},E\right)\right)$$

with $\{\underline{z},\overline{z}\} = \{x,y\}$ and the arc in $\vec{G}^*$ corresponding to edge $[xy;E]$ in G^* is oriented from $\underline{z}$ to $\overline{z}$. The cycle equations associated with **all** cycles of G^* are therefore satisfied if and only if α is in the **orthogonal complement** of C. For this, it suffices that α is orthogonal to each vector in a basis for C. The dimension of C gives the number of elements in a basis.

In summary, any basis for C has $C(X,\mathcal{M})$ elements, and CI is satisfied for $G(X,\mathcal{M})$ if and only if each of these elements is orthogonal to α. This is equivalent to cycle equation (††) being satisfied for the $C(X,\mathcal{M})$ cycles associated with the basis vectors.

A.4. Proof of Theorem 6

Suppose $\mathcal{M} = \{E_1, E_2, \ldots, E_m\}$. Let C_m denote the cardinality of (♠). Note that C_m is equal to:

$$\sum_{1\le i<j\le m} \left|E_i \cap E_j\right| \;-\; \frac{m(m-1)}{2} \qquad (C_m)$$

In the present scenario:

$$\kappa \;=\; 1 \;+\; \left|X \setminus \bigcup_{i=1}^{m} E_i\right|$$

so $C(X,\mathcal{M})$ simplifies to

$$\sum_{i=1}^{m} |E_i| - \left|\bigcup_{i=1}^{m} E_i\right| - (m-1) \qquad (N_m)$$

If $m=1$ then $N_1 \;=\; C_1 \;=\; 0$.
If $m=2$ then $N_2 \;=\; C_2 \;=\; |E_1 \cap E_2| - 1$.
We next show that $N_m \le C_m$ when $m>2$. Note that $N_m \le C_m$ if and only if

$$\left|\bigcup_{i=1}^{m} E_i\right| \;-\; \sum_{i=1}^{m} |E_i| \;+\; \sum_{1\le i<j\le m} \left|E_i \cap E_j\right| \;\ge\; \frac{(m-1)(m-2)}{2} \qquad (\#)$$

To prove (#) we first define $X_k \subseteq X$ to be the set of alternatives that appear in exactly k menus in $\mathcal{M}$. Thus $d \in X_m$, $X_k \cap X_{k'} = \emptyset$ whenever $k \neq k'$, and $\bigcup_{k=1}^{m} X_k = X$ (though some X_k may be empty). We may therefore re-write the left-hand side of (#) as follows:

$$\sum_{k=1}^{m}\left[\left|\bigcup_{i=1}^{m}\left(E_i \cap X_k\right)\right| \;-\; \sum_{i=1}^{m}\left|E_i \cap X_k\right| \;+\; \sum_{1\le i<j\le m}\left|E_i \cap E_j \cap X_k\right|\right]$$

The square-bracketed term is zero if $k \in \{1,2\}$ and strictly positive if $k>2$ (and $X_k \neq \emptyset$), since:

$$\left|\bigcup_{i=1}^{m}\left(E_i \cap X_k\right)\right| = \;|X_k|$$

$$\sum_{i=1}^{m}\left|E_i \cap X_k\right| \;=\; k\,|X_k|$$

and

$$\sum_{1\le i<j\le m}\left|E_i \cap E_j \cap X_k\right| \begin{cases} = \quad 0 & \text{if } k=1 \\ = \quad |X_k| & \text{if } k=2 \\ > \quad (k-1)\,|X_k| & \text{if } k>2 \end{cases}$$

Now, for each $k < m$, create k (hypothetical) "replicas" of each alternative in X_k – one for each menu in which the alternative appears – and treat these replicas as distinct alternatives. Do the same for all elements of X_m except for the default, d, which remains unreplicated. Let X^* denote the expanded universe of alternatives.[25] For each $E_i \in \mathcal{M}$ define $E_i^* \subseteq X^*$ by relabelling each non-default alternative to the corresponding replica,[26] and define the corresponding menu set $\mathcal{M}^* = \{E_1^*, \ldots, E_m^*\}$. Therefore, d is common to every menu in $\mathcal{M}^*$ but every other alternative in X^* appears in a unique menu. Finally, define X_k^* for each $k \in \{1,\ldots,m\}$ to be the alternatives in X^* that appear in exactly k menus in $\mathcal{M}^*$. Hence, $|X_m^*| = 1$ and $\left|X_k^*\right| = 0$ if $1<k<m$.

