Real Option Games with Stochastic Volatility

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Declaration

I hereby declare that this submission is my own work and that, to the best of my knowledge and belief, it contains no material previously published or written by another person (except where explicitly defined in the acknowledgements), nor material which to a substantial extent has been submitted for the award of any other degree or diploma of a university or other institution of higher learning.

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Abstract

This thesis presents several real option models to address investment-timing decision problems in various scenarios. The traditional NPV method only considers the difference between the future cash flow and the cost of a project, but ignores the future risk of the project. The concept of an American call option is used to improve the NPV method, and it is re-named as a real option. Classical real option problems are considered in a framework where the instantaneous volatility of the project value is given by a constant. Ting et al. [43] carried out an asymptotic approach in a single firm model by letting the volatility parameter be a stochastic process. In particular, they assumed that the project value is given by the Heston model. In this thesis, the project value is determined by Heston model, and a similar asymptotic approach is applied to classical real option models with two firms as well as another real option model in which suspending the project is allowed. Several numerical examples and comparisons are provided to show how the additional uncertainty in the volatility affects the investment thresholds and the payoffs of firms in different scenarios.

In addition, real option models with two firms can also be considered in competitive situations. Such models are also regarded as strategic real option games. This thesis presents several types of strategic real option games. In a standard framework of strategic real option games, 2-player non-cooperative games under complete information are considered, and both pure strategy equilibria and mixed strategy equilibria are obtained. If both firms agree to cooperate with each other in a game, then the game is called a 2-player cooperative game and the bargaining solution can be obtained. Lastly, we study mixed strategy equilibria of a strategic real option game with asymmetric information which is similar to that in [16]. Our result shows that the firm with complete information will always take the advantage.

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Chapter 1

Introduction

1.1 Historical Background

In economics and finance, an investment is defined as the act of incurring an immediate cost in the expectation of future rewards. A successful investment often brings a good profit in the future. Thus, calculating the profit in the unknown future will directly tell a firm whether a project is worth to invest. In 1938, motivated by the idea of discounted cash flows valuation, William [45] proposed that a stock is worth the present value of its future stream of dividends. Later on, the present value of a stock's future stream of dividends was introduced by Marshall as a concept into the analysis of investment decisions for situations such as when a firm budgets to take on capital projects or decides whether different projects should be considered. Nowadays, this concept is called the net present value (abbreviated as NPV), which is also known as the Marshall criterion. The NPV method has become the traditional measurement for the profit of a project. The classical economic theory of investments under uncertainty rests on the foundations of Marshall's analysis. According to the NPV method, if the net present value of a project is greater than zero, then the firm is recommended to invest in the project. However, this criterion can be very misleading, because it comes from the thinking that the choice is between acting right now and not investing at all. What happens if waiting for a while and then reassessing the decision is available? In fact, three features are common to most investment decisions. Firstly, an investment often has a sunk cost, which is irreversible. Secondly, the future cannot be predicted precisely, because things often have a probability to go better or worse. Thus, the best a firm can do is to calculate whether the mean value of a project gives a better return. Thirdly, a firm usually has some time to delay a project so that it can capture better information for the project. Therefore, a more appropriate approach on investment decisions under uncertainty is needed for a better and rigorous analysis.

In 1977, Myers [36] first realized that the opportunity of a firm holding to invest in a project could be viewed as an American call option. This concept provides a new viewpoint of investments, and also links the option pricing theory in modern finance to the irreversible investment analysis. Under his analysis, the value of a project is simply split into the present value already in place and the present value of the future growth. Nowadays, an option to invest in a project is called a real option, and this option has an intrinsic value that must be accounted for. By adopting the tools used to value financial options, the literature on real options has significantly improved our understanding of irreversible investment decisions under uncertainty. In financial option theory, Black and Scholes [3] proposed the most widely accepted model in 1973. In this model, the price of the underlying asset of an option is assumed to be a geometric Brownian motion. In 1985, Brennan and Schwartz [4] applied the real option analysis to the mining industry. They assumed that the value of a mine is a geometric Brownian motion. By applying the Black-Scholes pricing theory, Brennan and Schwartz developed a general mathematical model for their problem. Then in 1986, McDonald and Siegel [34] conducted further research on general investment problems. They assumed that both the present value and the cost of an investment project are geometric Brownian motions and gave a solution to the optimal investment timing problem. The pioneer work of Myers, McDonald and Siegel not only linked financial option theory to the study of irreversible investment analysis, but also led to a fertile area of research. Since then, many researchers in applied mathematics, economics and finance have been interested and worked in this area. A lot of work on real options has been summarised by Dixit and Pindyck in [11]. In a single firm case, also called a monopoly case, Dixit [9] modified the work of McDonald and Siegel by introducing the smooth pasting condition and gave a more rigorous result. However in his paper, Dixit also pointed out the assumption that the volatility of an asset price is constant in the geometric Brownian motion model is against many empirical studies. He suggested that geometric Brownian motions can be replaced by other stochastic processes. A popular extension is to use stochastic volatility model in valuing real options. In Finance, Heston [24] assumed that the volatility of the prices of an asset is an Ornstein-Uhlenbeck process, and gave a closed-form solution to the problem. Fouque et al. [12] provided a systematic treatment of the stochastic volatility for pricing derivatives in financial markets. Recently, they also studied the multiscale stochastic volatility for financial derivatives. Motivated by the work in [12], Ting et al. [43] presented an asymptotic analysis on real options under the Heston model.

Financial options usually grant the holder exclusive rights. In real option analysis, the framework of financial options has typically been applied to situations in which a single firm would have such rights to make the investment decision. In contrast, many firms operate within a competitive environment and their investment opportunities are not exclusive. Dixit and

Pindyck [11] claimed that a firm can treat the effect of competitors' investments on the path of its marginal profitability as an uncertainty, which is called the aggregate uncertainty. Suppose a firm's uncertainty is solely from its competitors. When the supply is fixed, the inverse demand (price) is proportional to the aggregate uncertainty. In particular, a high price implies a high level of demand. Then the firm's intuition suggests that there should be an upper threshold level, which if reached, will trigger new entry of other competitors. As soon as any new firm enters, the supply increases and price will drop back to a lower level. Thus the threshold becomes an upper barrier reflecting on the price process. This is different from a monopoly model because of the threat from the entry of rivals. In a monopoly model, there is a positive option value of waiting. Whereas under a competitive environment, the value of investing is negative for most of its price range and only just climbs to zero at the upper end of the price range. Thus the value of waiting is worth nothing. However, the trigger threshold for this extreme case is the same as that of the monopoly case. This is because for the monopoly case, the waiting leads up to an optimal entry threshold, for the competitive case, the threshold price bounds the firm's expectation in the future and will be the best price ever. The effect of the firm's current investment on the expected path of its marginal profitability of capital is called the firm-specific uncertainty. For the extreme case that only the firm-specific uncertainty exists, the value of waiting is the most important factor for the firm to decide when to invest. This is identical to the monopoly case. However, when determining an equilibrium, we must consider how many firms are in competition which results in a two-stage model. In the first stage, a firm has to pay for an entry cost to be able to learn its own demand shock. Once the demand exceeds the trigger threshold, the firm will invest in the project with the sunk cost. Then in the second stage, it either activates the project or dies out at the end of a certain period. These two features were embedded into the model, which are called the fresh entry and the Poisson death. In reality, both of these types of uncertainty often occur together. Thus, Caballero and Pindyck [6] combined both of these types of uncertainty into a general model and emphasized the effect of different sources of uncertainty such as price, demand or cost on the entry. It was shown that with the aggregate uncertainty, each firm knows that the others will enter the market and then result in a supply increase. Thus the firms' expected values of investments will be reduced. On the other hand, with the firm-specific uncertainty, each firm can take advantage of its own good fortune. However, the uncertainty makes the waiting more valuable, hence discourages the immediate investment. Therefore we need tools from not only quantitative finance but also economic theory, particularly, the theory of oligopoly. The amalgamation of real option theory with oligopoly results in a new exciting research area, namely strategic real option games. In a strategic real option game, we treat a firm's monopoly or oligopoly profit as a payoff to enter a particular market. Since this profit is time-dependent, the questions become when is the best time to exercise the option so that each firm would maximize its profit, and how strategic interactions within a competitive environment affect the value of the option. In reality, the market structure has another feature that we have to deal with: One firm may know more or less about the market than other firms. This asymmetry of information structure may significantly affect the predicted outcome, because although the net present value of the project seems identical across firms, its true value is in fact unknown. Dixit and Pindyck [11] also conducted a real option analysis in the situation where there are multiple firms to invest in one project. A typical case is a duopoly model in real options. Generally, there are two cases. In the first case, the leader and follower are pre-designed, and there are several investment thresholds depending on the initial value of the stochastic variable. In [17] the leader's real option has been studied and is also referred as the priority option. In the second case, two identical firms compete to be the leader. Suppose that the initial level of the stochastic variable is low. Then an unique competitive investment threshold at which the leader and the follower have the same payoff, can be determined. The equilibrium strategies turn out to be that one firm invests at that point and the other one will invest later or vice versa. Recently, Shackleton et al. [38] considered optimal strategies for two firms in a market that can only accommodate one active firm. In their model, the idle rival has the option to claim the market by paying a sunk cost. Assuming each firm's net operating profitability flow are mean reverting processes, they carried out a procedure similar to that in the standard analysis, which leads to solving a system of PDEs. In order to solve the system of PDEs, they used an optimal switching policy to reduce the problem's dimension and turned it to a system of ODEs. Moreover, they applied their model to a case study of Aircraft industry. In 2006, Marseguerra et al. [32] studied a duopoly model similar to the model in [11] but with a specified demand function. Their analysis has shown that the assumptions on both firms' ability to win the competition to enter the market and prospective pioneering leader advantages are crucial. In practice, it is hard to guarantee who wants to be the leader and who wants to be the follower. For example, suppose that two identical firms intend to enter a market and whoever is the first mover will take all profit, leaving the follower no profit at all. In this case, both firms will infinitely preempt another and destroy the option value of waiting. The problem with the above equilibria is that firms must somehow cooperate with each other in order to insure that both of them will not invest simultaneously. Such assumption was criticized by Huisman and Kort in [27]. In contrast, Huisman and Kort applied a mixed strategy approach developed by Fudenberg and Tirole [14] to analyse duopoly models. At such a mixed strategy equilibrium, both firms are allowed to invest simultaneously by chance. The original idea was due to Pitchik [37], who considered the equilibria of a similar game called "Two-person non-zerosum noisy game of timing". In this paper, each player's strategy is described by a cumulative probability distribution. However, this approach is not adequate for an investment game which is described in [14], as it does not specify the equilibrium for all the possible subgames. Moreover, when we take continuous-time games as a limit to discrete-time games, there will be some loss of information. Fudenberg and Tirole [14] modified this method by enlarging the strategy space to avoid the loss of information and determined a mixed strategy equilibrium for the game. As a consequence, a more realistic outcome can be found for classical strategic real option games using this concept. This result has also been applied in [2] and [15], where in [2] Bensoussan et al. used the approach of variational inequalities to investigate the strategic interaction between two firms competing for the opportunity to invest in the project with uncertain future values and in [15] Goto et al. investigated the entry and exit decisions for a symmetric duopoly real option game.

In the context of game theory, firms are assumed to have symmetric/asymmetric information about the others and are allowed to make decisions with their information. The above strategic real option games are based on complete information games. In the literature, duopoly models with incomplete information have also been considered in [16], [29], [31] and [35]. In [31], incomplete information was introduced to a strategic duopoly model in a way such that each of firms knows its own cost, but only knows that the cost of its rival is an independent draw from a probability distribution. Lambrecht and Perraudin [31] showed that each firm's entry threshold is situated between its Marshallian threshold and its non-strategic option threshold. The optimal investment threshold strikes a balance between the cost of being preempted and the option value of delaying to invest. Thus, both firms preserve some option value of delay. A duopoly model slightly different from the one in [31] was considered by Hsu and Lambrecht in [29]. In this model, two firms are named entrant and incumbent. It is assumed that the potential entrant knows everything about the incumbent, whereas the incumbent does not know the entrant's exact cost but only the probability distribution it is drawn from. This is also called a duopoly model with asymmetric information. Nevertheless, Hsu and Lambrecht [29] found that under this asymmetric information assumption, the incumbent acts at its Marshallian trigger, whereas the entrant epsilon preempts the incumbent if it is profitable to do so. It follows that the incumbent always sacrifices its option value of waiting. The cost of the informational disadvantage therefore corresponds to the option value of waiting. In contrast to this model, Graham [16] investigated the case where two firms have incomplete information with respect to the demand, and carried out a similar analysis. In Graham's model, one firm is fully aware of the true state of the demand whereas the other firm only knows the probability distribution of the demand parameter. Thus the uninformed firm can only observe the true state from the actions of the informed firm. Then he discovered that an equilibrium point might not exist within the standard continuous framework. When an equilibrium point does exist the competitive pressures from the uninformed firm are weak, i.e., when the uninformed firm can not do better by preempting the informed firm. In addition to these models, various approaches have also been employed to extend the theory of strategic real option games from complete markets to incomplete markets. For examples, Bensoussan et al. [2] also used the approach of variational inequalities to investigate the strategic interaction between two firms competing for the opportunity to invest in the project with uncertain future values, while Grasselli et al. [17] applied the utility-indifference arguments for similar purposes.

1.2 Research Questions

It is clear from the historical background that the following research questions are still unsolved.

Question 1.1. *Consider the monopoly real option model with stochastic volatility in [43]. Can it be extended to a duopoly model or a oligopoly model?*

Question 1.2. *How can we derive a result for real options with entry and exits decisions using an approach similar to that in [43] with stochastic volatility?*

Question 1.3. *Consider duopoly real option models in competitive situations. What are the outcomes in non-cooperative or cooperative games?*

Question 1.4. *Consider the game with incomplete information in [16]. Can we use the mixed strategy approach in [14] to find mixed strategy equilibria of the game?*

Question 1.5. *Consider the multiscale stochastic volatility model in pricing financial derivatives in [13]. Can we use a similar volatility model for real options?*

1.3 Thesis Contributions and Organization

The contributions of this thesis can be expressed in answering the questions given in Section 1.2. These answers are included in subsequent chapters, which are organized as follows.

Chapter 2: The intention of this chapter is to introduce mathematical preliminaries and economic models for studies of the research questions proposed previously. In particular, some notions and results in probability theory, stochastic calculus, dynamic programming, and perturbation theory are introduced. In addition, some basics in Economics and game theory are also introduced in order to extend simple models into more complicated scenarios.

Chapter 3: This chapter investigates various real option models for the monopoly case. We begin with a basic real option model and then extend it to a more complex case in which entry and exit decisions are considered. Then we recall the recent work of Ting and Ewald on real options with stochastic volatility in [43]. These models provide a background of our proposed topics. The approach in [43] can be used to study the real option model with entry and exit decisions under the framework of real options with stochastic volatility. This provides partial solution to Question 1.2. In the monopoly model, asymptotic solutions for both the real option value and the investment threshold of a firm were found. In addition, discussions on the effect of parameters of the model are obtained with several numerical examples and figures plotted by MATLAB.

Chapter 4: The aim of this chapter is to investigate duopoly real option models in which the leader and the follower are pre-determined. Similar to Chapter 3, we present a basic duopoly model with constant volatility. We first determine the follower's optimal investment threshold and its corresponding profit, then work backwards to determine the leader's counterparts. Following the procedure same as that in Chapter 3, we can extend the model to the case in which stochastic volatility is involved. By the existing results in Chapter 3, we obtain an asymptotic solution to the duopoly model. A similar procedure can be applied to the oligopoly model as well. We also discuss the effect of the parameters on the leader's option value and its optimal investment thresholds. This analysis provides a solution to Question 1.1.

Chapter 5: This chapter is designed to analyse duopoly real option models in competitive situations. A duopoly model can be embedded into several kinds of games. Firstly, we consider a basic duopoly model of 2-player non-cooperative games, which are the most common situations. Two approaches were established in the literature, namely the pure strategy equilibrium approach and the mixed strategy equilibrium approach. We also consider a duopoly model in a 2-player cooperative game. Then we analyse a duopoly model with entry and exit decisions in the context of these game situations. This analysis provides answers to Question 1.2 and Question 1.3. In the last section of Chapter 5, we answer Question 1.4 by providing a different approach for the duopoly model in [16]. Specifically, we apply the mixed strategy equilibrium approach to the duopoly model with asymmetric information.

Chapter 6: This is the last chapter of the thesis and is devoted to present conclusions and some future research directions.

1.4 Bibliographic Notes

This thesis is aimed to contribute the relevant research and deliver new knowledge to a wider scope of audiences by producing publications in leading journals and attending local and international conferences/workshops. During Bing Huang's study, the following papers, which are also listed as [7, 8, 26] in Bibliography, were written.

[**A**] Huang, B., Cao, J., and Chung, H., 2014. Strategic real options with stochastic volatility in a duopoly model. Chaos, Solitons and Fractals **58**, 40-51.

[**B**] Cao, J., Huang, B., 2014. Strategic real options with entry and exit decisions under stochastic volatility. Preprint.

[**C**] Cao, J., Huang, B., 2014. Strategic real options under asymmetric information: an mixed strategy approach. Preprint.

Some initial ideas were demonstrated by Huang at the 2012 New Zealand Mathematical Society Colloquium 2012, held at Massy University, Palmerston North. In 2013, Huang attended the 31st Australasian Economic Theory Workshop, held at the University of Queensland, Brisbane, Australia, and presented paper [**A**]. Furthermore, he also presented partial results from paper [**C**] at the 2013 Conference on Quantitative Methods in Finance, held in Sydney, Australia. All of these research discoveries provide a platform to enhance the research activities currently conducted at the university.

Chapter 2

Mathematical and Economics Preliminaries

In this chapter, some mathematical and economical terminologies and preliminaries are introduced. These include basic notations, definitions and many important facts, which will be used in the subsequent chapters. In Section 2.1, we build up the ground knowledge to our models by introducing some mathematical tools. These mathematical tools include stochastic calculus, dynamic programming, and perturbation theory, etc. In Section 2.2, we introduce some key concepts in Economics which helps us to understand the scenarios of our models. In Section 2.3, we study various kinds of games which can be used to analyse competitive situations.

2.1 Mathematics

In this section, we introduce some mathematical tools that are used in our analysis. First, we refer to [5], [39] and [30] for most of our definitions and important facts in probability theory and stochastic calculus. Some Poisson equations and invariant distributions are recalled from [12] and [44]. Moreover, we also introduce the method of perturbation as a tool to obtain asymptotic solutions for some problems. We refer to [25], [28], and [40] for the basics of this theory.

2.1.1 Basic Concepts in Probability Theory

In order to understand the concept of stochastic processes, it is necessary to understand some basic concepts in the probability theory, such as sample spaces, σ-fields and probability measures.

Definition 2.1. A set Ω collecting the outcomes of a particular experiment is called a *sample space*.

Definition 2.2. A *σ-field* $\mathcal{F}$ on a sample space Ω is a family of subsets of Ω such that

(1) $\emptyset \in \mathcal{F}$;

(2) if $A \in \mathcal{F}$, then so does $\Omega \setminus A$;

(3) if $\{A_n : n \geq 1\} \subseteq \mathcal{F}$, then $\bigcup_{n\geq 1} A_n \in \mathcal{F}$.

The ordered pair $(\Omega, \mathcal{F})$ is called a *measurable space*, and elements of $\mathcal{F}$ are called *events*.

In the probability theory, a σ-field is regarded as an information structure, and it tells the information about what events are likely to occur.

Definition 2.3. The σ-field generated on Ω by a collection of subsets $\mathcal{A}$ of Ω, denoted by $\sigma(\mathcal{A})$, is defined as

$$\sigma(\mathcal{A}) := \bigcap \{\mathcal{G} : \mathcal{A} \subseteq \mathcal{G} \text{ and } \mathcal{G} \text{ is a } \sigma\text{-field on } \Omega\}.$$

This means that the σ-field generated by $\mathcal{A}$ is the smallest σ-field that contains $\mathcal{A}$. The σ-field generated by the family of all open intervals on the set of real numbers $\mathbb{R}$ is denoted by $\mathcal{B}(\mathbb{R})$. Sets in $\mathcal{B}(\mathbb{R})$ are called *Borel sets*.

Definition 2.4. Let $(\Omega, \mathcal{F})$ be a measurable space. A set function $\mathbb{P} : \mathcal{F} \to [0, 1]$ is called a *probability measure* on $(\Omega, \mathcal{F})$ if the following holds:

(1) $\mathbb{P}(\Omega) = 1$;

(2) $\mathbb{P}\left(\bigcup_{n=1}^{\infty} A\right) = \sum_{n=1}^{\infty} \mathbb{P}(A_n)$ for a mutually disjoints subfamily $\{A_n : n \geq 1\} \subseteq \mathcal{F}$.

The triple $(\Omega, \mathcal{F}, \mathbb{P})$ is called a *probability space*. An event A is said to occur *almost surely* (abbreviated as a.s.) whenever $\mathbb{P}(A) = 1$. Two probability measures $\mathbb{P}$ and $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ are said to be *equivalent* if they agree which sets in $\mathcal{F}$ have probability zero.

Definition 2.5. Let $\mathcal{F}$ be a σ-field on Ω. A function $\xi : \Omega \to \mathbb{R}$ is called *$\mathcal{F}$-measurable* if $f^{-1}(B) \in \mathcal{F}$ for every Borel set $B \in \mathfrak{B}(\mathbb{R})$. If $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, then an $\mathcal{F}$-measurable function $\xi : (\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ is called a *random variable*.

For a random variable $\xi : (\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$, the σ-field $\sigma(\xi)$ generated by ξ is given by

$$\sigma(\xi) = \{\xi^{-1}(B) : B \in \mathfrak{B}(\mathbb{R})\}.$$

It is clear that $\sigma(\xi) \subseteq \mathcal{F}$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Two events $A, B \in \mathcal{F}$ are called *independent* if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B).$$

The *conditional probability* of A given B is defined as

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}.$$

By the mathematical manipulation of conditional probabilities, we can check that

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(B|A) \times \mathbb{P}(A)}{\mathbb{P}(B)}.$$

This is called the *Bayes' rule*. We can extend these concepts to random variables. Two random variables ξ and η are called *independent* if for any Borel sets $A, B \in \mathfrak{B}(\mathbb{R})$, the events $\xi^{-1}(A)$ and $\xi^{-1}(B)$ are independent. If two integrable random variables $\xi, \eta : \Omega \to \mathbb{R}$ are independent, then they are uncorrelated, i.e., $\mathbb{E}(\xi\eta) = \mathbb{E}(\xi)\mathbb{E}(\eta)$, provided that the product $\xi\eta$ is also integrable. Two σ-fields $\mathcal{G}$ and $\mathcal{H}$ contained in $\mathcal{F}$ are called *independent* if any two events $A \in \mathcal{G}$ and $B \in \mathcal{H}$ are independent. Finally, a random varibale ξ is *independent* of a σ-field $\mathcal{G}$ if the σ-fields $\sigma(\xi)$ and $\mathcal{G}$ are independent.

Definition 2.6. For any integrable random variable ξ and any event $B \in \mathcal{F}$ such that $\mathbb{P}(B) \neq \emptyset$, the *conditional expectation* of ξ given B is

$$\mathbb{E}(\xi|B) = \frac{1}{\mathbb{P}(B)} \int_B \xi \, d\mathbb{P}.$$

Definition 2.7. Let ξ be an integrable random variable and let η be a random variable. The conditional expectation of ξ given η is defined to be a random variable $\mathbb{E}(\xi|\eta)$ such that

(1) $\mathbb{E}(\xi|\eta)$ is $\sigma(\eta)$-measurable;

(2) for any $A \in \sigma(\eta)$, $\displaystyle\int_A \mathbb{E}(\xi|\eta) \, d\mathbb{P} = \int_A \xi \, d\mathbb{P}$.

Lemma 2.1. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $\mathcal{G}$ be a σ-field contained in $\mathcal{F}$. If ξ is a $\mathcal{G}$-measurable random variable and for any $B \in \mathcal{G}$,*

$$\int_B \xi \, d\mathbb{P} = 0,$$

then $\xi = 0$ a.s..

Definition 2.8. Let ξ be an integrable random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $\mathcal{G}$ be a σ-field contained in $\mathcal{F}$. The conditional expectation of ξ given $\mathcal{G}$ is defined to be a random variable $\mathbb{E}(\xi|\mathcal{G})$ such that

(1) $\mathbb{E}(\xi|\mathcal{G})$ is $\mathcal{G}$-measurable;

(2) for any $A \in \mathcal{G}$, $\int_A \mathbb{E}(\xi|\mathcal{G})\, d\mathbb{P} = \int_A \xi\, d\mathbb{P}$.

The existence and uniqueness of the conditional expectation $\mathbb{E}(\xi|\mathcal{G})$ is guaranteed by Lemma 2.1 and the following theorem.

Theorem 2.1 (Radon-Nikodym Theorem)**.** *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $\mathcal{G}$ be a σ-field contained in $\mathcal{F}$. Then for any integrable random variable ξ, there exists a $\mathcal{G}$-measurable random variable ζ such that*

$$\int_A \zeta\, d\mathbb{P} = \int_A \xi\, d\mathbb{P}$$

for each $A \in \mathcal{G}$.

Furthermore, some general properties of conditional expectations are given as follows:

(1) $\mathbb{E}(a\xi + b\zeta|\mathcal{G}) = a\mathbb{E}(\xi|\mathcal{G}) + b\mathbb{E}(\zeta|\mathcal{G})$ (*linearity*);

(2) $\mathbb{E}\left(\mathbb{E}(\xi|\mathcal{G})\right) = \mathbb{E}(\xi)$;

(3) $\mathbb{E}(\xi\zeta|\mathcal{G}) = \xi\mathbb{E}(\zeta|\mathcal{G})$ if ξ is $\mathcal{G}$-measurable (*taking out what is known*);

(4) $\mathbb{E}(\xi|\mathcal{G}) = \mathbb{E}(\xi)$ if ξ is independent of $\mathcal{G}$ (*an independent condition drops out*);

(5) $\mathbb{E}\left(\mathbb{E}(\xi|\mathcal{G})|\mathcal{H}\right) = \mathbb{E}(\xi|\mathcal{H})$ if $\mathcal{H} \subseteq \mathcal{G}$ (*tower property*);

(6) If $\xi \geq 0$, then $\mathbb{E}(\xi|\mathcal{G}) \geq 0$ (*positivity*).

2.1.2 Stochastic Processes

Definition 2.9. A *stochastic process* on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a family of random variables ξ_t parametrized by $t \in T$, where $T \subseteq \mathbb{R}$. For each $\omega \in \Omega$, the function

$$T \ni t \mapsto \xi_t(\omega)$$

is called a *sample path* of $\{\xi_t : t \in T\}$.

A stochastic process is often used to handle the changes of quantities under uncertainty. When $T = \mathbb{N}$, we shall say $\{\xi_t : t \in T\}$ is a stochastic process in *discrete time* (i.e., a sequence of random variables). A discrete time process $\{\xi_t : t \in \mathbb{N}\}$ is also called a *random walk*. There are different types of random walks and they are often assumed to have the *Markov property* which says the conditional probability distribution of the future states of the process depends only upon the present state. One type of random walks are called *discrete-time discrete-state random walks*. Suppose $\{\xi_t : t \in \mathbb{N}\}$ starts with a given value ξ_0 and at time $t = 1, 2, 3, \cdots$, takes a jump of size 1 either up or down with probability p or $1 - p$ respectively. Technically, one can describe the dynamics of ξ_t by

$$\xi_t = \xi_{t-1} + \varepsilon_t, \tag{2.1}$$

where

$$\mathbb{P}\{\varepsilon_t = 1\} = p \quad \text{and} \quad \mathbb{P}\{\varepsilon_t = -1\} = 1 - p. \tag{2.2}$$

Another type of random walks are called *discrete-time continuous-state stochastic processes*. They are similar to discrete-state random walks but with a jump size of any real number rather than an integer.

When T is an interval in $\mathbb{R}$ (typically, $T = [0, +\infty)$), we shall say $\{\xi_t : t \in T\}$ is a stochastic process in *continuous time*. One of the most typical continuous-time stochastic processes is called a *Brownian motion* or sometimes is also referred as a *Wiener process*. More precisely, a Brownian motion $\{z_t : t \geq 0\}$ satisfies the following properties:

(1) $z_0 = 0$ a.s.;

(2) the sample paths $t \to z_t$ are a.s. continuous;

(3) for any $0 \leq s < t$, $z_t - z_s$ has the normal distribution with mean 0 and variance $t - s$;

(4) $\{z_t : t \geq 0\}$ has independent increments, i.e., for $0 \leq t_1 < t_2 < \cdots < t_n$, $z_{t_1}, z_{t_2} - z_{t_1}, z_{t_3} - z_{t_2}, \cdots, z_{t_n} - z_{t_{n-1}}$ are independent.

