# The $\alpha$-Hypergeometric Stochastic Volatility Model

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# The Positivity Problem in Finance (and other fields)

Two simple ways to have positivity

$$x^2 \qquad e^x$$

Positivity is important in finance for:

- Volatility.
- Interest rates.
- Stock price.

and Noise is given by the Gaussian distribution, hence in $\mathbb{R}$.

# Positivity in Econometrics

The GARCH:

$$r_t = \sigma_t \epsilon_t$$
$$\sigma_t^2 = \alpha_0 + \alpha_1 \sigma_{t-1}^2 + \beta_1 \epsilon_{t-1}^2$$

The EGARCH:

$$r_t = \sigma_t \epsilon_t$$
$$\ln \sigma_t^2 = \alpha_0 + \alpha_1 g(\epsilon_{t-1}) + \beta_1 \ln \sigma_{t-1}^2$$

# Positivity in Interest rates

Zero coupon bond

$$B(t,T) = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r_u du}\right]$$

Vasicek (Ornstein-Ulhenbeck):

$$dr_t = \kappa(\theta - r_t)dt + \sigma dw_t$$

easy but Gaussian!

Dothan:

$$dr_t = \kappa r_t dt + \sigma r_t dw_t$$

positive but much more complicated.

# Equity Derivatives

For the stochastic volatility models:

$$ds_t = s_t \sigma_t dw_t^1 \tag{1}$$

and

$$d\sigma_t = a\sigma_t dt + b\sigma_t dw_t^2 \tag{2}$$

$$d \ln \sigma_t = a(b - \ln \sigma_t)dt + \alpha dw_t^2 \tag{3}$$

$$d\sigma_t = a(b - \sigma_t)dt + \alpha dw_t^2 \tag{4}$$

- Hull & White (2): volatility non stationary but exponential so positive!
- Chesney & Scott (3): logarithm of volatility Ornstein-Ulhenbeck so Gaussian but volatility is exponential so positive!
- Stein & Stein (4): volatility is Ornstein-Ulhenbeck so Gaussian, volatility is negative.

but

- Hull & White not good because volatility is a geometric Brownian motion.
- Chesney & Scott, we don't know the stock density or its characteristic function. Cannot calibrate the model.
- Stein & Stein (4), we don't know the stock characteristic function (option pricing by FFT) but volatility is Gaussian!

# Equity Derivatives

$$ds_t = s_t \sqrt{\sigma_t} dw_t^1$$

and

$$d\sigma_t = a(b - \sigma_t)dt + \alpha \sqrt{\sigma_t} dw_t^2 \tag{5}$$

- The volatility is positive and we know the characteristic function of the stock.
- The Feller condition $2ab > \alpha^2$ ensures that $\sigma_t > 0$.

Option contains integrated volatility

$$E_t^Q \left[ \left( s_t e^{-\frac{1}{2}\int_t^T \sigma_u du + \int_t^T \sigma_u dw_u^1} - K \right)_+ \right]$$

*Whether the volatility oscillates a lot (large $a$) or not (small $a$) option convey little (no) information on that aspect.*

# Equity Derivatives

The Feller condition is not satisfied in practice:

1. The volatility can touch $0$.
2. The volatility distribution is too close to $0$.

In fact the square root process is positivity using the $x^2$ function.

Positivity using $e^x$ doesn't work but the exponentiation is appealing.

# The Hypergeometric Stochastic Volatility Model

The forward price dynamic:

$$df_t = f_t e^{v_t} dw_{1,t} \quad (6)$$

$$dv_t = (a - b e^{\alpha v_t})dt + \sigma dw_{2,t} \quad (7)$$

with $dw_{1,t}.dw_{2,t} = \rho dt$ (controls the leverage).

- Volatility $v_t$ looks like an OU process.
- Stock volatility $e^{v_t}$ is positive by construction.

For $\alpha = 1$ we know how to compute the Mellin transform of the stock (so option pricing is possible).

# The Hypergeometric Stochastic Volatility Model

$$\mathbb{E}\left[\left(\frac{f_t}{f_0}\right)^\lambda\right] = \mathbb{E}\left[\exp\left(-\frac{\lambda}{2}\int_0^t e^{2v_u}du + \lambda\int_0^t e^{2v_u}dw_{1,u}\right)\right]$$
$$= e^{-\frac{\lambda\rho}{\sigma}e^{v_0}}\mathbb{E}\left[\exp\left(\alpha_0 e^{v_t} + \alpha_1\int_0^t e^{v_s}ds - \frac{\alpha_2^2}{2}\int_0^t e^{2v_s}ds\right)\right]$$

with

$$\alpha_0 = \frac{\lambda\rho}{\sigma} \quad \alpha_1 = -\frac{\lambda\rho}{\sigma}\left(a + \frac{\sigma^2}{2}\right) \quad \alpha_2^2 = -\lambda^2(1-\rho^2) - \frac{2b\rho\lambda}{\sigma} + \lambda.$$

and $dv_t = (a - be^{v_t})dt + \sigma dw_{2,t}$.

