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A Variational Formulation of European Option Prices in the 1-Hypergeometric Stochastic Volatility Model

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ABSTRACT

The paper proposes a variational analysis of the 1-hypergeometric stochastic volatility model for pricing European options. The methodology involves the derivation of estimates of the weak solution in a weighted Sobolev space. The weight is closely related to the stochastic volatility dynamic of the model. The solution is further analyzed using semigroup theory applied to the pricing operator and leads to certain constraints on the model parameters. An implementation of the model using a finite element method library is carried out and illustrates how the model works.

1 | Introduction

Since the seminal work of Black and Scholes [1] on option pricing, many extensions have been proposed in the literature. An important contribution is certainly the Heston model [2], which has the important property that the implied volatility associated with option prices is not flat but actually shows a smile or smirk, see Gatheral [3]. A property of the Heston model is that the Feller's condition must be satisfied to ensure that the volatility is positive. In practice, this condition is often not satisfied. To solve this problem, the α-hypergeometric model was introduced in Da Fonseca and Martini [4]. When $\alpha = 1$ the model corresponds to a model first proposed in Henry-Labordère [5]. These papers derived two different expressions for the characteristic function of the logarithm of the stock price, so that the pricing option using the Fourier transform, as in Carr and Madan [6], is feasible but rather complicated. Therefore, the use of the finite element method seems to be an interesting alternative for pricing European (or more exotic) options. The contribution of this paper to the literature is as follows. Using the important contribution of Achdou et al. [7], which builds on Achdou and Tchou [8], we develop a variational analysis of the α-hypergeometric stochastic volatility model when $\alpha = 1$. This amounts to introducing a weighted Sobolev space and performing the standard analysis in this space. Using elliptic estimates, we also derive an existence result using semigroup theory applied to the pricing operator. Finally, once the variational formulation is obtained, we apply the finite element method to the pricing problem and provide an implementation of the model using the finite element method library FreeFem++ (*i.e.*, Hecht [9]). The last section contains some concluding remarks.

2 | The Option Pricing Model

In the α-hypergeometric stochastic volatility model, the stock and volatility dynamic is given by the following:

$$ds_t = s_t r dt + s_t e^{y_t} dw_{1,t} \tag{1}$$

$$dy_t = (a - be^{\alpha y_t})dt + \sigma dw_{2,t} \tag{2}$$

with $(w_{1,t}, w_{2,t})_{t\geq 0}$ a two-dimensional Brownian motion such that $d\langle w_{1,.}, w_{2,.}\rangle_t = \rho dt$ and $a > 0$, $b > 0$ and $\sigma > 0$. The α-hypergeometric model is defined in [4], when $\alpha = 1$ the model corresponds to a model first proposed in [5], where a series expansion of the characteristic function of the logarithm of the stock price is given. In [4], an alternative series expansion of the characteristic function of the logarithm of the stock price is provided, so that the pricing of the European option could be in principle carried out. In both cases, the expression of this characteristic function is difficult to implement. Therefore, option pricing using the Fourier transform as presented in [6] is rather complicated. The use of the finite element method to price European (or more exotic) options is worth investigating. From now on, we will assume that $\alpha = 1$.[1]

Define $x_t = \log s_t$ and $v_t = e^{y_t}$, then Itô's Lemma implies that

$$dx_t = \left(r - \frac{v_t^2}{2}\right)dt + v_t dw_{1,t} \tag{3}$$

$$dv_t = (cv_t - bv_t^2)dt + \sigma v_t dw_{2,t} \tag{4}$$

with $c = a + \sigma^2/2$. Without loss of generality, we assume $r = 0$; it amounts to working with the forward price.

Following Karlin and Taylor [13, Eqs. (5.31,5.32)], the asymptotic distribution associated with v_t is as follows:

$$p(v) = \left(\frac{2b}{\sigma^2}\right)^{\frac{2c}{\sigma^2}-1} \frac{v^{\frac{2c}{\sigma^2}-2} e^{-\frac{2bv}{\sigma^2}}}{\Gamma\left(\frac{2c}{\sigma^2}-1\right)} \tag{5}$$

It is a gamma distribution with shape $\frac{2c}{\sigma^2} - 1$ and rate $\frac{2b}{\sigma^2}$. We need to have $\frac{2c}{\sigma^2} - 1 > 0$ which leads to $a > 0$ that is satisfied by hypothesis. We further assume that $\frac{2c}{\sigma^2} - 1 > 1$ or equivalently $a > \frac{\sigma^2}{2}$. For notational convenience, we define $v_0 = \frac{2a}{\sigma^2} - 1$ and $v_1 = \frac{2b}{\sigma^2}$ so that as a function of v_0 and v_1, the above function rewrites the following:

$$p(v) = v_1^{v_0+1} \frac{v^{v_0} e^{-v_1 v}}{\Gamma(v_0 + 1)} \tag{6}$$

and we have $\lim_{v\to 0} p(v) = 0$ since $v_0 > 0$ by hypothesis on the parameters.

Denote by $P(t, x_t, v_t)$ the European put (forward) price at time t when the stock log (forward) price is x_t and the volatility is v_t, then it satisfies the partial differential equation (PDE) for $t \in [0, T]$, $x \in \mathbb{R}$, $v \in \mathbb{R}_+$

$$\begin{aligned}\partial_t P + \frac{1}{2}v^2\partial_{xx}^2 P + \rho\sigma v^2 \partial_{xv}^2 P + \frac{1}{2}v^2\sigma^2\partial_{vv}^2 P \\ - \frac{v^2}{2}\partial_x P + (cv - bv^2)\partial_v P = 0\,,\end{aligned} \tag{7}$$

with terminal condition $P(T, x, v) = (K - e^x)_+$. Expressed in terms of the forward price s_t and volatility v_t, the payoff is $P(T, s, v) = (K - s)_+$ with $s \in \mathbb{R}_+$, $v \in \mathbb{R}_+$. The put is often used instead of a call option in the variational formulation because the function $s \to (K - s)_+$ is in $L^2(\mathbb{R}_+)$. In Hilber et al. [14] and Hilber et al. [15, p. 110], to price this product, the authors instead consider the put option price minus its discounted payoff, so that the resulting function decays to zero as the stock value converges to either zero or plus infinity. This property is useful because the problem is approximated by a function that satisfies Dirichlet conditions. This transformation introduces into the PDE a source term given by the second derivative of the payoff, which contains a Dirac function. As explained in these works, the advantage of the finite element method is that it can accommodate such an irregular function.

2.1 | Decomposing the Payoff of a Put With a Portfolio of Butterflies

We propose an alternative to deal with the fact that the put payoff is not zero on one of its boundary (i.e., $P(T, 0, v) = K$). Define $\varsigma = \frac{K}{m}$ with $m \in \mathbb{N}_+$ and K a given strike (of a put or call), then define $K_i = i\varsigma$ $i = 0, \dots, m$. Define $Cs(t, T, K_i)$ a butterfly with strikes K_{i-1}, K_i, and K_{i+1} (for $i = 1, \dots, m-1$), it consists in a long call with strike K_{i-1}, two short calls with strike K_i, and a long call with strike K_{i+1}. The maturity of the butterfly matches the maturity of the put (or call) one wishes to price. Consider the following sum of butterflies: $\sum_{j=1}^{m-1}\sum_{i=1}^{j} Cs(t, T, K_i) = \sum_{i=1}^{m-1} Cs(t, T, K_i)(m - i)$. It can be seen that it almost replicates the put payoff. The difference is given by the set of options $P(t, T, K = \varsigma) + (m-1)C_1(t, T)$ where $P(t, T, K = \varsigma)$ is a put option with strike $K = \varsigma$ and $C_1(t, T)$ is an option with payoff $s \to (s)_+ \mathbf{1}_{\{s<\varsigma\}}$ (with the function $\mathbf{1}_{\{s<\varsigma\}} = 1$ if $s < \varsigma$ and 0 otherwise). For a sufficiently large m and a given stock value s_t, the put option $P(t, T, K = \varsigma)$ is deep out of the money and therefore its price is small. Similarly, the option $C_1(t, T)$ is deep out of the money, so its price is also small. This implies that it is sufficient to price the butterflies $Cs(t, T, K_i)$ $i = 1, \dots, m-1$. Taking into account the fact that a call option payoff is a homogeneous function of degree one with respect to the stock and the strike, the model will generate option prices that also have this property, so without loss of generality, we can assume $s_0 = 1$ where s_0 stands for the initial value of the stock.[2] The butterfly payoff has a compact support along the x axis (where x is the logarithm of the stock forward price), while it is integrable along the v axis if the payoff of the butterfly is multiplied by a function that only depends on v, is integrable, and converges to 0 when $v \to 0$ and $v \to +\infty$, so we can approximate the butterfly option price with a function that satisfies a Dirichlet condition. Overall, it achieves the same purpose as the tansformation in [14] without introducing an irregular function into the PDE.

The decomposition can be better understood by considering a graphical representation. Suppose that we wish to decompose a put option with strike $K = 4$, using butterflies with support either equal to [0, 2] (butterfly centered on 1 or with strike 1), [1, 3] (butterfly centered on 2 or with strike 2) and [2, 4] (butterfly centered on 3 or with strike 3), so that the interval [0, 4] is divided by $m = 4$. Then the portfolio of butterflies is illustrated in Figures 1–4.

