Provocative mathematics questions: Drawing attention to a lack of attention

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The paper investigates the role of attention in the reflective thinking of school mathematics teachers. It analyses teachers' ability to pay attention to detail and *use* their mathematical knowledge. The vast majority of teachers can be expected to have an excellent knowledge of mathematical techniques. The question examined here is whether this kind of knowledge might structure their attention in such a way that the emphasis on procedures deflects their attention from the essential details. Four groups of participant teachers from New Zealand, Hong Kong, Germany and Ukraine were given a mini-test containing seven simple mathematics questions. Most questions in the test were provocative in the sense that they looked like routine questions but in fact had some catch. The results of the test were startling – the vast majority of the participants gave incorrect answers to most questions in the test. After the test the participants were given a short questionnaire to reflect on their performance on the test. Their responses were analysed using the theories of selective, divided and focused attention and Mason's concept of the *discipline of noticing.* Implementations of the results of the study in assessment and professional development are discussed.

Keywords: attention, assessment, knowledge, professional development

INTRODUCTION AND THEORETICAL FRAMEWORKS

In this paper, an attempt is made to investigate the role of attention in the case of school mathematics teachers *using* their existing mathematical knowledge while doing simple but non-routine mathematical tasks. We are confident that the vast majority of teachers have excellent mathematics knowledge of *knowing-that* (factual) and *knowing-how* (techniques and skills) as described by Mason and Spence (1999). Most of the formulas, rules and theorems however are not always applicable but have certain conditions and constraints. Often assessment questions focused on techniques are selected in such a way that the conditions/constraints of the relevant formula or rule are met. Students might therefore develop a habit of applying formulas or rules without checking the conditions/constraints. But in real-life problems not all functions and equations behave so nicely and ignoring conditions and constraints might lead to significant and costly errors. Another reason for teachers to pay attention to mathematical tasks offered in teaching materials is that some textbooks contain mathematical inaccuracies and mistakes. Mathematics textbooks and their potential role in supporting misconceptions were discussed in Kajander and Lovric (2009). The authors presented analysis of incorrect definitions of some fundamental concepts from introductory calculus based on examination of secondary and university textbooks. They wrote in the conclusion that "situations leading to potential misconceptions occurred consistently in multiple sources. Acknowledging that textbooks remain a fundamental teaching resource, we suggest that more attention be paid to the presentation of mathematics. Furthermore, analyses of textbooks should include developmental as well as subject matter scrutiny". (p.180).

In this paper, the following theories of attention are employed as theoretical frameworks: the late selection theory of selective attention (Deutsch and Deutsch 1963) based on the idea that all information is routinely processed and selection of response depends on the level of alertness; Kahneman's (1973) model of divided attention based on the idea of mental efforts and the level of arousal or state of alertness; and the feature-integration theory of attention (Treisman and Gelade 1980) based on the idea that putting different features into a coherent object demands focused attention. Other theoretical considerations are based on research by

Mason (2000, 2002, 2004). Mason has proposed that when we look at a mathematics question the focus of our attention may vary depending on whether we are *looking at* the symbols or *looking through* them. The idea is that we need to structure our attention, to be conscious of our awareness of different elements. Mason describes a number of elements that we may focus attention on, including: the whole, the details, the relationships between the parts, the properties of the whole or the parts and deductions (2004). One of the goals in teaching mathematics is developing and enhancing students' mathematical way of thinking while helping them to learn a variety of concepts, techniques and procedures. The mathematical way of thinking is concerned with the analytical thinking so that an individual analyses any situation, doesn't take anything for granted and always looks for evidence, proof and justification - which are the essence of mathematics. We should encourage students to pay attention to every detail, for example - conditions, constraints, locality, properties and relationships. The ability to pay attention, or the *discipline of noticing* as described by Mason (2002), is as equally important to develop as mathematical techniques. It needs to be a natural part of their mathematical culture. Students can see that the ability to carefully analyse a mathematics question enhances their skills to critically analyse other situations outside mathematics. In order to develop such skills in their students, teachers should possess those skills themselves.

In this paper, the concern is not in testing teachers' knowledge of mathematical techniques, procedures and algorithms but their skills of paying attention and analysing the question before applying a relevant formula or technique, that is the ability to 'question the question'. It is argued that attention plays a crucial role in doing non-routine mathematical tasks. As Mason and Spenser (1999) propose "*knowing-to act* in the moment depends on the structure of attention in the moment, depends on what one is aware of". (p.135).