A *filtration* $\mathbb{F} = \{\mathcal{F}_t : t \in T\}$ on Ω is a collection of σ-fields parametrized by $t \in T$ such that $\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}$ for any $s, t \in T$, and $s \leq t$. Here, $\mathcal{F}_t$ represents the knowledge or information at time t, and it contains all events A such that at time t it is possible to decide whether A has occurred or not. As t increases, there will be more such events A, i.e., the family $\mathcal{F}_t$ representing the knowledge will become larger. A stochastic process $\{\xi_t : t \in T\}$ is called *adapted* to filtration $\mathbb{F} = \{\mathcal{F}_t : t \in T\}$ if ξ_t is $\mathcal{F}_t$-measurable for each $t \in T$. Given a probability space $(\Omega, \mathcal{F}, P)$ and a stochastic process $\{\xi_t : t \in T\}$ such that each ξ_t is $\mathcal{F}$-measurable, let

$$\mathcal{F}_t = \sigma\left(\{\xi(s) : 0 \leq s \leq t\}\right).$$

This is all the information available from the observation of the process up to time t. Clearly, $\mathcal{F}_s \subseteq \mathcal{F}_t$ whenever $s \leq t$. These σ-fields form a filtration, which is called the *natural filtration* of the process $\{\xi_t : t \in T\}$.

Definition 2.10. A stochastic process $\{\xi_t : t \in T\}$ is called a *martingale* with respect to a filtration $\mathbb{F} = \{\mathcal{F}_t : t \in T\}$ if

(1) ξ_t is integrable for each $t \in T$;

(2) $\{\xi_t : t \in T\}$ is adapted to $\mathbb{F}$;

(3) $\xi_s = \mathbb{E}(\xi_t | \mathcal{F}_s)$ for every $s, t \in T$ such that $s \leq t$.

Lemma 2.2. *Let $\{z_t : t \geq 0\}$ be a Brownian motion, and $\mathbb{F} = \{\mathcal{F}_t : t \geq 0\}$ be its natural filtration. Then*

(1) $\{z_t : t \geq 0\}$ *is a martingale with respect to* $\mathbb{F}$*;*

(2) $\{z_t^2 - t : t \geq 0\}$ *is a martingale with respect to* $\mathbb{F}$*.*

Definition 2.11. If $f : [a, b] \to \mathbb{R}$ is a function of one real variable, its variation over the interval $[a, b]$ is

$$V_f[a, b] = \sup_{\Delta s \to 0} \sum_{i=0}^{n-1} |f(s_{i+1}) - f(s_i)|,$$

where $a = s_0 < s_1 < \cdots < s_n = b$, and $\Delta s = \max\{s_{i+1} - s_i : 0 \leq i \leq n-1\}$. If $V_f[a, b] < \infty$, then f is said to be a function of *bounded* (or *finite*) *variation*.

Definition 2.12. If $f : [a, b] \to \mathbb{R}$ is a function of one real variable, the *quadratic variation* of f over the interval $[a, b]$ (when it exists) is the limit

$$[f][a, b] = \lim_{\Delta s \to 0} \sum_{i=0}^{n-1} \left(f(s_{i+1}) - f(s_i)\right)^2,$$

where $a = s_0 < s_1 < \cdots < s_n = b$, and $\Delta s = \max\{s_{i+1} - s_i : 0 \leq i \leq n-1\}$.

Note that if f is continuous and of finite variation, then its quadratic variation is zero, see [30] for details. In addition, the quadratic covariation $[f,g][a,b]$ of two functions f and g on $[a,b]$ can be defined as follows.

Definition 2.13. Let f and g be functions of one real variable, the *quadratic covariation* of f and g on $[a,b]$ is defined by

$$[f,g][a,b] = \lim_{\Delta s \to 0} \sum_{i=0}^{n-1} (f(s_{i+1}) - f(s_i))\,(g(s_{i+1}) - g(s_i))\,,$$

where $a = s_0 < s_1 < \cdots < s_n = b$, and $\Delta s = \max\{s_{i+1} - s_i : 0 \leq i \leq n-1\}$.

Similarly, if f is continuous and g has finite variation on $[0,t]$, then $[f,g][0,t] = [f,g](t) = 0$. The definition of quadratic variation and covariation can be easily extended to stochastic processes. Given that a Brownian motion $\{z_t : t \geq 0\}$ is a martingale with respect to its filtration, the *quadratic variation* of the Brownian motion over $[0,t]$ is t.

Definition 2.14. A *generalized Brownian motion* $\{x_t : t \geq 0\}$ can be expressed as

$$dx_t = a(x_t,t)dt + b(x_t,t)dz_t, \tag{2.3}$$

where $a(x_t,t), b(x_t,t)$ are known functions and dz_t is the increment of a Brownian motion $\{z_t : t \geq 0\}$.

An important special case of generalized Brownian motion is the *geometric Brownian motion*, which satisfies

$$dx_t = \alpha x_t dt + \sigma x_t dz_t, \tag{2.4}$$

where the *drift parameter* α and the *volatility parameter* σ are constants. Geometric Brownian motions are useful because the Central Limit Theorem shows that the limit of a properly scaled binomial asset-pricing model leads to a stock price with a log-normal distribution. This is the basis of the well-known Black-Scholes model in the option pricing theory. It can be shown that the expected value of x_t given a initial value x_0, is

$$\mathbb{E}\,[x_t] = x_0 e^{\alpha t},$$

and the variance is given by

$$\mathbb{V}ar\,[x_t] = x_0^2 e^{2\alpha t}(e^{\sigma^2 t} - 1).$$

Another type of important stochastic processes that need to be introduced are the *mean reverting processes*. The simplest mean reverting process is called the *Ornstein-Uhlenbeck process*, which is in the following form

$$dx_t = k(m - x_t)dt + \sigma dz_t,$$

where k is called the *mean reverting rate*, m is the *mean reverting level* and σ is the *volatility parameter*. However we will need to use a slightly more complicated mean reverting process which is in the following form

$$dx_t = k(m - x_t)dt + \sigma \sqrt{x_t} dz_t. \tag{2.5}$$

This mean reverting process is also referred as the *Cox-Ingersoll-Ross process* or the CIR process. The difference between these two processes is that the variance in the CIR process is also affected by x_t. Unlike the *Ornstein-Uhlenbeck process*, when x_t in the CIR process is close to zero, the standard deviation also becomes very small and thus it would avoid the possibility of negative states. It is also worth to mention that for the CIR process $\{x_t : t \geq 0\}$, the expectation and the variation can be expressed as

$$\mathbb{E}\left[x_t|x_0\right] = x_0 e^{-kt} + m(1 - e^{-kt}), \tag{2.6}$$

$$\mathbb{V}ar\left[x_t|x_0\right] = x_0 \frac{\sigma^2}{k}(e^{-kt} - e^{-2kt}) + \frac{b\sigma^2}{2k}(1 - e^{-kt})^2, \tag{2.7}$$

respectively. The CIR process is very popular in modelling the interest rate and the volatility. In finance, empirical studies show that the volatility of a stock price is not a constant. Thus the stochastic volatility models have been introduced. One of such models is called the *Heston model*, which combines the geometric Brownian motion and the CIR process as follows:

$$dx_t = \alpha x_t dt + \sqrt{y_t} x_t dz_t, \tag{2.8}$$

$$dy_t = k(m - y_t)dt + \sigma \sqrt{y_t} dw_t, \tag{2.9}$$

where $\{z_t : t \geq 0\}$ and $\{w_t : t \geq 0\}$ are correlated Brownian motions with $[dz_t, dw_t] = \rho dt, (-1 \leq \rho \leq 1)$ and other parameters are similar to those defined previously. In this thesis, the Heston model will be used frequently.

2.1.3 Stochastic Integrals

Given a fixed time T, let $\{x_t : 0 \leq t \leq T\}$ be a stochastic process adapted to the filtration up to T, $\mathbb{F} = \{\mathcal{F}_t : 0 \leq t \leq T\}$, such that

$$\mathbb{E}\left[\int_0^T x_t^2 dt\right] < \infty. \tag{2.10}$$

By some mathematical manipulation and properties of Brownian motions, one can check that for $0 = t_0 < t_1 < \cdots < t_n = T$,

$$\mathbb{E}\left[\left(\sum_{i=0}^{n-1} x_{t_i}(z_{t_{i+1}} - z_{t_i})\right)^2\right] = \mathbb{E}\left[\sum_{i=0}^{n-1} x_{t_i}^2(t_{i+1} - t_i)\right], \tag{2.11}$$

which converges to $\mathbb{E}\left[\int_0^T x_t^2 dt\right]$. The stochastic integral of $\{x_t : 0 \leq t \leq T\}$ with respect to $\{z_t : 0 \leq t \leq T\}$ is defined as

$$\int_0^T x_t dz_t = \lim_{n\to\infty} \sum_{i=0}^{n-1} x_{t_i}\left(z_{t_{i+1}} - z_{t_i}\right). \tag{2.12}$$

From (2.11) we can get

$$\mathbb{E}\left[\left(\int_0^T x_t dz_t\right)^2\right] = \mathbb{E}\left[\int_0^T x_t^2 dt\right]. \tag{2.13}$$

The stochastic integral also satisfies the martingale property, i.e., for any $0 \leq s < t \leq T$

$$\mathbb{E}\left[\int_0^t x_\tau dz_\tau \middle| \mathcal{F}_s\right] = \int_0^s x_\tau dz_\tau,$$

and it's quadratic variation over $[0, T]$ satisfies

$$\lim_{n\to\infty} \sum_{i=0}^{n} \left(\int_0^{t_{i+1}} x_t dz_t - \int_0^{t_i} x_t dz_t\right)^2 = \int_0^T x_t^2 dt.$$

In stochastic calculus, a function of a Brownian motion defines a new stochastic process, which is continuous in time but is not differentiable. To work with such functions, we need to use Itô's lemma to differentiate or integrate them. Itô's lemma is somewhat similar to the chain rule in Calculus.

Theorem 2.2 (Itô's formula for Brownian motions)**.** *Let f be a twice differentiable function* *and* $\{z_t : t \geq 0\}$ *be a Brownian motion. Then for any* $t \geq 0$,

$$f(z_t) = f(0) + \int_0^t f'(z_s)\, dz_s + \frac{1}{2}\int_0^t f''(z_s)\, ds,$$

or

$$df(z_t) = \frac{df}{dz}\, dz_t + \frac{1}{2}\frac{d^2 f}{dz^2}\, dt.$$

As a stochastic version of the classical chain rule of differentiation, Itô's formula describes how a function of Brownian motion $f(z_t)$ changes stochastically as time progresses.

Theorem 2.3 (Itô's formula for generalized Brownian motions)**.** *Let* $\{x_t : t \geq 0\}$ *be a stochastic process which is given by* (2.3)*. Suppose that* $f(x_t, t)$ *is at least twice differentiable in* x *and once in* t*. Then the stochastic differential of the process* $\{f(x_t, t) : t \geq 0\}$ *exists and is given by*

$$df(x_t, t) = \left[a(x_t, t)\frac{\partial f}{\partial x} + \frac{\partial f}{\partial t} + \frac{1}{2}b^2(x_t, t)\frac{\partial^2 f}{\partial x^2}\right] dt + b(x_t, t)\frac{\partial f}{\partial x}dz_t. \tag{2.14}$$

Moreover, we may extend Itô's formula to a function that consists of two correlated stochastic processes $\{x_t : t \geq 0\}$ and $\{y_t : t \geq 0\}$ satisfying

$$\begin{aligned} dx_t &= a(x_t, t)dt + b(x_t, t)dz_t, \\ dy_t &= c(y_t, t)dt + d(y_t, t)dw_t, \end{aligned}$$

where $[dz_t, dw_t] = \rho dt$. Note that

$$d(x_t y_t) = x_t\, dy_t + y_t\, dx_t + \rho b(x_t, t)d(y_t, t)dt.$$

Theorem 2.4. If $f(x, y, t)$ is twice continuously differentiable in x, y, and continuously differentiable in t, then

$$\begin{aligned} df(x_t, y_t, t) &= \left[\frac{1}{2}b^2(x_t, t)\frac{\partial^2 f}{\partial x^2} + \frac{1}{2}d^2(x_t, t)\frac{\partial^2 f}{\partial y^2} + \rho b(x_t, t)d(y_t, t)\frac{\partial^2 f}{\partial x \partial y}\right] dt \\ &\quad + \left[\frac{\partial f}{\partial t} + a(x_t, t)\frac{\partial f}{\partial x} + c(y_t, t)\frac{\partial f}{\partial y}\right] dt + b(x_t, t)\frac{\partial f}{\partial x}dz_t + d(y_t, t)\frac{\partial f}{\partial y}dw_t. \end{aligned}$$

Itô's lemma is the most important tool in stochastic calculus. The difference between Itô's lemma and the ordinary chain rule is that it gives one extra term which is caused by the quadratic variation.

2.1.4 Invariant Distributions and Infinitesimal Generators

In this subsection, we introduce the invariant distribution of a stochastic process. Let $\{x_t : t \geq 0\}$ be a stochastic process. An initial distribution for x_0 is called *the invariant distribution* of $\{x_t : t \geq 0\}$ if for any $t > 0$, x_t has the same distribution. In other words, the invariant distribution of $\{x_t : t \geq 0\}$ must satisfy that for any $t > 0$,

$$\frac{d}{dt}\mathbb{E}[g(x_t)] = \frac{d}{dt}\mathbb{E}[\mathbb{E}[g(x_t)|x_0]] = 0, \tag{2.15}$$

where g is an arbitrary function.

Definition 2.15. The *infinitesimal generator* $\mathfrak{L}$ of $\{x_t : t \geq 0\}$ is defined by

$$\mathfrak{L}g(x) = \lim_{t\to 0}\frac{\mathbb{E}[g(x_t)|x_0 = x] - g(x)}{t},$$

for any arbitrary function g.

Suppose $\{x_t : t \geq 0\}$ is a *time-homogeneous* diffusion process, i.e.,

$$dx_t = a(x_t)dt + b(x_t)dz_t,$$

and g is a twice continuously differentiable function. By Itô's formula, we see that

$$dg(x_t) = \left[a(x_t)\frac{\partial g(x_t)}{\partial x} + \frac{1}{2}b^2(x_t)\frac{\partial^2 g(x_t)}{\partial x^2}\right]dt + b(x_t, t)\frac{\partial g(x_t)}{\partial x}dz_t.$$

Then

$$M_t = g(x_t) - \int_0^t \left[a(x_s)\frac{\partial g(x_s)}{\partial x} + \frac{1}{2}b^2(x_s)\frac{\partial^2 g(x_s)}{\partial x^2}\right]ds \tag{2.16}$$

defines a martingale. Taking the expectation on both sides of (2.16) and rearranging the terms give

$$\mathbb{E}[g(x_t)|x_0 = x] - g(x_0) = \mathbb{E}\left[\int_0^t \left[a(x_s)\frac{\partial g(x_s)}{\partial x} + \frac{1}{2}b^2(x_s)\frac{\partial^2 g(x_s)}{\partial x^2}\right]ds\middle|x_0 = x\right]. \tag{2.17}$$

By Definition 2.15, we can divide both sides of (2.17) by t and take the limit as $t \to 0$ to get

$$\begin{aligned}
\mathfrak{L}g(x) &= \mathbb{E}\left[\lim_{t\to 0}\frac{1}{t}\int_0^t \left[a(x_s)\frac{\partial g(x_s)}{\partial x} + \frac{1}{2}b^2(x_s)\frac{\partial^2 g(x_s)}{\partial x^2}\right]ds\middle| x_0 = x\right], \\
&= \mathbb{E}\left[\lim_{t\to 0}\left[a(x_t)\frac{\partial g(x_t)}{\partial x} + \frac{1}{2}b^2(x_t)\frac{\partial^2 g(x_t)}{\partial x^2}\right]\middle| x_0 = x\right], \\
&= a(x)\frac{\partial g}{\partial x} + \frac{1}{2}b^2(x)\frac{\partial^2 g}{\partial x^2}.
\end{aligned} \tag{2.18}$$

Since $\{x_t : t \geq 0\}$ is time-homogeneous, its statistics depends only on the time for which it has been running and not on the specific time. Thus by Definition 2.15, we can rewrite (2.15) as

$$\begin{aligned}
\frac{d}{dt}\mathbb{E}[g(x_t)] &= \lim_{\Delta t\to 0}\frac{\mathbb{E}[g(x_{t+\Delta t})] - \mathbb{E}[g(x_t)]}{\Delta t} \\
&= \mathbb{E}\left[\lim_{\Delta t\to 0}\frac{\mathbb{E}[g(x_{t+\Delta t}|x_t)] - g(x_t)}{\Delta t}\right] \\
&= \mathbb{E}\left[\lim_{\Delta t\to 0}\frac{\mathbb{E}[g(x_{\Delta t}|x_0)] - g(x_0)}{\Delta t}\right] \\
&= \mathbb{E}[\mathfrak{L}g(x_0)] = 0.
\end{aligned} \tag{2.19}$$

In the following chapters we are going to deal with the CIR process. Thus we will find the invariant distribution for CIR process $\{y_t : t \geq 0\}$ described by (2.9). By (2.18), we know that the infinitesimal generator of $\{y_t : t \geq 0\}$ is

$$\mathfrak{L} = k(m - y)\frac{\partial}{\partial y} + \frac{1}{2}\sigma^2 y\frac{\partial^2}{\partial y^2}. \tag{2.20}$$

The condition (2.19) implies that we must find the invariant distribution such that $\mathbb{E}[\mathfrak{L}g(y_0)] = 0$, which is also the distribution for y_0. Let us denote the density function of this distribution by $\Phi(y)$. Then

$$\mathbb{E}[\mathfrak{L}g(y_0)] = \int_0^\infty \Phi(y)\mathfrak{L}g(y)dy = 0, \tag{2.21}$$

where $\Phi(y)$ also satisfies the following conditions

$$\Phi(0) = 0, \tag{2.22}$$

$$\Phi(y) \to 0, \qquad \text{as } y \to \infty, \tag{2.23}$$

$$y\Phi(y) \to 0, \qquad \text{as } y \to \infty, \tag{2.24}$$

$$y\Phi'(y) \to 0, \qquad \text{as } y \to \infty, \tag{2.25}$$

and g is a bounded function whose derivative is also bounded. By using integration by parts and (2.22)−(2.25), we can rewrite (2.21) as

$$\int_0^\infty g(y)\mathfrak{L}^*\Phi(y)dy = 0, \tag{2.26}$$

where $\mathfrak{L}^*$ is the adjoint operator of $\mathfrak{L}$ defined by

$$\mathfrak{L}^* = -k\frac{\partial}{\partial y}((m-y)\cdot) + \frac{1}{2}\sigma^2\frac{\partial^2}{\partial y^2}(y\cdot). \tag{2.27}$$

If (2.26) holds for any smooth test function g, then it implies that the invariant distribution of $\{y_t : t \geq 0\}$ follows Gamma distribution with scale $\lambda = 2mk/\sigma^2$ and slope $\theta = \sigma^2/2k$, i.e.,

$$\Phi(y) = \frac{e^{-y/\theta}y^{\lambda-1}}{\Gamma(\lambda)\theta^\lambda}. \tag{2.28}$$

We can see this by checking

$$\mathfrak{L}*\Phi(y) = -k\frac{\partial}{\partial y}((m-y)\Phi(y)) + \frac{1}{2}\sigma^2\frac{\partial^2}{\partial y^2}(y\Phi(y)) = 0,$$

and that

$$\int_0^\infty \Phi(y) = 1.$$

Note that $\Phi(y)$ satisfies all the conditions (2.22)−(2.25) only when $\lambda > 1$. Furthermore, in the following chapters we will use $\langle\cdot\rangle$ to indicate taking the average with respect to the invariant distribution, whose density function is denoted by $\Phi_{inv}(y)$, of a process $\{y_t : t \geq 0\}$, i.e.,

$$\langle g(y)\rangle = \int_0^\infty g(y)\Phi_{inv}(y)dy.$$

2.1.5 Poisson Equations

Let $g : [0,\infty) \to \mathbb{R}$ be a smooth and bounded function with bounded derivative, and $\mathfrak{L}$ be the operator defined by (2.20). In this subsection, we shall consider the following equation,

$$\mathfrak{L}\chi(y) + g(y) = 0, \tag{2.29}$$

which is known as a Poisson equation for $\chi(y)$ with respect to the operator $\mathfrak{L}$ in the variable y. It is known that (2.29) does not have a solution unless the function $g(y)$ is centered with respect to the invariant distribution of the Markov process $\{y_t : t \geq 0\}$ whose infinitesimal generator is $\mathfrak{L}$, i.e.,

$$\langle g(y) \rangle = 0. \tag{2.30}$$

Equation (2.30) is called the *centering condition* and can be derived by using $\mathfrak{L}^*\Phi(y) = 0$, where $\mathfrak{L}^*$ is given by (2.27). If we assume the centering condition (2.30), a formal solution to (2.29) can be given by

$$\chi(y) = \int_0^\infty \mathbb{E}[g(y_t)|y_0 = y]dt. \tag{2.31}$$

All the solutions of the Poisson equation (2.29) can be obtained by adding constants to (2.31). These results are shown in [12] and [44].

2.1.6 Risk-neutral Probability Measures

Let $\{x_t : t \geq 0\}$ be a geometric Brownian motion. Usually, under the real world probability measure $\mathbb{P}$, there exists an arbitrage opportunity unless the drift parameter α equals to the risk-free interest rate r. The key results in [19] and [20] are the established connection between the concepts of absence of arbitrage and the existence of an equivalent martingale measure. In what follows, we will introduce the concept of equivalent martingale measures.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\{\mathcal{F}_t : t \geq 0\}$ be a filtration. Furthermore, suppose that Z is an almost surely positive random variable such that $\mathbb{E}_{\mathbb{P}}[Z] = 1$. We define $\mathbb{Q}$ such that

$$\mathbb{Q}(A) := \int_A Z(\omega) d\mathbb{P}(\omega) \qquad \text{for all} \quad A \in \mathcal{F}.$$

Then $\mathbb{Q}$ is a probability measure generated by Z on $(\Omega, \mathcal{F})$. It can be easily checked that $\mathbb{P}$ and $\mathbb{Q}$ are equivalent probability measures. Moreover, $\mathbb{P}$ and $\mathbb{Q}$ are related by the formula

$$\mathbb{E}_{\mathbb{Q}}[X] = \mathbb{E}_{\mathbb{P}}[XZ].$$

We call Z the *Radon-Nikodým derivative* of $\mathbb{Q}$ with respect to $\mathbb{P}$, and we write

$$Z = \frac{d\mathbb{Q}}{d\mathbb{P}}.$$

The *Radon-Nikodým derivative process* $\{Z_t : 0 \leq t \leq T\}$ is defined by

$$Z_t = \mathbb{E}_{\mathbb{P}}[Z|\mathcal{F}_t], \qquad 0 \leq t \leq T.$$

Since for any $0 \le s \le t \le T$,

$$\mathbb{E}_{\mathbb{P}}[Z_t|\mathcal{F}_s] = \mathbb{E}_{\mathbb{P}}[\mathbb{E}_{\mathbb{P}}[Z|\mathcal{F}_t]|\mathcal{F}_s] = \mathbb{E}_{\mathbb{P}}[Z|\mathcal{F}_s] = Z_s,$$

$\{Z_t : 0 \le t \le T\}$ is a martingale with respect to $\{\mathcal{F}_t : 0 \le t \le T\}$.

Theorem 2.5 (One-dimensional Girsanov theorem)**.** *Let $\{z_t : 0 \le t \le T\}$ be a Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\{\mathcal{F}_t : 0 \le t \le T\}$ be a filtration for this Brownian motion. Let $\{\xi_t : 0 \le t \le T\}$ be an adapted process. Define*

$$Z_t = \exp\left(-\int_0^t \xi_s dz_s - \frac{1}{2}\int_0^t \xi_s^2 ds\right),$$

$$\tilde{z}_t = z_t + \int_0^t \xi_s ds,$$

and assume that

$$\mathbb{E}_{\mathbb{P}}\left[\int_0^T \xi_s^2 Z_s^2 ds\right] < \infty.$$

Set $Z = Z_T$. Then $\mathbb{E}_{\mathbb{P}}[Z] = 1$, and under the equivalent probability measure $\mathbb{Q}$ generated by Z, the process $\{\tilde{z}_t : 0 \le t \le T\}$ is a Brownian motion.

Theorem 2.6 (Multi-dimensional Girsanov theorem)**.** *Let $\{\mathbf{z}_t : 0 \le t \le T\}$ be an n-dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\{\mathcal{F}_t : 0 \le t \le T\}$ be its filtration. Let ζ_t be an n-dimensional adapted process. Define*

$$Z_t = \exp\left(-\int_0^t \zeta_s \cdot d\mathbf{z}_s - \frac{1}{2}\int_0^t ||\zeta_s||^2 ds\right),$$

$$\tilde{\mathbf{z}}_t = \mathbf{z}_t + \int_0^t \zeta_s ds,$$

and assume that

$$\mathbb{E}_{\mathbb{P}}\left[\int_0^T ||\zeta_s||^2 Z_s^2 ds\right] < \infty.$$

Set $Z = Z_T$. Then $\mathbb{E}_{\mathbb{P}}(Z) = 1$, and under the probability measure $\mathbb{Q}$ generated by Z, the process $\{\tilde{\mathbf{z}}_t : 0 \le t \le T\}$ is an n-dimensional Brownian motion.

2.1.7 The Feynman-Kac Theorem

In general, stochastic differential equations can not be solved directly. A traditional way to solve stochastic differential equations is to transfer them into partial differential equations first, and then apply either analytical or numerical methods to solve the resulted PDEs. The Feynman-Kac theorem is a powerful tool that relates stochastic differential equations and partial differential equations.

Theorem 2.7 (One-dimensional case)**.** *Let $\{x_t : t \geq 0\}$ be a stochastic process satisfying the following one-dimensional stochastic differential equation*

$$dx_t = a(x_t, t)dt + b(x_t, t)d\tilde{z}_t,$$

where $\{\tilde{z}_t : 0 \leq t \leq T\}$ is a one-dimensional Brownian motion under the risk-neutral probability measure $\mathbb{Q}$. Let $h(x)$ be a Borel-measurable function and r be a constant representing the interest rate. Fix $T > 0$, for $t \in [0, T], x_t = x$, define $F(x, t)$ as the following conditional expectation,

$$F(x, t) = \mathbb{E}_{\mathbb{Q}}\left[e^{-r(T-t)}h(x_T)\middle|\mathcal{F}_t\right]. \tag{2.32}$$

Then $F(x, t)$ satisfies the following PDE

$$\frac{\partial F}{\partial t} + a(x, t)\frac{\partial F}{\partial x} + \frac{1}{2}b(x, t)^2\frac{\partial^2 F}{\partial x^2} - rF(x, t) = 0, \tag{2.33}$$

subject to the terminal condition $F(x, T) = h(x)$ for all x.

Theorem 2.8 (Multi-dimensional case)**.** *Let $\{\mathbf{x}_t : t \geq 0\}$ be an n-dimensional stochastic process satisfying the following n-dimensional stochastic differential equation*

$$d\mathbf{x}_t = \mathbf{a}(\mathbf{x}_t, t)dt + \mathbf{b}(\mathbf{x}_t, t)d\tilde{\mathbf{z}}_t, \tag{2.34}$$

where $\{\tilde{\mathbf{z}}_t : 0 \leq t \leq T\}$ is an m-dimensional Brownian motion, $\mathbf{a}(\mathbf{x}_t, t)$ is an n-dimensional vector and $\mathbf{b}(\mathbf{x}_t, t)$ is an $n \times m$ matrix under the risk-neutral probability measure $\mathbb{Q}$. Let $h(\mathbf{x})$ be a Borel-measurable function and r be a constant representing the interest rate. Fix $T > 0$, for $t \in [0, T], \mathbf{x}_t = \mathbf{x}$, define $F(\mathbf{x}, t)$ as the following conditional expectation,

$$F(\mathbf{x}, t) = \mathbb{E}_{\mathbb{Q}}\left[e^{-r(T-t)}h(\mathbf{x}_T)\middle|\mathcal{F}_t\right]. \tag{2.35}$$

Then $F(x, t)$ satisfies the following PDE

$$\frac{\partial F}{\partial t} + \sum_{i=1}^{n} a_i(x, t)\frac{\partial F}{\partial x_i} + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\mathbf{b}(\mathbf{x}, t)\mathbf{b}(\mathbf{x}, t)^T\right)_{ij}\frac{\partial^2 F}{\partial x_i \partial x_j} - rF(\mathbf{x}, t) = 0, \tag{2.36}$$

subject to the terminal condition $F(\mathbf{x}, T) = h(\mathbf{x})$ for all $\mathbf{x}$.

