Attenuation of ocean waves due to random perturbations in the seabed profile

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KOZWAVES 2014 - Newcastle

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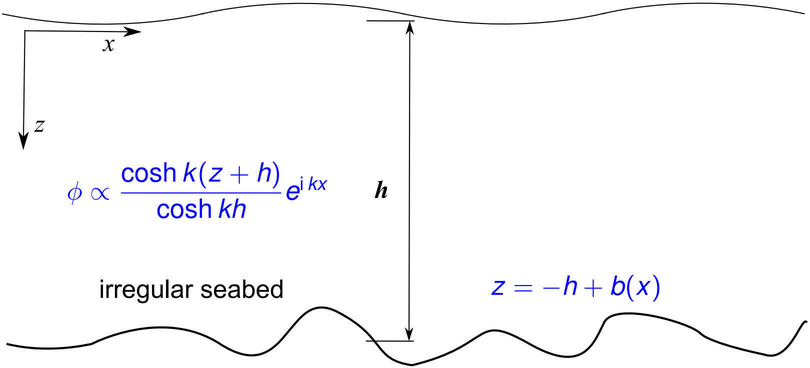


x
z
$\phi \propto \frac{\cosh k(z+h)}{\cosh kh} e^{\mathrm{i}\,kx}$
h
irregular seabed
$z = -h + b(x)$

Background

1. Linear PDEs and boundary conditions
2. Multi-scale expansion: slow variables
3. Attenuation of waves by randomly irregular seabed
4. Ensemble-average and realization-dependent solutions
5. Finding the above within the linear wave theory

Mathematics of waves over irregular seabed

Linear time-harmonic waves in incompressible fluid

- $\phi(x, z)$: velocity potential of water
- $\frac{\omega^2}{g}\phi = \partial_z\phi$: free surface condition

$$
\begin{aligned}
(\partial_x^2 + \partial_z^2)\phi = 0 \quad & \text{for} \quad -h + b(x) < z < 0 \\
(g\partial_z - \omega^2)\phi = 0 \quad & \text{for} \quad z = 0 \\
\partial_n\phi = 0 \quad & \text{for} \quad z = -h + b(x)
\end{aligned}
$$

Scaling regime

Scaling regime based on wavenumber k and small ε

1. The seabed shape given by smooth random process $b(x)$
2. $kh = O(1)$ and $kl_g = O(1)$, l_g is the correlation length of $b(x)$
3. $l_g/h = O(1)$, the seabed shape $b(x) = O(\varepsilon)$, the slope of the seabed $b'(x) = O(\varepsilon)$

Realizations of seabed

Stationary process

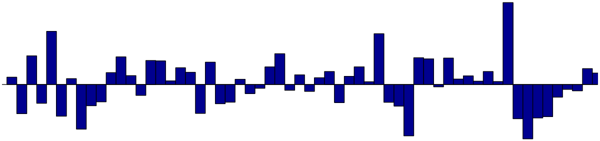
$$b(x) = \sigma \sqrt{\frac{2}{M}} \sum_{m=1}^{M} \cos\left(A_m x + B_m\right)$$

A_m and B_m are random variables that are determined by the prescribed probability density and auto-correlation functions.

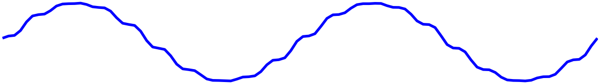
- PDF : $b(x)$ has the same normal distribution at any x
- Auto-correlation : Gaussian function
- Correlation/characteristic length l_g is the standard deviation (width) of the auto-correlation function

Other examples of $b(x)$

- Step-functions : series of random numbers



- Deterministic deviation from a periodic function: $\sin(x + \varepsilon g(x))$



In both cases, diffusion of a pulse over the seabed has been observed.