THE STUDY

The study was conducted with four groups of teachers – in-service teachers from New Zealand, Hong Kong and Germany and mathematics students from Ukraine. The New Zealand (the first) group comprised 14 experienced upper secondary school mathematics teachers who attended a workshop during a one day conference as part of their voluntary professional development. The Hong Kong (the second) group comprised 26 secondary school mathematics teachers who attended a 5-week full-time compulsory training course. The German (the third) group comprised 10 experienced school mathematics teachers who attended a compulsory professional development seminar. The Ukraine (the fourth) group comprised 26 year 3-4 mathematics students training to become secondary school mathematics teachers with the majority having had teaching experience as part of their training. A combination of two non-probability sampling methods – convenience and judgment – was used to select the participants. All groups were given the mini-test containing 7 simple mathematical questions. Most questions in the test were provocative in the sense that they looked like routine questions but in fact they had some catch. In some cases it was an extraneous root of an equation because of the restricted domain, in others the rule was inapplicable because the conditions were not met. The participants were aware that some questions in the test were routine questions and some had a catch and they had to decide which was which. The purpose of the test and the study was to check the level of attention of the teachers, not to trick them. It was expected that many participants would not notice a catch and solve some questions incorrectly. The intention was to discuss the solutions of the questions after the test and draw teachers' attention to the importance of attention while doing mathematical tasks. The main research questions were to check how the level of attention affects teachers' abilities to do simple mathematical tasks and identify the reasons for not solving all questions correctly. The environment in all four groups was very friendly and professional with mutual respect among the participants and the person giving the test. The teachers had 15 minutes for the test. The questions from the test are below.

The mini-test

1. Find the area of the right-angled triangle if its hypotenuse is 10cm and the height dropped on the hypotenuse is 6cm.

2. Find the domain of the function $y = f(g(x))$ if $f(x) = x^2 + 1$, $g(x) = \sqrt{x-2}$.

3. Solve the equation $\ln(x^2 + 17x - 18) - \ln(x^2 + 5x - 6) = 0$.

4. Prove the identity $\sin x = \sqrt{(1-\cos^2 x)}$.

5. Show that the equation $\dfrac{x^2 + \sqrt{x} + 1}{x-1} = 0$ has a solution on the interval [0, 2].

6. Find the derivative of the function $y = \ln(2\sin(3x) - 4)$.

7. Find the integral $\int_{-1}^{1} \frac{1}{x} dx$.

The results and discussion of the solutions of the mini-test

The results of the test are presented in Table 1.

Table 1. Percentage of correct answers to the test questions.

	Q1.	Q2.	Q3.	Q4.	Q5.	Q6.	Q7.
Correct answers group 1	0%	7%	21%	7%	0%	8%	0%
Correct answers group 2	23%	12%	27%	19%	12%	15%	12%
Correct answers group 3	0%	60%	30%	N/A	20%	0%	0%
Correct answers group 4	0%	19%	31%	15%	12%	0%	8%

After the test there was a detailed discussion of the solutions of every question from the test.

Question 1. The correct answer is: there is no sense to talk about the area as the triangle doesn't exist. By the Thales' theorem the hypotenuse in a right-angled triangle is a diameter of its semicircle so in this case the height cannot be bigger than 5cm.

In the first group there were no correct answers out of 14 solutions with 12 teachers giving either 30cm^2 or 24cm^2. In the second group there were 6 correct answers out of 26 with some teachers arriving to the correct answer after checking their initial incorrect answer of 30cm^2 or 24cm^2 and rejecting it. In the third and fourth groups there were no correct answers. Most participants applied the familiar formula $A = \frac{ah}{2}$ ignoring the important piece of information about the right angle.

Question 2. The correct answer is: $x \ge 2$. The composite function $y = f(g(x))$ is defined whenever both $g(x)$ and $f(g(x))$ are defined.

In the first group there was just one correct answer out of 14. In the second group there were 3 correct answers out of 26. In the third group there were 6 correct answers out of 10. In the fourth group there were 5 correct answers out of 26. Most participants did not pay attention to the definition of the domain of a composite function.

Question 3. The correct answer is: no solutions ($x = 1$ is outside of the domain of both log functions which can be easily checked by substitution).

In the first group there were 3 correct answers out of 14. In the second group there were 7 correct answers out of 26. In the third group there were 3 correct answers out of 10. In the fourth group there were 8 correct answers out of 26. Most participants ignored the restricted domain.

Question 4. The correct answer is: the 'identity' is not true. Squaring both sides doesn't prove it because this operation is irreversible. It is not an identity but an equation with infinitely many solutions $x \in [2\pi n, \pi(2n+1)]$.

In the first group there was just one correct answer out of 14. In the second group there were 5 correct answers out of 26. This question was note offered in the third group. In the fourth group there were 4 correct answers out of 26. Most participants assumed that it was provable because of the wording of the question and did not pay attention that squaring is an irreversible operation. Some participants 'proved' it geometrically by considering only the first quadrant.