Note that when pricing a put option with a given strike of K, the underlying stock value is often around K (in fact, in practice, it is the opposite, options are introduced to the market with strikes around the spot value, i.e., typically ±20% of the spot value, with this percentage increasing with the maturity of the options). Thus, in the above example, the spot value will be around 4 and therefore the difference between the put option and its decomposition by a portfolio of butterflies will consist of options that are

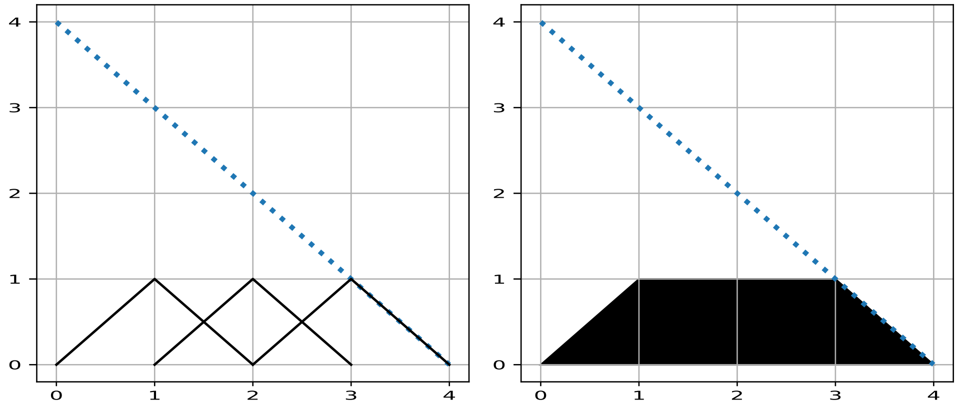

FIGURE 1 | The left figure displays the payoff of a put option with strike $K = 4$ in dotted blue along with three butterflies in black with strikes 1–3. On the right figure, the black-shaded surface shows the corresponding part of the put payoff covered by the combined butterflies. [Colour figure can be viewed at wileyonlinelibrary.com]

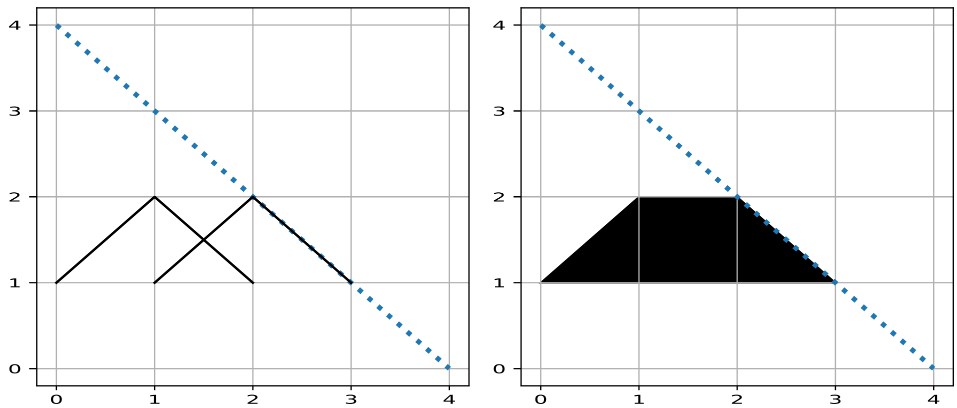

FIGURE 2 | The left figure displays the payoff of a put option with strike $K = 4$ in dotted blue along with two butterflies in black with strikes 1 and 2 (they are shifted up by 1 unit). On the right figure, the black-shaded surface shows the corresponding part of the put payoff covered by the combined butterflies (shifted up by 1 unit). [Colour figure can be viewed at wileyonlinelibrary.com]

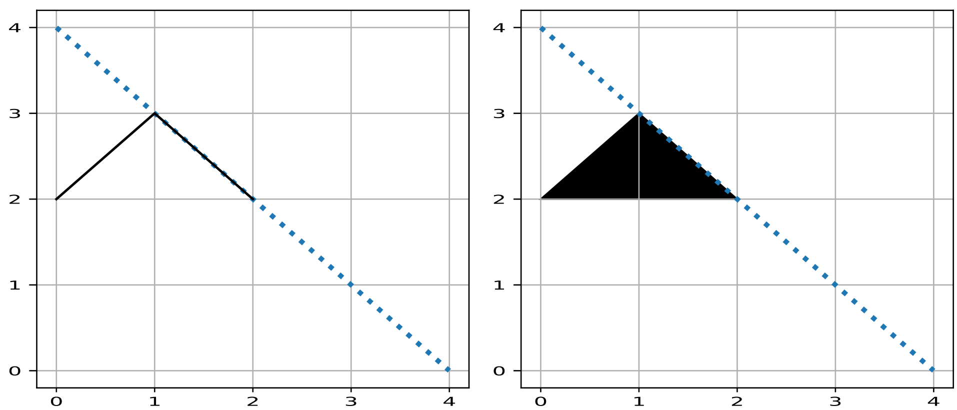

FIGURE 3 | The left figure displays the payoff of a put option with strike $K = 4$ in dotted blue along with one butterfly in black with strike 1 (shifted up by 2 units). On the right figure, the black-shaded surface shows the corresponding part of the put payoff covered by the butterfly (shifted up by 2 units). [Colour figure can be viewed at wileyonlinelibrary.com]

all out of the money. As m increases, these options become deep out of the money and therefore their price must converge to zero.

Note that the transformation is specific to the European option pricing problem. From now on, $P(t, x, v)$ will stand for the butterfly option price at time $t < T$ (T is the maturity), with strike price K when the log stock forward price is x (or the stock forward price is $s = e^x$) and the volatility is v. It is known that $P(t, x, v)$ satisfies the pricing PDE (7) with the appropriate terminal condition.

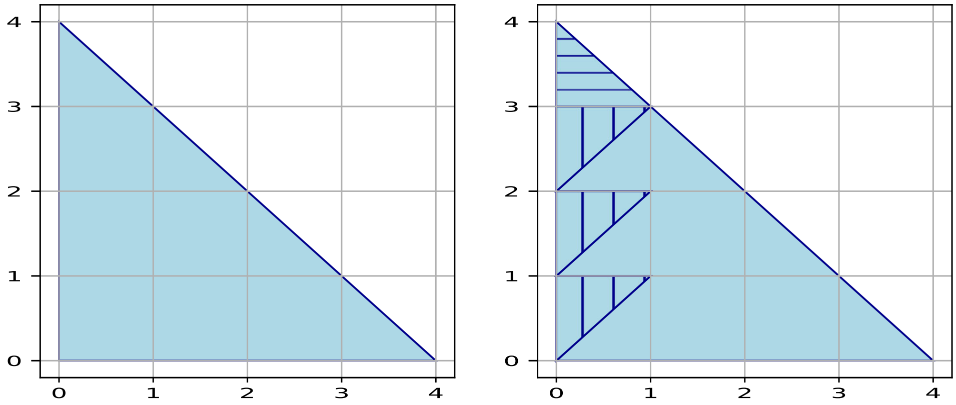

FIGURE 4 | The left figure displays the payoff of a put option with strike $K = 4$, while the right figure displays the surface covered by the portfolio of butterflies (in plain blue) that is the sum of the butterflies of Figures 1–3, while the uncovered part that is composed of three identical payoffs of the type $s \to (s)_+ \mathbf{1}_{\{s<1\}}$ (with the function $\mathbf{1}_{\{s<1\}} = 1$ if $s < 1$ and 0 otherwise) in blue with vertical lines and the payoff of a put option with strike $K = 1$ in blue with horizontal lines. [Colour figure can be viewed at wileyonlinelibrary.com]

Remark 2.1. Regarding the existence and uniqueness of the system of SDEs as well as a probabilistic approach of the European option price and its Feynman–Kac PDE representation, these questions are addressed in the paper Sousa et al. [11]. In this paper, the authors show for the 2-hypergeometric stochastic volatility model that, although the standard assumptions are not satisfied, the domain is unbounded (Baldi [16, Theorem 10.3, p. 315] and Baldi [16, Theorem 10.4, p. 315]) the coefficients do not satisfy the linear growth conditions (Baldi [16, Assumption A, p. 260]); there exists a unique strong solution to the SDEs, and the option price satisfies the PDE given by the Feynman–Kac representation (with the standard regularity conditions).[3] To this end, the authors follow Rubio [17], who considers the Cauchy-Dirichlet problem for a class of linear parabolic differential equations with unbounded coefficients in an unbounded domain and shows that the results of that paper still hold under weaker conditions, which are satisfied by the 2-hypergeometric stochastic volatility model. Their verification, whose details appear in Sousa [18], relies crucially on many other results and on Da Fonseca and Martini [4, Proposition 6], which proves the martingality of the forward price in this model. Still in Da Fonseca and Martini [4, Proposition 6], it is proved that in the 1-hypergeometric stochastic volatility model, the forward price is a martingale if (and only if) $b \geq \rho\sigma$. If $\rho > 0$, this means that the mean-reverting parameter must be "strong" enough relative to the volatility of the volatility parameter σ. If we assume that $\rho = 0$, then the forward price is a martingale and the conclusions of Sousa et al. [11] still hold for the 1-hypergeometric stochastic volatility, thus providing the existence and uniqueness of the SDEs and the Feynman–Kac representation of the option price.

In the current work, we follow the point of view of Achdou and Tchou [8] and Achdou et al. [7] and develop of variational approach of the pricing problem.

3 | Variational Formulation and Analysis

Given an option price function $(t, x, v) \to P(t, x, v)$ and following the above discussion when implementing the model, it will be a butterfly option; define the scaled option price by the following:

$$f(t, x, v) = P(T - t, x, v) v^{\eta\nu_0} e^{-\eta\nu_1 v} \tag{8}$$

with η such that $0 < \eta < 1$. Note that the scaling function is the power of the density function (6) (up to a multiplicative constant). Then $P(t, x, v) = f(T - t, x, v) v^{-\eta\nu_0} e^{\eta\nu_1 v}$ and if P satisfies (7), then f satisfies the PDE

$$\partial_t f - \mathcal{A} f = 0 \tag{9}$$

with

$$\begin{aligned} \mathcal{A} = {} & \frac{1}{2} v^2 \partial^2_{xx} + \rho\sigma v^2 \partial^2_{xv} + \frac{1}{2} v^2 \sigma^2 \partial^2_{vv} \\ & - \left(\alpha_1 v + \alpha_2 v^2\right) \partial_x - \left(\gamma_1 v + \gamma_2 v^2\right) \partial_v \\ & - \mu_0 - \mu_1 v - \mu_2 v^2, \end{aligned} \tag{10}$$

with the constants $\alpha_1 = \rho\sigma\eta\nu_0$, $\alpha_2 = -\rho\sigma\eta\nu_1 + 1/2$, $\gamma_1 = \sigma^2\eta\nu_0 - c$, $\gamma_2 = b - \sigma^2\eta\nu_1 = b(1 - 2\eta)$, $\mu_0 = -c\eta\nu_0 + \frac{1}{2}\sigma^2\nu_0\eta(\nu_0\eta + 1)$, $\mu_1 = \sigma^2\eta^2\nu_0\nu_1 - c\eta\nu_1 - b\eta\nu_0$ and $\mu_2 = -\frac{1}{2}\sigma^2\eta^2\nu_1^2 + b\eta\nu_1$ and the initial condition $f(0, x, v) = P(T, x, v) v^{\eta\nu_0} e^{-\eta\nu_1 v}$.