We now observe that the left-hand side of (#) weakly exceeds

$$\left|\bigcup_{i=1}^{m}\left(E_i \cap X_m\right)\right| \;-\; \sum_{i=1}^{m}\left|E_i \cap X_m\right| \;+\; \sum_{1\le i<j\le m}\left|E_i \cap E_j \cap X_m\right|$$

which in turn weakly exceeds

$$\left|\bigcup_{i=1}^{m}\left(E_i^* \cap X_m^*\right)\right| \;-\; \sum_{i=1}^{m}\left|E_i^* \cap X_m^*\right| \;+\; \sum_{1\le i<j\le m}\left|E_i^* \cap E_j^* \cap X_m^*\right|$$

[22] Modern treatments of graph theory (e.g., Diestel, 2010; Wallis, 2000) encode the cycles in a vector space defined over the two-element field $\mathbb{Z}_2$ comprising the integers $\{0,1\}$ together with addition and multiplication modulo-2, rather than a real vector space. However, the log of the cycle equation (†) is linear in the Euclidean sense, so the older set-up of Berge (1962) is more convenient for our purposes.

[23] Let $\{x^1, x^2, \ldots, x^K\}$ be a linearly dependent collection of cycle vectors for a particular orientation. Then there exists $\beta \in \mathbb{R}^K$ with $\beta \neq 0$ and $\sum_{k=1}^{K} \beta_k x^k = 0$. If we change orientations, we simply reverse the signs of the x^k vectors in specified components; the corresponding rows of $\sum_{k=1}^{K} \beta_k x^k$ get multiplied by -1. Therefore, letting the resulting set of vectors be denoted $\{\hat{x}^1, \hat{x}^2, \ldots, \hat{x}^K\}$ we still have $\sum_{k=1}^{K} \beta_k \hat{x}^k = 0$. Thus, a given set of cycles has linearly dependent cycle vectors for one orientation if and only if it has a linearly dependent set of cycle vectors for all orientations.

[24] Berge considers the space of vectors associated with all *circuits*, not just cycles. For circuits that are not cycles, the $\imath([xy;E])$ coordinate of the associated vector is equal to (zero plus) the number of times that edge $[xy;E]$ is traversed in the positive orientation less the number of times it is traversed in the opposite direction. It is evident that this vector is the sum of the vectors associated with sub-cycles into which the original circuit may be decomposed. Therefore, the dimension of the cycle subspace is the same whether or not we include the vectors associated with circuits.

[25] Thus, $|X^*| = \left(\sum_{k=1}^{m} k\,|X_k|\right) - (m-1)$.

[26] Thus, $|E_i^*| = |E_i|$.

$$= \sum_{k=1}^{m}\left[\left|\bigcup_{i=1}^{m}\left(E_i^*\cap X_k^*\right)\right| - \sum_{i=1}^{m}\left|E_i^*\cap X_k^*\right| + \sum_{1\le i<j\le m}\left|E_i^*\cap E_j^*\cap X_k^*\right|\right]$$
$$= \left|\bigcup_{i=1}^{m} E_i^*\right| - \sum_{i=1}^{m}\left|E_i^*\right| + \sum_{1\le i<j\le m}\left|E_i^*\cap E_j^*\right|$$

By the inclusion-exclusion formula

$$\left|\bigcup_{i=1}^{m} E_i^*\right| - \sum_{i=1}^{m}\left|E_i^*\right| + \sum_{1\le i<j\le m}\left|E_i^*\cap E_j^*\right| = \sum_{\substack{J\subseteq\{1,2,\dots,m\}\\ |J|\ge 3}} (-1)^{|J|+1}\left|\bigcap_{j\in J} E_j^*\right|.$$