2.1.8 Dynamic Optimization

Dynamic programming is a very useful tool for dynamic optimization, especially when we deal with uncertainty. It breaks a whole sequence of decisions into two components: the immediate decision and the value of continuation which encapsulates the consequences of all subsequent decisions. In a finite time horizon case, the terminal payoff is usually given. Dynamic programming suggests that we start from the very last time interval since there is nothing following it and then apply standard optimization methods to determine the optimal decision. We can work from backwards all the way to the initial condition by using the results in precedent steps. However in an infinite time horizon case, it is impossible to work in the same way, but a theoretical characterization of the solution can be determined in some cases. The idea behind the dynamic programming is formally stated as follows

Bellman's principle of optimality: *An optimal policy has the property that, whatever the initial action is, the remaining choices constitute an optimal policy with respect to the subproblem starting at the state that results from the initial action.*

The theory of dynamic programming is perfectly general. However, the purpose of this subsection is to provide an example to show how dynamic programming can be applied to a firm's decision analysis. In particular, suppose that a firm considers an investment opportunity which involves uncertainty in the future payoff $V(x_t, t)$. The uncertainty is driven by a factor x_t. At time t the current value of x_t is given, but the future values of x_t are unknown. Thus for simplicity, we can describe x_t by a stochastic process. At any time t the firm can decide to pay a constant sunk cost I to invest in the project and get an immediate payoff $V(x_t, t) - I$ or continue to wait for a better chance to invest. Let $F(x_t, t)$ denote the payoff at time t. The continuation value is denoted by $F(x_{t+dt}, t + dt)$. When the immediate payoff has been taken, there are no future values of waiting. On the other hand, if the firm chooses to wait, there will not be any immediate payoff but only the discounted continuation value $e^{-rdt}F(x_{t+dt}, t + dt)$. Based on the Bellman's principle of optimality, the problem given here forms a special case of the *Bellman equation*. The aim of the firm is to maximise its payoff,

$$F(x_t, t) = \max\left\{V(x_t, t) - I, e^{-rdt}F(x_{t+dt}, t + dt)\right\}, \tag{2.37}$$

where a sequence of controls $\{u_t : t \geq 0\}$ for each period t are hidden. This is because u_t here is set as a binary variable such as 1 for investing and 0 for waiting. For some values of x_t, the immediate payoff is better than the continuation value and the converse is true for other values of x_t and u_t maps x_t to 0, 1. Thus the problem becomes to find the optimal investment threshold x^* at which $u_t = 1$.

2.1.9 Basics of Perturbation Theory

Generally, an exact solution to a problem cannot be found in a closed form. Perturbation theory is a collection of methods used to find an approximate solution to such a problem. The basic idea of perturbation theory is to introduce a parameter ε into the problem, and express the solution in terms of a power series in ε. Then the behaviour of the solution can be determined by letting $\varepsilon \to 0$. Motivated by the classical Taylor series, a smooth function $f(\varepsilon)$ at $\varepsilon = \varepsilon_0$ can be expressed as

$$f(\varepsilon) = \sum_{n=0}^{\infty} \frac{f^{(n)}(\varepsilon_0)}{n!}(\varepsilon - \varepsilon_0)^n. \tag{2.38}$$

In perturbation theory, we are interested in

$$\lim_{\varepsilon \to \varepsilon_0} f(\varepsilon), \qquad \varepsilon - \varepsilon_0 > 0.$$

The first goal of perturbation theory is to construct an *asymptotic solution* so that it is close to the exact solution. In order to understand the concept of an asymptotic solution, it is necessary to introduce the concept of *gauge functions*. The simplest examples of gauge functions are the powers of $\varepsilon, 0 < \varepsilon < 1$. By using the gauge function, denoted by $g(\varepsilon)$, it is sufficient to determine whether $f(\varepsilon)$ converges faster or slower than the given function $g(\varepsilon)$ as $\varepsilon \to \varepsilon_0$. To do this, we must have the following definitions

Definition 2.16. The function $f(\varepsilon)$ is said to *be of order* $g(\varepsilon)$, denoted by $f(\varepsilon) = O(g(\varepsilon))$ as $\varepsilon \to \varepsilon_0$, if there are constants M and ε_1 (independent of ε) such that

$$|f(\varepsilon)| \leq M|g(\varepsilon)|, \qquad \varepsilon_0 < \varepsilon < \varepsilon_1.$$

Definition 2.17. The function $f(\varepsilon)$ is said to *be less than order* $g(\varepsilon)$, denoted by $f(\varepsilon) = o(g(\varepsilon))$ as $\varepsilon \to \varepsilon_0$, if for every positive δ there is an ε_2 (independent of ε) such that

$$|f(\varepsilon)| \leq \delta|g(\varepsilon)|, \qquad \varepsilon_0 < \varepsilon < \varepsilon_2.$$

Moreover, it is easily to check that as $\varepsilon \to \varepsilon_0$

$$f(\varepsilon) = o(g(\varepsilon)) \Rightarrow f(\varepsilon) = O(g(\varepsilon))$$

Then to see the asymptotic behaviour of the function when $\varepsilon \to \varepsilon_0$, we shall introduce the asymptotic series.

Definition 2.18. The series $\sum_{n=0}^{N} c_n f_n(\varepsilon)$ is said to be an *asymptotic series* of $f(\varepsilon)$ at $\varepsilon = 0$ if $f_n(\varepsilon) = o(f_{n-1}(\varepsilon))$ for $n = 1, \cdots, (N+1)$ and

$$f(\varepsilon) = \sum_{n=0}^{N} c_n f_n(\varepsilon) + O(f_{N+1}(\varepsilon))$$

as $\varepsilon \to 0$.

In [25], the asymptotic series is defined using $o(f_N(\varepsilon))$ instead of $O(f_{N+1}(\varepsilon))$. It is equivalent to Definition 2.18 if the series is a convergent power series. This asymptotic series gives a asymptotic solution to the problem. In order to find the asymptotic series, we must use the following theorem [40, p12].

Theorem 2.9 (The fundamental theorem of perturbation theory)**.** *If an asymptotic series satisfies*

$$A_0 + \varepsilon A_1 + \varepsilon^2 A_2 + \cdots + \varepsilon^N A_N + O(\varepsilon^{N+1}) \equiv 0 \tag{2.39}$$

for all sufficiently small $\varepsilon > 0$ and $\{A_i : 0 \leq i \leq N\}$ are independent of ε, then

$$A_0 = A_1 = \cdots = A_N = 0. \tag{2.40}$$

Perturbation theory and asymptotic analysis are particularly useful in solving differential equations, but they can also be applied to a broad class of other problems. Furthermore, we should also make an imprecise distinction between *regular perturbation problems* and *singular perturbation problems*. A regular perturbation problem is the one for which the perturbed problem for small, nonzero values of ε is qualitatively the same as the unperturbed problem for $\varepsilon = 0$. A singular perturbation problem is the one for which the perturbed problem is qualitatively different from the unperturbed problem. Although singular perturbation problems may appear atypical, they are the most interesting problems to study because they allow us to understand qualitatively new phenomena. In the sequel chapters, we will also be dealing with singular perturbation problems when the stochastic volatility models are considered.

2.2 Economics

In this section, we first introduce some basic concepts on investment decision-making. Then we also discuss various concepts on market structures. These concepts are essential to build up our models. The detailed explanations can be found in [11], [32] and [41].

2.2.1 Basics of Investment Decision-making

When a firm tries to decide whether to invest in a project, it must consider several factors such as cost, profit and risk, etc. There are several types of cost that an investment can involve. When an investment has not yet occurred, there is usually a cost to start it. Then during the production, the project may also have some running cost. Sometimes, if the project is no longer profitable, the firm may also be able to suspend it. This may also require a cost. Most importantly, these costs are irreversible. Thus firms must be very careful when they make such decisions. Profit is the key to measure an investment. However, it is usually uncertain over the future. One may assume that the profit P follows

$$P = XYD, \tag{2.41}$$

where X and Y represent the demand shocks, D represents the demand function which is shocked by X and Y. Although there are general studies in the demand function itself in Economics, in this thesis, only a simple form of demand function will be considered. We assume that the demand function $D(Q)$ follows

$$D(Q) = N - SQ, \tag{2.42}$$

where N denotes a measure of the market dimension and S denotes the sensitivity of the price to the quantity Q of a product. For simplicity, throughout this thesis, we will assume that a firm only produces one unit output if it invests in a project. Thus the quantity of the product can indicate how many firms have invested in the project. The demand shocks X and Y represent the firm-specific shock and industry-wide shock respectively. The firm-specific shock takes effect of the firm's current investment on the expected path of its marginal profitability of capital. It is the risk that is unique to the firm itself, for example, bad management decisions of the firm. The industry-wide shock represents the risk that comes from the entire industry. Typically, in this thesis the industry-wide shock represents the uncertainty that comes from whether or not other competitors will enter the market and drag the price down.

To value an investment, the concept of discounted cash flow valuation, namely the net present value (NPV), has been introduced. More precisely, the NPV can be expressed mathematically as follows

$$NPV = C_0 + \sum_{t=1}^{\infty} \frac{P_t}{(1+r)^t}, \tag{2.43}$$

where C_0 denotes the expenditure that an investment requires and $\{P_t : t \geq 1\}$ denotes the steam of profits that are revived. The intuition behind the NPV method is that if the sum of the discounted profit flows exceeds the expenditure, then the firm is recommended to invest in the project. However, one suggest that the future uncertainty should be considered when calculating the NPV and analysing the decision at each time interval. Unfortunately, the uncertainty can not be simply determined by a probability distribution. To overcome the problem, the concept of real options was introduced and has become a generalized method for investment decision analysis. The real option model uses a stochastic process to represent the uncertainty in the future, then determines a best investing time for a firm. The purpose of this thesis is to study real options under different scenarios.

2.2.2 Market Structures

In Economics, one type of ideal markets are those consisting of many agents or firms, but none of whom would be able to control the price of the product. Such markets are called *markets with perfect competition*. In real option analysis for markets with perfect competition, it usually assumes that the industry-wide shock follows a particular probability distribution and a firm is able to calculate its upper threshold level to invest in the project. If this level is reached, it will trigger a new entry of other competitors. As soon as any new firm enters the market, the price will drop back to a lower level. Thus the threshold becomes an upper barrier reflecting the price process and the value of investing is negative for most part of its price range and only just climbs to zero at the upper end of the price range, see [11] for details. However, it is more practical to investigate the real option analysis under imperfect competition situations.

The fundamental case of markets with imperfect competition, in which there is just one firm of a good or service, is called *monopoly*. The firm has a high degree of control over the price of the product and can take advantage of its own good fortune. In monopoly, a firm only needs to concern about the firm-specific risk which makes the waiting more valuable. Thus the firm will invest at the best investing time based on the given information and earn a positive value of waiting. However in reality, monopoly markets hardly exist. In markets with imperfect competition, the fear of rivals must be considered. In order to do this, we can consider a simple case, namely *duopoly*. A duopoly involves exactly two firms in the analysis, in which firms must consider the firm-specific risk as well as the competition from the rival. The value of preempting its rival often offsets the value of waiting of a project. Markets with similar characteristics but more than two firms are called *oligopoly*. A special model of duopoly, called the *Stackelberg model*, specifies a leader and a follower in a market in order to avoid the competition. Most of concepts for a duopoly model can be extended naturally to a oligopoly model. Thus we will focus on duopoly models in this thesis.

2.3 Game Theory

An ideal tool to analyse firms' behaviour in a market will be to use game theory. In this section, based on [33], [18] and [42], we introduce some key concepts and results on several types of games with complete information. Since we only focus on duopoly models in this thesis, we will describe some of the solution concepts in terms of 2-player games. However, most of the theory can be easily extended to n-player games which correspond to oligopoly models. Then we also study games with incomplete information. The basics of games with incomplete information was developed by Harsanyi in his papers [21], [22] and [23]. His work was generalised in [33] and [46], which includes several aspects in games with incomplete information. We will also investigate real option games with incomplete information in this thesis.

2.3.1 Extensive Form and Strategic Form Games

Game theory is typically applied in the situation where there are conflicts of interests between different parties. These parties are identified as *players* in a game. The results of a game are usually denoted by the payoffs of each player. There are two typical ways to form a game mathematically, namely the extensive form and the strategic form. In this subsection, we introduce basic concepts of extensive form games. We start with the concept of a game tree.

Definition 2.19. A triple (V, E, x_0) is called a *game tree*, where the distinguished vertex x_0 is called the *root*, V is a finite collection of nodes, called *vertices*, and E denotes a finite collection of *edges* each of which connects two vertices.

We say a vertex $\hat{x}$ *follows* another vertex x if there is an unique path from x_0 to $\hat{x}$ passing through x. If $\hat{x}$ follows another vertex x and there is an edge joining x and $\hat{x}$, we say $\hat{x}$ follows x *immediately*. Each edge that leads from a vertex x to one of its immediate followers is called a *possible action* at x. The set of all actions at x can be denoted by $A(x)$. A vertex x is called a *terminal vertex* if no vertex follows it.

Definition 2.20. An *extensive form game* is an ordered vector

$$\Gamma = (N, V, E, x_0, \{P_i : i \in N\}, \mathcal{U}, f),$$

where

(1) $N = \{1, 2, \cdots, n\}$ is a finite set of players;

(2) (V, E, x_0) is a game tree that is defined in Definition 2.19;

(3) $\{P_i : i \in N\}$ is a partition of the non-terminal vertices of a game tree into pairwise disjoint sets $P_1, \cdots, P_n$.

(4) $\mathcal{U}$ is the set of all possible outcomes;

(5) f is a payoff function associating every terminal vertex with a game outcome in the set $\mathcal{U}$.

For each $i \in N$, P_i can be partitioned into m_i many subsets $P_i^1, P_i^2, \cdots, P_i^{m_i}$ such that for each $1 \leq j \leq m_i$, each vertex $x_i^{jk} \in P_i^j, k = \{1, 2, \cdots, l_{ij}\}$, there are exact same number (say y_{ij}) of immediate followers and no vertex can follow another. Then $P_i^1, P_i^2, \cdots, P_i^{m_i}$ are called *information sets* of player i. For each information set P_i^j, let $A(P_i^j)$ be a partition of $\bigcup_{k=1}^{l_{ij}} A(x^{jk})$ (which has $l_{ij} * y_{ij}$ edges) into y_{ij} many disjoint sets, each of which contains one element from the sets $\{A(x^{jk}) : k = 1, 2, \cdots, l_{ij}\}$.

Definition 2.21. Player i is said to have *perfect information* in a game if each of his information set P_i^j, contains only one vertex. An extensive form game is called a *game with perfect information* if all of the players have perfect information in the game.

We can define a strategy as a plan for playing a game which is mathematically defined as follows.

Definition 2.22. A *strategy* of player i is a function which assigns, to each of his information sets P_i^j, one of the actions available at that information set, i.e.,

$$s_i : \mathcal{P}_i \to \bigcup_{j=1}^{m_i} A(P_i^j), \tag{2.44}$$

where $\mathcal{P}_i = \{P_i^1, P_i^2, \cdots, P_i^{m_i}\}$ is the collection of player i's information sets, and for each information set $P_i^j \in \mathcal{P}_i$,

$$s_i(P_i^j) \in A(P_i^j). \tag{2.45}$$

This is also called a *pure strategy*. Generally speaking, a pure strategy is a description of the actions that a player chooses at all the possible vertices in a game tree. However, in many cases, drawing a game tree can be difficult, especially for a large game. Hence, we may represent games in a more convenient form, called the *strategic form*.

Definition 2.23. A game in strategic form is an ordered triple $G = (N, (S_i)_{i \in N}, (f_i)_{i \in N})$, in which:

(1) $N = \{1, 2, \cdots, n\}$ is a finite set of players.

(2) S_i is the set of strategies of player i, for every player $i \in N$. We denote the set of all vectors of strategies by $S = S_1 \times S_2 \times \cdots \times S_n$.

(3) $f_i : S \to \mathbb{R}$ is a function associating each vector of strategies $s = (s_i)_{i \in N}$ with the payoff $f_i(s)$ to player i, for every player $i \in N$.

In fact, once a game is transformed into the strategic form, it is easy to observe that some strategies do worse in every situations than the others. These strategies are identified as dominated strategies. Moreover, strategies that give the exact same payoff as the other strategies are identified as duplicated strategies. The elimination of dominated and duplicated strategies is used to reduce a game in the strategic form, so that it can be easier to analyse. In the rest of this thesis, we will use the strategic form.

2.3.2 Non-corporative Games and Nash Equilibria

Most of real-life situations can be described as non-cooperative games, in which players, namely a and b, are only interested in maximizing their own payoffs U^a and U^b. A special form of non-cooperative games, in which the sum of the players' payoffs is zero no matter what strategies are used, are called *2-person zero sum games*. In such games, if player a tries to maximize its own payoff, it also minimizes player b's payoff. Thus the payoff of player a also reflects player b's payoff. Without loss of generality, we can denote the maximum of the minimum of U^a by V_l, and call it the *maximin value*. Similarly, the minimum of the maximum of U^b can be denoted by V_u, and called the *minimax value*. If $V_l = V_u$, then this value is called *the value of the game* and is denoted by V. Any pair of strategies corresponding to the value of the game is called an *equilibrium*. An equilibrium point together with the value of the game is called a *solution* of the game.

A classical result in Game Theory claims that a 2-player non-competitive game with finite many strategies and perfect information possesses an equilibrium point in pure strategies. However, players usually do not have perfect information in a game. Such games are called *imperfect information games*. In imperfect information games the existence of a pure strategy equilibrium can not be guaranteed. Thus we need to introduce the concept of mixed-strategies. Simply speaking, a *mixed-strategy* for a player is a convex combination of her/his pure strategies. All the pure strategies that appears in a mixed strategy with a positive probability are called the *worthwhile strategies*.

Definition 2.24. In a 2-player game, let X denote the set of all mixed strategies for player A and Y denote the set of all mixed strategies for player B. A pair of strategies $x^* \in X, y^* \in Y$ is called a *Nash equilibrium* for the game if for any $x \in X, y \in Y$, the payoffs for players U^a and U^b satisfy

$$U^a(x, y^*) \leq U^a(x^*, y^*) \text{ and } U^b(x^*, y) \leq U^b(x^*, y^*).$$

Theorem 2.10 (Nash's Theorem)**.** *Any 2-player game with a finite number of pure strategies has at least a Nash equilibrium.*

Furthermore, it is also worth to mention that there is another special form of non-cooperative games, called *repeated games*. Unlike ordinary games, a repeated game has one or more entry in the payoff such that if the players choose the corresponding strategy, they will replay the game until they choose another strategy or reach the limit of the number of times that game can be played. If there exists a terminal payoff, then the very last sub-game can be solved by the standard method. In this case, the solution to the whole game can be obtained by working from backwards. However, there are situations in which games can be played infinitely many times. In these cases, the whole games have to be solved recursively. Note that sometimes it may not be possible to construct recursive equations.

2.3.3 Cooperative Games

In reality, players can usually communicate and cooperate with each other in order to do better in a game. In a 2-player cooperative game, we assume that the players can discuss the game before the game starts, and then cooperate with each other. This means that binding contracts can be made, and that utility can be transferred from one player to the other. Thus the problem becomes how to preplay the negotiation instead of actually playing the game. In a non-cooperative game, players choose their own randomised strategies to confuse the opponent, whereas in a cooperative game, players choose their randomised strategies to use as a way of arbitrating between outcomes. These are called the *jointly randomised strategies*, which can be used by players to maximize their payoffs together. Let (x, y) and (x', y') be two pairs of jointly randomised strategies which gives the payoffs (u, v) and (u', v') respectively. We say (x, y) is *jointly dominated* by (x', y') if $u' \geq u, v' \geq v$ and $(u', v') \neq (u, v)$. Furthermore, (x', y') is said to be *Pareto optimal* if it is not jointly dominated by any other randomised strategy. One of the major solution concepts is called the maximin bargaining solution. Let (u_0, v_0) be the maximin payoff, i.e., the payoff if players play the game non-cooperatively. Then the *bargaining set* $\mathcal{B}$ can be defined by

$$\mathcal{B} = \{(u, v) : u \geq u_0, v \geq v_0, \quad (u, v) \text{ is Pareto optimal}\}$$

This says that players should only be interested in Pareto optimal payoffs and the bargaining payoff should be at least as much as (u_0, v_0). Here (u_0, v_0) is called the *status quo point*. Now we should determine which point of the bargaining set should be chosen. It can be checked that the payoff region, denoted by $\mathcal{P}$, of a finite cooperative game is always closed, bounded and convex. To find the optimal point, denoted by (u^*, v^*), we define a arbitration procedure such that

$$\psi((u_0, v_0), \mathcal{P}) = (u^*, v^*). \tag{2.46}$$

Moreover, the procedure must satisfy the following axioms which are called *Nash's bargaining axioms*:

(1) $u^* \geq u_0, v^* \geq v_0$.

(2) $(u^*, v^*) \in \mathcal{P}$.

(3) (u^*, v^*) must be Pareto optimal.

(4) Suppose that (u^*, v^*) is the solution of the procedure for $((u_0, v_0), \mathcal{P}_1)$. If $(u^*, v^*) \in \mathcal{P}_2$ and $\mathcal{P}_2 \subset \mathcal{P}_1$, then (u^*, v^*) is also the solution of the procedure for $((u_0, v_0), \mathcal{P}_2)$.

(5) For any $(\lambda_1, \lambda_2) > 0$ and (γ_1, γ_2), let $\mathcal{P}'$ be

$$\mathcal{P}' = \{(\lambda_1 u + \gamma_1, \lambda_2 v + \gamma_2) : (u, v) \in \mathcal{P}\}.$$

If (u^*, v^*) is the solution of the procedure for $((u_0, v_0), \mathcal{P})$, then $(\lambda_1 u + \gamma_1, \lambda_2 v + \gamma_2)$ is the the solution of the procedure for $((\lambda_1 u_0 + \gamma_1, \lambda_2 v_0 + \gamma_2), \mathcal{P}')$

(6) If $(u, v) \in \mathcal{P}$ implies $(v, u) \in \mathcal{P}$, then $u_0 = v_0$ implies $u^* = v^*$.

Next, we shall find the maximin bargaining solution. Consider an arbitration procedure ψ which satisfies the following.

(1) If there exists some $(u, v) \in \mathcal{P}$ with $u > u_0, v > v_0$, define $f(u, v) = (u - u_0)(v - v_0)$ over $(u, v) \in \mathcal{P}$. Then there always exists a unique point (u^*, v^*) which maximises $f(u, v)$, and we can define the procedure to be $\psi((u_0, v_0), \mathcal{P}) = (u^*, v^*)$.

(2) If there is no $(u, v) \in \mathcal{P}$ satisfying $u > u_0, v > v_0$, then there is either $v = v_0, u > u_0$ or $u = u_0, v > v_0$. If there exists a u^* which maximises $f(u, v_0)$, then we can define the procedure to be $\psi((u_0, v_0), \mathcal{P}) = (u^*, v_0)$.

(3) Similarly, for the other case where there exists a v^* such that $f(u_0, v)$ is maximised, we can define the procedure to be $\psi((u_0, v_0), \mathcal{P}) = (u_0, v^*)$.

The above arbitration procedure ψ satisfies Nash's bargaining axioms, and will produce an unique solution to the game, called the *maximin bargaining solution*.

Furthermore, players may also choose to play threaten strategies rather than the maximin strategies if both players can not agree on the payoff they request. This means that they will only be able to get the threaten payoff which is even less than the maximin value. In this case, the game will still have an unique equilibrium. The corresponding bargaining solution is called the *Nash bargaining solution*.

Theorem 2.1. *Any finite 2-player cooperative game possesses at least one equilibrium of threaten strategies and all equilibria lead to the same value.*

How to find the optimal threats depends on the form of the negotiation set. When the negotiation set is of the form $\{(u, v) : \lambda u + v = \gamma, c_0 \leq u \leq c_1\}$, the objective function f can be written as

$$f(u) = (u - e^a)(\gamma - \lambda u - e^b),$$

where e^a and e^b are the threat payoffs for Player a and Player b respectively. To find the maximum value of $f(u)$, we can differentiate $f(u)$ and set the derivative $f'(u)$ equal to 0. This gives

$$u = \frac{1}{2\lambda}(\gamma + \lambda e^a - e^b). \tag{2.47}$$

Suppose that there are n many strategies for Player a and m many strategies for Player b. Let e^a_{ij} and e^b_{ij} be the payoffs of the players a and b when a uses his ith strategy and b uses his jth strategy respectively. Then in terms of a game with finite many strategies, (2.47) can be represented as

$$u = \frac{1}{2\lambda}\left(\gamma + \sum_{i=1}^{n}\sum_{j=1}^{m} x_i(\lambda e^a_{ij} - e^b_{ij})y_j\right),$$

where $x = (x_1, x_2, \cdots, x_n)$ and $y = (y_1, y_2, \cdots, y_m)$ are mixed strategies of the players a and b respectively. To make u as large as possible, a needs to maximise $\sum_{i=1}^{n}\sum_{j=1}^{m} x_i(\lambda e^a_{ij} - e^b_{ij})y_j$, given y as a mixed strategy that is chosen by b. Similarly for Player b, we get

$$\begin{aligned} v &= \frac{1}{2}(\gamma - (\lambda e^a - e^b)), \\ &= \frac{1}{2}\left(\gamma - \sum_{i=1}^{n}\sum_{j=1}^{m} x_i(\lambda e^a_{ij} - e^b_{ij})y_j\right). \end{aligned} \tag{2.48}$$

Then to make v as large as possible, b needs to choose a mixed strategy $y = (y_1, y_2, \cdots, y_m)$ such that $\sum_{i=1}^{n}\sum_{j=1}^{m} x_i(\lambda e^a_{ij} - e^b_{ij})y_j$ is minimised for any x chosen by a. Thus in order to find the optimal threats, the players need to play a zero-sum game. Let w denote the value of the zero-sum game, and let (x^*, y^*) be the corresponding optimal strategies. We can substitute this into (2.47) and (2.48) so that the bargaining solution is

$$u = \frac{1}{2\lambda}(\gamma + w), \tag{2.49}$$

$$v = \frac{1}{2}(\gamma - w), \tag{2.50}$$

provided that $c_0 \le u \le c_1$. Otherwise, the bargaining solution must be at one of the end points of the negotiation set. In the subsequent chapters, we will show how a cooperative game can be used for a competitive real option situation to improve the payoffs of both firms.

2.3.4 Incomplete Information Games

Games discussed in the previous subsections concern situations where players have complete information, i.e., everything in a game is common knowledge to every player. Games with incomplete information are used to describe situations where players have some private information about their own payoffs or available choices. There are many kinds of private information that a player can have. For instance, there could be private information about the actual outcome produced by each strategy of the game; There could be private information on the utility function that gives the actual outcome; Or there could be private information about available strategies for other players. However, John Harsanyi [21] generalized the above situations to private information on the payoff functions by introducing the type of players. Thus game with incomplete information can be defined as follows

Definition 2.25. A *game with incomplete information* is a vector $(N, (T_i)_{i\in N}, \mathbb{P}, S, (s_t)_{t\in \times_{i\in N} T_i})$ where:

(1) $N = \{1, \cdots, n\}$ is a finite set of players.

(2) T_i is a finite set of types for player i, for each $i \in N$, and the set of type vectors is denoted by $T = \bigtimes_{i \in N} T_i$.

(3) $\mathbb{P} \in \triangle(T)$ is a probability distribution over the set of type vectors that satisfies $p(t_i) := \sum_{t_{-i} \in T_{-i}} p(t_i, t_{-i}) > 0$ for every player $i \in N$ and every type $t_i \in T_i$, called the *prior probability distribution* ($t_{-i} = (t_j)_{j \neq i}$, and $T_{-i} = (T_j)_{j \neq i}$).

(4) S is a set of states of nature, which will be called *state games*. Every state of nature $s \in S$ is a vector $s = (N, (A_i)_{i \in N}, (u_i)_{i \in N})$, where A_i is a nonempty set of actions (pure strategies) of player i and $u_i : \bigtimes_{i \in N} A_i \to \mathbb{R}$ is the payoff function of player i.