Let us denote $Q = \mathbb{R} \times \mathbb{R}_+$ and let $H = L^2(Q)$ the set of functions $f : Q \to \mathbb{R}$ such that

$$\|f\|^2_H := \int_Q f^2 < +\infty \tag{11}$$

Following [15, Eqs.(9.23)-(9.24)], we define the following norm:

$$\|f\|_V = \left(\int_Q (1 + v^2) f^2 + v^2 f_x^2 + v^2 f_v^2 \right)^{\frac{1}{2}} \tag{12}$$

where $f_x = \frac{\partial f}{\partial x}$ and $f_v = \frac{\partial f}{\partial v}$ are defined in the sense of distributions and the weighted Sobolev space V as the closure of $\mathcal{D}(Q)$, the space of smooth and compactly supported real valued functions (i.e., $C_0^\infty(Q)$), with respect to that norm, that is, $V := \overline{\mathcal{D}(Q)}^{\|\cdot\|_V}$. In (11) and (12), the integrals and weak derivatives are understood with respect to the Lebesgue measure. This space with the norm is a Hilbert space.[4]

Denote the domain of the operator $\mathcal{A}$ by $D = \{f \in V : \mathcal{A} f \in H\}$. Since $\mathcal{A}$ given by (10) can be decomposed into $\mathcal{A} = \mathcal{A}_1 +$

$\mathcal{A}_2$ with

$$\mathcal{A}_1 = \frac{1}{2}v^2\partial_{xx}^2 + \rho\sigma v^2\partial_{xv}^2 + \frac{1}{2}v^2\sigma^2\partial_{vv}^2 - \alpha_2 v^2\partial_x - \gamma_2 v^2\partial_v - \mu_2 v^2 \tag{13}$$

$$\mathcal{A}_2 = -\alpha_1 v\partial_x - \gamma_1 v\partial_v - \mu_0 - \mu_1 v \tag{14}$$

and since $\mathcal{A}_2$ is a bounded operator from V to H as we have that there exists $c_1 > 0$ such that

$$\|\mathcal{A}_2 f\|_H^2 \le c_1 \int_Q (vf_x)^2 + (vf_v)^2 + (1+v^2)f^2 = c_1\|f\|_V^2,$$

we conclude that $D = \{f \in V : \mathcal{A}_1 f \in H\}$.

Remark 3.1. In [8], they scale the option price using the Gaussian density; see Achdou and Tchou [8, Eq. (13)], and this is consistent with the fact that they are analyzing the Stein and Stein model [19], whose volatility dynamic is Gaussian (it is an Ornstein–Uhlenbeck process). Regarding the weights involved in the norm of their weighted Sobolev space, they are consistent with the operator involved in the option pricing formula; see Achdou and Tchou [8, Eqs. (18), (15)]. In [7], the authors perform the same choice since it is the same model; see Achdou et al. [7, Eqs. (9), (14) and (11)]. A Gaussian distribution is also used to scale the option price in Canale et al. [20]; see Canale et al. [20, Eq. (1.5)], but for the weights involved in the norm of their weighted Sobolev space, they are consistent with the operator used in the option pricing which is associated to the Heston model; see Canale et al. [20, Eq. (1.7)] for the operator and the first unnumbered equation above the Canale et al. [20, Eq. (2.9)]. In Hilber et al. [15], the option price is scaled by a Gaussian density function; see Hilber et al. [15, Eq. (9.17)], while the weights involved in the norm of their weighted Sobolev space are consistent with the operator of the pricing formula; see Hilber et al. [15, Eqs. (9.24), (9.21)]. In Lamberton and Terenzi [21], the authors also provide a variational analysis for (American) option prices using a weighted Sobolev space. In this paper, the function that scales the option price is a gamma density (for the volatility component of the scaling function), and this is consistent with the fact that in the Heston model [2], which is the model used in [21], the stock volatility is a square root stochastic process whose asymptotic distribution is a gamma distribution. Still in [21], for the weight involved in the norm of their weighted Sobolev space, it is consistent with the operator involved in the option pricing, that is, it is consistent with the Heston model. These choices explain our scaling function in (8), which is consistent with (6), and the choice of the norm (12), which is consistent with (10).[5]

We embed the problem in functions with complex values. The notation H (i.e., $L^2(Q)$) is still used for complex-valued functions whose modulus is square integrable and the scalar product of complex-valued functions in H is the sesquilinear form defined by $\langle f, g\rangle = \int_Q f\overline{g}$. The modulus of a complex number $u \in \mathbb{C}$ is denoted $|u|$, its real part $\mathfrak{R}(u)$, its imaginary part $\mathfrak{I}(u)$, and $\bar{u}$ its conjugate.

Let $a(.,.)$ be defined on $V \times V$ by the following:

$$a(f,g) = \langle -\mathcal{A}f, g\rangle \tag{15}$$

with

$$\begin{aligned} a(f,g) = &\int_Q \frac{1}{2}v^2 f_x\overline{g}_x + \int_Q \frac{\rho\sigma}{2}v^2 f_v\overline{g}_x \\ &+ \int_Q \frac{\rho\sigma}{2}v^2 f_x\overline{g}_v + \int_Q \frac{1}{2}\sigma^2 v^2 f_v\overline{g}_v \\ &+ \int_Q (\tilde{\alpha}_1 v + \alpha_2 v^2) f_x\overline{g} + \int_Q (\tilde{\gamma}_1 v + \gamma_2 v^2) f_v\overline{g} \\ &+ \int_Q (\mu_0 + \mu_1 v + \mu_2 v^2) f\overline{g}, \end{aligned} \tag{16}$$

with the constants $\tilde{\alpha}_1 = \alpha_1 + \rho\sigma$, $\tilde{\gamma}_1 = \gamma_1 + \sigma^2 = (\eta - 1/2)(2a - \sigma^2)$.

Proposition 3.2. *The operator $\mathcal{A}$ is a bounded operator from V to V'.*

Proof. With $\langle -\mathcal{A}f, g\rangle = a(f,g)$ for $f, g \in V$ and the expression for $a(.,.)$ given by (16), using Cauchy–Schwarz's inequality, we get that there exists a constant c_1 such that $|a(f,g)| \le c_1\|f\|_V\|g\|_V$. □

Remark 3.3. We will use repeatedly the following inequality:

$$\mathfrak{R}(a_1\overline{b}_1) \le \frac{\zeta}{2}|a_1|^2 + \frac{1}{2\zeta}|b_1|^2, \text{ for } a_1, b_1 \in \mathbb{C},\ \zeta > 0 \tag{17}$$

Proposition 3.4. *The Gårding inequality holds; that is, there exist two positive constants C_M and c_m and $\eta \in]0\ ,1[$ such that for any $f \in V$,*

$$\mathfrak{R}(a(f,f)) \ge C_M\|f\|_V^2 - c_m\|f\|_H^2 \tag{18}$$

Proof. We proceed as in Achdou and Tchou [8, Proposition 2], Achdou et al. [7, Proposition 2] or Hilber et al. [15, p. 111, Theorem 9.3.1], that is, by density and considering a function $f \in \mathcal{D}(Q)$. As $\langle -\mathcal{A}f, f\rangle = a(f,f)$, the sum of the first four integrals in the right-hand side of the equality (16), which we denote I_1, leads, after using (17), to

$$\mathfrak{R}(I_1) \ge \int_Q \frac{1}{2}(1-|\rho|)v^2|f_x|^2 + \int_Q \frac{1}{2}\sigma^2 v^2(1-|\rho|)|f_v|^2 \tag{19}$$

The fifth integral in the right-hand side of equality (16), which we denote I_2, leads, after an integration by part, to

$$\mathfrak{R}(I_2) = \mathfrak{R}\left(\int_Q (\tilde{\alpha}_1 v + \alpha_2 v^2)\partial_x f\overline{f}\right) = \int_Q (\tilde{\alpha}_1 v + \alpha_2 v^2)\frac{1}{2}\frac{\partial|f|^2}{\partial x} = 0 \tag{20}$$

since $\tilde{\alpha}_1$ and α_2 are constant and $f \in \mathcal{D}(Q)$.

The sixth integral in the right-hand side of equality (16), which we denote I_3, leads, after an integration by part, to

$$\begin{aligned} \mathfrak{R}(I_3) &= \mathfrak{R}\left(\int_Q (\tilde{\gamma}_1 v + \gamma_2 v^2)\partial_v f\overline{f}\right) \\ &= \int_Q (\tilde{\gamma}_1 v + \gamma_2 v^2)\frac{1}{2}\frac{\partial|f|^2}{\partial v} = \int_Q (-\tilde{\gamma}_1/2 - \gamma_2 v)|f|^2, \end{aligned} \tag{21}$$

since $\tilde{\gamma}_1$ and γ_2 are constant and $f \in \mathcal{D}(Q)$.