Since $\bigcap_{j\in J} E_j^* = \{d\}$ for all $J\subseteq\{1,2,\dots,m\}$ with $|J|\ge 3$ we have:

$$\sum_{\substack{J\subseteq\{1,2,\dots,m\}\\ |J|\ge 3}} (-1)^{|J|+1}\left|\bigcap_{j\in J} E_j^*\right| = (-1)\sum_{\substack{J\subseteq\{1,2,\dots,m\}\\ |J|\ge 3}} (-1)^{|J|}$$
$$= (-1)\left[\sum_{J\subseteq\{1,2,\dots,m\}} (-1)^{|J|} - 1 + m - \frac{m(m-1)}{2}\right]$$
$$= \frac{(m-1)(m-2)}{2} + (-1)\sum_{J\subseteq\{1,2,\dots,m\}} (-1)^{|J|}$$
$$= \frac{(m-1)(m-2)}{2}$$

where the final equality uses Lemma 2.1 of Shafer (1976). We therefore deduce the required inequality (#).

A.5. Proof of Theorem 7

We already observed the "if" part. For the "only if" part, let v and u be Luce models for p_k, and let x and y be two distinct elements of X_k. Let $\left\{\left(x_i, x_{i+1}, E_i\right)\right\}_{i=1}^{m}$ be a connected sequence with $x_1 = x$ and $x_{m+1} = y$. We therefore have:

$$\frac{p_k\left(x_1, E_1\right)}{p_k\left(x_2, E_1\right)}\frac{p_k\left(x_2, E_2\right)}{p_k\left(x_3, E_2\right)}\cdots\frac{p_k\left(x_m, E_m\right)}{p_k\left(x_{m+1}, E_m\right)} = \frac{v(x)}{v(y)} = \frac{u(x)}{u(y)}$$

Since this holds for every distinct x and y in X_k, the result follows.

Appendix B. Identifying a "cycle basis"

Let $\mathcal{C}\subseteq\mathcal{V}$ be the cycle space for a given version, $G^* = (X, \mathcal{E}^*)$, of $G(X, \mathcal{M})$. A *cycle basis* is a basis for the subspace $\mathcal{C}$. Any collection of $C(X, \mathcal{M})$ linearly independent vectors in $\mathcal{V}$ will constitute such a basis.

Suppose G^* is connected (i.e., $\kappa = 1$) and consider applying the algorithm in Section 3.3 to G^*, with $\left\{C_k\right\}_{k=1}^{K}$ the family of cycles identified in the process and $\left\{e_k\right\}_{k=1}^{K}\subseteq\mathcal{E}^*$ the edges removed. Let $T = (X, \hat{\mathcal{E}})$ be the subgraph of G^* that is obtained at the end of this process. This subgraph is known as a *spanning tree* for G^* (Wallis, 2000, Section 4.2). If $z^k\in\mathcal{C}$ is the vector associated with C^k, then the family $\left\{z^k\right\}_{k=1}^{K}$ is linearly independent: $\left|\hat{z}_{i_k}^k\right| = 1$ and $\hat{z}_{i_k}^j = 0$ for all $j > k$. By a well-known result on trees (Wallis, 2000, Theorem 4.2), we have $|\hat{\mathcal{E}}| = |X| - 1$ so

$$K = \left|\mathcal{E}^*\right| - |X| + 1 = C(X, \mathcal{M}).$$

It follows that the $\left\{C^k\right\}_{k=1}^{K}$ forms a basis for $\mathcal{C}$.

Now suppose $\kappa > 1$. The components of G^* determine subgraphs which are individually connected and together partition G^*. Any cycle in G^* will be contained in one of those subgraphs. Fixing a cycle basis for each subgraph, their union will therefore be a cycle basis for G^*.

Data availability

No data was used for the research described in the article.

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