(5) $s_t = (N, (A_i(t_i))_{i \in N}, (u_i(t))_{i \in N}) \in S$ is the state game for every $t \in T$. Player i's action set in the state game s_t depends on his type t_i only, and is independent of types of the other players.

For a game with incomplete information, we can also introduce another factor called the state of world, denoted by $Y \subseteq S \times T$. Before a game starts, there is a state of nature ω representing a chance move. Then players are assigned with a type vector $t = (t_1(\omega), t_2(\omega), \cdots, t_n(\omega)) \in T$ according to the prior probability distribution $\mathbb{P}$. Let $A(t) := \bigtimes A_i(t_i)$ denote the set of available actions to all players. Each player i only knows the type of himself t_i but does not know the exact type of other players. Instead, he has an initial believe about the distribution of the types of others. According the Bayes' rule, players can update their prior probability distribution as

$$\mathbb{P}(t_{-i}|t_i) = \frac{\mathbb{P}(t_i, t_{-i})}{\mathbb{P}(t_i)}.$$

Then the game can be transformed into an equivalent game with complete information, in which the payoff function can be represented by $u_i(t; a)$, where $a = (a_1, a_2, \cdots, a_n) \in A(t)$. However, a new notion of strategies, called the normalized strategies, should be introduced.

Definition 2.26. A *normalized strategy* (pure strategy) of player i is a function $s_i : T_i \to \bigcup_{t_i \in T_i} A_i(t_i)$ which specifies an action for the player for each type $t_i \in T_i$ such that

$$s_i(t_i) \in A_i(t_i), \ \forall t_i \in T_i.$$

Definition 2.27. A *behaviour strategy* x_i of player i is a function mapping each type $t_i \in T_i$ to a probability distribution over the actions available to that type, i.e., for each $t_i \in T_i$,

$$x_i(t_i) = (x_i(t_i; a_i))_{a_i \in A_i(t_i)} \in \triangle(A_i(t_i)).$$

Now, let $x = (x_1, x_2, , \cdots, x_n)$ be a vector of the players' behaviour strategies. We can denote the conditional expected payoff to player i of type t_i as

$$U_i(x|t_i) := \sum_{t_{-i}\in T_{-i}} \mathbb{P}(t_{-i}|t_i)u_i((t_i, t_{-i}); x).$$

Then an equilibrium of a game with incomplete information, called a Bayesian Nash equilibrium, can be defined as follows.

Definition 2.28. A strategy vector $x^* = (x_1^*, x_2^*, \cdots, x_n^*)$ is a *Bayesian Nash equilibrium* if for each player $i \in N$, each type $t_i \in T_i$, and each possible action $a_i \in A_i(t_i)$,

$$U_i((x_i, x_{-i}^*)|t_i) \leq U_i(a_i^*|t_i).$$

By the existing theorem of the Nash equilibrium, we also have the following theorem for games with incomplete information.

Theorem 2.2. *Every game with incomplete information in which the set of types is finite and the set of actions of each type is finite has a Bayesian Nash equilibrium.*

In subsequent chapters, several kinds of real option games with incomplete information will be studied.

Chapter 3

Real Option Models under Monopoly

In real option analysis, the classical NPV method has been modified by involving future uncertainty and irreversibility of the sunk cost, and generalized in a continuous time frame. The aim of this chapter is to introduce various real option models in monopoly. In Section 3.1, we first present a basic real option model and then extend it to a real option model with entry and exit decisions. In Section 3.2, we study the models obtained in Section 3.1 with stochastic volatility involved. Furthermore, to confirm our theoretical results, we also study the effect of parameters on the firms' decisions and their corresponding profits. Throughout this chapter, we consider a situation where a firm has the potential to activate a project with the following assumptions.

(A1) The project requires a sunk cost I.

(A2) Let x be the demand shock which represents the firm's uncertainty and $D(\cdot)$ be the demand function which is given by (2.42). Then the profit flow of the firm can be expressed as $xD(\cdot)$.

(A3) Let r and δ be the risk-free interest rate and the dividend yield respectively with $r-\delta > 0$.

(A4) The investment decisions can be postponed infinitely so that the real option value is independent of time t.

3.1 Geometric Brownian Motion Real Option Models

In this section, we present a basic real option model for investing in a project which is similar to that in [11]. Since real options can also be considered to suspend a project, we also study a model similar to that in [15], which extends the basic model to a case where entry and exit decisions are involved.

3.1.1 Real Options to Market Entry

In order to study the basic real option model, we should make the following assumptions in addition to **(A1)−(A4)**:

(A5) There is no running cost for the project and the firm will operate the project forever once it invests.

(A6) The demand shock is a geometric Brownian motion $\{x_t : t \geq 0\}$.

Under **(A1)−(A6)**, according to the Girsanov theorem, there exists a probability measure $\mathbb{Q}$ equivalent to the real world probability measure $\mathbb{P}$. Under $\mathbb{Q}$, define

$$\tilde{z}_t = z_t + \frac{\alpha - (r - \delta)}{\sigma} t.$$

Then, (2.4) can be transformed to

$$dx_t = (r - \delta)x_t dt + \sigma x_t d\tilde{z}_t.$$

Given the fact that the firm's investment opportunity is equivalent to an American call option, the goal of this model is to determine when it is optimal for the firm to pay the sunk cost and activate the project so that the profit is maximized. Thus the decision problem has been transformed to an optimal stopping time problem. The following result is taken from [11].

Proposition 3.1. *In a monopoly market, a firm's option value, $F(x)$, is given by*

$$F(x) = \begin{cases} \left(\dfrac{x^* D}{\delta} - I\right)\left(\dfrac{x}{x^*}\right)^{\beta_1}, & \text{if} \quad x < x^*, \\ \\ \dfrac{xD}{\delta} - I, & \text{if} \quad x \geq x^*, \end{cases}$$

where β_1 and the optimal threshold x^* are given by

$$\beta_1 = \frac{1}{2} - \frac{r - \delta}{\sigma^2} + \sqrt{\left[\frac{r - \delta}{\sigma^2} - \frac{1}{2}\right]^2 + \frac{2r}{\sigma^2}}, \tag{3.1}$$

$$x^* = \frac{\beta_1 I \delta}{(\beta_1 - 1)D}, \tag{3.2}$$

respectively. The corresponding investment time T^* is given by

$$T^* = \inf\{t \geq 0 : x \geq x^*\}. \tag{3.3}$$

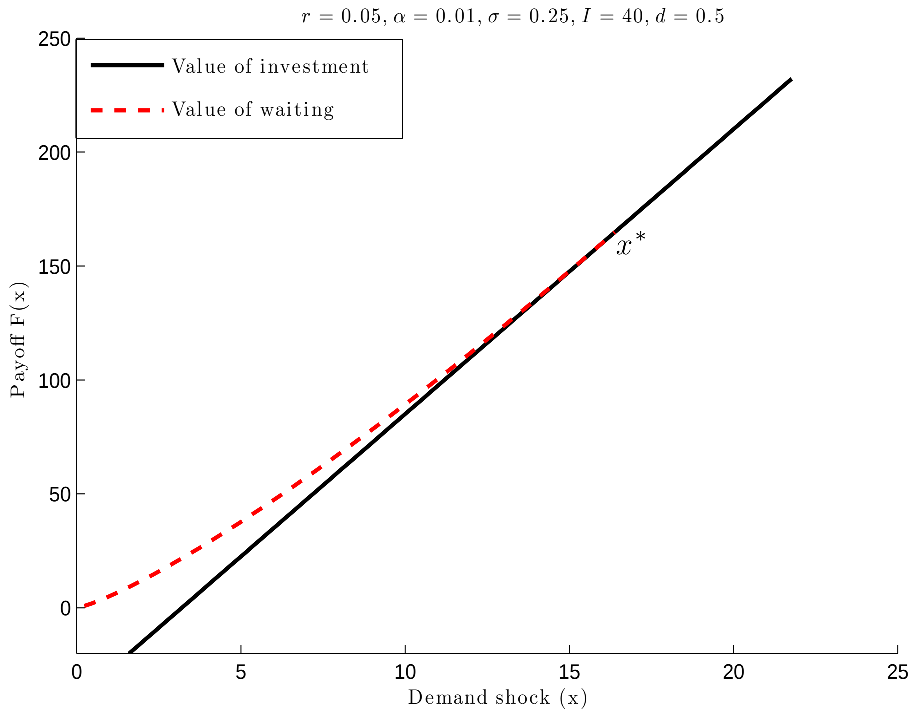

Figure 3.1: Real option value for investing in the project

In Figure 3.1, the dashed curve represents the value of waiting when $x < x^*$. The straight line represents the NPV of the project. It is clearly shown that it is much better for the firm to wait when the demand shock is low. For the sake of completeness, we present a detailed derivation for Proposition 3.1. In order to find the firm's real option value, we first determine how much the value of the project is worth. It is known that the value of project is not a one-time payoff but is uncertain in the future. By **(A2)** and **(A5)** once the firm invests in the project at time T^* with the current shock level $x_{T^*} = x^*$, the value of project is given by

$$V(x_{T^*}) = \int_{T^*}^{\infty} e^{-r(t-T^*)}\mathbb{E}[x_t D]dt. \tag{3.4}$$

For a small interval of time $(t, t + dt)$, the value of the project at time t can be expressed as the sum of the operating profit over the interval $(t, t + dt)$ and the continuation value as follows

$$V(x_t) = x_t D dt + \mathbb{E}[e^{-rdt}V(x_{t+dt})|x_t]. \tag{3.5}$$

By Taylor's expansion theorem, e^{rdt} can be expressed as

$$e^{rdt} = 1 + rdt + \frac{(rdt)^2}{2!} + \frac{(rdt)^3}{3!} + \cdots .$$

By substituting this into (3.5), expanding the right-hand side of (3.5) according to Itô's lemma, and omitting the higher order terms of dt, we obtain

$$\frac{1}{2}\sigma^2 x^2 \frac{d^2 V}{dx^2} + (r - \delta)x\frac{dV}{dx} - rV + xD = 0. \tag{3.6}$$

Equation (3.6) can be solved by finding a particular solution and the general solution to its associated homogeneous equation

$$\frac{1}{2}\sigma^2 x^2 \frac{d^2 V}{dx^2} + (r - \delta)x\frac{dV}{dx} - rV = 0. \tag{3.7}$$

To find such a particular solution, we recall that $\{x_t : t \geq 0\}$ is a geometric Brownian motion. So $\mathbb{E}[x_t D] = x^* D e^{\alpha t}$. Substituting this to (3.4) gives

$$V(x_{T^*}) = \frac{x^* D}{\delta}, \tag{3.8}$$

where $\delta = r - \alpha$. Equation (3.7) is a Cauchy-Euler equation. Suppose its solution has the form of $V = Ax^\beta$. By substituting $V = Ax^\beta$ back to (3.7), we obtain

$$\frac{1}{2}\sigma^2 \beta(\beta - 1) + (r - \delta)\beta - r = 0 \tag{3.9}$$

which is called the *fundamental quadratic equation* in [11]. Two roots of (3.9) are β_1 and β_2, where β_1 is given by (3.1) and β_2 is given by

$$\beta_2 = \frac{1}{2} - \frac{r - \delta}{\sigma^2} - \sqrt{\left[\frac{r - \delta}{\sigma^2} - \frac{1}{2}\right]^2 + \frac{2r}{\sigma^2}}. \tag{3.10}$$

To understand β_1 and β_2 from a geometric perspective, let $Q(\beta)$ be the quadratic function defined by

$$Q(\beta) = \frac{1}{2}\sigma^2 \beta(\beta - 1) + (r - \delta)\beta - r.$$

Since the coefficient of β^2 in $Q(\beta)$ is positive, the graph of $Q(\beta)$ is an upward-pointing parabola with $Q(1) = -\delta$ as indicated in Figure 3.2. Note that $\delta > 0$ implies $\beta_1 > 1$ and $Q(0) = -r$ implies $\beta_2 < 0$. These are very important facts to be used in the sequel.

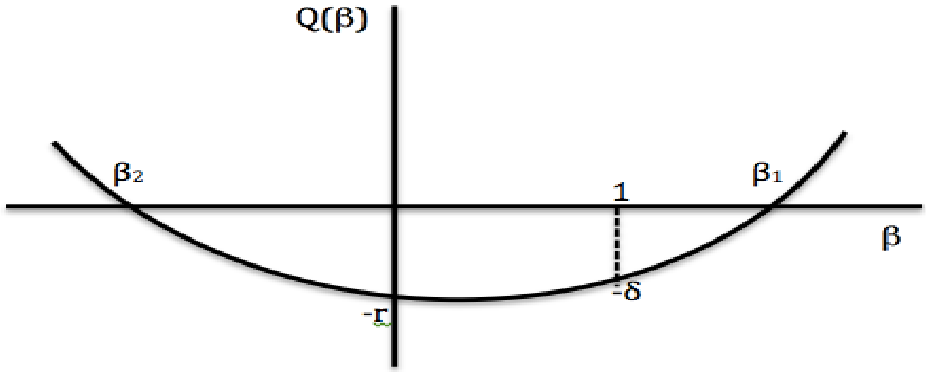

Figure 3.2: Roots of the fundamental quadratic equation

Now the general solution to (3.7) can be expressed as the following

$$V(x) = A_1 x^{\beta_1} + A_2 x^{\beta_2},$$

where A_1, A_2 are constants to be determined. However, following the explanation in [11], $A_1 x^{\beta_1}$ and $A_2 x^{\beta_2}$ can be treated as speculative bubbles and be eliminated by invoking economic consideration. Thus the value of project is just equal to the particular solution given in (3.8), i.e., $V(x) = \dfrac{xD}{\delta}$, which is called the *fundamental component of the value of the project*. In fact, this is just the expected present value of the future revenue stream P_t when the value at time of investment is P.

Once the value of project $V(x)$ is known, we are able to determine the value of real option $F(x)$. According to the Bellman's principle of optimality, the option value $F(x)$ to invest satisfies

$$F(x) = e^{-rdt}\mathbb{E}[F(x_{t+dt})|x]. \tag{3.11}$$

By applying the one-dimensional Feynman-Kac formula, we can obtain the following second order homogeneous differential equation,

$$\frac{1}{2}\sigma^2 x^2 \frac{d^2F}{dx^2} + (r-\delta)x\frac{dF}{dx} - rF = 0. \tag{3.12}$$

From the previous analysis, the solution to (3.12) is also of the form

$$F(x) = A_1 x^{\beta_1} + A_2 x^{\beta_2},$$

where β_1 and β_2 are given previously. The rest of the task is to determine coefficients A_1 and A_2. To do this, one must not only concern what happens in the short interval time dt, but also after the time $t+dt$. Firstly, note that as the demand shock level drops to 0, the option value becomes worthless. Since $\beta_2 < 0$, x^{β_2} goes to infinity as x goes to 0. Thus according to this condition, A_2 must be 0. Moreover, as this is a free boundary problem, at the time of investment, say T^*, the immediate investment payoff must equal to the option value, i.e.,

$$F(x_{T^*}) = V(x_{T^*})$$

for some terminal payoff $V(x_{T^*})$. This is called the *value-matching condition*. Note that the firm must also choose the optimal time T^* such that its payoff is always maximized. This implies that the function $F(x)$ must be smooth at the time of investment, i.e.,

$$\left.\frac{dF}{dx}\right|_{x=x_{T^*}} = \left.\frac{dV}{dx}\right|_{x=x_{T^*}} - I.$$

This is called the *smooth-pasting condition*. Now, substituting $V(x)$ to the value-matching condition and smooth pasting condition respectively gives the following boundary conditions

$$\begin{aligned} A_1 x^{*\beta_1} &= \frac{x^* D}{\delta} - I, \\ \beta_1 A_1 x^{*\beta_1 - 1} &= \frac{D}{\delta}. \end{aligned}$$

After solving the above system of equations, the optimal investment threshold and the coefficient are

$$\begin{aligned} x^* &= \frac{\beta_1 \delta I}{(\beta_1 - 1)D}, \\ A_1 &= \left(\frac{x^* D}{\delta} - I\right)\left(\frac{1}{x^*}\right)^{\beta_1}. \end{aligned}$$

Furthermore, we will also discuss the effect of the drift parameter α and the volatility parameter σ on the investment. From Figure 3.3, we see that an increase in α will decrease the investment threshold and the overall payoff of the firm, whereas in [11, p139], an increase in α will only increase the value of waiting. A meaningful explanation will be that α represent the revenue of the project, an increase in α means that the the project's value is increased, thus the value of waiting becomes relatively less, and the firm should invest earlier. From Figure 3.4, we see that an increase in σ will only increase the value of waiting and the optimal investment threshold but not the NPV of the project. Comparing with the effect of the drift parameter α, we find that

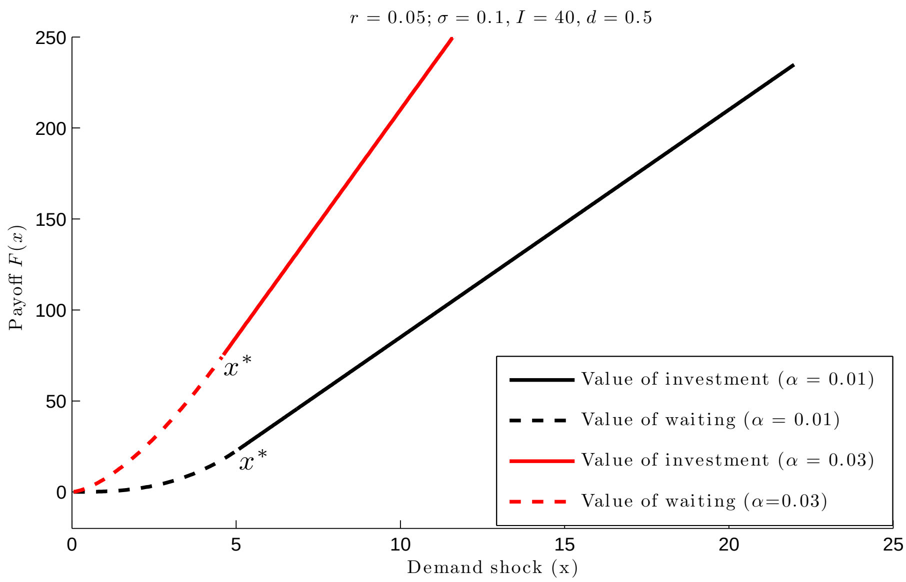

Figure 3.3: Effect of the drift parameter α

a change in the volatility parameter σ will affect both the investment threshold and the value of waiting but not the investment payoff. This can be interpreted as the greater volatility is the greater the option value will be. However, as the project becomes more risky, the firm must wait longer in order to capture the full value of waiting.

3.1.2 Real Options with Entry and Exit Decisions

In this subsection, following [11] and [15], we consider investment decisions on entry and exit a project. In order to do this, we shall adjust Assumption **(A5)** to

(A5') The firm needs to pay a running cost C in order to keep the project activated and if the demand drops the firm can suspend the project by paying another sunk cost E.

Under **(A1)**–**(A4)**, **(A5')** and **(A6)**, from discussions in the previous subsection, we know when the firm has not yet invested in the project, the option value $F_0(x)$ satisfies

$$\frac{1}{2}\sigma^2 x^2 \frac{d^2 F_0}{dx^2} + (r - \delta)x\frac{dF_0}{dx} - rF_0 = 0. \tag{3.13}$$

Given the fact that the option to invest becomes worthless as the demand shock $x \to 0$, we know that the solution is of the form

$$F_0(x) = A_1 x^{\beta_1},$$

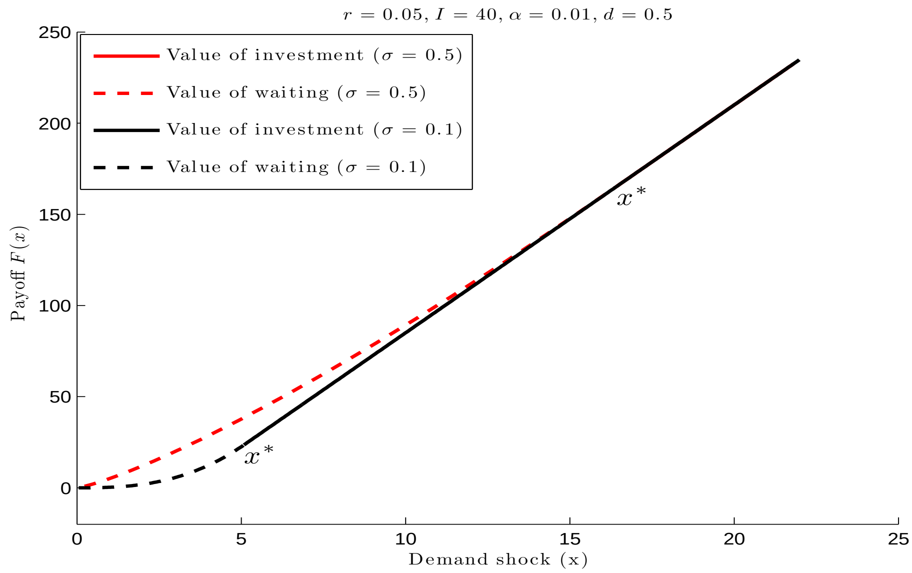

Figure 3.4: Effect of the volatility parameter σ

where A_1 is the constant to be determined.

Now let us suppose that the firm has already invested in the project. If the demand shock increases, then it is good for the firm to keep its position. However, keeping in mind that there is a running cost C, if the demand shock drops down to a certain level, then the project should be suspended. Thus we need to determine when it is optimal for the firm to suspend the project by paying the cost E. Applying the techniques similar to those before, we derive that the option value $F_1(x)$ to suspend the project satisfies

$$\frac{1}{2}\sigma^2 x^2 \frac{d^2 F_1}{dx^2} + (r - \delta)x\frac{dF_1}{dx} - rF_1 + xD - C = 0. \tag{3.14}$$

A simple substitution shows that the solution to (3.14) is of the form

$$F_1(x) = B_1 x^{\beta_1} + B_2 x^{\beta_2} + \frac{xD}{\delta} - \frac{C}{r}.$$

However it is worth to mention that $B_1 x^{\beta_1}$ and $B_2 x^{\beta_2}$ are not speculative bubbles in this case. When the demand shock becomes large, the option value to suspend the project becomes worthless. Since $\beta_1 > 0$, B_1 must be 0. If the demand shock falls down to a certain level, say x_1^*, then the firm will exercise the option to suspend the project and switch the position to waiting for better payoff to invest. Similarly, if the demand shock increases to a certain level x_0^*, the firm will invest in the project again. These facts imply the following boundary conditions: If the firm has not yet invested in the project then there exists an investment threshold $\overline{x}^*$ such that

$$\begin{aligned} A_1\overline{x}^{*\beta_1} &= B_2\overline{x}^{*\beta_2} + \frac{\overline{x}^* D}{\delta} - \frac{C}{r} - I, \\ \beta_1 A_1\overline{x}^{*\beta_1-1} &= \beta_2 B_2\overline{x}^{*\beta_2-1} + \frac{D}{\delta}. \end{aligned}$$

If the firm has invested in the project then there exists another threshold $\underline{x}^*$ such that

$$\begin{aligned} A_1\underline{x}^{*\beta_1} - E &= B_2\underline{x}^{*\beta_2} + \frac{\underline{x}^* D}{\delta} - \frac{C}{r}, \\ \beta_1 A_1\underline{x}^{*\beta_1-1} &= \beta_2 B_2\underline{x}^{*\beta_2-1} + \frac{D}{\delta}. \end{aligned}$$

The four unknowns $A_1, B_2, \overline{x}^*$ and $\underline{x}^*$ can be determined by solving this system of equations numerically. In Figure 3.5, the blue dashed curves represent the option value to invest in the

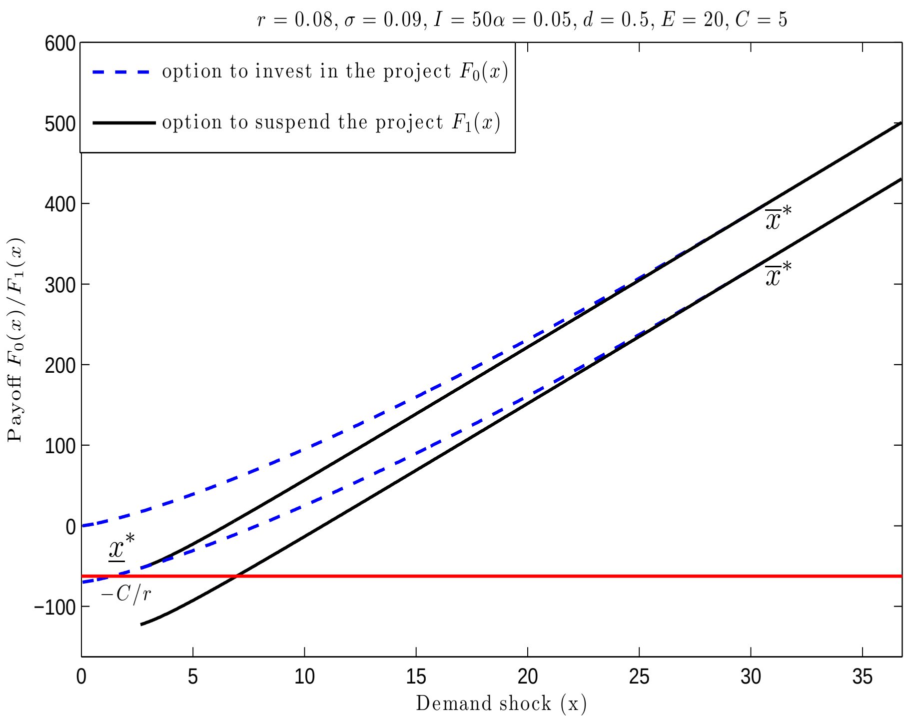

Figure 3.5: Real option value with entry and exit decisions

project and the black dashed curves represent the option value to suspend the project. If the firm has not yet invested in the project, then it is recommended to wait until $\overline{x}^*$ is reached. If the firm has already invested in the project and the demand shock drops below $\underline{x}^*$, then firm must pay the sunk cost E to suspend the project. However, it is worth to mention that, the firm may not suspend the project at all if the cost of suspending the project E is too big. This is because the maximum loss of the firm is $\frac{C}{r}$, and the firm may have already reached $-\frac{C}{r}$ by the time the demand shock x reaches $\underline{x}^*$. Since we have already discussed the investment

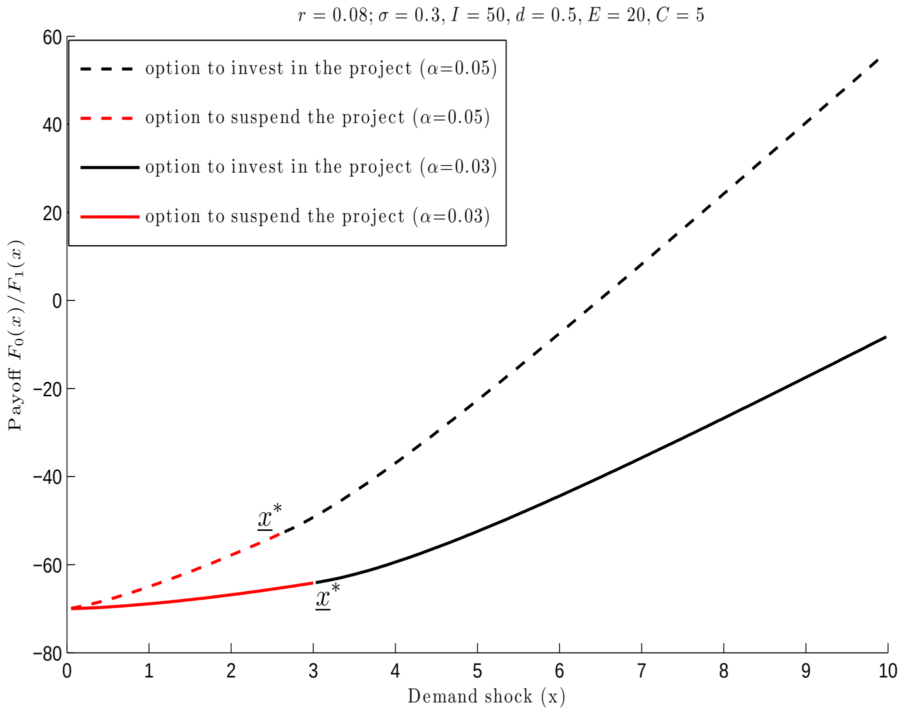

Figure 3.6: Effect of the drift parameter α

threshold $\overline{x}^*$ in Subsection 3.1.1, we will only focus on the suspension threshold $\underline{x}^*$. In Figure 3.6, again we see that as α increases, the entire payoff increases. Moreover, because the value of waiting is relatively smaller, the firm can suspend the project at a lower threshold. For the effect of the volatility on the investment, we get a conclusion similar to that for α. This is a result different from the case of the investment threshold $\overline{x}^*$. The threshold for suspending the project decreases as the volatility increases. This is due to the fact that the project value itself also includes a real option which gains its value when the volatility σ increases.