Collecting these results, we get the following:

$$\begin{aligned}\Re(a(f,f)) \geq & \int_Q \frac{(1-|\rho|)}{2} v^2 |f_x|^2 \\ & + \int_Q \frac{\sigma^2(1-|\rho|)}{2} v^2 |f_v|^2 \\ & + \int_Q (\beta_0 + \beta_1 v + \beta_2 v^2)|f|^2 ,\end{aligned} \tag{22}$$

with $\beta_0 = \mu_0 - \tilde{\gamma}_1/2$, $\beta_1 = \mu_1 - \gamma_2$, $\beta_2 = \mu_2$. The function $\eta \to \beta_2(\eta)$ is an inverse parabola, and the two roots are $\eta = 0$ and $\eta = 1$; therefore, $\forall \eta \in]0, 1[$; we have $\beta_2(\eta) > 0$ and the maximum is reached for $\eta = 1/2$. From now on, we suppose that $\eta = 1/2$ so that $\beta_2 = b^2/(2\sigma^2)$, $\beta_1 = -b\left(\frac{a}{\sigma^2} + \frac{1}{2}\right)$ and $\beta_0 = -\frac{8}{\sigma^2}\left(\frac{a^2}{8} - \frac{2a}{\sigma^2} - 1\right)$.

We have the following:

$$\int_Q \beta_1 v|f|^2 \geq -\frac{|\beta_1|\zeta}{2}\int_Q v^2|f|^2 - \frac{|\beta_1|}{2\zeta}\int_Q |f|^2 \quad \text{for } \zeta > 0 \tag{23}$$

and combining (22) and (23), we reach the following:

$$\begin{aligned}\Re(a(f,f)) \geq & \int_Q \frac{1}{2}(1-|\rho|)v^2|f_x|^2 \\ & + \int_Q \frac{1}{2}\sigma^2 v^2(1-|\rho|)|f_v|^2 \\ & + \left(\beta_2 - \frac{|\beta_1|\zeta}{2}\right)\int_Q v^2|f|^2 \\ & - \left(|\beta_0| + \frac{|\beta_1|}{2\zeta}\right)\int_Q |f|^2,\end{aligned} \tag{24}$$

and as $\beta_2 > 0$, we can choose ζ sufficiently small such that $\beta_2 - \frac{|\beta_1|\zeta}{2} = C_M$ with $C_M = \frac{1}{2}\min((1-|\rho|)/2, \sigma^2(1-|\rho|)/2, \beta_2)$, that is, $\zeta = 2(\beta_2 - C_M)/|\beta_1|$. Define $c_m = C_M + |\beta_0| + \frac{|\beta_1|}{2\zeta}$; then we have two positive constants such that (18) is satisfied. □

Remark 3.5. Note that if $\eta = 1/2$, then $\tilde{\gamma}_1 = 0$ and $\gamma_2 = 0$.

Remark 3.6. Note that if $\eta = 1$, then $\beta_2 = 0$ and $\beta_1 = -b$ ($b > 0$ by hypothesis) and the rightmost integral in (22) implies that in (18) the term $\|f\|_H^2$ has to be replaced by $\|\sqrt{1+v^2}f\|_H^2$; see [21] which falls in this case (but for a different model).

Remark 3.7. It is of interest to compare the results available in the literature even if the analyzed models are different. In Achdou and Tchou [8], the authors reach inequality (18) for the Stein and Stein model under the assumption that ρ is not too large but also that the mean-reverting parameter should be large enough compared to the volatility of volatility parameter; see Achdou and Tchou [8, Eqs. (49-50) of Proposition 5.]. In Hilber et al. [15], the authors obtain inequality (18) for the Heston model under similar assumptions, that is, ρ should not be too large and the mean-reverting parameter should be large enough compared to the volatility of volatility parameter; see Hilber et al. [15, Theorem 9.3.1]. The Heston model is also considered in Achdou and Pironneau [24], but under the assumption that ρ is zero, the authors obtain the Gårding inequality if the mean-reverting parameter is sufficient large compared to the volatility of volatility parameter. Interestingly, Lamberton and Terenzi [21] obtain a Gårding inequality for the Heston model without any constraint on the parameters, but as explained in the previous remark, one of the norms involved in the inequality has to be adjusted.

A consequence of Propositions 3.2 and 3.4 is the following result.

Proposition 3.8. *For all $t \in [0, T]$, the resolvent $R(\lambda : \mathcal{A}) = (\lambda I - \mathcal{A})^{-1}$ of $\mathcal{A}$ exists for $\Re(\lambda) \geq c_m$ with c_m defined in Proposition 3.4, and there exists a constant M independent of $t \geq 0$ such that*

$$\|R(\lambda : \mathcal{A})\|_{\mathcal{L}(H)} \leq \frac{M}{|\lambda - c_m| + 1} \tag{25}$$

Proof. The operator $\lambda I - \mathcal{A}$ is bounded from V to V' and if $\Re(\lambda) \geq c_m$, then thanks to (18), the operator satisfies the coercivity condition

$$\Re(\langle(\lambda I - \mathcal{A})f, f\rangle) \geq C_M\|f\|_V^2 + \Re(\lambda - c_m)\|f\|_H^2 \tag{26}$$

so that for $f \in H$, there exits a unique $g \in V$ such that $g = R(\lambda : \mathcal{A})f$ or $(\lambda I - \mathcal{A})g = f$, and therefore,

$$\lambda\|g\|_H^2 - \langle \mathcal{A}g, g\rangle = \langle f, g\rangle \tag{27}$$

and taking the real part of (27), we get the following:

$$(C_M + \Re(\lambda) - c_m)\|g\|_H \leq \|f\|_H \tag{28}$$

and

$$C_M\|g\|_V^2 \leq \|f\|_H\|g\|_H \tag{29}$$

while taking the imaginary part of (27) gives

$$\begin{aligned}|\Im(\lambda)|\|g\|_H^2 = & \left|\Im\left(\int g\overline{f} + \langle \mathcal{A}g, g\rangle\right)\right| \\ \leq & \|f\|_H\|g\|_H + c_1\|g\|_V^2, \\ \leq & (1 + c_1/C_M)\|f\|_H\|g\|_H,\end{aligned} \tag{30}$$

where c_1 appears in the proof of Proposition 3.2 and we use (29). Combining (28) and (30), we get the following:

$$\left(|\Im(\lambda)| + |\Re(\lambda) - c_m|\right)\|g\|_H \leq (2 + c_1/C_M)\|f\|_H \tag{31}$$

and the above equation along with (29) gives the result. □

Lemma 3.9. *The space $\mathcal{D}(Q)$ is dense in D with the graph norm.*

Proof. Consider the function $h_n(x)$ defined for $x \in \mathbb{R}$ such that

- $h_n(x) = 0$ if $|x| > 2n$,
- $h_n(x) = 1$ if $|x| < n$,
- $\|h_n'\|_{L^\infty(\mathbb{R})} \leq \frac{\bar{c}_1}{n}$, $\|h_n''\|_{L^\infty(\mathbb{R})} \leq \frac{\bar{c}_1}{n^2}$

with $\bar{c}_1 > 0$. Consider the function $l_n(v)$ defined for $v \in \mathbb{R}_+$ such that

- $l_n(v) = 0$ if $v \in]0, \frac{1}{n}[\cup]2n, +\infty[$,
- $l_n(v) = 1$ if $v \in]\frac{2}{n}, n[$,
- $\|l'_n\|_{L^\infty(]\frac{1}{n}, \frac{2}{n}[)} \le \bar{c}_2 n$, $\|l'_n\|_{L^\infty(]n, 2n[)} \le \frac{\bar{c}_2}{n}$,
- $\|l''_n\|_{L^\infty(]\frac{1}{n}, \frac{2}{n}[)} \le \bar{c}_2 n^2$, $\|l''_n\|_{L^\infty(]n, 2n[)} \le \frac{\bar{c}_2}{n^2}$,

and let $\varphi_n(x, v) = h_n(x) l_n(v)$. Then $\lim_{n\to+\infty} \|f - f\varphi_n\|_V = 0$ and

$$\begin{aligned}\mathcal{A}_1(f\varphi_n) - \mathcal{A}_1(f) = (\varphi_n - 1)\mathcal{A}_1(f) + \frac{v^2}{2}(2f_x\varphi_{n,x} + f\varphi_{n,xx}) \\ + \rho\sigma^2 v^2(f_x\varphi_{n,v} + f_v\varphi_{n,x} + f\varphi_{n,xv}) \\ + \frac{\sigma^2 v^2}{2}(2f_v\varphi_{n,v} + f\varphi_{n,vv}) \\ - \alpha_2 v^2(f_x\varphi_n + f\varphi_{n,x}) - \gamma_2 v^2(f_v\varphi_n + f\varphi_{n,v}).\end{aligned}$$

As we have that there exists a constant c_1 such that

$$|v^2 f_x\varphi_{n,x}| = |vf_x||v\varphi_{n,x}| \le c_1|vf_x| \left(\frac{v}{n}\mathbf{1}_{[n,\ 2n]}(x)\mathbf{1}_{[1/n,\ 2/n]}(v) + \frac{v}{n}\mathbf{1}_{[n,\ 2n]}(x)\mathbf{1}_{[n,\ 2n]}(v)\right),$$

and since both $\frac{v}{n}\mathbf{1}_{[n,\ 2n]}(x)\mathbf{1}_{[1/n,\ 2/n]}(v)$ and $\frac{v}{n}\mathbf{1}_{[n,\ 2n]}(x)\mathbf{1}_{[n,\ 2n]}(v)$ belong to $L^\infty(Q)$ and converge a.s. to zero when $n \to +\infty$, then thanks to Lebesgue's theorem and the fact that $vf_x \in H$, we get that $v^2 f_x\varphi_{n,x}$ converges to zero in H. Similar argument applies to all the other terms. Note that the term that is likely to cause a problem is $v^2 f\varphi_{n,vv}$ as over the interval $]1/n, 2/n[$ the function $l''_n(v)$ behaves like n^2. But since

$$|v^2 f\varphi_{n,vv}| = |f||v^2\varphi_{n,vv}| \le c_6|f| \left(v^2 n^2\mathbf{1}_{[n,\ 2n]}(x)\mathbf{1}_{[1/n,\ 2/n]}(v) + \frac{v^2}{n^2}\mathbf{1}_{[n,\ 2n]}(x)\mathbf{1}_{[n,\ 2n]}(v)\right),$$

with $v^2n^2\mathbf{1}_{[n,\ 2n]}(x)\mathbf{1}_{[1/n,\ 2/n]}(v)$ and $\frac{v^2}{n^2}\mathbf{1}_{[n,\ 2n]}(x)\mathbf{1}_{[n,\ 2n]}(v)$ belonging to $L^\infty(Q)$ and converging a.s. to zero when $n \to +\infty$, using again Lebesgue's theorem as in the previous case and the fact that $f \in H$, we conclude that $v^2 f\varphi_{n,vv}$ converges to zero in H when $n \to +\infty$. As a result, the functions in D with compact support are dense in D.