Question 5. The correct answer is: there are no solutions. The equation $x^2 + \sqrt{x} + 1 = 0$ has no solutions (the left-hand side is always positive). If one can try to apply the Intermediate Value Theorem it is not applicable because the function $f(x) = \frac{x^2 + \sqrt{x} + 1}{x - 1}$ is not continuous on [0, 2].

In the first group there were no correct answers out of 14. In the second group there were 3 correct answers out of 26. In the third group there were 2 correct answers out of 10. In the fourth group there were 3 correct answers out of 26. Most participants misused the Intermediate Value Theorem – they checked only that $f(0) < 0$ and $f(2) > 0$ and did not check the continuity condition.

Question 6. The correct answer is: the derivative doesn't exist because the function doesn't exist as the argument of the log function is always negative.

In the first group there was just one correct answer out of 13 (one teacher did not attempt the question). In the second group there were 4 correct answers out of 26. In the third and fourth groups there were no correct answers. Most participants failed to check the domain of the function and applied the familiar Chain Rule.

Question 7. The expected answer is: it is not a definite integral because the function $y = \frac{1}{x}$ is not continuous on [-1, 1] and for this reason the Newton-Leibniz formula is not applicable. It is beyond the secondary school curriculum (it is an improper integral and in this particular case it is not defined).

In the first group there were no correct answers out of 14. In the second group there were 3 correct answers out of 26. In the third group there were no correct answers out of 10. In the fourth group there were 2 correct answers out of 26. Most participants failed to check the continuity condition of the Newton-Leibnitz formula and applied it. Some used graphs to produce incorrect solutions.

The questionnaire and participants' responses

After the discussion of the solutions the participants were given a short questionnaire to reflect on their performance in the test. The response rate in all groups was 100%. The questionnaire is below.

Question 1. What are your feelings after you have learnt about the correct solutions to the test questions?

In the first group the feelings were polarized: 50% of the teachers expressed self-criticism – "disappointed', "without thinking", "embarrassing", "didn't apply critical at all", "felt stupid – oversimplified the questions" while the other 50% were happy to learn lessons from the test – "enlightened, a very good test and ensure reflection upon teaching practice", "attention to the words and wider picture", "feel OK", "like it – I should have known…', "fun, I love anything that knocks me out of academic boredom", "a bit more enlightened", "glad that I have the opportunity to see and think through these problems".

In the second group the vast majority of the teachers reported that they were embarrassed and uncomfortable about their performance on the test.

In the third group, similar to the first group, there were equally polarized feelings from "not good feelings" and "anger" to "happy about not falling to every trap" and "laughing about my stupidity".

In the fourth group, similar to the second group, the majority of the participants (20 out of 26) were disappointed, surprised and uncomfortable about their performance with 3 more participants expressing mixed feelings – "funny and sad", "fifty-fifty".

As most of the participants gave incorrect answers to the vast majority of the questions it was interesting to notice different attitudes in two clusters: in groups 1 and 3 roughly half of the participants were disappointed and embarrassed while the other half were more positive and saw the opportunity for improvements; whereas in groups 2 and 4 the vast majority were very disappointed and uncomfortable. The difference between the two clusters might be due to culture.

Question 2. What are the reasons for not solving all test questions correctly?

In the first group all 14 teachers gave comments on the lack of attention and careful thinking. The common responses are as follows:

> Not thinking carefully about whether my solution method was appropriate to that particular problem; I did not think critically; not paying attention, impulsive reaction; I did not rely on my

> understanding but jumped straight to applying the rule; lack of knowledge and 'testing' things and being programmed to look for 'set' answers'; possibly that's how I was taught back home (South Korea), learnt lots of techniques (some difficult) but not to question the questions; not looking at all conditions; not thinking carefully and not reading the questions carefully; applying skills but not applying knowledge; not thinking about the structure of the expressions, considering its conditions; I knew there was something more to check but did not check thoroughly enough.

In the second group the teachers reported that the main reasons for making mistakes were carelessness and the expectation that each test questions had an answer (often a certain number).

In the third group the teachers commented on their emphasis on calculations and the 'tricky' nature of the test questions.

In the fourth group the main reasons according to the participants were carelessness, lack of knowledge and a habit to solve standard questions without thinking.

There were no comments about lack of time to finish the test so it is assumed it was not a reason for poor performance. In fact, the vast majority of the participants in all groups finished the test earlier than the allocated 15 minutes.

Question 3. Would you make any changes in your teaching practice after doing the mini-test? If so – which changes? If not – why?