3.2 Real Option Models with Stochastic Volatility

One of the drawbacks of modelling with geometric Brownian motion is that the volatility parameter is assumed to be a constant. In [24], Heston considered the stochastic nature of volatil-

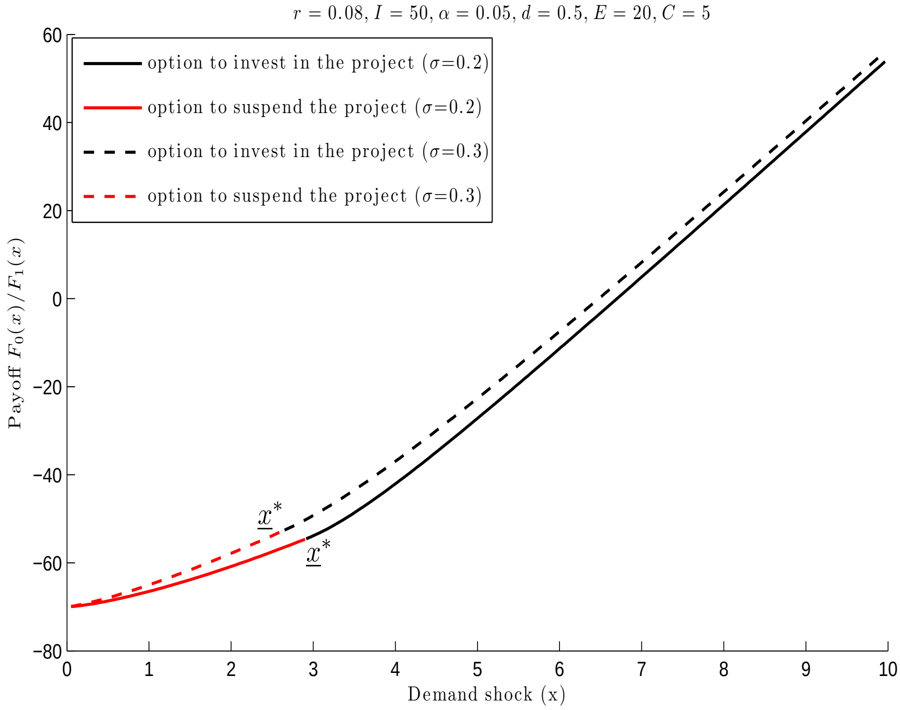

Figure 3.7: Effect of the volatility parameter σ

ity and proposed a model to avoid this drawback. Nowadays, this model is called the Heston model which has been popularly used by researches and practitioners. In this section, we study real opion models with stochastic volatility. The first part of this section presents the recent work of Ting and Ewald in [43]. The second part of this section, published in [7], is an extension of the model in Subsection 3.1.2 to the case where stochastic volatility is involved.

3.2.1 Real Options for Market Entry with Stochastic Volatility

To study real option models with stochastic volatility, we must adjust **(A6)** to the following assumption.

(A6') The demand shock is a diffusion process $\{x_t : t \geq 0\}$ with a stochastic instantaneous variance $\{y_t : t \geq 0\}$, as described by (2.8) and (2.9).

Under **(A1)**–**(A5)** and **(A6')**, by the multi-dimensional Girsanov theorem, we are able to change the real probability measure $\mathbb{P}$ to a risk-neutral probability measure $\mathbb{Q}$ and transform

(2.8) and (2.9) into

$$\begin{aligned} dx_t &= (r-\delta)x_t dt + \sqrt{y_t} x_t d\tilde{z}_t, & (3.15) \\ dy_t &= k^*(m^* - y_t)dt + \sigma\sqrt{y_t} d\tilde{w}_t, & (3.16) \end{aligned}$$

where $[d\tilde{z}_t, d\tilde{w}_t] = \rho dt$, and

$$k^* = k + \lambda,\ m^* = \frac{km}{k+\lambda}$$

are the risk-neutral parameters, and the new parameter λ is the premium of volatility risk, refer to [24]. For the rest of this section, our analysis will be based on the risk-neutral probability measure $\mathbb{Q}$.

Let x and y be the values of x_t and y_t. Assume that D is independent of x and y. Similar to those in the previous section, the project value, denoted by $V(x, y)$, can be expressed as the sum of the current profit in the time interval $[t, t + dt]$ and the continuation value, i.e.,

$$V(x,y) = xDdt + \mathbb{E}\left[e^{-rdt}V(x_{t+dt}, y_{t+dt})|x,y\right]. \tag{3.17}$$

By the multi-dimensional Feynman-Kac theorem, we can obtain the following PDE

$$\begin{aligned} &\frac{1}{2}\left[x^2 y\frac{\partial^2 V}{\partial x^2} + y\sigma^2\frac{\partial^2 V}{\partial y^2} + 2\rho xy\sigma\frac{\partial^2 V}{\partial x\partial y}\right] \\ &+ \alpha x\frac{\partial V}{\partial x} + k^*(m^* - y)\frac{\partial V}{\partial y} + xD - rV = 0. \end{aligned} \tag{3.18}$$

A meaningful particular solution is the fundamental component of the project value $V^p(y) = \frac{xD}{\delta}$. Note that this is also the only particular solution that is independent of the volatility. Other terms in the solution are speculative components as we discussed previously. For any time t, the firm can either invest and take the immediate payoff $V^p(y) - I$, or wait for a small amount of time dt and take the continuation value. Thus the option value $F(x, y)$ satisfies the following Bellman equation,

$$F(x,y) = \max\left\{V - I, \mathbb{E}\left[e^{-rdt}V(x_{t+dt}, y_{t+dt})|x,y\right]\right\}. \tag{3.19}$$

Let the free boundary $x^*(y)$ be the optimal threshold such that the continuation value is equal to the immediate payoff. Again by the multi-dimensional Feynman-Kac theorem, we conclude that before x reaches $x^*(y)$, F satisfies the following PDE

$$\frac{1}{2}\left[x^2 y\frac{\partial^2 F}{\partial x^2} + y\sigma^2\frac{\partial^2 F}{\partial y^2} + 2\rho xy\sigma\frac{\partial^2 F}{\partial x\partial y}\right] + \alpha x\frac{\partial F}{\partial x} + k^*(m^* - y)\frac{\partial F}{\partial y} - rF = 0 \tag{3.20}$$

with the boundary conditions

$$F(x^*(y), y) = V(x^*(y), y) - I, \tag{3.21}$$
$$F(0, y) = V(0, y), \tag{3.22}$$
$$\left.\frac{\partial F}{\partial x}\right|_{x=x^*(y)} = \left.\frac{\partial V}{\partial x}\right|_{x=x^*(y)}, \tag{3.23}$$
$$\left.\frac{\partial F}{\partial y}\right|_{x=x^*(y)} = \left.\frac{\partial V}{\partial y}\right|_{x=x^*(y)}, \tag{3.24}$$

where (3.21) is the value matching condition, (3.22) says that the value of option F becomes worthless if the firm does not experience any demand shock, (3.23) and (3.24) are the smooth pasting conditions. Substituting V^p into (3.21)–(3.24) gives

$$\begin{aligned}
F(x^*(y), y) &= \frac{x^*(y)D}{\delta} - I,\\
F(0, y) &= 0,\\
\left.\frac{\partial F}{\partial x}\right|_{x=x^*(y)} &= \frac{D}{\delta},\\
\left.\frac{\partial F}{\partial y}\right|_{x=x^*(y)} &= 0.
\end{aligned}$$

For the sake of notational simplicity and also in order to transform (3.20) to a compact form, let $\eta^2 = m^*\sigma^2/2k^*$, $\varepsilon = 1/k^*$. Following the asymptotic approach in [12] and [43], we define the following operators

$$\begin{aligned}
\mathfrak{L}_0 &= \frac{\eta^2 y}{m^*}\frac{\partial^2}{\partial y^2} + (m^* - y)\frac{\partial}{\partial y},\\
\mathfrak{L}_1 &= \frac{\rho\eta\sqrt{2}}{\sqrt{m^*}}xy\frac{\partial^2}{\partial x\partial y},\\
\mathfrak{L}_2 &= \frac{1}{2}yx^2\frac{\partial^2}{\partial x^2} + (r - \delta)x\frac{\partial}{\partial x} - r.
\end{aligned}$$

Here we would like to point out that $k^*\mathfrak{L}_0$ is the infinitesimal generator of the CIR process y_t. Then, (3.20) can be rewritten as the following form,

$$\left(\frac{1}{\varepsilon}\mathfrak{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathfrak{L}_1 + \mathfrak{L}_2\right)F = 0. \tag{3.25}$$

We expand $F(x,y)$ and $y^*(x)$ as

$$F(x,y) = F_0(x,y) + \sqrt{\varepsilon}F_1(x,y) + \varepsilon F_2(x,y) + \cdots, \tag{3.26}$$

$$x^*(y) = x_0^*(y) + \sqrt{\varepsilon}x_1^*(y) + \varepsilon x_2^*(y) + \cdots. \tag{3.27}$$

Substituting (3.26) and (3.27) into (3.25) and taking terms up to the order $\sqrt{\varepsilon}$ give

$$\begin{aligned}&\frac{1}{\varepsilon}\mathfrak{L}_0F_0 + \frac{1}{\sqrt{\varepsilon}}(\mathfrak{L}_0F_1 + \mathfrak{L}_1F_0) + (\mathfrak{L}_0F_2 + \mathfrak{L}_1F_1 + \mathfrak{L}_2F_0)\\ &+ \sqrt{\varepsilon}(\mathfrak{L}_0F_3 + \mathfrak{L}_1F_2 + \mathfrak{L}_2F_1) = 0.\end{aligned} \tag{3.28}$$

Substituting (3.26) and (3.27) into the boundary conditions (3.21)−(3.24) yields

$$F_0(x^*(y),y) + \sqrt{\varepsilon}\left(x_1^*(y)\frac{\partial F_0}{\partial x}\bigg|_{x=x_0^*} + F_1(x_0^*(y),y)\right) = \frac{x_0^*(y)D}{\delta} + \frac{\sqrt{\varepsilon}x_1^*(y)D}{\delta} - I, \tag{3.29}$$

$$F_0(0,y) + \sqrt{\varepsilon}F_1(0,y) = 0, \tag{3.30}$$

$$\frac{\partial F_0}{\partial x}\bigg|_{x=x_0^*} + \sqrt{\varepsilon}\left(x_1^*(y)\frac{\partial^2 F_0}{\partial x^2}\bigg|_{x=x_0^*} + \frac{\partial F_1}{\partial x}\bigg|_{x=x_0^*}\right) = \frac{D}{\delta}, \tag{3.31}$$

$$\frac{\partial F_0}{\partial y}\bigg|_{x=x_0^*} + \sqrt{\varepsilon}\left(x_1^*(y)\frac{\partial^2 F_0}{\partial x\partial y}\bigg|_{x=x_0^*} + \frac{\partial F_1}{\partial y}\bigg|_{x=x_0^*}\right) = 0. \tag{3.32}$$

Thus we derive the following proposition using the approach similar to that in [43].

Proposition 3.2. *Under* **(A1)−(A5)** *and* **(A6')***, the payoff of the firm is given asymptotically by*

$$F(x) \approx \begin{cases}\left(1 - \dfrac{\rho\sigma\beta_1^2(\beta_1-1)}{k^*(\beta_2-\beta_1)}\ell n\left(\dfrac{x_0^*}{x}\right)\right)\left(\dfrac{x_0^*D}{\delta} - I\right)\left(\dfrac{x}{x_0^*}\right)^{\beta_1}, & \textit{if} \quad x < x_0^*,\\[2ex] \dfrac{xD}{\delta} - I, & \textit{if} \quad x \geq x_0^*,\end{cases} \tag{3.33}$$

where

$$\beta_1 = \frac{1}{2} - \frac{r-\delta}{m^*} + \sqrt{\left(\frac{r-\delta}{m^*} - \frac{1}{2}\right)^2 + \frac{2r}{m^*}}, \tag{3.34}$$

$$\beta_2 = \frac{1}{2} - \frac{r-\delta}{m^*} - \sqrt{\left(\frac{r-\delta}{m^*} - \frac{1}{2}\right)^2 + \frac{2r}{m^*}}, \tag{3.35}$$

and

$$x_0^* = \frac{\beta_1 I\delta}{(\beta_1 - 1)D}. \tag{3.36}$$

The optimal threshold x^* of the firm is given asymptotically by

$$x^* \approx \frac{\beta_1 I\delta}{(\beta_1 - 1)D}\left(1 - \frac{\rho\sigma\beta_1}{k^*(\beta_2 - \beta_1)}\right). \tag{3.37}$$

Hence the optimal strategy for the firm is to invest at time T^* given by

$$T^* = \inf\{t \geq 0 : x \geq x^*\}. \tag{3.38}$$

Proof. The asymptotic solution consists of two parts, the zero order term and the correction term. First, we compare order $\frac{1}{\varepsilon}$ terms in (3.28) and get

$$\mathfrak{L}_0 F_0(x,y) = 0, \qquad \text{if} \quad x < x_0^*. \tag{3.39}$$

Then we collect order 1 terms from (3.29) and (3.31) to get the corresponding boundary conditions

$$F_0(x,y) = \frac{xD}{\delta} - I, \qquad \text{if} \quad x \geq x_0^*, \tag{3.40}$$

$$\left.\frac{\partial F_0}{\partial x}\right|_{x=x_0^*} = \frac{D}{\delta}. \tag{3.41}$$

(3.39) implies that F_0 is independent of y when $x < x_0^*$, since $\mathfrak{L}_0$ takes derivatives with respect to y only. (3.40) shows that F_0 is also independent of y on other side of the x_0^*. F_0 being independent of y implies that x_0^* is also independent of y. Then we collect all the order $\frac{1}{\sqrt{\varepsilon}}$

terms and get

$$\mathfrak{L}_0 F_1(x, y) = 0, \qquad \text{if} \quad x < x_0^*, \tag{3.42}$$

A direct result from (3.29) and (3.41) gives us

$$F_1(x, y) = 0, \qquad \text{if} \quad x \geq x_0^*. \tag{3.43}$$

From order $\sqrt{\varepsilon}$ terms in (3.32), we also get

$$x_1^* \frac{\partial^2 F_0}{\partial x^2}\bigg|_{x=x_0^*} + \frac{\partial F_1}{\partial x}\bigg|_{x=x_0^*} = 0. \tag{3.44}$$

(3.42) is the result of F_0 being independent of y, and thus $\mathfrak{L}_1 F_0 = 0$. Therefore F_1 is also independent of y. Similar to F_0, from (3.43), we conclude that $\mathfrak{L}_1 F_1 = 0$. Furthermore, we want to make all the order 1 and order $\sqrt{\varepsilon}$ zero. With the above results, we find that

$$\mathfrak{L}_0 F_2 + \mathfrak{L}_2 F_0 = 0. \tag{3.45}$$

We can see that (3.45) is a Poisson equation with respect to the operator $\mathfrak{L}_0$ in the variable y. From Subsection 2.1.5, a solution F_2 to (3.45) exists if and only if $\mathfrak{L}_2 F_0$ is centred with respect to the invariant distribution of the diffusion process whose infinitesimal generator is $\mathfrak{L}_0$. Thus we have

$$\langle \mathfrak{L}_2 F_0 \rangle = 0.$$

The angled brackets indicate taking the average of the argument with respect to the invariant distribution of the diffusion. Since F_0 does not depend on y, the centering condition becomes $\langle \mathfrak{L}_2 \rangle F_0 = 0$, which is equivalent to

$$\frac{1}{2} m^* x^2 \frac{d^2 F_0}{dx^2} + (r - \delta) x \frac{dF_0}{dx} - r F_0 = 0. \tag{3.46}$$

Note that the invariant distribution is in fact a Gamma distribution and hence $\langle y \rangle = m^*$. This ODE is similar to that in the classical real option problem where the volatility is given by a constant $\sqrt{m^*}$. Thus following [11], the zero order term of the problem is given by

$$F_0 = \begin{cases} \left(\dfrac{x_0^* D}{\delta} - I\right)\left(\dfrac{x}{x_0^*}\right)^{\beta_1}, & \text{if} \quad x < x_0^*, \\ \\ \dfrac{xD}{\delta} - I, & \text{if} \quad x \geq x_0^*, \end{cases}$$

where x_0^* and β_1 are given in (3.36) and (3.34) respectively. Since $\mathfrak{L}_1 F_1 = 0$, for the correction term, order 1 terms of (3.28) give

$$\begin{aligned} \mathfrak{L}_0 F_2 &= -\mathfrak{L}_2 F_0 \\ &= -(\mathfrak{L}_2 F_0 - \langle \mathfrak{L}_2 F_0 \rangle) \\ &= -\frac{1}{2}(y - m^*)x^2 \frac{d^2 F_0}{dx^2}. \end{aligned} \tag{3.47}$$

Then we can use (3.47) to find F_2. In order to do this, we must first find the solution $\phi(y)$ to the Poisson equation $\mathfrak{L}_0(\phi) = y - m^*$. Since $k^* \mathfrak{L}_0$ is the infinitesimal generator of the CIR process y_t, $\phi(y)$ is also the solution to

$$k^* \mathfrak{L}_0(\phi) = k^*(y - m^*).$$

Then by (2.29), we have

$$\begin{aligned} \phi(y) &= \int_0^\infty \mathbb{E}(k^*(m^* - y_t)|y_0 = y)dt \\ &= k^* \int_0^\infty \left[m^* - y_0 e^{-k^* t} - m^*(1 - e^{-k^* t})\right] dt \\ &= k^* \int_0^\infty (m^* - y)e^{-k^* t} dt \\ &= m^* - y, \end{aligned} \tag{3.48}$$

where we use (2.6) to calculate the expectation of y_t. Now let $\overline{c}(t, x)$ be a function that is independent of y. From (3.47) and (3.48), we get

$$\begin{aligned} F_2(t, x, y) &= -\frac{1}{2}\mathfrak{L}_0^{-1}(y - m^*)x^2 \frac{d^2 F_0}{dx^2} \\ &= -\frac{1}{2}(\phi(y) + \overline{c}(t, x))x^2 \frac{d^2 F_0}{dx^2} \\ &= \frac{1}{2}(y - m^* - \overline{c}(t, x))x^2 \frac{d^2 F_0}{dx^2} \\ &= \frac{1}{2}(y + c(t, x))x^2 \frac{d^2 F_0}{dx^2}, \end{aligned} \tag{3.49}$$

where $c(t,x) = -(\overline{c}(t,x) + m^*)$ is also independent of y. Order $\sqrt{\varepsilon}$ terms of (3.28) give

$$\mathfrak{L}_0 F_3 + \mathfrak{L}_1 F_2 + \mathfrak{L}_2 F_1 = 0,$$

which leads to a Poisson equation for F_3. The corresponding centering condition is

$$\langle \mathfrak{L}_1 F_2 + \mathfrak{L}_2 F_1 \rangle = 0.$$

Since $\mathfrak{L}_1$ takes derivatives with respect to y, and $c(t,x)$ is independent of y, we can use (3.49) to get

$$\begin{aligned} \mathfrak{L}_2(\sqrt{m^*})F_1 &= \langle \mathfrak{L}_2 F_1 \rangle \\ &= -\langle \mathfrak{L}_1 F_2 \rangle \\ &= \langle \mathfrak{L}_1 x y^2 \frac{d^2 F_0}{dx^2} \rangle \\ &= -\frac{\rho\sigma m^*}{2\sqrt{k^*}} \left(2x^2 \frac{d^2 F_0}{dx^2} + x^3 \frac{d^3 F_0}{dx^3} \right). \end{aligned} \tag{3.50}$$

Let $\overline{F}_1 = \sqrt{\varepsilon} F_1$. Substituting $\mathfrak{L}_2$ and F_0 into (3.50) gives

$$\frac{1}{2} m^* x^2 \frac{d^2 \overline{F}_1}{dx^2} + (r - \delta) x \frac{d\overline{F}_1}{dx} - r\overline{F}_1 = BAx^{\beta_1}, \tag{3.51}$$

where

$$\begin{aligned} B &= -\frac{\rho\sigma m^*}{2k^*} \beta_1^2 (\beta_1 - 1), \\ A &= \left(\frac{x_{M0} D_M}{\delta} - I \right) \left(\frac{1}{x_{M0}} \right)^{\beta_1}. \end{aligned}$$

The homogeneous equation associated with (3.51) is identical to (3.46). To solve (3.51), we only need to find a particular solution by applying the method of variation of parameters. Let the particular solution, $\overline{F}_{1p}$, be

$$\overline{F}_{1p} = C(x) x^{\beta_1} + D(x) x^{\beta_2},$$

where β_1 and β_2 are the same as those in (3.34) and (3.35), and $C(x)$ and $D(x)$ are functions of x yet to be determined. Taking the first derivative gives

$$\frac{d\overline{F}_{1p}}{dx} = \beta_1 C(x) x^{\beta_1 - 1} + \beta_2 D(x) x^{\beta_2 - 1},$$

where we make

$$\frac{dC}{dx}x^{\beta_1} + \frac{dD}{dx}x^{\beta_2} = 0.$$

Taking the second derivative gives

$$\begin{aligned}\frac{d^2\overline{F}_{1p}}{dx^2} &= \beta_1(\beta_1 - 1)C(x)x^{\beta_1-2} + \beta_2(\beta_2 - 1)D(x)x^{\beta_2-2} \\ &\quad + \beta_1\frac{dC}{dx}x^{\beta_1-1} + \beta_2\frac{dD}{dx}x^{\beta_2-1}.\end{aligned}$$

We substitute these derivatives back to (3.51) and get

$$\begin{aligned}\frac{dC}{dx}x^{\beta_1} + \frac{dD}{dx}x^{\beta_2} &= 0, \\ \frac{1}{2}m^*\left(\beta_1\frac{dC}{dx}x^{\beta_1+1} + \beta_2\frac{dD}{dx}x^{\beta_2+1}\right) &= BAx^{\beta_1}.\end{aligned}$$

Solving this system of equations gives

$$\begin{aligned}C(x) &= \frac{-2BA}{m^*}\frac{\ell n(x)}{\beta_2-\beta_1}, \\ D(x) &= \frac{-2BA}{m^*}\frac{x^{\beta_1-\beta_2}}{(\beta_2-\beta_1)^2}.\end{aligned}$$

Thus the general solution to (3.51) is of the form

$$\begin{aligned}\overline{F}_1(x) &= C_1x^{\beta_1} + C_2x^{\beta_2} + C(x)x^{\beta_1} + D(x)x^{\beta_2} \\ &= C_1x^{\beta_1} + C_2x^{\beta_2} - \frac{2BA}{m^*}\frac{x^{\beta_1}}{\beta_2-\beta_1}\left(\ell n(x) + \frac{1}{\beta_2-\beta_1}\right),\end{aligned}$$

where C_1 and C_2 are arbitrary constants. The boundary conditions for $\overline{F}_1$ requires that $\overline{F}_1(0) \to 0$ as $x \to 0$. This leads to $C_2 = 0$. At $x = x_0^*$, we must have $\overline{F}_1(x_0^*) = 0$. This leads to

$$C_1 = \frac{2BA}{m^*(\beta_2-\beta_1)}\left(\ell n(x_0^*) + \frac{1}{\beta_2-\beta_1}\right).$$

Hence, the correction term is given by

$$\sqrt{\varepsilon}F_1 = \overline{F}_1 = \frac{2BA}{m^*(\beta_2-\beta_1)}\ell n\left(\frac{x_0^*}{x}\right)x^{\beta_1}.$$

It is not hard to check that the error is of order $O(\varepsilon)$. For the threshold expansion, x_1^*, we isolate it from the boundary condition (3.31) and get

$$x_1^* = -\frac{dF_1}{dy}\bigg|_{x=x_0^*}\bigg/\frac{d^2F_0}{dy^2}\bigg|_{x=x_0^*}.$$

It is easy to verify that the above expression of x_1^* is equivalent to

$$\sqrt{\varepsilon}x_1^* = -\frac{d\overline{F}_1}{dx}\bigg|_{x=x_0^*}\bigg/\frac{d^2F_0}{dx^2}\bigg|_{x=x_0^*}.$$

Thus we can get

$$\sqrt{\varepsilon}x_1^* = -\frac{\rho\sigma\beta_1 x_0^*}{k^*(\beta_2-\beta_1)},$$

by some simple algebraic operations. Combining F_0, $\overline{F}_1$ and x_0^*, $\overline{x}_1^*$ gives the result in Proposition 3.2 . □

From Proposition 3.2, we see that the first term of the asymptotic solution is given by the classical solution with constant volatility m^*. Thus we will omit the effects of the drift parameter α and the mean reversion level m^* as we have discussed them in the previous section. By adding stochastic volatility, we have a few more parameters remained in the solution which are the mean reversion rate k^*, the correlation ρ between the project's value and the variance process, and the volatility parameter σ of the variance process. The first parameter we would like to discuss is the correlation parameter ρ. In Figure 3.8, we can see that the sign of ρ plays a very important role in affecting the investment threshold. A positive ρ leads an increase in the investment threshold and a negative ρ leads to a decrease in the investment threshold. We can also observe that when ρ does not equal to 0 the value of waiting and the investment threshold do not seem to be optimal any more. However, keeping in mind that the solution is an approximation only, it does not necessary mean that investing at x^* is not an optimal choice. Moreover, we see that when $\rho = 1$ the value of waiting is greater than that in the classical solution. However, when x is greater than x_0^*, the correction term becomes negative and the value of waiting is less than that in the classical solution. For the case when $\rho = -1$, the value of waiting is always less than that in the classical solution since the firm will invest earlier. We can explain the above observations as that positive correlations imply more variance in the value of project. Thus the firm should set the investment threshold at a higher level than that in the classical solution. As the result shows, we expect the effect of the extra parameters to be small, since they only sit in the correction terms in the expansion. Thus to see their effects more clearly we will keep $\rho = -1$. In Figure 3.9, we show how the volatility parameter σ of

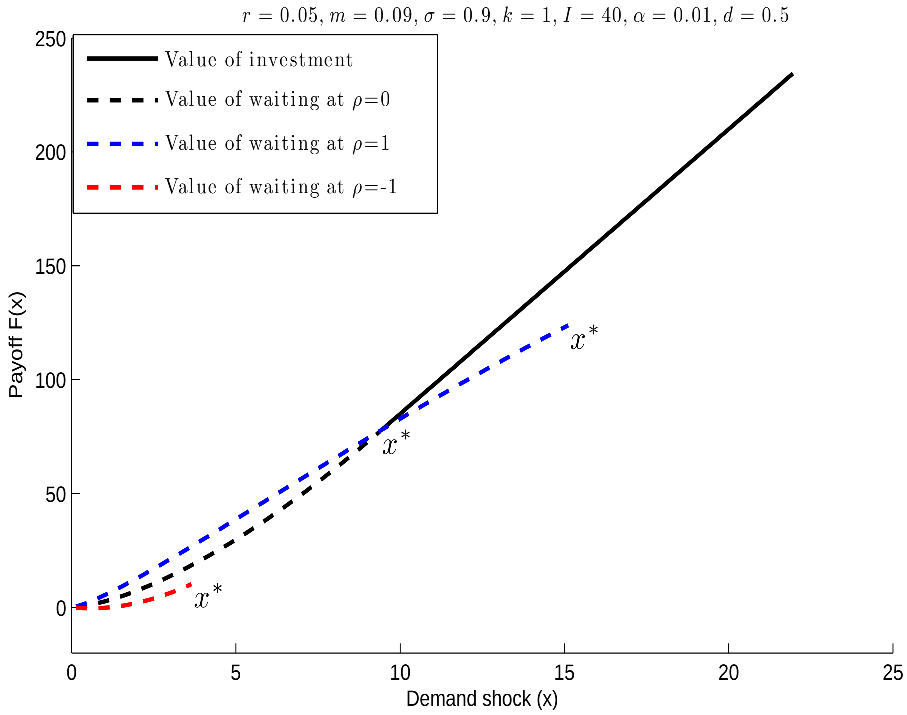

Figure 3.8: The effect of the correlation ρ

the variance process affects the solutions. When σ is small, the effect of the correction terms is small. When there is a negative correlation between the processes, the greater σ is, the less risky the variance process becomes. Thus we see that both the value of waiting and the investment threshold decrease as σ increases. Note that an opposite result holds when there is a positive correlation between the processes. Finally, we check the effect of the mean reverting rate k. From Figure 3.10, we see that the asymptotic solution converges to the classical solution as k increases. The difference in the real option value is barely distinguishable after $k > 10$. For more details on the effects of these parameters in monopoly case, refer to [43]. In Chapter 4 we will show how these parameters affect both firms in the duopoly case.