Still following [7], we consider a function f with compact support such that $f \in D$, and since it has a compact support, it is sufficient to restrict to $f \in \hat{D}$ with $\hat{D} = \{f : f, v\partial_x f, v\partial_v f, \hat{\mathcal{A}}_1 f \in H\}$ with

$$\hat{\mathcal{A}}_1 f = \frac{1}{2}v^2\partial^2_{xx} + \rho\sigma v^2\partial^2_{xv} + \frac{1}{2}v^2\sigma^2\partial^2_{vv} \tag{32}$$

and define the norm for $\hat{D}$ as $\|f\|^2_{\hat{D}} = \|f\|^2_H + \|v\partial_x f\|^2_H + \|v\partial_v f\|^2_H + \|\hat{\mathcal{A}}_1 f\|^2_H$. We also consider the space $\hat{V} = \{f : f, v\partial_x f, v\partial_v f \in H\}$ with the norm $\|f\|^2_{\hat{V}} = \|f\|^2_H + \|v\partial_x f\|^2_H + \|v\partial_v f\|^2_H$. Let $\varphi(x)$ be a smooth function $\mathbb{R} \to \mathbb{R}_+$ with support in $[-1, 1]$ such that $\int_{\mathbb{R}} \varphi = 1$. The aim is to show that f can be approximated in $\hat{D}$ by a sequence of smooth functions with compact support. Define the function from $\mathbb{R}^2 \to \mathbb{R}_+$ $\varphi_\epsilon(x, v) = \frac{1}{\epsilon^2}\varphi\left(\frac{x}{\epsilon}\right)\varphi\left(\frac{v}{\epsilon}\right)$; its support is in $]-\epsilon, \epsilon[\times]-\epsilon, \epsilon[$. Consider $f_\epsilon = f * \varphi_\epsilon$, which is smooth with compact support, then $\|f - f_\epsilon\|_H \to 0$ if $\epsilon \to 0$. But in fact, there is the stronger result $\|f - f_\epsilon\|_{\hat{V}} \to 0$ if $\epsilon \to 0$ so that $\mathcal{D}(Q)$ is dense in $\hat{V}$. Let us prove that $\lim_{\epsilon\to 0}\|\hat{\mathcal{A}}_1 f - \hat{\mathcal{A}}_1 f_\epsilon\|_H = 0$. Define the integral operator $J_\epsilon f = f * \varphi_\epsilon$; then we have $\|\hat{\mathcal{A}}_1 f - \hat{\mathcal{A}}_1 f_\epsilon\|_H \le \|\hat{\mathcal{A}}_1 f - J_\epsilon\hat{\mathcal{A}}_1 f\|_H + \|\hat{\mathcal{A}}_1 J_\epsilon f - J_\epsilon\hat{\mathcal{A}}_1 f\|_H$, and since $\lim_{\epsilon\to 0}\|\hat{\mathcal{A}}_1 f - J_\epsilon\hat{\mathcal{A}}_1 f\|_H = 0$, we are thus left to prove that $\lim_{\epsilon\to 0}\|\hat{\mathcal{A}}_1 J_\epsilon f - J_\epsilon\hat{\mathcal{A}}_1 f\|_H = 0$. Define the operator $H_\epsilon f = \hat{\mathcal{A}}_1 J_\epsilon f - J_\epsilon\hat{\mathcal{A}}_1 f$, that is,

$$H_\epsilon f(x, v) = \int_Q h_\epsilon(\tilde{x}, \tilde{v}, x, v) f(\tilde{x}, \tilde{v}) d\tilde{x} d\tilde{v} \tag{33}$$

with

$$\begin{aligned}h_\epsilon(\tilde{x}, \tilde{v}, x, v) &= \frac{v^2}{2}\varphi_{\epsilon,xx}(x - \tilde{x}, v - \tilde{v}) - \frac{\tilde{v}^2}{2}(\varphi_\epsilon(x - \tilde{x}, v - \tilde{v}))_{\tilde{x}\tilde{x}} \\ &+ \rho\sigma\left(v^2\varphi_{\epsilon,xv}(x - \tilde{x}, v - \tilde{v}) - (\tilde{v}^2\varphi_\epsilon(x - \tilde{x}, v - \tilde{v}))_{\tilde{x}\tilde{v}}\right) \\ &+ \frac{\sigma^2}{2}\left(v^2\varphi_{\epsilon,vv}(x - \tilde{x}, v - \tilde{v}) - (\tilde{v}^2\varphi_\epsilon(x - \tilde{x}, v - \tilde{v}))_{\tilde{v}\tilde{v}}\right) \\ &= h^1_\epsilon(\tilde{x}, \tilde{v}, x, v) + h^2_\epsilon(\tilde{x}, \tilde{v}, x, v) + h^3_\epsilon(\tilde{x}, \tilde{v}, x, v),\end{aligned}$$

and inserting the above equation into (33), we define the corresponding $H^1_\epsilon f(x, v)$, $H^3_\epsilon f(x, v)$, and $H^3_\epsilon f(x, v)$. Note that $h_\epsilon(\tilde{x}, \tilde{v}, x, v) = 0$ if $\max(|x - \tilde{x}|, |v - \tilde{v}|) > \epsilon$ thanks to the fact that the support of φ_ϵ is in $]-\epsilon, \epsilon[\times]-\epsilon, \epsilon[$. Also, $H_\epsilon 1 = 0$ since $\hat{\mathcal{A}}_1 J_\epsilon 1 = 0$ and $J_\epsilon\hat{\mathcal{A}}_1 1 = 0$. Since we have $(\varphi_\epsilon(x - \tilde{x}, v - \tilde{v}))_{\tilde{x}\tilde{x}} = \varphi_{\epsilon,xx}(x - \tilde{x}, v - \tilde{v})$, we get the following:

$$\begin{aligned}H^1_\epsilon f(x, v) &= \int_Q \frac{1}{2} f(\tilde{x}, \tilde{v})\left(v^2 - \tilde{v}^2\right)\varphi_{\epsilon,xx}(x - \tilde{x}, v - \tilde{v}) d\tilde{x} d\tilde{v} \\ &= \int_Q \frac{1}{2} f(\tilde{x}, \tilde{v})(v - \tilde{v})^2\varphi_{\epsilon,xx}(x - \tilde{x}, v - \tilde{v}) d\tilde{x} d\tilde{v} \\ &+ \int_Q \tilde{v} f(\tilde{x}, \tilde{v})(v - \tilde{v})\varphi_{\epsilon,xx}(x - \tilde{x}, v - \tilde{v}) d\tilde{x} d\tilde{v},\end{aligned} \tag{34}$$

and Young's convolution inequality, see Hörmander [25, Eq. (4.5.5)], implies that $\|H^1_\epsilon f\|_H \le \|f\|_H\|v^2\varphi_{\epsilon,xx}\|_{L^1(\mathbb{R}_2)} + \|vf\|_H\|v\varphi_{\epsilon,xx}\|_{L^1(\mathbb{R}_2)}$. Further to this, we have the following:

$$\begin{aligned}H^2_\epsilon f(x, v) &= \int_Q \sigma\rho f(\tilde{x}, \tilde{v})(v^2 - \tilde{v}^2)\varphi_{\epsilon,xv}(x - \tilde{x}, v - \tilde{v}) d\tilde{x} d\tilde{v} \\ &+ \int_Q 2\sigma\rho\tilde{v} f(\tilde{x}, \tilde{v})\varphi_{\epsilon,x}(x - \tilde{x}, v - \tilde{v}) d\tilde{x} d\tilde{v},\end{aligned} \tag{35}$$

and Young's convolution inequality, see Hörmander [25, Eq. (4.5.5)], implies that there exists a constant c_1 such that $\|H^2_\epsilon f\|_H \le c_1(\|f\|_H\|v^2\varphi_{\epsilon,xv}\|_{L^1(\mathbb{R}_2)} + \|vf\|_H\|v\varphi_{\epsilon,xv}\|_{L^1(\mathbb{R}_2)} + \|vf\|_H\|\varphi_{\epsilon,x}\|_{L^1(\mathbb{R}_2)})$, we use the fact that the first integral in (35) has the same structure as the integral in (34). Lastly, we have the following:

$$\begin{aligned}H^3_\epsilon f(x, v) &= \int_Q \frac{\sigma^2}{2} f(\tilde{x}, \tilde{v})(v^2 - \tilde{v}^2)\varphi_{\epsilon,vv}(x - \tilde{x}, v - \tilde{v}) d\tilde{x} d\tilde{v} \\ &+ \int_Q \sigma^2\tilde{v} f(\tilde{x}, \tilde{v})\varphi_{\epsilon,v}(x - \tilde{x}, v - \tilde{v}) d\tilde{x} d\tilde{v} \\ &- \int_Q 2\sigma^2 f(\tilde{x}, \tilde{v})\varphi_\epsilon(x - \tilde{x}, v - \tilde{v}) d\tilde{x} d\tilde{v},\end{aligned}$$

and using similar arguments, we get that there exists a constant c_1 such that $\|H_\epsilon^3 f\|_H \le c_1(\|f\|_H \|v^2\varphi_{\epsilon,vv}\|_{L^1(\mathbb{R}_2)} + \|vf\|_H\|v\varphi_{\epsilon,vv}\|_{L^1(\mathbb{R}_2)} + \|vf\|_H\|\varphi_{\epsilon,v}\|_{L^1(\mathbb{R}_2)} + \|f\|_H\|\varphi_\epsilon\|_{L^1(\mathbb{R}_2)})$.
Combining these bounds, we conclude that there exists a constant c_1 such that $\|H_\epsilon f\|_H \le c_1\|f\|_{\hat{V}}$ for $f \in \hat{D}$ with compact support, so we have a continuity property. To prove that $\lim_{\epsilon\to 0}\|H_\epsilon f\|_H = 0$, it is sufficient to establish that relation for $f \in \mathcal{D}(Q)$ thanks to the continuity property. Using $H_\epsilon 1 = 0$, we get $H_\epsilon f = \int_Q h_\epsilon(\tilde{x},\tilde{v},x,v)(f(\tilde{x},\tilde{v}) - f(x,v))d\tilde{x}d\tilde{v}$ and conclude that $\lim_{\epsilon\to 0}\|H_\epsilon f\|_H = 0$ derives from the continuity of f. It concludes the proof. □