Multi-scale expansion

Introduction of slow variables

$$x_0 = x, x_1 = \varepsilon x, x_2 = \varepsilon^2 x, ...$$

Perturbation method

$$\phi = \phi_0 + \varepsilon\phi_1 + \varepsilon^2\phi_2 + \cdots$$

$$\partial_x = \partial_{x_0} + \varepsilon\partial_{x_1} + \varepsilon^2\partial_{x_2} + \cdots$$

Approximation of the seabed condition

$$\partial_n\phi(x, -h + b(x)) = \partial_z\phi - b'\partial_x\phi = 0$$

$$\phi_z + b\phi_{zz} + \frac{b^2}{2}\phi_{zzz} = b'\left\{\phi_x + b\phi_{xz} + \frac{b^2}{2}\phi_{xzz}\right\}$$

Attenuation in the leading order wave

Slow-attenuating wave

ϕ_0 satisfies the homogeneous BVP w.r.t. the fast variable x_0.

$$\phi_0(x_0, x_1, x_2) = \frac{\mathrm{i}\, gA(x_1, x_2)}{\omega^2} \frac{\cosh k(z+h)}{\cosh kh} e^{\mathrm{i}\, kx_0}$$

where k is the real root of the dispersion equation

$$gk \tanh kh = \omega^2$$

It turns out $A(x_2)$

$$A(x_2) \sim \exp\left(-\beta_i + \mathrm{i}\, \beta_r\right) x_2$$

Exponentially decaying w.r.t. the slow variable x_2.

Expression of ϕ_1

Seabed condition for ϕ_1

$$\partial_z \phi_1 = \partial_{x_0} \left(b(x_0) \partial_{x_0} \phi_0 \right), \qquad \text{for } z = -h$$

Expression of ϕ_1

$$\phi_1 = \int_{-\infty}^{\infty} \partial_{x'} (b(x') \partial_{x'} \phi_0) G(|x - x'|, -h)\, dx'$$

$G(|x - x'|, -h)$ is a Green's function for the Laplace equation with the seabed condition $\partial_z G = \delta(x - x')$ at $z = -h$.

Green's function

Green's function for the BVP of ϕ_1

$$G(\xi,-h)=\frac{\mathrm{i}\,\omega^2 e^{\mathrm{i}\,k|\xi|}}{\omega^2 kh+gk\sinh^2 kh}-\sum_{n=1}^{\infty}\frac{\mathrm{i}\,\omega^2 e^{\mathrm{i}\,k_n|\xi|}}{\omega^2 k_n h+gk\sin^2 k_n h}$$

where $\{\mathrm{i}\,k_n\}$ are the imaginary roots of the dispersion equation.

Expression of ϕ_2

Deriving the equation for $A(x_1, x_2)$

The ensemble average/coherent $\langle\cdot\rangle$ part of the equation for ϕ_2

$$
\begin{aligned}
(\partial_{x_0}^2 + \partial_z^2)\langle\phi_2\rangle &= 2\,\mathrm{i}\,k\partial_{x_2}\phi_0 \quad &\text{for} \quad & -h < z < 0 \\
(g\partial_z - \omega^2)\langle\phi_2\rangle &= 0 \quad &\text{for} \quad & z = 0 \\
\partial_z\langle\phi_2\rangle = \langle\partial_{x_0}(b(x_0)&\partial_{x_0}\phi_1)\rangle \quad &\text{for} \quad & z = -h
\end{aligned}
$$

$\langle\phi_2\rangle$ is expressed using the same $G(|x - x'|, -h)$. Then ϕ_0 and ϕ_1 are used to derive the equation for $A(x_2)$ w.r.t. the slow variable x_2

$$C_g \frac{\partial A}{\partial x_2} = \frac{\mathrm{i}(\beta_r + \mathrm{i}\,\beta_i)}{2\cosh kh} A(x_2)$$

Attenuation amplitude

Attenuation in the slow variable regime

$$A(x_2) = A(0)\exp\left[(-\beta_i + \mathrm{i}\,\beta_r)x_2/C_g\right]$$

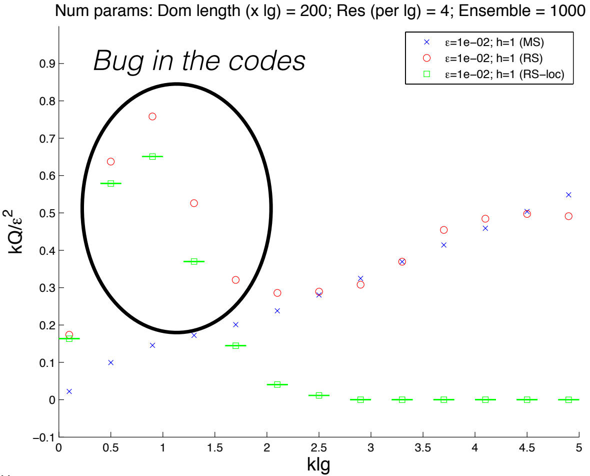
Attenuation parameters

$$Q_{\text{att}} = \beta_i \qquad \text{Attenuation rate}$$

$$L_{\text{loc}} = \frac{C_g}{\varepsilon^2\beta_i} \qquad \text{Localization length}$$

Attenuation happens at ε^{-2} order, and is sensitive to the range of parameters.