3.2.2 Real Options with Entry and Exit Decisions under Stochastic Volatility

In this subsection, under **(A1)**–**(A4)**, **(A5')** and **(A6')**, we shall extend the above model to the case where entry and exit decisions are involved. Recall that the option value to invest $F_0(x, y)$

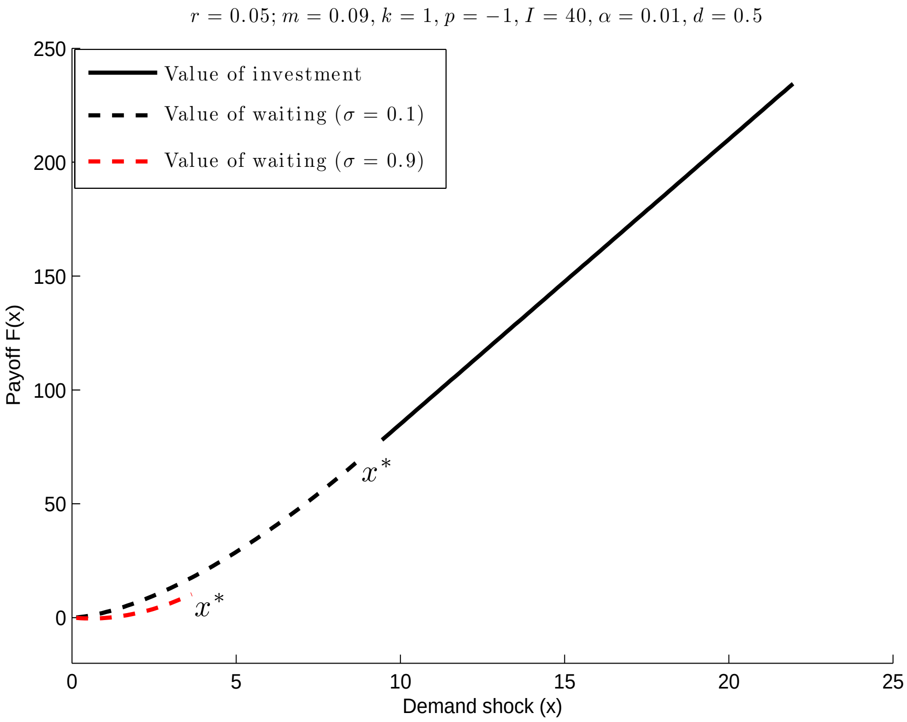

Figure 3.9: The effect of the volatility parameter σ of the variance process

follows the following PDE,

$$\frac{1}{2}\left[x^2 y \frac{\partial^2 F_0}{\partial x^2} + y\sigma^2 \frac{\partial^2 F_0}{\partial y^2} + 2\rho x y \sigma \frac{\partial^2 F_0}{\partial x \partial y}\right] + \alpha x \frac{\partial F_0}{\partial x} + k^*(m^* - y)\frac{\partial F_0}{\partial y} - rF_0 = 0. \tag{3.52}$$

Let F_1 denote the project value when the the project has been already activated by the firm. Then, F_1 satisfies the following PDE

$$\frac{1}{2}\left[x^2 y \frac{\partial^2 F_1}{\partial x^2} + y\sigma^2 \frac{\partial^2 F_1}{\partial y^2} + 2\rho x y \sigma \frac{\partial^2 F_1}{\partial x \partial y}\right] + \alpha x \frac{\partial F_1}{\partial x} + k^*(m^* - y)\frac{\partial F_1}{\partial y} + xD - rF_1 = 0. \tag{3.53}$$

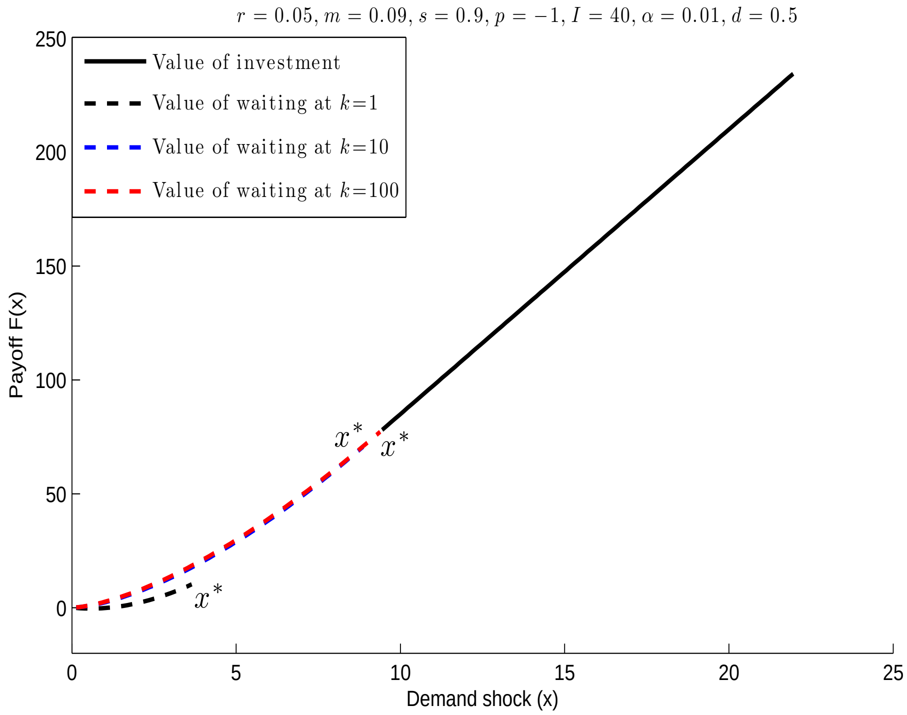

Figure 3.10: The effect of the mean reverting rate k

The boundary conditions corresponding to (3.52) and (3.53) are

$$F_0(x^*(y), y) = F_1(x^*(y), y) - I, \tag{3.54}$$

$$F_1(x^*(y), y) = F_0(x^*(y), y) - E, \tag{3.55}$$

$$F_0(0, y) = 0, \tag{3.56}$$

$$F_1(\infty, y) - \frac{xD}{\delta} = 0, \tag{3.57}$$

$$\left.\frac{\partial F_0}{\partial x}\right|_{x=x^*(y)} = \left.\frac{\partial F_1}{\partial x}\right|_{x=x^*(y)}, \tag{3.58}$$

$$\left.\frac{\partial F_0}{\partial y}\right|_{x=x^*(y)} = \left.\frac{\partial F_1}{\partial y}\right|_{x=x^*(y)}. \tag{3.59}$$

Then (3.52) and (3.53) can be rewritten in the following compact forms,

$$\left(\frac{1}{\varepsilon}\mathfrak{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathfrak{L}_1 + \mathfrak{L}_2\right) F_0 = 0, \tag{3.60}$$

$$\left(\frac{1}{\varepsilon}\mathfrak{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathfrak{L}_1 + \mathfrak{L}_2\right) F_1 + xD = 0. \tag{3.61}$$

We expand $F_0(x,y)$, $F_1(x,y)$ and $x^*(y)$ by

$$F_0(x,y) = F_{0,0}(x,y) + \sqrt{\varepsilon}F_{0,1}(x,y) + \varepsilon F_{0,2}(x,y) + \cdots, \tag{3.62}$$

$$F_1(x,y) = F_{1,0}(x,y) + \sqrt{\varepsilon}F_{1,1}(x,y) + \varepsilon F_{1,2}(x,y) + \cdots, \tag{3.63}$$

$$\overline{x}^*(y) = \overline{x}_0^*(y) + \sqrt{\varepsilon}\overline{x}_1^*(y) + \varepsilon\overline{x}_2^*(y) + \cdots, \tag{3.64}$$

$$\underline{x}^*(y) = \underline{x}_0^*(y) + \sqrt{\varepsilon}\underline{x}_1^*(y) + \varepsilon\underline{x}_2^*(y) + \cdots. \tag{3.65}$$

Substituting (3.62) and (3.63) into (3.60) and (3.61) respectively, and taking terms up to order $\sqrt{\varepsilon}$ give

$$\begin{aligned}
&\frac{1}{\varepsilon}\mathfrak{L}_0F_{0,0} + \frac{1}{\sqrt{\varepsilon}}(\mathfrak{L}_0F_{0,1} + \mathfrak{L}_1F_{0,0}) + (\mathfrak{L}_0F_{0,2} + \mathfrak{L}_1F_{0,1} + \mathfrak{L}_2F_{0,0})\\
&\qquad + \sqrt{\varepsilon}(\mathfrak{L}_0F_{0,3} + \mathfrak{L}_1F_{0,2} + \mathfrak{L}_2F_{0,1}) = 0,\\
&\frac{1}{\varepsilon}\mathfrak{L}_0F_{1,0} + \frac{1}{\sqrt{\varepsilon}}(\mathfrak{L}_0F_{1,1} + \mathfrak{L}_1F_{1,0}) + (\mathfrak{L}_0F_{1,2} + \mathfrak{L}_1F_{1,1} + \mathfrak{L}_2F_{1,0})\\
&\qquad + \sqrt{\varepsilon}(\mathfrak{L}_0F_{1,3} + \mathfrak{L}_1F_{1,2} + \mathfrak{L}_2F_{1,1}) + xD = 0.
\end{aligned}$$

The boundary conditions become

$$\begin{aligned}
&F_{0,0}(\overline{x}^*(y),y) + \sqrt{\varepsilon}\left(\overline{x}_1^*(y)\frac{\partial F_{0,0}}{\partial x}\bigg|_{x=\overline{x}_0^*} + F_{0,1}(\overline{x}_0^*(y),y)\right)\\
&= F_{1,0}(\overline{x}^*(y),y) + \sqrt{\varepsilon}\left(\overline{x}_1^*(y)\frac{\partial F_{1,0}}{\partial x}\bigg|_{x=\overline{x}_0^*} + F_{1,1}(\overline{x}_0^*(y),y)\right) - I,\\
&F_{1,0}(\underline{x}^*(y),y) + \sqrt{\varepsilon}\left(\underline{x}_1^*(y)\frac{\partial F_{1,0}}{\partial x}\bigg|_{x=\underline{x}_0^*} + F_{1,1}(\underline{x}_0^*(y),y)\right)\\
&= F_{0,0}(\underline{x}^*(y),y) + \sqrt{\varepsilon}\left(\underline{x}_1^*(y)\frac{\partial F_{0,0}}{\partial x}\bigg|_{x=\underline{x}_0^*} + F_{0,1}(\underline{x}_0^*(y),y)\right) - E,\\
&\frac{\partial F_{0,0}}{\partial x}\bigg|_{x=\overline{x}_0^*} + \sqrt{\varepsilon}\left(\overline{x}_1^*(y)\frac{\partial^2 F_{0,0}}{\partial x^2}\bigg|_{x=\overline{x}_0^*} + \frac{\partial F_{0,1}}{\partial x}\bigg|_{x=\overline{x}_0^*}\right)\\
&= \frac{\partial F_{1,0}}{\partial x}\bigg|_{x=\overline{x}_0^*} + \sqrt{\varepsilon}\left(\overline{x}_1^*(y)\frac{\partial^2 F_{1,0}}{\partial x^2}\bigg|_{x=\overline{x}_0^*} + \frac{\partial F_{1,1}}{\partial x}\bigg|_{x=\overline{x}_0^*}\right),\\
&\frac{\partial F_{0,0}}{\partial y}\bigg|_{x=\underline{x}_0^*} + \sqrt{\varepsilon}\left(\underline{x}_1^*(y)\frac{\partial^2 F_{0,0}}{\partial x\partial y}\bigg|_{x=\underline{x}_0^*} + \frac{\partial F_{0,1}}{\partial x}\bigg|_{x=\underline{x}_0^*}\right)\\
&= \frac{\partial F_{1,0}}{\partial y}\bigg|_{x=\underline{x}_0^*} + \sqrt{\varepsilon}\left(\underline{x}_1^*(y)\frac{\partial^2 F_{1,0}}{\partial x\partial y}\bigg|_{x=\underline{x}_0^*} + \frac{\partial F_{1,1}}{\partial x}\bigg|_{x=\underline{x}_0^*}\right).
\end{aligned}$$

From the leading terms we get

$$
\begin{aligned}
\mathfrak{L}_0 F_{0,0}(x,y) &= 0, && \text{if} \quad x < \overline{x}_0^*,\\
\mathfrak{L}_0 F_{1,0}(x,y) &= 0, && \text{if} \quad x \geq \underline{x}_0^*,\\
F_{0,0}(\overline{x}_0^*, y) &= F_{1,0}(\overline{x}_0^*, y) - I, &&\\
F_{1,0}(\underline{x}_0^*, y) &= F_{0,0}(\underline{x}_0^*, y) - E, &&\\
\left.\frac{\partial F_{0,0}}{\partial x}\right|_{x=\overline{x}_0^*} &= \left.\frac{\partial F_{1,0}}{\partial x}\right|_{x=\overline{x}_0^*}, &&\\
\left.\frac{\partial F_{1,0}}{\partial x}\right|_{x=\underline{x}_0^*} &= \left.\frac{\partial F_{0,0}}{\partial x}\right|_{x=\underline{x}_0^*}. &&
\end{aligned}
$$

These conditions imply that $F_{0,0}$ and $F_{1,0}$ are independent from y. Then we collect the next order terms of F_0 and F_1 and get

$$\mathfrak{L}_0 F_{0,1}(x,y) = 0, \qquad \text{if} \quad x < \overline{x}_0^*, \tag{3.66}$$

$$\mathfrak{L}_0 F_{1,1}(x,y) = 0, \qquad \text{if} \quad x \geq \underline{x}_0^*, \tag{3.67}$$

$$F_{0,1}(\overline{x}_0^*, y) = F_{1,1}(\overline{x}_0^*, y), \tag{3.68}$$

$$F_{1,1}(\underline{x}_0^*, y) = F_{0,1}(\underline{x}_0^*, y),, \tag{3.69}$$

$$\overline{x}_1^* \left.\frac{\partial^2 F_{0,0}}{\partial x^2}\right|_{x=\overline{x}_0^*} + \left.\frac{\partial F_{0,1}}{\partial x}\right|_{x=\overline{x}_0^*} = \overline{x}_1^* \left.\frac{\partial^2 F_{1,0}}{\partial x^2}\right|_{x=\overline{x}_0^*} + \left.\frac{\partial F_{1,1}}{\partial x}\right|_{x=\overline{x}_0^*}, \tag{3.70}$$

$$\underline{x}_1^* \left.\frac{\partial^2 F_{0,0}}{\partial x^2}\right|_{x=\underline{x}_0^*} + \left.\frac{\partial F_{0,1}}{\partial x}\right|_{x=\underline{x}_0^*} = \underline{x}_1^* \left.\frac{\partial^2 F_{1,0}}{\partial x^2}\right|_{x=\underline{x}_0^*} + \left.\frac{\partial F_{1,1}}{\partial x}\right|_{x=\underline{x}_0^*}. \tag{3.71}$$

Thus $F_{0,0}$ and $F_{1,0}$ are also independent from y and the following conditions hold.

$$
\begin{aligned}
\mathfrak{L}_0 F_{0,2} + \mathfrak{L}_2 F_{0,0} &= 0, && \text{if} \quad x < \overline{x}_0^*,\\
\mathfrak{L}_0 F_{1,2} + \mathfrak{L}_2 F_{1,0} &= 0, && \text{if} \quad x \geq \underline{x}_0^*.
\end{aligned}
$$

Then the centering conditions are

$$
\begin{aligned}
\langle \mathfrak{L}_2 F_{0,0} \rangle &= 0, \quad \text{if} \quad x < \overline{x}_0^*,\\
\langle \mathfrak{L}_2 F_{1,0} \rangle &= 0, \quad \text{if} \quad x \geq \underline{x}_0^*,
\end{aligned}
$$

where the angled brackets indicate taking the average of the argument with respect to the invariant distribution of the diffusion whose infinitesimal generator is $\mathfrak{L}_0$. Following the analysis from Subsection 3.1.2 for the zero order terms, we get

$$\begin{aligned} F_{0,0} &= A_0 x^{\beta_1}, \\ F_{1,0} &= A_1 x^{\beta_2} + \frac{xD}{\delta} - \frac{C}{r}, \end{aligned}$$

where β_1 and β_2 are same as in (3.1) and (3.10), and the following boundary conditions

$$\begin{aligned} A_0 \overline{x}_0^{*\beta_1} &= A_1 \overline{x}_0^{*\beta_2} + \frac{\overline{x}_0^* D}{\delta} - \frac{C}{r} - I, \\ A_1 \underline{x}_0^{*\beta_2} + \frac{\underline{x}_0^* D}{\delta} - \frac{C}{r} &= A_0 \underline{x}_0^{*\beta_1} - E, \\ \beta_1 A_0 \overline{x}_0^{*\beta_1 - 1} &= \beta_2 A_1 \overline{x}_0^{*\beta_2 - 1} + \frac{D}{\delta}, \\ \beta_2 A_1 \underline{x}_0^{*\beta_2 - 1} + \frac{D}{\delta} &= \beta_1 A_0 \underline{x}_0^{*\beta_1 - 1}, \end{aligned}$$

which are sufficient to determine A_0, A_1 and the optimal thresholds $\overline{x}_0^*, \underline{x}_0^*$.

Comparing $\sqrt{\varepsilon}$ order terms gives

$$\begin{aligned} &\mathfrak{L}_0 F_{0,3} + \mathfrak{L}_1 F_{0,2} + \mathfrak{L}_2 F_{0,1} = 0, && \text{if} \quad x < \overline{x}_0^*, \\ &\mathfrak{L}_0 F_{1,3} + \mathfrak{L}_1 F_{1,2} + \mathfrak{L}_2 F_{1,1} = 0, && \text{if} \quad x \geq \underline{x}_0^*, \end{aligned}$$

which lead to a Poisson equation for F_3. The corresponding centering conditions are

$$\begin{aligned} \langle \mathfrak{L}_1 F_{0,2} + \mathfrak{L}_2 F_{0,1} \rangle = 0, \\ \langle \mathfrak{L}_1 F_{1,2} + \mathfrak{L}_2 F_{1,1} \rangle = 0. \end{aligned}$$

Let $\overline{F}_{0,1} = \sqrt{\varepsilon} F_{0,1}$ and $\overline{F}_{1,1} = \sqrt{\varepsilon} F_{1,1}$. Then the above centering conditions imply

$$\begin{aligned} \frac{1}{2} m x^2 \frac{d^2 \overline{F}_{0,1}}{dx^2} + (r-\delta) x \frac{d\overline{F}_{0,1}}{dx} - r\overline{F}_{0,1} &= B_0 A_0 x^{\beta_1}, \\ \frac{1}{2} m x^2 \frac{d^2 \overline{F}_{1,1}}{dx^2} + (r-\delta) x \frac{d\overline{F}_{1,1}}{dx} - r\overline{F}_{1,1} &= B_1 A_1 x^{\beta_2}, \end{aligned}$$

where

$$B_0 = -\frac{\rho\sigma m}{2k} \beta_1^2 (\beta_1 - 1) \quad \text{and} \quad B_1 = -\frac{\rho\sigma m}{2k} \beta_2^2 (\beta_2 - 1).$$

By applying the conditions (3.56) and (3.57), the general solutions of $F_{0,1}$ and $F_{1,1}$ are of the forms

$$\begin{aligned}\overline{F}_{0,1} &= C_0 x^{\beta_1} - \frac{2B_0A_0}{m}\frac{x^{\beta_1}}{\beta_2-\beta_1}\left(\ell n(x) + \frac{1}{\beta_2-\beta_1}\right),\\ \overline{F}_{1,1} &= C_1 x^{\beta_2} - \frac{2B_1A_1}{m}\frac{x^{\beta_2}}{\beta_1-\beta_2}\left(\ell n(x) + \frac{1}{\beta_1-\beta_2}\right).\end{aligned}$$

From boundary conditions (3.68) and (3.69), we get

$$\overline{F}_{0,1}(\overline{x}_0^*) = \overline{F}_{1,1}(\overline{x}_0^*) \quad \text{and} \quad \overline{F}_{1,1}(\underline{x}_0^*) = \overline{F}_{1,1}(\underline{x}_0^*).$$

Again, we are only able to determine C_0 and C_1 numerically. For the threshold expansions $\overline{x}_1^*$ and $\underline{x}_1^*$, we use conditions (3.70) and (3.71) and get

$$\overline{x}_1^* = \left(\left.\frac{\partial F_{1,1}}{\partial x}\right|_{x=\overline{x}_0^*} - \left.\frac{\partial F_{0,1}}{\partial x}\right|_{x=\overline{x}_0^*}\right)\Big/\left(\left.\frac{\partial^2 F_{0,0}}{\partial x^2}\right|_{x=\overline{x}_0^*} - \left.\frac{\partial^2 F_{1,0}}{\partial x^2}\right|_{x=\overline{x}_0^*}\right), \tag{3.72}$$

$$\underline{x}_1^* = \left(\left.\frac{\partial F_{1,1}}{\partial x}\right|_{x=\underline{x}_0^*} - \left.\frac{\partial F_{0,1}}{\partial x}\right|_{x=\underline{x}_0^*}\right)\Big/\left(\left.\frac{\partial^2 F_{0,0}}{\partial x^2}\right|_{x=\underline{x}_0^*} - \left.\frac{\partial^2 F_{1,0}}{\partial x^2}\right|_{x=\underline{x}_0^*}\right). \tag{3.73}$$

In what follows we will show the effect of the additional parameters on the model. For similar reason in Subsection 3.1.2, we only focus on $\underline{x}^*$. Again, we start with the correlation parameter ρ. In Figure 3.11, the solid line and the middle $\underline{x}^*$ represent the result of the classical solution. The graph shows that when ρ is positive (resp. negative), $\underline{x}_1^*$ is positive (resp. negative). Moreover, the correction term of the real option to invest in the project is negative (resp. positive) and the correction term of the real option to suspend the project is positive (resp. negative). To see the effect of the volatility parameter σ of the variance process, we fix $\rho = 1$. From Figure 3.12, we see that a larger σ leads to an increase in the suspension threshold. Moreover, the real option to invest in project increases and the real option to suspend the project decreases as σ increases. Finally, since we have checked that under fast mean reversion of the variance process, the asymptotic solution converges to the classical solution, we will omit the discussion for the mean reverting rate k here.

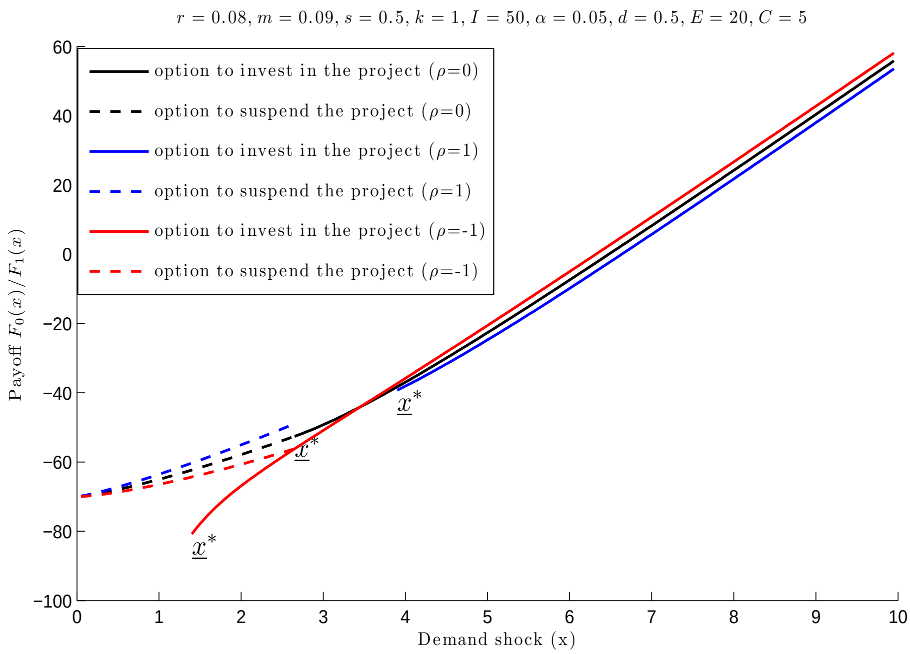

Figure 3.11: The effect of the correlation ρ

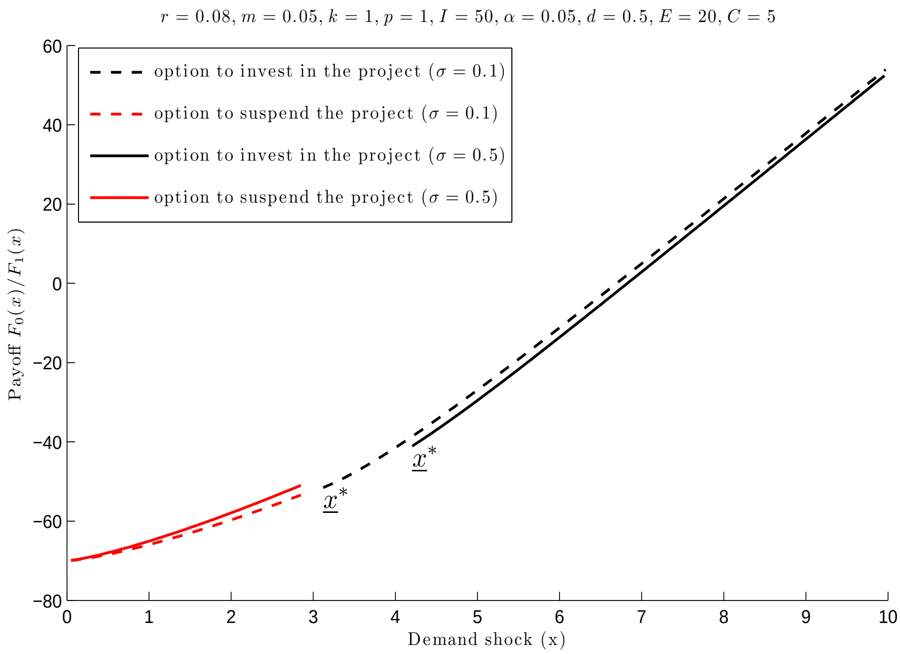

Figure 3.12: The effect of the volatility parameter σ of the variance process

Chapter 4

Real Option Duopoly Models

In this chapter, we investigate real option models for the duopoly case where the leader and the follower are fixed. In Section 4.1, we first present a basic duopoly model with constant volatility. We do this by first determining the follower's optimal investment threshold and its corresponding profit, then working backwards to determine the leader's counterparts. In Section 4.2, we extend the model in Subsection 3.2.1 to a duopoly model. Finally, we also discuss the effect of the parameters on the leader's option value and its optimal investment thresholds.

4.1 A Basic Duopoly Model

In this section, we consider a duopoly model similar to that in [11]. Suppose there are two firms, namely a and b who have potentials to enter the market. Let a be the leader and b be the follower. The follower is not allowed to invest until the leader does. This is in the same manner as the Stackelberg model. We also assume that the industry output, denoted by Q, can take values 0, 1 or 2, depending on the number of active firms in the market, and the demand function $D(Q)$ is defined as that in (2.42). Thus Firm a earns the monopoly profit flow $xD(1)$. Then after Firm b invests, they share the market and each of them earns the duopoly profit flow $xD(2)$. According to the concepts in Dynamic Programming, we should start the analysis from Firm b, because there is no other firm following it. To calculate Firm b's option value, we first assume that Firm a has already invested. This means that we can treat Firm b as it is in a monopoly case with less profit. The following proposition is a slight modification of Proposition 3.1.