Remark 3.10. The cutoff functions defined above are standard and defined in Hörmander [25, Section 1.4] while for the L^∞ norm of these functions (and their derivatives), see Hörmander [25, Eq. (1.4.2)]

Remark 3.11. The above lemma implies that $\mathcal{D}(Q)$ is a core for the extension of $\mathcal{A}$; see Engel and Nagel [26, Definition 1.6, page 39].

Remark 3.12. It is possible to prove a regularity result similar to Achdou et al. [7, Theorem 1] or Achdou and Pironneau [24, Theorem 2.9]. The proof is essentially the same as the one presented in Achdou et al. [7, Theorem 1]. In the notations of our work, this requires assuming that $\eta = 1/2$, $\rho = 0$, and $b > \sigma/\sqrt{2}$. Under these assumptions, it is possible to prove that $D = \left\{f \in V : v^2 f_{xx}, v^2 f_{xv}, v^2 f_{vv}, v^2 f_x, v^2 f_v, v^2 f \in H\right\}$. Note that these parameter constraints are similar to those in Achdou et al. [7, Theorem 1] or Achdou and Pironneau [24, Theorem 2.9]. However, the fact that $\rho = 0$ is somewhat problematic since, in practice, this parameter is negative and large in absolute value.[6]

Those results being established, the European option pricing problem can be solved. A first consequence of Proposition 3.4 is the next result.

Theorem 3.13. *For any $f_0 \in H$, there exists a unique $f \in L^2(0,T;V) \cap C^0([0,\ T];H)$ with $\frac{\partial f}{\partial t} \in L^2(0,T;V')$ such that, in the sense of distributions in time, the following equation holds:*

$$\frac{d}{dt}(f,g) - \langle \mathcal{A}f, g\rangle = 0 \ \forall g \in V \tag{36}$$

and with initial condition $f(t=0) = f_0$.

The mapping $f_0 \to f$ is continuous from H to $L^2(0,T;V) \cap C^0([0,\ T];H)$.

Theorem 3.14. *Theorem 3.8 and Lemma 3.9 imply that $\mathcal{A}$ is a quasi-contractive analytic continuous semigroup.*

Proof. The proof is the one of [7] and reads as follows. The domain of $\mathcal{A}$ is dense in H; the resolvent $R(\lambda : \mathcal{A})$ of $\mathcal{A}$ exists for $\Re(\lambda) \ge c_m$ (where c_m is defined in Proposition 3.4) and we have (25) then Pazy [27, Chap. 1, Corollary 3.8] or Engel and Nagel [26, Chap. 2, condition (c) of Corollary 3.6] can be applied and gives the result. □

Remark 3.15. One can also consider Canale et al. [20, Theorem 3.3] that reads as follows. Define the following norm:

$$\|f\|_a := \sqrt{\Re(a(f,f)) + c_m\|f\|_H^2} \tag{37}$$

with $a(.,.)$ given by (16) and c_m given by (18). Thanks to Propositions 3.2 and 3.4, this norm is equivalent to norm $\|.\|_V$. The sesquilinear form $a(.,.)$ with domain V is closed. Following Ouhabaz [28, Definition 1.21, page 13], these authors define the operator, denoted A, associated with $a(.,.)$ as follows:

$$\tilde{D}(A) = \{f \in V \text{ such that } \exists\, g \in H : a(f,\varphi) = \langle g,\varphi\rangle \ \forall\, \varphi \in \mathcal{D}(Q)\}$$
$$-Af = g.$$

Since $a(.,.)$ with domain V is densely defined, closed, continuous, and quasi-accretive form on H, then Canale et al. [20, Theorem 3.3] implies that A is a quasi-contractive analytic semigroup on H thanks to Ouhabaz [28, Theorem 1.52].

As mentioned in [7], the pricing problem is specified on an unbounded domain, while the parabolic problem (36) is usually formulated on a bounded domain. It needs to be localized, and this is done in Achdou and Tchou [8] for the Stein and Stein model or Hilber et al. [14, Section 3.4] and Hilber et al. [15, Section 9.4] for the Heston model. For completeness, we report below the corresponding result for the 1-hypergeometric stochastic volatility model.

Consider the cutoff function φ_n of (3.9) and define $Q_n =]-2n, 2n[\times]1/n, 2n[$ and V_n the Hilbert space that is the closure of $\mathcal{D}(Q_n)$ with respect to the norm $\|.\|_V$, that is, $V_n := \overline{\mathcal{D}(Q_n)}^{\|.\|_V}$ then the bounds in Propositions 3.2 and 3.4 hold in V_n with the same constants, and we have that there exists a unique $f_n \in L^2(0,T;V_n) \cap C^0([0,\ T];L^2(Q_n))$ with $\frac{\partial f}{\partial t} \in L^2(0,T;V_n')$ such that in the sense of distributions, the following equation holds:

$$\partial_t f_n - \mathcal{A}f_n = 0 \tag{38}$$

and

$$f_n(t=0) = f_0.$$

We extend f_n to Q and assume that $f_n = 0$ on $Q \setminus Q_n$ (we still denote f_n the extension). Using (36) and (38) together with the coercive conditions, we get that there exists $c_1 > 0$ such that

$$\|f(t)\|_H^2 \ + \int_0^t \|f(s)\|_V^2 ds \le c_1\|f_0\|_H^2,$$
$$\|f_n(t)\|_H^2 \ + \int_0^t \|f_n(s)\|_V^2 ds \le c_1\|f_0\|_H^2.$$

Define $e_n = f_n - f$ with f_n and f as above. Following Hilber et al. [14, Theorem 3.6] or Hilber et al. [15, Theorem 9.4.1], the next proposition holds.

Proposition 3.16. *Let φ_n be the cut-off function of Lemma 3.9. Then there exists a sequence $(\overline{c}_n)_{n\ge 0}$ such that $\lim_{n\to+\infty}\overline{c}_n = 0$ and*

$$\|\varphi_n e_n(t)\|_H^2 + \int_0^t \|\varphi_n e_n(s)\|_V^2 ds \le \overline{c}_n \tag{39}$$

Proof. From (36) and (38), we get the following:

$$\frac{d}{dt}(e_n(t), g) + a(e_n(t), g) = 0\ , \quad \forall g \in V\ ,$$

and by considering $g = \varphi_n^2 e_n(t)$ in the above question, we reach the following:

$$\frac{1}{2}\frac{d}{dt}\|\varphi_n e_n(t)\|_H + a(\varphi_n e_n(t), \varphi_n e_n(t)) = h_n(t) \tag{40}$$

where $h_n(t) = a(\varphi_n e_n(t), \varphi_n e_n(t)) - a(e_n(t), \varphi_n^2 e_n(t))$ and taking into account (16) and the fact that $\tilde{\gamma}_1 = 0$ and $\gamma_2 = 0$ for $\eta = 1/2$ (*i.e.*, Remark 3.5), it leads to the following:

$$h_n(t) = \int_Q \frac{v^2}{2}\varphi_{n,x}^2 e_n^2(t) + \rho\sigma v^2 \varphi_{n,x}\varphi_{n,v} e_n^2(t) + \frac{\sigma^2 v^2}{2}\varphi_{n,v}^2 e_n^2(t)$$
$$+ \int_Q (\tilde{\alpha}_1 v + \alpha_2 v^2)\varphi_{n,x}\varphi_n e_n^2(t)\ ,$$

so

$$|h_n(t)| \le c_1 \int_Q v^2(|\varphi_{n,x}| + |\varphi_{n,v}|)^2 e_n^2(t) + c_1 \int_Q (v + v^2)|\varphi_{n,x}|\varphi_n e_n^2(t)\ ,$$

with a constant c_1 that does not depend on n. Using the properties of φ_n, we reach

$$|h_n(t)| \le c_1 \int_{Q_n \backslash]-n,n[\times]2/n,n[} v^2 e_n^2(t)\ ,$$

for another constant c_1 that does not depend on n. As $f_n(t)$ and $f(t)$ belong to V, then $ve_n(t)$ is bounded in H and $\mathbf{1}_{\{Q_n \backslash]-n,n[\times]2/n;n[\}} v^2 e_n^2(t) \to 0$ a.s. so $\mathbf{1}_{\{Q_n \backslash]-n,n[\times]2/n,n[\}} ve_n(t) \to 0$ in H. It implies that there is a sequence $(\overline{c}_n)_{n\ge 0}$ such that $\lim_{n\to+\infty} \overline{c}_n = 0$ and such that $|h_n(t)| \le \overline{c}_n$. Then using (40) along with the coercivity of $a(.,.)$ (after being regularized), we get the announced result. □