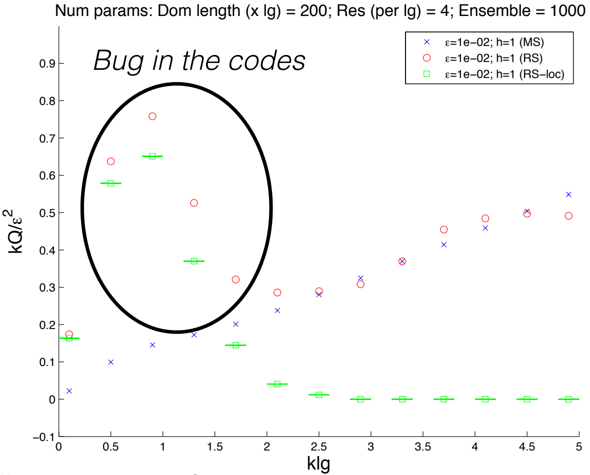
Numerical results



H

- M-Scale method $\sim$ numerical ensemble average

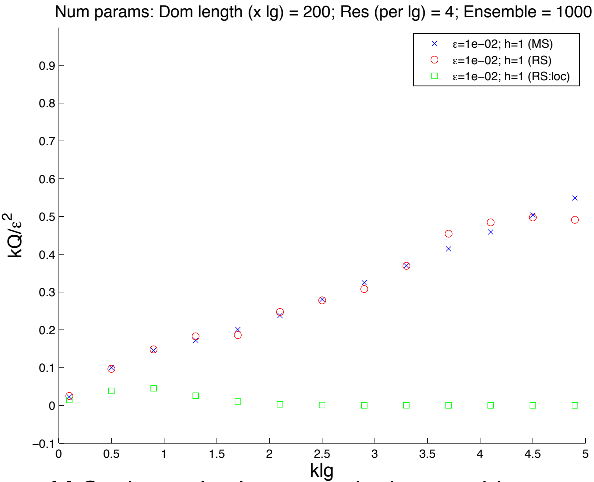
Numerical results



H

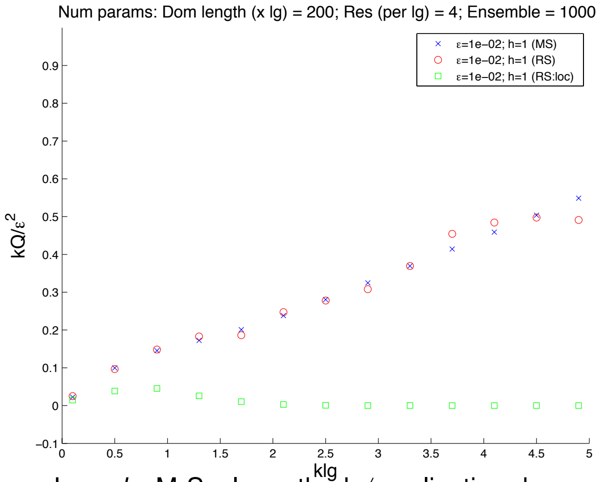
- Long l_g: M-Scale method $\neq$ realization dependent
- Short l_g: M-Scale method $\sim$ realization dependent

Numerical results



- M-Scale method $\sim$ numerical ensemble average

Numerical results



- Long l_g: M-Scale method $\neq$ realization dependent
- Short l_g: M-Scale method $\sim$ realization dependent

Summary

1. Linear wave equations can lead to attenuation in the ensemble average sense
2. The random seabed is simulated using harmonic random process satisfying the conditions of multi-scale expansion
3. There is a big discrepancy between the ensemble average solution and the realization dependent solution for weakly random seabed