In the first group all 14 teachers reported that they would make changes in their teaching practice after doing the test. The common responses were as follows:

> Introduce tricks like this to class to make them think; keep encouraging and creating environment where a deep conceptual knowledge is cultivated; encourage and reward checking of answers; more emphasis on the validity of solutions; teach them to examine the question thoroughly; give students more questions that will force them to think about the conditions surrounding the questions; I would encourage students to think through questions carefully; students need to understand, observe and consider answer to ensure they make sense and think before you solve; give students questions to challenge their knowledge; I try to make my students think more about restricted domains, check solutions and not trust graphical calculators; encourage kids to think about their solutions in light of the original question; give them problems occasionally that will 'trip' them up if they have not gone back and re-assessed their solutions; more emphasis on the nature of problem solving; stop answering impulsively, think before respond; I will expose students to such questions to get them to think more deeply about the conditions.

One teacher however, along with his/her positive response, made the following comment regarding the changes: "unless it is an element of the assessment I might not have time".

In the second group 13 out of 26 teachers reflected that they would change their teaching practice. Some of them reported that they would take the test back to their school and use it as teaching material. The other 13 teachers reported that they would not change their teaching practice as such test questions are not common.

In the third group all 10 teachers reported that they would make changes in their teaching practice after doing the test. The common responses were as follows: "re-think exercises", "discuss more special cases", "implementing exercises with surprising answers", "more emphasis on self-control".

In the fourth group 19 out of 26 participants commented that they would make changes in their teaching practice. The typical comments were: "stimulate student's thinking", "solving the questions not only according to an algorithm", "be attentive rather than solving automatically", "develop logical thinking", "develop thoughtfulness and reasoning".

ANALYSIS AND CONCLUSIONS

The study illustrated dialectic relationship between knowledge and attention. Is knowledge valuable when it is not applicable or applied incorrectly? In their performance on the test and their responses to the questionnaire, a majority of participants demonstrated a serious lack of attention and careful thinking that led them to fail most of the questions in the test. According to Mason and Spence (1999) the participants, in spite of having good *knowing-that* and *knowing-how* skills, need to enhance their *knowing-to act* skills in order to perform better: "active, practical knowledge, knowledge that enables people to act creatively rather than merely react to stimuli with trained or habituated behaviour involves knowing-to act, in the moment" (p.136). Knowing-to act in many cases is a multistep activity and each step needs attention. We are absolutely confident that the participants of the study had the relevant knowledge (e.g. they knew the domain of the log function, the range of the sine function, the conditions of the Intermediate Value Theorem, the definition of a definite integral, and so on). So the question was about their ability to *use* their knowledge on the test. Theories of attention developed by psychologists might be helpful in analysing the relationship between knowledge and attention. Deutsch and Deutsch (1963) argue that "however alert or responsive we may be, there is a limit to the number of things to which we can attend at any one time" (p.80). Kahneman's (1973) model of divided attention (when attention is divided between two or more concurrent tasks) suggests that attention can be flexibly allocated between tasks based on processing priority. Treisman & Gelade (1980) went further claiming that "without focused attention, features cannot be related to each other" (p.98). In solving mathematical questions attention is required to be given to each step and often the priority of allocation of attention to different steps is very important. In many cases attention is required for the 'analysis of the question' step (e.g. checking conditions of the rule, domain of the functions, type of the equation, locality of the statement, and so on) before switching attention to the next steps – procedure, verification, etc. Ignoring the 'analysis of the question' or 'question the question' step can lead to incorrect solutions especially in non-routine questions, as the study shows. In some cases, however, the order of steps can be changed. For example, one way to solve Question 3 of the test is to find the common domain of both log functions by first solving a system of two quadratic inequalities, then performing calculations using the log rules and finally checking whether the solution belongs to the common domain. An easier way however, is to just notice that we are dealing with the restricted domain without finding it (which can be time consuming), perform calculations using the log rules and then verify the resulting solution by substitution into the original equation. Feedback from the participants of the study show that one of the main reasons for poor performance on the test was not the priority of the steps but the ignoring of some of the crucial steps, in most cases the 'analysis of the question' step. The majority of the participants reported that they would definitely make changes in their teaching practice after the test by putting more emphasis on the analysis of the question before applying a certain formula or theorem. Those participants, who reported that they would not change their teaching practice because the test questions were uncommon, probably tend to 'teach to the test'. Including the type of questions from the mini-test into the assessment would encourage those teachers to pay more attention to details and analysis and enhance such skills in their students. After all, many situations in real life don't have a single 'correct' answer as is the case with routine questions from traditional assessments in mathematics. Solving non-routine, non-standard questions would better prepare students for the real world. Enhancing their own and their students' *discipline of noticing* by paying attention to details can also be a useful addition to teachers' professional development.

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