Proposition 4.1. *Under the assumptions* **(A1)−(A6)***, suppose that Firm a has already invested* *in the project, Firm b's option value, denoted by $F^b(x)$, is given by*

$$F^b(x) = \begin{cases} \left(\frac{x^b D(2)}{\delta} - I\right)\left(\frac{x}{x^b}\right)^{\beta_1}, & \text{if} \quad x < x^b, \\ \\ \frac{xD(2)}{\delta} - I, & \text{if} \quad x \geq x^b, \end{cases} \tag{4.1}$$

where β_1 and β_2 are the same as those in (3.1) *and* (3.10) *respectively, and Firm b's optimal threshold x^b is given by*

$$x^b = \frac{\beta_1 I \delta}{(\beta_1 - 1)D(2)}.$$

Then the corresponding investing time T^b is

$$T^b = \inf\left\{t \geq 0 : x \geq x^b\right\}. \tag{4.2}$$

By Proposition 4.1, Firm b only has one strategy, i.e., to invest at time T^b. Since Firm a has already invested in the project, its project value will change depending on Firm b's action. Specifically, when Firm b is waiting, Firm a earns the monopoly profit flow. After Firm b invests in the project, both firms earn the duopoly profit. Firm a's project value, denoted by V^a, is given in the following proposition

Proposition 4.2. *Given that Firm b will invest in the project at time T^b, Firm a's project value is*

$$V^a(x) = \begin{cases} \frac{xD(1)}{\delta} + \left(\frac{x}{x^b}\right)^{\beta_1} \frac{x^b(D(2) - D(1))}{\delta}, & \text{if} \quad x < x^b, \\ \\ \frac{xD(2)}{\delta}, & \text{if} \quad x \geq x^b. \end{cases} \tag{4.3}$$

Proof. To see how V^a is derived, note that Firm a's project value can be expressed as the sum of the expected present value of the monopoly future profits and the expected present value of the duopoly future profits, i.e.,

$$V^a(x) = \begin{cases} \mathbb{E}\left[\int_0^{T^b} e^{-rt} xD(1)dt\right] + \mathbb{E}\left[\int_{T^b}^{\infty} e^{-rt} xD(2)dt\right] & \text{if} \quad x < x^b, \\ \\ \frac{xD(2)}{\delta}, & \text{if} \quad x \geq x^b. \end{cases}$$

To calculate $V^a(x)$ when $x < x^b$, we can break the formula into two parts. First, we let

$$L(x) = \mathbb{E}\left[\int_0^{T^b} e^{-rt} xD(1)dt\right].$$

Using Bellman's optimality principal, we get the following ODE

$$\frac{1}{2}x^2\sigma\frac{d^2L}{dx^2} + (r-\delta)x\frac{dL}{dx} + xD(1) - rL = 0. \tag{4.4}$$

The boundary conditions corresponding to (4.4) are

$$L(x^b) = 0, \tag{4.5}$$

$$L(0) = 0. \tag{4.6}$$

Condition (4.5) says that as x approaches to x^b, T^b is likely to be small and so $L(x^b) = 0$. Condition (4.6) says that when x is very small, T^b is likely to be large and therefore $L(0) = 0$. Thus (4.4) can be solved by the standard method to yield

$$L(x) = -\frac{x^b D(1)}{\delta}\left(\frac{x}{x^b}\right)^{\beta_1} + \frac{xD(1)}{\delta}. \tag{4.7}$$

Similarly, let

$$G(x) = \mathbb{E}\left[\int_{T^b}^{\infty} e^{-rt} xD(2)dt\right].$$

Then the following ODE holds

$$\frac{1}{2}x^2\sigma\frac{d^2G}{dx^2} + (r-\delta)x\frac{dG}{dx} - rG = 0, \tag{4.8}$$

whose corresponding boundary conditions are

$$G(0) = 0, \tag{4.9}$$

$$G(x^b) = \frac{x^b D(2)}{\delta}. \tag{4.10}$$

Condition (4.9) says when x is very small, T^b is likely to be large and thus $G(0) = 0$. Condition (4.10) says that as x approaches x^b, T^b is likely to be small and thus $G(x^b) = \frac{x^b D(2)}{\delta}$. Solving (4.8) by a method similar to that for (4.4) gives

$$G(x) = \left(\frac{x}{x^b}\right)^{\beta_1} \frac{x^b D(2)}{\delta}. \tag{4.11}$$

Combining (4.7) and (4.11) gives (4.3). □

Now we are ready to calculate the real option value for Firm a and its investment thresholds. Again, it is obvious that the option value F^a satisfies

$$\frac{1}{2}y^2 v\frac{d^2F^a}{dx^2} + (r-\delta)x\frac{dF^a}{dx} - rF^a = 0. \tag{4.12}$$

However, for Firm a, there are three investment thresholds that must be determined. The first investment threshold, denoted by x^{a_1}, is used when the initial level of the demand shock x_0 is low. In this case, the corresponding boundary conditions are

$$\begin{aligned} F^a(x^{a_1}) &= V^a(x^{a_1}) - I, && (4.13)\\ F^a(0) &= 0, && (4.14)\\ \left.\frac{dF^a}{dx}\right|_{x=x^{a_1}} &= \left.\frac{dV^a}{dx}\right|_{x=x^{a_1}}. && (4.15) \end{aligned}$$

By a method similar to that for (4.4), we can solve (4.12) to get the following result.

Proposition 4.3. *When $x_0 < x^{a_1}$, Firm a's profit is given by*

$$F^a(x) = \begin{cases} \left(\dfrac{x^{a_1}D(1)}{\delta} - I\right)\left(\dfrac{x}{x^{a_1}}\right)^{\beta_1} + \dfrac{x^b\left(D(2)-D(1)\right)}{\delta}\left(\dfrac{x}{x^b}\right)^{\beta_1}, & \text{if} \quad x < x^{a_1}, \\ V^a(x) - I, & \text{if} \quad x \geq x^{a_1}, \end{cases} \tag{4.16}$$

where $V^a(x)$ is given in (4.3). *The corresponding optimal strategy for Firm a is to wait until x first reaches the optimal investment threshold x^{a_1} and invest in the project, where*

$$x^{a_1} = \frac{\beta_1 I\delta}{(\beta_1 - 1)D(1)}. \tag{4.17}$$

The optimal investment threshold x^{a_1} is in fact the same as that in the monopoly case. The fixed priority gives Firm a the ability to fully capture the value of waiting without being preempted by the follower. Moreover, the fixed priority also guarantees another demand shock region in which the value of waiting is greater than the value of project, since Firm b's investment will decrease Firm a's profit. Such region is defined by the second and the third investment thresholds, namely x^{a_2} and x^{a_3} respectively, which we will consider together when the initial demand shock is high. For a range of x that includes neither zero nor infinity, the solution of the option value takes the form of

$$F^a(x) = J(1)x^{\beta_1} + J(2)x^{\beta_2}, \tag{4.18}$$

where β_1 and β_2 are the same as those in (3.1) and (3.10). When $x^{a_1} \le x < x^b$, there is a boundary point x^{a_2} satisfying the boundary conditions,

$$F^a(x^{a_2}) = V^a(x^{a_2}) - I, \tag{4.19}$$

$$\left.\frac{dF^a}{dx}\right|_{x=x^{a_2}} = \left.\frac{\partial V^{a_0}}{\partial x}\right|_{x=x^{a_2}}. \tag{4.20}$$

When $x \ge x^b$, there is a boundary point x^{a_3} at which F^a satisfies the following boundary conditions

$$F^a(x^{a_3}) = \frac{x^{a_3} D(2)}{\delta} - I, \tag{4.21}$$

$$\left.\frac{dF^a}{dx}\right|_{x=x^{a_3}} = \frac{D(2)}{\delta}. \tag{4.22}$$

Since this system of equations are non-linear, the investment thresholds x^{a_2}, x^{a_3} and the constants $J(1), J(2)$ can only be determined numerically. The previous analysis can be summarised into the following proposition.

Proposition 4.4. *Let x^{a_2} and x^{a_3} satisfy* (4.19)–(4.22) *respectively, and the initial level of the demand shock x_0 be such that $x_0 \ge x^{a_1}$. The optimal investing strategy for Firm a is to wait if $x \in [x^{a_2}, x^{a_3})$ and invest otherwise.*

In Figure 4.1, we show both firms' profits and all the corresponding investment thresholds. The dashed curves denote the option values of waiting, which means that firms should not invest in the project. In the pre-determined leader-follower case, if x falls in the range $[0, x^{a_1}) \cap [x^{a_2}, x^{a_3})$, we can see that waiting is more profitable than investing in the project for Firm a. Firm b will invest if x^b is reached. While Firm b is waiting, its profit is denoted by the red dash curve. If both firms invest in the project, they share the market and enjoy the duopoly profit, which is denoted by the black curve. The reason that Firm a has to wait when $x \in [x^{a_2}, x^{a_3})$ is that if the current demand shock gets too close to Firm b's optimal investment threshold, Firm a may not earn the monopoly profit. Even if Firm a can earn the monopoly profit, the earning period is very short, since Firm b will invest soon. Thus, Firm a is recommended to wait until an even higher demand shock level is reached, and both firms invest simultaneously and enjoy a higher duopoly profit. Again, we exam how the drift parameter α can affect Firm a. From Figure 4.2, we see the effect is similar to that in the monopoly case. An increase in α

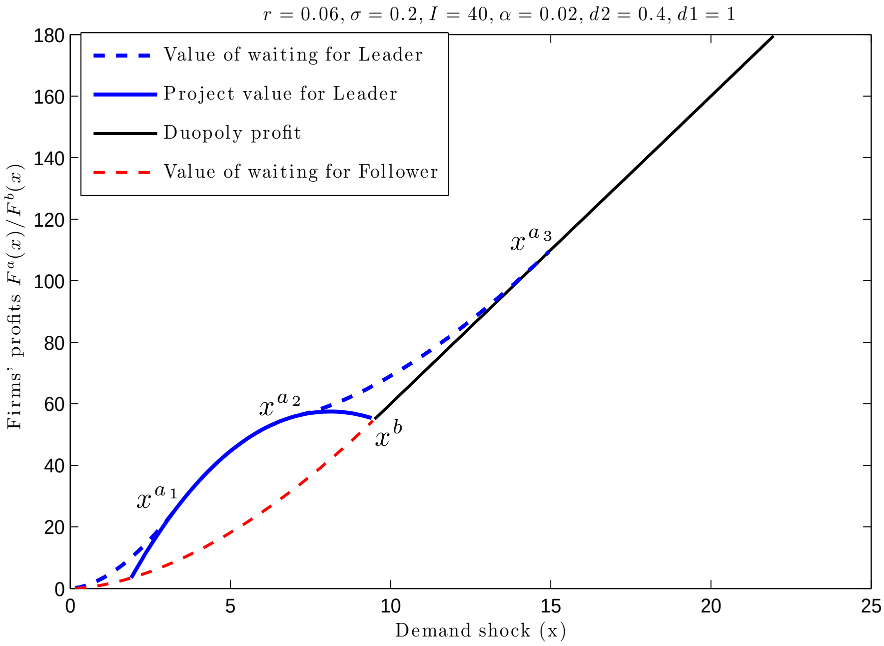

Figure 4.1: Leader vs Follower

will not only increase the profit of Firm a but also decrease the optimal investment thresholds of Firm a. Another result we see from the graph is that the gap between the profits of firms get smaller which means that being a leader becomes less attractive when α gets bigger. For the effect of volatility parameter σ, we also get a conclusion similar to that in the monopoly case. An increase in the volatility parameter σ will increase the option value, which will result in increases of all the investment thresholds of both firms. Furthermore, in the monopoly case, the project value is not effected by any change in the value of σ whereas in the duopoly model, the Firm a's project value is increased significantly.

4.2 A Duopoly Model with Stochastic Volatility

Next, we shall again extend the analysis in Section 4.1 to the stochastic volatility case. The work presented here is mainly from [26]. It should be clear that Firm b's real option value is similar to that in the monopoly case. Thus we refer to Proposition 3.2 and get the following proposition

Proposition 4.5. *Under* **(A1)–(A5)** *and* **(A6')**, *Firm b's option value, denoted by* $F^b(x)$, *is*

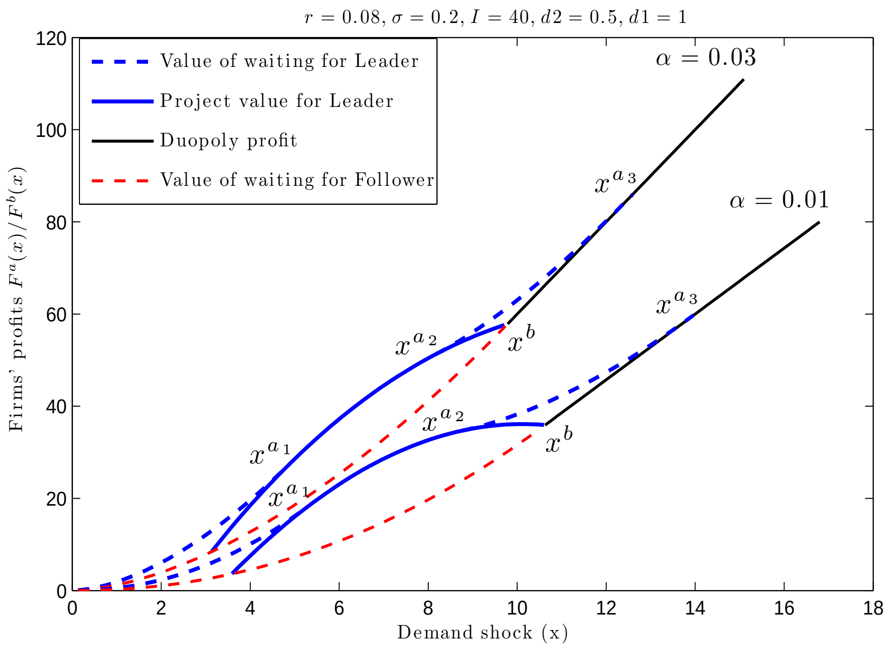

Figure 4.2: Effect of the drift parameter α on Leader and Follower

given asymptotically by

$$F^b(x) \approx \begin{cases} \left(1 - \dfrac{\rho\sigma\beta_1^2(\beta_1-1)}{k^*(\beta_2-\beta_1)}\ell n\left(\dfrac{x_0^b}{x}\right)\right)\left(\dfrac{x_0^b D}{\delta} - I\right)\left(\dfrac{x}{x_0^b}\right)^{\beta_1}, & \textit{if} \quad x < x_0^b, \\[2ex] \dfrac{xD}{\delta} - I, & \textit{if} \quad x \geq x_0^b, \end{cases} \tag{4.23}$$

where β_1 *and* β_2 *are given in* (3.1) *and* (3.10) *and*

$$x_0^b = \frac{\beta_1 I\delta}{(\beta_1-1)D}. \tag{4.24}$$

The optimal threshold x^b *of Firm b is given asymptotically by*

$$x^b \approx \frac{\beta_1 I\delta}{(\beta_1-1)D}\left(1 - \frac{\rho\sigma\beta_1}{k^*(\beta_2-\beta_1)}\right). \tag{4.25}$$

Hence the optimal strategy for Firm b is to invest at time T^b *which is given by*

$$T^b = \inf\left\{t \geq 0 : x \geq x^b\right\}. \tag{4.26}$$

Next, we will discuss the real option value and investment strategies of Firm a. Similar to Section 4.1, by knowing the optimal investment threshold of Firm b, we can express Firm a's project value V^a as

$$V^a(x,y) = \begin{cases} \mathbb{E}\left[\int_0^{T^b} e^{-rt} xD(1)dt\right] + \mathbb{E}\left[\int_{T^b}^{\infty} e^{-rt} xD(2)dt\right] & \text{if} \quad x < x_0^b, \\ \dfrac{xD(2)}{\delta}, & \text{if} \quad x \geq x_0^b. \end{cases}$$

If we denote $L'(x,y) = \mathbb{E}\left[\int_0^{T^b} e^{-rt} xD(1)dt\right]$ and $G'(x,y) = \mathbb{E}\left[\int_{T^b}^{\infty} e^{-rt} xD(2)dt\right]$, from the previous analysis, it is not difficult to derive the following PDEs.

$$\begin{aligned} \frac{1}{2}\left[\frac{\partial^2 L'}{\partial x^2}x^2 y + \frac{\partial^2 L'}{\partial y^2}y\sigma^2 + 2\frac{\partial^2 L'}{\partial x \partial y}\rho xy\sigma\right] & \\ +\frac{\partial L'}{\partial x}\alpha x + k(m-S)\frac{\partial L'}{\partial y} + xD(1) - rL' &= 0, \end{aligned} \tag{4.27}$$

and

$$\begin{aligned} \frac{1}{2}\left[\frac{\partial^2 G'}{\partial x^2}x^2 y + \frac{\partial^2 G'}{\partial y^2}y\sigma^2 + 2\frac{\partial^2 G'}{\partial x \partial y}\rho xy\sigma\right] & \\ +\frac{\partial G'}{\partial x}\alpha x + k(m-S)\frac{\partial G'}{\partial y} - rG' &= 0. \end{aligned} \tag{4.28}$$

The boundary conditions corresponding to (4.27) and (4.28) are

$$L'(x_0^b, y) = 0, \tag{4.29}$$

$$L'(0, y) = 0, \tag{4.30}$$

$$\left.\frac{\partial L'}{\partial y}\right|_{x=x_0^b} = 0, \tag{4.31}$$

and

$$G'(0, y) = 0, \tag{4.32}$$

$$G'(x_0^b, y) = \frac{x_0^b D(2)}{\delta}, \tag{4.33}$$

$$\left.\frac{\partial G'}{\partial y}\right|_{x=x_0^b} = 0, \tag{4.34}$$

respectively. By following the steps similar to those in Section 3.2, we know that the first two terms in the asymptotic expansions of $L'(x,y)$ and $G'(x,y)$ are independent of y. Thus in the sequel, we will only find the first two terms of the asymptotic expansions of $L'(x,y)$ and $G'(x,y)$, which are functions of x only. For the zero order terms, we get

$$L_0'(x) = -\frac{x_0^b D(1)}{\delta}\left(\frac{x}{x_0^b}\right)^{\beta_1} + \frac{xD(1)}{\delta}, \tag{4.35}$$

and

$$G_0'(x) = \left(\frac{x}{x_0^b}\right)^{\beta_1} \frac{x_0^b D(2)}{\delta}. \tag{4.36}$$

The correction terms for $L'(x,y)$ and $G'(x,y)$ are given by

$$\sqrt{\varepsilon}L_1'(x) = \frac{\rho\sigma\beta_1^2(\beta_1-1)}{k^*(\beta_2-\beta_1)}\ell n\left(\frac{x_0^b}{x}\right)\frac{x_0^b D(1)}{\delta}\left(\frac{x}{x_0^b}\right)^{\beta_1}, \tag{4.37}$$

and

$$\sqrt{\varepsilon}G_1'(x) = -\frac{\rho\sigma\beta_1^2(\beta_1-1)}{k^*(\beta_2-\beta_1)}\ell n\left(\frac{x_0^b}{x}\right)\left(\frac{x}{x_0^b}\right)^{\beta_1}\frac{x_0^b D(2)}{\delta}, \tag{4.38}$$

respectively. Combining (4.35)-(4.38), we get the following asymptotic result for Firm a's project value

$$V^a(x,y) \approx \begin{cases} \dfrac{xD(1)}{\delta} + \left(\dfrac{x}{x_0^b}\right)^{\beta_1}\left(1 - \dfrac{\rho\sigma\beta_1^2(\beta_1-1)}{k^*(\beta_2-\beta_1)}\ell n\left(\dfrac{x_0^b}{x}\right)\right)\dfrac{x_0^b(D(2)-D(1))}{\delta}, & \text{if} \quad x < x_0^b, \\ \dfrac{xD(2)}{\delta}, & \text{if} \quad x \geq x_0^b. \end{cases} \tag{4.39}$$

Again we have to determine three investment thresholds and the corresponding real option value $F^a(x,y)$. To solve the following PDE

$$\frac{1}{2}\left[x^2 y\frac{\partial^2 F^a}{\partial x^2} + y\sigma^2\frac{\partial^2 F^a}{\partial y^2} + 2\rho xy\sigma\frac{\partial^2 F^a}{\partial x\partial y}\right] + \alpha x\frac{\partial F^a}{\partial x} + k^*(m^*-y)\frac{\partial F^a}{\partial y} - rF^a = 0,$$

whose boundary conditions are

$$\begin{aligned}
F^a(x^a(y), y) &= V^a(x^a(y), y) - I, && (4.40)\\
F^a(0, y) &= 0, && (4.41)\\
\left.\frac{\partial F^a}{\partial x}\right|_{x=x^a(y)} &= \left.\frac{\partial V^a}{\partial x}\right|_{x=x^a(y)}, && (4.42)\\
\left.\frac{\partial F^a}{\partial y}\right|_{x=x^a(y)} &= \left.\frac{\partial V^a}{\partial y}\right|_{x=x^a(y)}, && (4.43)
\end{aligned}$$

we put

$$V_0^a(x) = \frac{xD(1)}{\delta} + Hx^{\beta_1} \text{ and } \sqrt{\varepsilon}V_1^a(x) = \frac{2BH}{m^*(\beta_2 - \beta_1)}\ell n\left(\frac{x_0^b}{x}\right)x^{\beta_1},$$

where

$$B = -\frac{\rho\sigma m^*}{2k^*}\beta_1^2(\beta_1 - 1) \text{ and } H = \left(\frac{1}{x_0^b}\right)^{\beta_1}\frac{x_0^b(D(2) - D(1))}{\delta}.$$

Then, we can rewrite (4.39) in the following form,

$$V^a(x) \approx \begin{cases} V_0^a(x) + \sqrt{\varepsilon}V_1^a(x), & \text{if } \quad x < x_0^b, \\ \dfrac{xD(2)}{\delta}, & \text{if } \quad x \geq x_0^b. \end{cases}$$

Applying techniques similar to those in Section 3.2 gives

$$\frac{1}{2}m^* x^2 \frac{d^2F_0^a}{dx^2} + (r - \delta)x\frac{dF_0^a}{dx} - rF_0^a = 0,$$

with the boundary conditions

$$\begin{aligned}
F_0^a(x_0^a) &= V_0^a(x_0^a) - I,\\
F_0^a(0) &= 0,\\
\left.\frac{dF_0^a}{dx}\right|_{x_0^a} &= \left.\frac{dV_0^a}{dx}\right|_{x=x_0^a}.
\end{aligned}$$

For the range of x that includes zero, $F_0^a = Jx^{\beta_1}$, where β_1 is given in (3.1) and

$$J = \left(\frac{x_0^{a_1}D(1)}{\delta} - I\right)\left(\frac{1}{x_0^{a_1}}\right)^{\beta_1} + \frac{x_0^b(D(2) - D(1))}{\delta}\left(\frac{1}{x_0^b}\right)^{\beta_1}. \tag{4.44}$$

The first optimal threshold $x_0^{a_1}$ is given by

$$x_0^{a_1} = \frac{\beta_1 I\delta}{(\beta_1 - 1)D(1)}. \tag{4.45}$$

Let $\overline{F}_1^a = \sqrt{\varepsilon}F_1^a$. Then $\overline{F}_1^a$ satisfies the following equation

$$\frac{1}{2}m^* x^2 \frac{d^2\overline{F}_1^a}{dx^2} + (r-\delta)x\frac{d\overline{F}_1^a}{dx} - r\overline{F}_1^a = BJx^{\beta_1}, \tag{4.46}$$

whose boundary conditions are

$$\begin{aligned} \overline{F}_1^a(x_0^{a_1}) &= \overline{V}_1^a(x_0^{a_1}), & (4.47)\\ \overline{F}_1^a(0) &= 0. & (4.48) \end{aligned}$$

Solving (4.46) gives

$$\overline{F}_1^a(x) = \frac{2BJ}{m^*(\beta_2 - \beta_1)}\ell n\left(\frac{x_0^{a_1}}{x}\right)x^{\beta_1} + \frac{2BH}{m^*(\beta_2 - \beta_1)}\ell n\left(\frac{x_0^b}{x_0^{a_1}}\right)x^{\beta_1}. \tag{4.49}$$

Now we determine the correction term $x_1^{a_1}$. Let $\overline{V}_1^a = \sqrt{\varepsilon}V_1^a$. Applying Taylor's expansion theorem to equation (4.42) yields

$$\overline{x}_1^{a_i} = \left(\frac{\partial \overline{V}_1^a}{\partial x}\bigg|_{x_0^{a_i}} - \frac{\partial \overline{F}_1^a}{\partial x}\bigg|_{x_0^{a_i}}\right)\bigg/\left(\frac{\partial^2 F_0^a}{\partial x^2}\bigg|_{x_0^{a_i}} - \frac{\partial^2 V_0^a}{\partial x^2}\bigg|_{x_0^{a_i}}\right), \quad i = 1,2,3 \tag{4.50}$$

Substituting $\overline{V}_1^a$ and $\overline{F}_1^a$ into (4.50) gives

$$\sqrt{\varepsilon}x_1^{a_1} = \overline{x}_1^{a_1} = -\frac{\rho\sigma\beta_1 x_0^{a_1}}{k^*(\beta_2 - \beta_1)}. \tag{4.51}$$

We summarize the above results in the following proposition.

Proposition 4.6. *Let $x^{a_1} \approx x_0^{a_1} + \sqrt{\varepsilon}x_1^{a_1}$, where $x_0^{a_1}$ and $\sqrt{\varepsilon}x_1^{a_1}$ are given in* (4.45) *and* (4.51) *respectively. When $x < x^{a_1}$, the optimal strategy for Firm a is to wait until x first reaches the trigger level x^{a_1} and invest in the project.*

For a range of x that includes neither zero nor infinity, the solution to the option value takes the form of

$$F_0^a(x) = J(1)x^{\beta_1} + J(2)x^{\beta_2}, \tag{4.52}$$

where β_1 and β_2 are the same as those in (3.1) and (3.10). We now determine two free boundaries $x^{a_2}(y)$ and $x^{a_3}(y)$ of the problem. When $x^{a_1} \leq x < x^b$, there is a boundary point $x_0^{a_2}$ satisfying the following boundary conditions,

$$F_0^a(x_0^{a_2}) = V_0^a(x_0^{a_2}) - I, \tag{4.53}$$

$$\left.\frac{dF_0^a}{dx}\right|_{x=x_0^{a_2}} = \left.\frac{dV_0^a}{dx}\right|_{x=x_0^{a_2}}. \tag{4.54}$$

When $x \geq x^b$, there is a boundary point $x^{a_3}(y)$, whose zero-order term $x_0^{a_3}$ satisfies the following boundary conditions

$$F_0^a(x_0^{a_3}) = \frac{x_0^{a_3} D(2)}{\delta} - I, \tag{4.55}$$

$$\left.\frac{\partial F_0^a}{\partial y}\right|_{x=x_0^{a_3}} = \frac{D(2)}{\delta}. \tag{4.56}$$

We can find the correction term F_1^a once $x_0^{a_2}$, $x_0^{a_3}$, $J(1)$ and $J(2)$ are determined. For a range of x that includes neither zero nor infinity, $\overline{F}_1^a$ satisfies

$$\frac{1}{2}m^* x^2 \frac{d^2\overline{F}_1^a}{dx^2} + (r-\delta)x\frac{d\overline{F}_1^a}{dx} - r\overline{F}_1^a = B(\beta_1)J(1)x^{\beta_1} + B(\beta_2)J(2)x^{\beta_2}. \tag{4.57}$$

We change the constant B to $B(\beta) = -\frac{\rho\sigma m^*}{2k^*}\beta^2(\beta-1)$, since it depends on which β is chosen. The general solution to (4.57) takes the form of

$$\begin{aligned}
\overline{F}_1^a(x) \quad &= \quad L(1)x^{\beta_1} + L(2)x^{\beta_2} \\
&\quad + \frac{2\ell n(x)}{m^*(\beta_1-\beta_2)}(B\left(\beta_1)J(1)x^{\beta_1} + B(\beta_2)J(2)x^{\beta_2}\right) \\
&\quad + \frac{2}{m^*(\beta_1-\beta_2)^2}\left(B(\beta_1)J(1)x^{\beta_1} - B(\beta_2)J(2)x^{\beta_2}\right).
\end{aligned} \tag{4.58}$$

We can determine the constants $L(1)$ and $L(2)$ by the following boundary conditions

$$\begin{aligned}
\overline{F}_1^a(x_0^{a_2}) &= \frac{2BH}{m^*(\beta_2-\beta_1)}\ell n\left(\frac{x_0^b}{x_0^{a_2}}\right)(x_0^{a_2})^{\beta_1}, \\
\overline{F}_1^a(x_0^{a_3}) &= 0.
\end{aligned}$$

Again we use (4.50) to determine the trigger point expansions $x_1^{a_2}$ and $x_1^{a_3}$ numerically. The above analysis can be summarized in the following proposition.