4 | Numerical Results

We demonstrate the evaluation of a European put option $P(t, T, K)$ with the following model parameters:

$$s_0 = 1, v_0 = 0.3, r = 0, a = 0.6, b = 2.0, \rho = -0.5, \sigma = 0.2\,.$$

The first part is to approximate the butterfly options $P(t, x, v)$ for various strike values of K_i with $K_i = i\frac{K}{m}$ and K a given strike price, $m = 20$ and $1 \le i \le 19$. The domain of (x, v) is specified as $[\ln(s_{\min}), \ln(s_{\max})] \times [v_{\min}, v_{\max}]$ with $s_{\min} = 0.05$, $s_{\max} = 4.0$, $v_{\min} = 0.0$, $v_{\max} = 5$. We use a regular mesh on the domain, with 180 nodes in space (for both the stock and the volatility). We split the time interval [0, 1] with 100 points (regularly spaced). We use artificial Dirichlet boundary conditions on $x = \ln(s_{\min})$, $x = \ln(s_{\max})$, $v = v_{\min}$ and $v = v_{\max}$. These boundary conditions, which are not exactly satisfied by butterfly options $P(t, x, v)$, may induce very small errors. We find the solutions of these butterfly options utilizing the free software FreeFem++. For an implementation of the Heston model in FreeFem++, see Pironneau [29]. Once the butterfly options are priced, we recover the put option prices for different maturities and different strike prices using the decomposition of the put price as a sum of butterflies mentioned in Section 2 and report the values in Figure 5 where we can check that for a given maturity and initial stock value, the put option price increases with the strike price K, while for a given strike price and initial stock value, the option price increases with the maturity. In terms of implementation, if we consider Figures 1–3, in the sum of butterflies $\sum_{j=1}^{m-1}\sum_{i=1}^{j} Cs(t, T, K_i)$, with $Cs(t, T, K_i)$, the butterfly with maturity T and payoff support $[K_{i-1},\ K_{i+1}]$, the outer sum is composed of the following: one butterfly with strike 1 (cf. Figure 3); two butterflies, one with strike 1 and one with strike 2 (cf. Figure 2); and three butterflies, one with strike 1, one with strike 2, and one with strike 3 (cf. Figure 1). It is preferable to rewrite this sum as $\sum_{i=1}^{m-1} Cs(t, T, K_i)(m - i)$ and compute it by reversing the sum, that is, with i running from $i = m - 1$ to $i = 1$ and stop as soon as $Cs(t, T, K_i)(m - i)$ is small enough. In the present implementation, we stop as soon as $Cs(t, T, K_i)(m - i) < \epsilon$ with $\epsilon = 0.001$. We check the put option price values with a Monte Carlo method, without using the butterfly decomposition, using a simple Euler scheme with 70,000 paths, and a discretization of the time interval [0, 1] of 250 regularly spaced points. We compute the root mean square error (RMSE) between the finite element method and the Monte Carlo prices over a range of strike prices from 0.8 to 1.2 and maturities from 0.2 to 1.0, as shown in Figure 5 and find it to be 1.87%. In terms of computation time, pricing a single option with maturity $T = 1$ and strike $K = 1$ takes 310.25 s using the finite element method, compared to 556.67 s using the Monte Carlo method. Note that the Monte Carlo pricing is implemented in Python.

To illustrate how the pricing error (relative to the put price obtained via the Monte Carlo method) depends on the number of spatial steps, we compute the RMSE for various space step counts and report the errors (in %) in Figure 6. Regarding the dependence of the pricing error on the option's moneyness we obtain for in-the-money options ($K \ge 1.1$), the error is 0.92%, for at-the-money options ($K \in]0.9,\ 1.1[$), the error is 0.85%, and for out-of-the-money options ($K \le 0.9$), the error is 2.98%.

Note that the value of m has been chosen arbitrarily; however, it should be clear that it is related to the mesh size of the finite elements used to partition the space. A detailed error analysis is

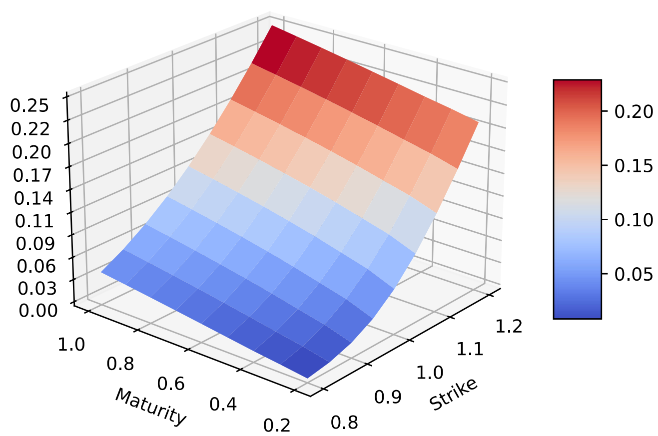

FIGURE 5 | Put option prices $P(t, T, K)$ for $T - t \in [0, 1]$ and strike $K \in [0.8, 1.2]$ with $s_0 = 1$. [Colour figure can be viewed at wileyonlinelibrary.com]

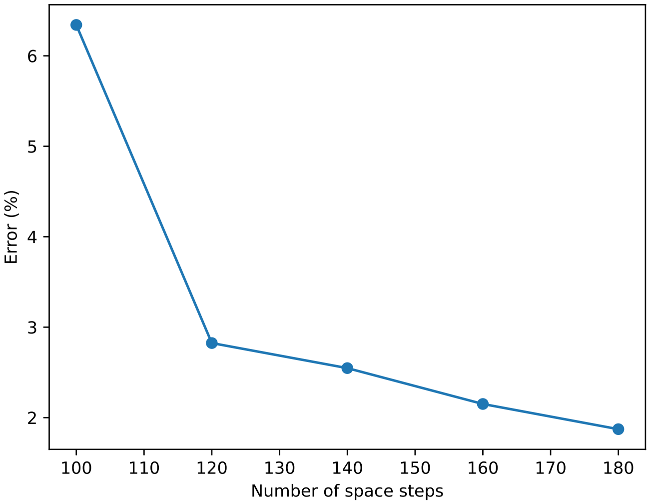

FIGURE 6 | Put option pricing error (RMSE) compared to the Monte Carlo prices for different number of space steps. Put option prices $P(t, T, K)$ for $T - t \in [0, 1]$ and strike $K \in [0.8, 1.2]$ with $s_0 = 1$. [Colour figure can be viewed at wileyonlinelibrary.com]

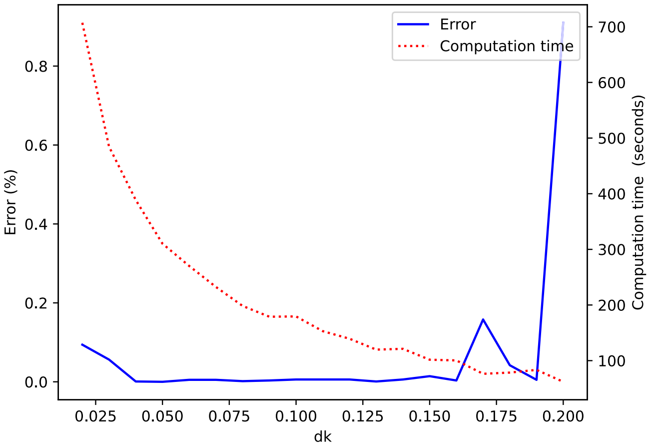

FIGURE 7 | Put option absolute pricing error relative to the put price obtained for $dK = 0.05$, controlling the support of the butterflies used to decompose the put, for different values of dK ranging from 0.02 to 0.2. The put option strike is 1, and the maturity is 1 year (the stock value is 1). The model parameters are those reported above. The solid blue curve reports the absolute pricing error (in %), while the dotted red curve reports the computation time (in seconds). [Colour figure can be viewed at wileyonlinelibrary.com]

required to clarify this relationship, which is a study in itself, but its impact can be assessed numerically. To this end, we focus on a single put option with a strike of 1 (with the stock price also equal to 1) and a maturity of 1 year. We price this option using butterfly spreads of varying widths. Specifically, we consider butterflies with regularly spaced strikes—that is, the three strikes K_{i-1}, K_i, and K_{i+1} that define the support of the butterfly satisfy $K_{i+1} - K_i = K_i - K_{i-1} = dK$. The previous pricing was performed with $dK = 0.05 = \frac{1}{m} = \frac{1}{20}$. We now price the option for values of dK ranging from 0.02 to 0.20 and report both the absolute pricing error (relative to the price obtained with $dK = 0.05$) and the computation time. The results, shown in Figure 7, indicate that increasing dK can reduce computation time without significantly increasing the pricing error–up to a certain threshold.

Note that we use a regular mesh. In Achdou and Tchou [8], the authors also consider an adaptive mesh that reduces the computational cost for a given level of accuracy; this mesh adaptation algorithm is derived from the Black–Scholes model (see Pironneau and Hecht [30]). An analysis of the adaptivity of the mesh to the structure of the model presented here is certainly of interest. Finally, the decomposition of the put option as a portfolio of butterflies is not perfect as Figure 4 clearly shows; there is a difference, and we claim (and verify numerically) that it converges to zero, but a formal proof is needed.

4.1 | Payoff Subtraction

In Hilber et al. [14 , p. 6] and Hilber et al. [15 , p. 110], the authors formulate the variational analysis of the option price minus the (discounted) option payoff. This approach allows them to control the behavior at the boundaries of the domain along the logarithmic stock price axis. In our case, we employed the butterfly decomposition of the put option price. The payoff of a butterfly option resembles the hat function used in the FEM, and therefore, the accuracy of the approximate price depends on both the mesh size used in the FEM and the number of butterflies employed to decompose the put option payoff. As a result, it is difficult to disentangle the errors arising from these two components. However, the results obtained so far can be adapted at minimal cost to the approach proposed in Hilber et al. [14 , p. 6] and Hilber et al. [15 , p. 110].