Girsanov's theorem to cancel the drift of the volatility

$$\mathbb{E}\left[\left(\frac{f_t}{f_0}\right)^\lambda\right] = e^{-\frac{a}{\sigma^2}v_0 + (\frac{b}{\sigma^2} - \frac{\lambda\rho}{\sigma})e^{v_0}}e^{-\frac{a^2t}{2\sigma^2}}\mathbb{E}^Q\left[\exp\left(\frac{av_t}{\sigma^2} + \beta_0 e^{v_t} + \beta_1\int_0^t e^{v_s}ds - \frac{\beta_2^2}{2}\int_0^t e^{2v_s}ds\right)\right]$$

with

$$\beta_0 = \frac{\lambda\rho\sigma - b}{\sigma^2} \quad \beta_1 = (b - \lambda\rho\sigma)\left(\frac{a}{\sigma^2} + \frac{1}{2}\right) \quad \beta_2^2 = -\lambda^2(1-\rho^2) + \lambda\left(1 - \frac{2b\rho}{\sigma}\right) + \frac{b^2}{\sigma^2}.$$

and $dv_t = \sigma d\tilde{w}_{2,t}$

# The Hypergeometric Stochastic Volatility Model

$$F(t,v) = \mathbb{E}^Q\left[\exp\left(\frac{av_t}{\sigma^2} + \beta_0 e^{v_t} + \beta_1 \int_0^t e^{v_s} ds - \frac{\beta_2^2}{2}\int_0^t e^{2v_s} ds\right)\right] \quad (8)$$

and $F(0,v) = \exp\left(\frac{av}{\sigma^2} + \beta_0 e^v\right)$. $F(t,v)$ solves the PDE:

$$\begin{aligned}\partial_t F &= \frac{\sigma^2}{2}\frac{d^2F}{dv^2} - \frac{\beta_2^2}{2}e^{2v}F + \beta_1 e^v F, \\ &= -HF\end{aligned}$$

so $F(t) = e^{-Ht}F(0)$ and in itegral form:

$$F(t,v_0) = \int_{-\infty}^{+\infty} q(\sigma^2 t, v_0, y)F(0,y)dy$$

- $q$ is the heat kernel.
- $-\frac{\beta_2^2}{2}e^{2v} + \beta_1 e^v$ is the potential (well known): Morse potential.

# The Hypergeometric Stochastic Volatility Model

The Laplace transform of the HK is known

$$G(v,y;s^2/2) = \int_0^{+\infty} e^{-\frac{s^2}{2}t} q(t,v,y)dt = \int_0^{+\infty} e^{-\frac{s^2}{2}t} e^{-Ht} dt.$$
$$= \left(\frac{s^2}{2} + H\right)^{-1}$$

$G$ is the fundamental solution (the Green function, or the resolvant) of $H + \frac{s^2}{2} = 0$ that is to say $G$ solves:

$$-\frac{\sigma^2}{2}\frac{d^2G}{dv^2} + \frac{\beta_2^2}{2}e^{2v}G - \beta_1 e^v G + \frac{s^2}{2} = \delta_y \tag{9}$$

$$G(v,y;\eta^2/2) = \frac{\Gamma\left(\eta - \frac{\nu_1}{\nu_2} + \frac{1}{2}\right)}{\nu_2 \Gamma(1+2\eta)} e^{-(v+y)/2} W_{\frac{\nu_1}{\nu_2},\eta}\left(2\nu_2 e^{y_>}\right) M_{\frac{\nu_1}{\nu_2},\eta}\left(2\nu_2 e^{y_<}\right)$$

with $\nu_1, \nu_2$ related to $\beta_1, \beta_2, \eta$ to $s$ and $y_> = \max(v,y)$, $y_< = \min(v,y)$, $W_{\kappa,\eta}$ and $M_{\kappa,\eta}$ are the Whittaker functions (related to confluent hypergeometric functions):

$$W_{\kappa,\eta}(z) = z^{\eta+\frac{1}{2}} e^{-z/2} \Psi\left(\eta - \kappa + \frac{1}{2}, 1+2\eta; z\right)$$
$$M_{\kappa,\eta}(z) = z^{\eta+\frac{1}{2}} e^{-z/2} \Phi\left(\eta - \kappa + \frac{1}{2}, 1+2\eta; z\right).$$

# The Hypergeometric Stochastic Volatility Model

1. $G$ is know.
2. $q$ is the inverse Laplace transform of $G$.
3. We integrate $q$ over $F(0, v)$ it gives the Mellin transform of the spot.
4. We compute the inverse Mellin transform of the spot to get the option price.

# Conclusions

- we develop a stochastic volatility model with positive volatility
- we provide the main results to perform option pricing

# Open Problems

- all the problems are open....