Effective versus individual waves for water wave and thin plate problems

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Abstract

Numerical results are presented to show that, for small-amplitude roughness, individual wave elevations attenuate at a far slower rate than the corresponding effective wave elevation for both ocean waves travelling over a rough seabed in intermediate depth and waves in a thin plate in vacuo. Use of the effective wave elevation, therefore, results in misleading predictions of attenuation.

Keywords: Wave attenuation, random media, effective wave field, multiple scattering

1 Introduction

Ocean surface waves attenuate with distance travelled into the sea-ice covered ocean. Wave propagation in the ice-covered ocean is conventionally modelled using linear potential-flow theory for the water and thin-plate theory for the ice cover. Bennetts & Peter [1] conducted a preliminary investigation of wave attenuation in the ice-covered ocean due to ice roughness. They modelled the roughness as random variations in stiffness and mass of the ice and derived a semi-analytic expression for the attenuation rate of the effective wave field, i.e. the mean wave field with respect to realisations.

In a recent paper, Bennetts et al. [2] showed that individual wave fields attenuate far slower than the effective wave field for the related problem of free-surface waves over a rough seabed in intermediate depth, originally considered by Mei & Hancock [3]. They used large ensembles of numerical solutions for randomly generated realisations of the bed profile. Further, they review the existing literature on wave propagation over a rough seabed.

Here, we extend the study of Bennetts et al. [2] to problems involving thin plates, with the aim of establishing whether effective media theory is valid to study wave propagation in the ice-covered ocean. We summarise the method and results of [2] in §2 and apply the method to a rough thin plate in vacuo in §3.

2 Free-surface/rough-bed problem

Consider a linear monochromatic wave propagating in the positive x-direction. In open water, the wavenumber, k, is related to the angular frequency, ω, via the dispersion relation $k\tanh(kh) = K$, where $K = \omega^2/g$, h is the fluid depth and g is the gravitational acceleration.

The seabed fluctuates about $z = -\bar{h}$, where $\bar{h}$ is an intermediate depth, i.e. $k\bar{h} = \mathrm{O}(1)$. The fluctuations have a known correlation length, l, and root-mean-square amplitude, $\varepsilon \ll 1$. The function $z = -h(x)$, where $h(x) = \bar{h} - \varepsilon p(x)$ and $p = \mathrm{O}(1)$ is an autocorrelated random function, denotes the location of the bed.

The water velocity field is defined as the gradient of $\mathrm{Re}\{(g/\mathrm{i}\omega)\phi(x,z)\mathrm{e}^{-\mathrm{i}\omega t}\}$. The velocity potential, ϕ, satisfies

$$\partial_x^2\phi + \partial_z^2\phi = 0 \qquad (-h < z < 0), \qquad (1a)$$

$$\partial_z\phi + h'(\partial_x\phi) = 0 \qquad (z = -h), \qquad (1b)$$

and is coupled to the wave elevation, denoted $z = \mathrm{Re}\{\eta(x)\mathrm{e}^{-\mathrm{i}\omega t}\}$, via

$$\phi = \eta \quad \text{and} \quad \partial_z\phi = K\eta \quad (z = 0). \qquad (1c)$$

Consider the problem in which the roughness extends over a long, finite interval $x \in (0, L)$. For given $h(x)$, we approximate the rough bed profile by a piece-wise constant function on $M \gg$

1 sub-intervals and solve for the velocity potential using an iterative algorithm.

Wave elevations are calculated for a large ensemble of randomly generated realisations of the bed profile, cf. [2]. Two measures of the attenuation rate are obtained from the ensemble of wave elevations. First, an attenuation rate, $Q_{\text{eff}}^{(\text{rs})}$, is extracted from the effective wave elevation via

$$|\langle\eta\rangle| \propto \mathrm{e}^{-Q_{\text{eff}}^{(\text{rs})}x} \quad (0 < x < L). \tag{2}$$

Second, an attenuation rate, $Q_{\text{ind}}^{(\text{rs})}$, is calculated as the ensemble average of attenuation rates of individual wave elevations. The attenuation rate is defined as $Q_{\text{ind}}^{(\text{rs})} = \langle Q_i \rangle$, where Q_i is extracted from the individual wave elevation $\eta = \eta_i$, i.e.

$$|\eta_i| \propto \mathrm{e}^{-Q_i x} \quad (0 < x < L). \tag{3}$$

It turns out that both rates are proportional to the bed amplitude squared but differ by orders of magnitude for a large range of parameters [2].

Figure 1 shows example individual wave elevations and corresponding effective wave elevations for $\bar{k}l = 0.9$ and 5. The wavenumber $\bar{k}$ corresponds to the mean depth $\bar{h}$, and $\bar{k}\bar{h} = 1$ is set. The smaller correlation length is chosen to produce visible (though weak) attenuation of the individual wave elevation. The corresponding effective wave elevation attenuates slightly more rapidly than the individual wave elevation. The largest correlation length is chosen to produce maximal attenuation of the effective wave elevation. The corresponding individual elevation does not attenuate (on the scale shown). Attenuation of the effective elevation is, therefore, not related to the individual elevations.

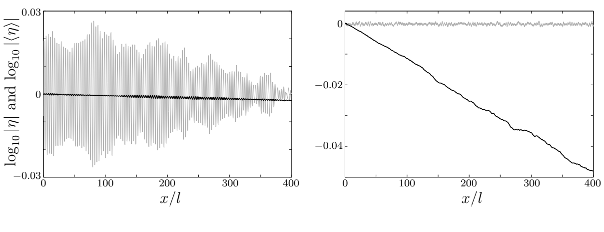

Fig. 1: Example individual wave elevations (grey) and corresponding effective wave elevations (black), for $\bar{k}\varepsilon = 10^{-2}$ and $\bar{k}l = 0.9$ (left) and 5 (right).

3 In vacuo plate problem

Next, we consider an infinitely long rough thin plate in vacuo. The problem is one-dimensional in the horizontal coordinate x. The spatial part $u(x)$ of the plate deflection $\mathrm{Re}\{u(x)\mathrm{e}^{-\mathrm{i}\omega t}\}$ satisfies the thin plate equation

$$\beta\partial_x^4 u - \gamma\omega^2 u = 0 \quad (-\infty < x < \infty), \tag{4}$$

where β is the constant plate stiffness and $\gamma(x)$ is its varying mass.

We use an analogous iterative algorithm together with a step approximation as in the rough bed problem, where the local wavenumber κ_m is $\kappa(x) = (\omega^2\gamma(x)/\beta)^{\frac{1}{4}}$, evaluated at the midpoint of the mth sub-interval.

Again, solutions are calculated for large ensembles of different realisations of the varying wavenumber function, which share a common correlation length and roughness amplitude. Figure 2 shows the results for the in vacuo plate, in analogy to figure 1 for the rough bed. As can be seen, the behaviour is very similar and the analogous conclusions are drawn.

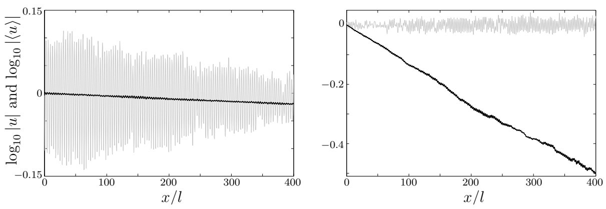

Fig. 2: As in Fig. 1 but for in vacuo plate problem.

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