Consider $f_0(x, v) = P(0, x, v)v^{\eta\nu_0}e^{-\eta\nu_1 v}$ with $P(0, x, v) = \max(K - e^x, 0)$ the put payoff and assume that in (8) $P(T - t, x, v)$ stands for the put option price at time t. Define $h(t, x, v) = f(t, x, v) - f_0(x, v)$ then since $f(t, x, v)$ satisfies (9), with $\mathcal{A}$ given by (10), we deduce that $h(t, x, v)$ solves the following:

$$\begin{aligned} \partial_t h - \mathcal{A}h &= l_0 , \\ h(0, x, v) &= 0 , \end{aligned} \tag{41}$$

with $l_0 = \mathcal{A}f_0$. Since Propositions 3.2 and 3.4 remain valid, Theorem 3.13 applies and the localization of Proposition 3.16 also applies. In Theorem 3.13, the right-hand side of (36) is $\langle l_0, g\rangle = -a(f_0, g)$ with $a(\cdot,\cdot)$ given by (15). Implementing this approach, using the parameters and mesh grid from the previous section, results in a RMSE of 1.59% compared to the Monte Carlo prices. In terms of computation time, to price one option with maturity $T = 1$ and strike $K = 1$, the finite element method takes 382.27 s. Regarding the dependence of the pricing error to the degree of the moneyness of the option, for in-the-money options ($K \geq 1.1$), it is 0.55%; for at-the-money options ($K \in]0.9,\ 1.1[$), it is 1.71%; and for out-of-the-money options ($K \leq 0.9$), it is 2.08%.

5 | Conclusion

This paper proposes a variational formulation for the European option pricing problem in the 1-hypergeometric stochastic volatility model proposed in Da Fonseca and Martini [4]. Following the work of Achdou et al. [7], which builds on Achdou and Tchou [8], we derive the main ingredients, namely the Gårding inequality, which allow us to solve the European option pricing problem. Still following [7], we derive estimates that allow us to use results from the theory of semigroups. As in these works, the analysis involves a weighted Sobolev space, and following a critical remark in Lamberton and Terenzi [21], the weight is closely related to the dynamic of the volatility. We provide an implementation of the model using the finite element method library FreeFem++. Our work focuses on European options, but as shown in Hilber et al. [15], other products could also be analyzed. Extending the model to include jumps is also an interesting possibility.

Author Contributions

José Da Fonseca: conceptualization, methodology, writing – original draft, visualization, resources, investigation, formal analysis, software, project administration. **Wenjun Zhang:** formal analysis, project administration, software, methodology, conceptualization, validation, visualization, writing – review and editing.

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Conflicts of Interest

The authors declare no conflicts of interest.

Endnotes

[1] In the case $\alpha = 2$, Privault and She [10] propose an approximation of the implied volatility of a European option price while Sousa et al. [11] propose an approximation of the Barrier option price, both use a volatility of volatility expansion of the pricing operator. In a different context, that is optimal portfolio choice, Cipriano et al. [12] analyze the α-hypergeometric stochastic volatility model.

[2] The stock log return volatility solely depends on the volatility process, which is autonomous.

[3] In Baldi [16, Remark 10.4, p. 319] it is proved that when the domain is unbounded it is still possible to obtain similar results, but it requires a boundedness of the diffusion coefficients which is not satisfied by the α-hypergeometric stochastic volatility models.

[4] We follow the presentation of Hilber et al. [15, Chap. 9.3] instead of Achdou and Tchou [8, Section 3].

[5] See also [22] and Hozman and Tichý [23] for alternative weighted Sobolev spaces to formulate the variational option pricing problem.

[6] The corresponding proof is available upon request.

References

1. F. Black and M. Scholes, "The Pricing of Option and Corporate Liabilities," *Journal of Political Economy* 81, no. 3 (1973): 637–654, https://www.jstor.org/stable/1831029.

2. S. L. Heston, "A Closed-Form Solution for Options With Stochastic Volatility With Applications to Bond and Currency Options," *Review of Financial Studies* 6, no. 2 (1993): 327–343, https://doi.org/10.1093/rfs/6.2.327.

3. J. Gatheral, *The Volatility Surface: A Practitioner's Guide* (Wiley Finance, 2006).

4. J. Da Fonseca and C. Martini, "The α-Hypergeometric Stochastic Volatility Model," *Stochastic Processes and their Applications* 126, no. 5 (2016): 1472–1502, https://doi.org/10.1016/j.spa.2015.11.010.

5. P. Henry-Labordère, "Solvable Local and Stochastic Volatility Models: Supersymmetric Methods in Option Pricing," *Quantitative Finance* 7, no. 5 (2007): 525–535, https://doi.org/10.1080/14697680601103045.

6. P. Carr and D. Madan, "Option Valuation Using the FFT," *Journal of Computational Finance* 2 (1999): 61–73.

7. Y. Achdou, B. Franchi, and N. Tchou, "A Partial Differential Equation Connected to Option Pricing With Stochastic Volatility: Regularity Results and Discretization," *Mathematics of Computation* 74, no. 251 (2005): 1291–1322, https://doi.org/10.1090/S0025-5718-04-01714-4.

8. Y. Achdou and N. Tchou, "Variational Analysis for the Black and Scholes Equation With Stochastic Volatility," *Mathematical Modelling and Numerical Analysis* 36, no. 3 (2002): 373–395, https://doi.org/10.1051/m2an:2002018.

9. F. Hecht, "New Development in FreeFem++," *Journal of Numerical Mathematics* 20, no. 3-4 (2012): 251–265, https://freefem.org/.

10. N. Privault and Q. She, "Option Pricing and Implied Volatilities in a 2-Hypergeometric Stochastic Volatility Model," *Applied Mathematics Letters* 53 (2016): 77–84, https://doi.org/10.1016/j.aml.2015.09.008.

11. R. Sousa, A. B. Cruzeiro, and M. Guerra, "Barrier Option Pricing Under the 2-Hypergeometric Stochastic Volatility Model," *Journal of Computational and Applied Mathematics* 328 (2018): 197–213, https://doi.org/10.1016/j.cam.2017.06.034.

12. F. Cipriano, N. F. M. Martins, and D. Pereira, "Optimal Portfolio for the α-Hypergeometric Stochastic Volatility Model," *SIAM Journal on Financial Mathematics* 12, no. 1 (2021): 226–253, https://doi.org/10.1137/19M1299165.

13. S. Karlin and H. Taylor, *A Second Course on Stochastic Processes*, 1st ed. (Academic Press, 1981).

14. N. Hilber, O. Reichmann, and C. Schwab, "Sparse Wavelet Methods for Option Pricing Under Stochastic Volatility," *Journal of Computational Finance* 8 (2005): 1–42, https://doi.org/10.21314/JCF.2005.131.

15. N. Hilber, O. Reichmann, C. Schwab, and C. Winter, *Computational Methods for Quantitative Finance* (Springer-Verlag, 2013), https://doi.org/10.1007/978-3-642-35401-4.

16. P. Baldi, *Stochastic Calculus: An Introduction Through Theory and Exercises. Universitext*, 1st ed. (New York: Springer-Verlag, 2017), https://doi.org/10.1007/978-3-319-62226-2.

17. G. Rubio, "The Cauchy-Dirichlet Problem for a Class of Linear Parabolic Differential Equations With Unbounded Coefficients in an Unbounded Domain," *International Journal of Stochastic Analysis* 2011, no. 1 (2011): 469806, https://doi.org/10.1155/2011/469806.

18. R. Sousa, "Barrier Option Pricing Under the 2-Hypergeometric Stochastic Volatility Model" (Master's thesis *Instituto Superior Técnico, Universidade de Lisboa*, 2016), URL https://fenix.tecnico.ulisboa.pt/cursos/mma/dissertacao/283828618789741.

19. E. M. Stein and J. C. Stein, "Stock Price Distribution With Stochastic Volatility: An Analytic Approach," *Review of Financial Studies* 4 (1991): 727–752, https://doi.org/10.1093/rfs/4.4.727.

20. A. Canale, R. M. Mininni, and A. Rhandi, "Analytic Approach to Solve a Degenerate Parabolic PDE for the Heston Model," *Mathematical Methods in the Applied Sciences* 40, no. 13 (2017): 4982–4992, https://doi.org/10.1002/mma.4363.

21. D. Lamberton and G. Terenzi, "Variational Formulation of American Option Prices in the Heston Model," *SIAM Journal on Financial Mathematics* 10, no. 1 (2019): 261–368, https://doi.org/10.1137/17M1158872.

22. J. F. Bonnans and A. Kröner, "Variational Analysis for Options With Stochastic Volatility and Multiple Factors," *SIAM Journal on Financial Mathematics* 9, no. 2 (2018): 465–492, https://doi.org/10.1137/17M1130836.

23. J. Hozman and T. Tichý, "DG Framework for Pricing European Options Under One-Factor Stochastic Volatility Models," *Journal of Computational and Applied Mathematics* 344 (2018): 585–600, https://doi.org/10.1016/j.cam.2018.05.064.

24. Y. Achdou and O. Pironneau, "Partial Differential Equations for Option PRICING". *Working paper*, 2011.

25. L. Hörmander, *The Analysis of Linear Partial Differential Operators I*, 2nd ed. (Springer-Verlag, 2003), https://doi.org/10.1007/978-3-642-61497-2.

26. K.-J. Engel and R. Nagel, *A Short Course on Operator Semigroups. Universitext*, 1st ed. (Springer-Verlag, 2006), https://doi.org/10.1007/0-387-36619-9.

27. A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations. Applied Mathematical Sciences*, vol. 44, 1st ed. (Springer-Verlag, 1983), https://doi.org/10.1007/978-1-4612-5561-1.

28. E.-M. Ouhabaz, *Analysis of Heat Equations on Domains. London Mathematical Society Monographs*, vol. 31, 1st ed. (Princeton University Press, 2005).

29. O. Pironneau, "Calibration of Heston Model With Keras". Working Paper or Preprint, Aug. 2019, https://hal.sorbonne-universite.fr/hal-02273889.

30. O. Pironneau and F. Hecht, "Mesh Adaption for the Black and Scholes Equations," *East-West Journal of Numerical Mathematics* 2000, no. 8 (2000): 25–35.