Proposition 4.7. *Let* $x^{a_2} \approx x_0^{a_2} + \sqrt{\varepsilon}x_1^{a_2}$ *and* $x^{a_3} \approx x_0^{a_3} + \sqrt{\varepsilon}x_1^{a_3}$*, where* $x_0^{a_2}$ *and* $x_0^{a_3}$ *satisfy* (4.53)−(4.56) *respectively, and* $\sqrt{\varepsilon}x_1^{a_2}$, $\sqrt{\varepsilon}x_1^{a_3}$ *are determined by* (4.50). *If* $x \geq x^{a_1}$*, the optimal investing strategy for Firm a is to wait if* $x \in [x^{a_2}, x^{a_3})$ *and invest otherwise.*

From Chapter 3, we see that the effect of the volatility parameter σ of the variance process works as an amplification factor of the correction terms similar to the mean reverting rate k which we have also discussed before. The effects of the drift parameter α and the mean reverting level m are similar to those in Section 4.1. Thus we will omit discussions on these parameters. Next we will discuss the effect of the correlation parameter ρ on the investment thresholds of Firm a.

By comparing Figure 4.5 to Figure 4.1, we see that when ρ decreases from 0 to -1, both x^{a_1} and x^b decrease, but x^{a_2} and x^{a_3} increase. This means that the region in which Firm a should hold the investment gets smaller and Firm a is more likely to invest in the project. Whereas when ρ increases from 0 to 1, by comparing Figure 4.6 to Figure 4.1, we see that x^{a_1} and x^{a_2} approach to each other, which means that there could be some cases in which Firm a will never invest if $x < x^{a_3}$.

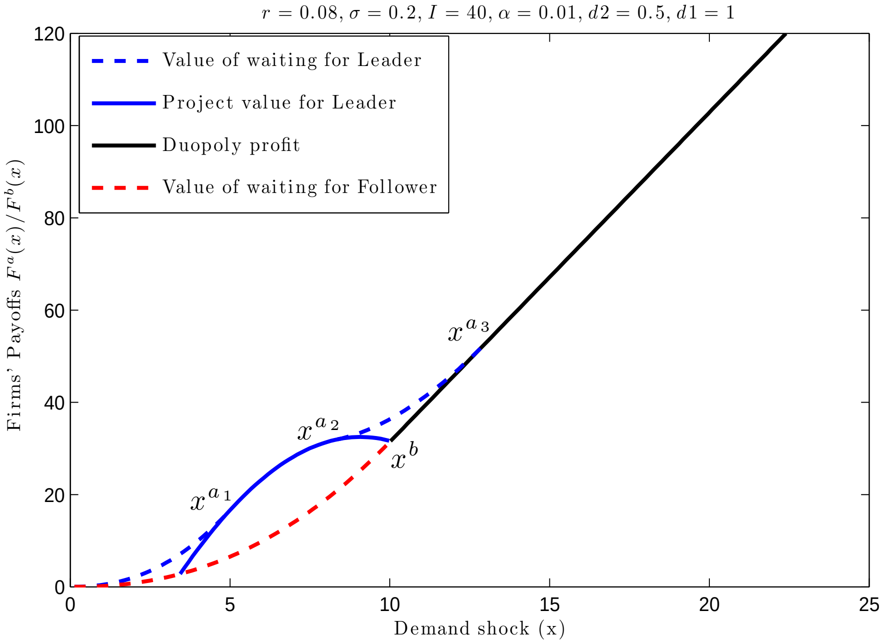

Figure 4.3: Firms' payoffs with volatility $\sigma = 0.2$

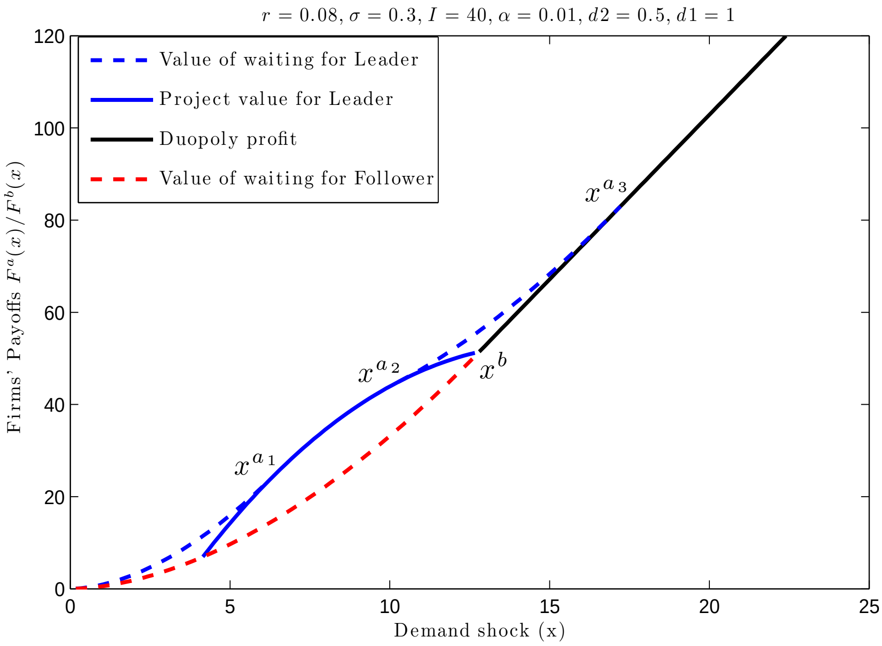

Figure 4.4: Firms' payoffs with volatility $\sigma = 0.3$

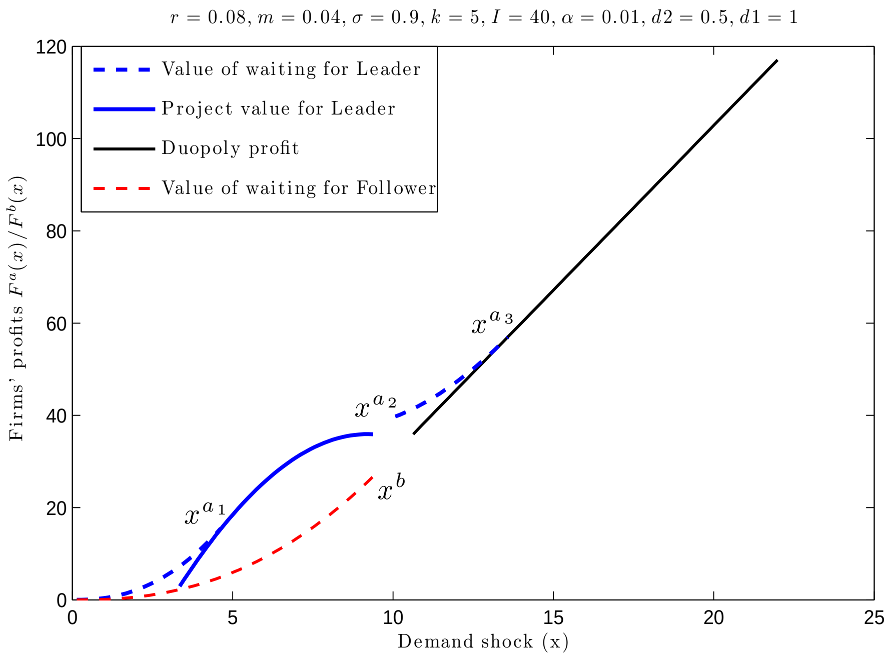

Figure 4.5: The effect on Leander and Follower with correlation $\rho = -1$

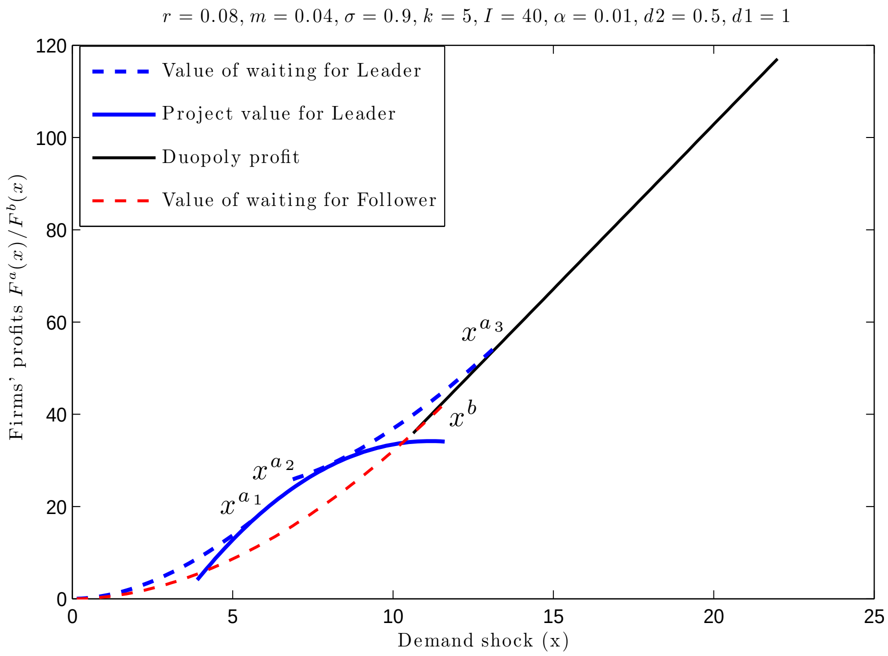

Figure 4.6: The effect on Leander and Follower with correlation $\rho = 1$

Chapter 5

Real Option Duopoly Games

In this chapter, we investigate duopoly real option models in various competitive scenarios, which are also called strategic real option games. The basic duopoly model can be treated as a 2-player non-cooperative game. We consider two approaches to solve the game, namely the pure strategy equilibrium approach and the mixed strategy equilibrium approach. Moreover, we also consider a duopoly model of a 2-player cooperative game. Then, we present a duopoly model with entry and exit decisions in the context of these game situations. Finally, we consider a strategic real option game with asymmetric information. In this model, the pure strategy equilibrium approach may not be optimal. Thus, we apply the mixed strategy equilibrium approach to this model.

5.1 Real Option Games with Complete Information

In this section, we use a basic duopoly model as an example to illustrate different solution concepts for real option games, namely pure strategy equilibria, mixed strategy equilibria, and bargaining solutions. Pure strategy equilibria are commonly considered in many real option literatures such as [11], [32], etc. Huisman and Kort in [27] claimed that the mixed strategy equilibrium approach is more appropriate to model firms' behaviours. The approach is first introduced by Fudenberg and Tirole in [14]. Finally, we think if firms can cooperate with each other, their payoffs are better. Thus we refer to the classical theory in cooperative games to find firms' bargaining payoffs.

5.1.1 Pure Strategy Equilibria

In this subsection, we consider investment decisions in a game situation under **(A1)**–**(A6)**. Specifically, we assume that the demand shock starts at a low level and each firm picks its investment threshold to invest in the project. Whoever invests first will be the leader and earn the monopoly profit for a period of time. Thus both firms have potentials and intentions to become the leader in this game. In order to analyse the game, we need to know the payoff of being each role in the game. The payoff and the optimal investment time of the follower are given in Proposition 3.1. Thus once a firm invests, the game is over. We only need to determine at which point a firm will invest and become the leader. Let V represent the leader's project value, which is given by (4.3). Let F represent the follower's option value, which is given by (4.1). From results in Chapter 4, we know that when the demand shock is low, it is best for the leader, say Firm a, to invest at x^{a_1} to fully capture the value of waiting. However, when Firm a competes with Firm b, Firm a is very likely to be preeampted by Firm b, because Firm b can do better by investing just before Firm a does. Thus Firm a is also forced to invest earlier and the process repeats. This leads us to the following proposition.

Proposition 5.1. *In a competitive situation where both firms can become the leader, there exists a unique investment threshold* $x^P \in (0, x^b)$ *with the following properties:*

$$\begin{aligned} V(x) - I < F(x) \quad &\textit{for} \quad x < x^P, \\ V(x) - I = F(x) \quad &\textit{for} \quad x = x^P, \\ V(x) - I > F(x) \quad &\textit{for} \quad x^P < x < x^b, \\ V(x) - I = F(x) \quad &\textit{for} \quad x \geq x^b. \end{aligned}$$

The proof of Proposition 5.1 is similar to that in [32]. In Proposition 5.1, x^P is called *the pure strategy equilibrium*, at which only one firm will invest. The other firm will wait and invest at x^b, which is given in (4.2).

5.1.2 Mixed Strategy Equilibria

At the pure strategy equilibrium, if both firms choose to invest, it is assumed that firms must somehow cooperate with each other or use random chances to pick one of the firms to be the leader to avoid the simultaneous investment. However, it is more appropriate to show that there exists a probability for the simultaneous investment. To allow mixed-strategies to be used in the above game, a traditional way appeared in [37] suggests that a firm chooses a cumulative probability $G(x)$ as a strategy to represent the probability of investing in the project, given that its rival has not yet invested. Thus for any firm $i \in \{a, b\}$, its payoff can be expressed as

$$U^i(G_a, G_b) = \int_0^\infty [(V(x) - I)(1 - G_j(x))dG_i(x) + F(x)(1 - G_i(x))dG_j(x)] + \sum z_i(x)z_j(x)M(x),$$

where $z_i(y) = \lim_{\epsilon \to 0}[G_i(x) - G_i(x - |\epsilon|)]$ denotes the size of the jump in G_i at x and $M(x)$ denotes the duopoly profit $\frac{xD(2)}{\delta} - I$. We know that the probability distribution $G_i(x)(i \in \{a, b\})$ is a non-decreasing function of x and goes to 1 for a certain level of x. One may ask what happens if the demand shock x suddenly goes over that certain level. If x reaches a level such that both $G_a(x)$ and $G_b(x)$ are equal to 1, then both firms should invest in the project for sure. This can lead both firms to the worst payoff. To overcome this problem, instead of using one single function $G_i(x), (i \in \{a, b\})$, Fudenberg and Tirole [14] introduced following concepts.

Definition 5.1. A *simple strategy* for firm i in the game starting at demand shock x_0 is a pair of real-valued functions $(G_i, \alpha_i) : [x_0, \infty) \times [x_0, \infty) \to [0, 1] \times [0, 1]$ satisfying

(1) G_i is non-decreasing and right-continuous;

(2) $\alpha_i(x) > 0 \Rightarrow G_i(x) = 1$;

(3) α is right-differentiable;

(4) if $\alpha_i(x) = 0$ and $x = \inf\{s \geq x : \alpha_i(s) > 0\}$; then $\alpha_i(\cdot)$ has positive right derivative at x.

Let x_i be the investment threshold for Firm $i \in \{a, b\}$, i.e, $x_i = \inf\{x : \alpha_i(x) > 0\}$, and define

$$T(x) = \inf\{t : x \geq \min\{x_a, x_b\}\}.$$

Following the same argument in [14], the payoff of Firm a can be written as

$$\begin{aligned} U^a(x, G_a, G_b) \quad = \quad & \int_0^{T(x)^-} [(V(x) - I)(1 - G_b(x))dG_a(x) + F(x)(1 - G_a(x))dG_b(x)] \\ & + \sum z_a(x)z_b(x)M(x) \\ & +(1 - G_a^-(x))(1 - G_b^-(x))W^a(x, (G_a, \alpha_a), (G_b, \alpha_b)), \end{aligned}$$

where

$$W^a(x,(G_a,\alpha_a),(G_b,\alpha_b)) = \begin{cases} \dfrac{G_b(x)-G_b^-(x)}{1-G_b^-(x)}\left[(1-\alpha_a(x))F(x)+\alpha_a(x)M(x)\right], & \\ +\dfrac{1-G_b(x)}{1-G_b^-(x)}(V(x)-I) & \text{if} \quad x_a < x_b, \\ \dfrac{G_b(x)-G_b^-(x)}{1-G_b^-(x)}\left[(1-\alpha_a(x))(V(x)-I)+\alpha_a(x)M(x)\right], & \\ +\dfrac{1-G_b(x)}{1-G_b^-(x)}F(x) & \text{if} \quad x_a > x_b, \end{cases}$$

while if $x_a = x_b$,

$$W^a(x,(G_a,\alpha_a),(G_b,\alpha_b)) = \begin{cases} M(x), & \text{if} \qquad \alpha_a(x)=\alpha_b(x)=1, \\ \dfrac{(\alpha_a(x)(1-\alpha_b(x))(V(x)-I))}{\alpha_a(x)+\alpha_b(x)-\alpha_a(x)\alpha_b(x)} & \\ +\dfrac{(\alpha_b(x)(1-\alpha_a(x))F(x))}{\alpha_a(x)+\alpha_b(x)-\alpha_a(x)\alpha_b(x)} & \text{if} \quad 0<\alpha_a(x)+\alpha_b(x)<2, \\ +\dfrac{\alpha_b(x)M(x)}{\alpha_a(x)+\alpha_b(x)-\alpha_a(x)\alpha_b(x)}, & \\ \dfrac{\alpha_a'(x)(V(x)-I)+\alpha_b'(x)F(x)}{\alpha_a'(x)+\alpha_b'(x)}, & \text{if} \qquad \alpha_a(x)=\alpha_b(x)=0. \end{cases}$$

The payoff of Firm b can be defined similarly.

Definition 5.2. A *closed-loop strategy* for a firm i, $(i \in \{a,b\})$ is a collection of simple strategies $\left\{\left(G_i^{x_0}(\cdot),\alpha_i^{x_0}(\cdot)\right) : x_0 \geq 0\right\}$ satisfying the intertemporal consistency conditions:

$$\begin{aligned} G_i^{x_0}(y) &= G_i^{x_0}(s) + \left(1-G_i^{x_0}(s)\right)G_i^x(y), \\ \alpha_i^{x_0}(y) &= \alpha_i^s(y) = \alpha_i(y), \end{aligned}$$

where $x_0 \leq s \leq y$.

Definition 5.3. A pair $\left(\{(G_a^{x_0}(\cdot),\alpha_a^{x_0}(\cdot)) : x_0 \geq 0\}, \left\{\left(G_b^{x_0}(\cdot),\alpha_b^{x_0}(\cdot)\right) : x_0 \geq 0\right\}\right)$ of closed-loop strategies is a *perfect equilibrium* if for every x_0, $\left((G_a^{x_0}(\cdot),\alpha_a^{x_0}(\cdot)), \left(G_b^{x_0}(\cdot),\alpha_b^{x_0}(\cdot)\right)\right)$ is a Nash equilibrium.

Table 5.1: Repeated investment game

		Firm b	
		invest	wait
Firm a	invest	$M(x), M(x)$	$V(x) - I, F(x)$
	wait	$F(x), V(x) - I$	Repeat

Using the above definitions, for each x occurring in the range, in which both $G_a(x)$ and $G_b(x)$ equal to 1, we form a repeated game that is shown in Table 5.1. Each firm has two strategies, investing or waiting. If any firm invests, then the game ends. However, if both firms choose to wait, then the game will repeat until a firm invests. In this game, the demand shock x is fixed, both firms choose probabilities α_a, α_b to invest in the project. Since the game is symmetrical, it is sufficient to calculate the optimal strategy of one firm only. Let $R^a(\alpha_a, \alpha_b)$ be the payoff of Firm a. Then $R^a(\alpha_a, \alpha_b)$ satisfies

$$R^a(\alpha_a, \alpha_b) = \frac{\alpha_a \alpha_b M(x) + \alpha_a(1 - \alpha_b)(V(x) - I) + \alpha_b(1 - \alpha_a)F(x)}{\alpha_a \alpha_b + \alpha_a(1 - \alpha_b) + (1 - \alpha_a)\alpha_b}.$$

Then maximizing $R^a(\alpha_a, \alpha_b)$ with respect to α_a gives

$$\alpha_a = \frac{V(x) - I - F(x)}{V(x) - I - M(x)}.$$

Since the game is symmetrical, $\alpha_b = \alpha_a$. Referring to [14], we obtain the following proposition.

Proposition 5.2. *The equilibrium strategy for each firm $i \in \{a, b\}$ is given by*

$$G_i^{x_0}(x) = G(x) = \begin{cases} 0, & \text{if} \quad x < x^P, \\ 1, & \text{if} \quad x \geq x^P, \end{cases} \tag{5.1}$$

$$\alpha_i^{x_0}(x) = \alpha(x) = \begin{cases} 0, & \text{if} \quad x < x^P, \\ \dfrac{V(x) - I - F(x)}{V(x) - I - M(x)}, & \text{if} \quad x^P \leq x < x^b, \\ 1, & \text{if} \quad x \geq x^b. \end{cases} \tag{5.2}$$

Thus according to (5.1) and (5.2), if $x \in \left(0, x^P\right]$, there are two possible outcomes. The first outcome is that Firm a invests at x^P, and Firm b becomes the follower. The second is the symmetric counterpart, and the probability of each outcome is $\frac{1}{2}$. Note that the probability of simultaneous investment is 0 in this region. If $x \in \left(x^P, x^b\right]$, the probability for Firm a to invest at time 0 is

$$\frac{V(x) - I - F(x) - M(x)}{V(x) - I + F(x) - 2M(x)},$$

and Firm b becomes the follower or vice-versa. In addition, both firms may invest simultaneously at time 0 with probability

$$\frac{V(x) - I - F(x)}{V(x) - I + F(x) - 2M(x)}.$$

If $x \in \left(x^b, \infty\right)$, then both firms invest at once with probability 1.

5.1.3 Bargaining Solutions

In the above analysis, we have considered non-cooperative games. At either a pure strategy equilibrium or a mixed strategy equilibrium, we have shown that the best a firm can expect in this game is to get the same payoff as being the follower. In this subsection, we consider how both firms can do better by cooperating with each other. In particular, we consider bargaining solutions to the repeated game shown in Table 5.1. Since we are only interested in the case where both firms intend to invest, we assume that $G_a(x)$ and $G_b(x)$ all equal to 1 and the demand shock x is fixed at its current level throughout this subsection. By following the concept in Subsection 2.3.3, we can first set the status quo point to the payoff of the non-cooperative game which is $(F(x), F(x))$. Then we can define the bargaining set $\mathcal{B}$ as

$$\mathcal{B} = \{(u, v) : u + v = F(x) + V(x) - I, F(x) \leq u \leq V(x) - I\},$$

and define the objective function f as

$$f(u, v) = (u - F(x))(v - F(x)).$$

By applying the method similar to that in Subsection 2.3.3, we can get

$$u = v = \frac{V(x) - I + F(x)}{2}. \tag{5.3}$$

We may also be interested in a threat bargaining solution, in which the status quo is replaced by the threaten payoff. By following the discussion in Subsection 2.3.3, we can define a zero-sum game Γ as in Table 5.2.

Table 5.2: Zero-sum game between firms

		Firm b	
		invest	wait
Firm a	invest	0	$V(x) - I - F(x)$
	wait	$F(x) - V(x) + I$	0

It is easy to see that both firms will choose to invest with probability 1 to threaten the other and the value of the game Γ is 0. By substituting the value of Γ into (2.49) and (2.50), we also get

$$u = v = \frac{V(x) - I + F(x)}{2}.$$

Thus in either case, firms can cooperate with each other by choosing a pair of strategies (α_a, α_b) such that their payoffs are (u, v). However, the pair of investment strategies (α_a, α_b) need not to be unique.

5.2 Real Option Games with Entry and Exit Decisions

In this section, we consider a strategic real option game in which entry and exit decisions are involved, specifically, we mean a model under the assumptions **(A1)–(A4)**, **(A5')** and **(A6')**. The results in this section are taken from [7]. Just like other duopoly real option models, we need to calculate the follower's payoff first. From the analysis in Subsection 3.2.2 for the monopoly case, we can get the following proposition for the follower.

Proposition 5.3. *Suppose that the one firm has already invested in the project, the follower's payoff consists of two real options the option to invest and the option to invest.*

$$F(x) \approx \begin{cases} A_0 x^{\beta_1} + C_0 x^{\beta_1} - \dfrac{2B_0 A_0}{m} \dfrac{x^{\beta_1}}{\beta_2 - \beta_1}\left(\ell n(x) + \dfrac{1}{\beta_2 - \beta_1}\right), & \text{if} \quad x < \overline{x}_0^b, \\[2ex] A_1 x^{\beta_2} + \dfrac{xD}{\delta} - \dfrac{C}{r} + C_1 x^{\beta_2} - \dfrac{2B_1 A_1}{m} \dfrac{x^{\beta_2}}{\beta_1 - \beta_2}\left(\ell n(x) + \dfrac{1}{\beta_1 - \beta_2}\right), & \text{if} \quad x \geq \overline{x}_0^b, \end{cases} \tag{5.4}$$

where the constants A_0, A_1, C_0, C_1 and the investment threshold $\overline{x}^b$ are determined numerically. Thus we know that the follower will invest at time T^b such that

$$T^b = \inf\left\{t \geq 0 : x \geq \overline{x}_0^b\right\}.$$

To calculate the leader's value, we first assume that the follower will invest at time T^b as given in Proposition 5.3. In the meantime, the leader's project value can be expressed as

$$V = \begin{cases} \mathbb{E}\left[\int_0^{T^b} e^{-rt} xD(1)dt\right] + \mathbb{E}\left[\int_{T^b}^{\infty} e^{-rt} xD(2)dt\right] - I & \text{if} \quad x < \overline{x}_0^b, \\ A_1 x^{\beta_2} + \frac{xD}{\delta} - \frac{C}{r} - I, & \text{if} \quad x \geq \overline{x}_0^b. \end{cases}$$

Following the steps that are similar to those in Chapter 4, we can get

$$\mathbb{E}\left[\int_0^{T^b} e^{-rt} xD(1)\right] = \frac{xD(1)}{\delta} - \frac{C}{r} - \frac{\overline{x}_0^b D(1)}{\delta}\left(\frac{x}{\overline{x}^b 0}\right)^{\beta_1}\left(1 - \frac{\rho\sigma\beta_1^2(\beta_1 - 1)}{k(\beta_2 - \beta_1)}\ell n\left(\frac{\overline{x}_0^b}{x}\right)\right),$$

and

$$\mathbb{E}\left[\int_{T^b}^{\infty} e^{-rt} xD(2)dt\right] = \left(A_1 \overline{x^b}_0^{\beta_2} + \frac{\overline{x}_0^b D(2)}{\delta}\right)\left(\frac{x}{\overline{x}_0^b}\right)^{\beta_1}\left(1 - \frac{\rho\sigma\beta_1^2(\beta_1 - 1)}{k(\beta_2 - \beta_1)}\ell n\left(\frac{y_0^b}{y}\right)\right).$$

Note that the boundary conditions are different from those in Chapter 4. Thus the leader's project value is

$$V(x) = \begin{cases} \frac{xD(1)}{\delta} - \frac{C}{r} + \left(A_1 \overline{x^b}_0^{\beta_2} + \frac{\overline{x}_0^b\,(D(2) - D(1))}{\delta}\right)\left(\frac{x}{\overline{x}^b 0}\right)^{\beta_1} & \text{if} \quad x < \overline{x}_0^b \\ -\left(A_1 \overline{x^b}_0^{\beta_2} + \frac{\overline{x}_0^b\,(D(2) - D(1))}{\delta}\right)\left(\frac{x}{\overline{x}^b 0}\right)^{\beta_1}\left(\frac{\rho\sigma\beta_1^2(\beta_1 - 1)}{k(\beta_2 - \beta_1)}\ell n\left(\frac{\overline{x}_0^b}{x}\right)\right) & \\ A_1 x^{\beta_2} + \frac{xD}{\delta} - \frac{C}{r}, & \text{if} \quad x \geq \overline{x}_0^b. \end{cases} \tag{5.5}$$

Now we can refer to Section 5.1.1, and find a break-even point $\overline{x}^P$, which satisfies

$$\begin{aligned} V(x) - I &< F(x) \quad \text{for} \quad x < \overline{x}^P, \\ V(x) - I &= F(x) \quad \text{for} \quad x = \overline{x}^P, \\ V(x) - I &> F(x) \quad \text{for} \quad \overline{x}^P < x < \overline{x}_0^b. \end{aligned}$$

Next, we consider the mixed-strategy equilibrium. Suppose that both firms have not yet invested. By the existing result in Subsection 5.1.2, we can conclude the following:

(1) There exists a unique investment threshold $\overline{x}^P$ such that if $0 \leq x < \overline{x}^P$, there are two possible outcomes. There is $\frac{1}{2}$ probability that Firm a invests in the project as soon as x reaches $\overline{x}^P$ and becomes the leader. The other outcome is the symmetric counterpart,

and the probability that both firms invest simultaneously is 0.

(2) If $\overline{x}^P \leq x < \overline{x}_0^b$, then there are three possible outcomes. In the first, Firm a invests at time 0 and Firm b is the follower with probability $\frac{F(x)-M(x)}{V(x)-I+F(x)-2M(x)}$. The second is the symmetric counterpart. In the third, there is $\frac{V(x)-I-F(x)}{V(x,y)-I+F(x)-2M(x,y)}$ probability that both firms invest simultaneously.

(3) If $x \geq \overline{x}_0^b$, then both firms invest simultaneously with probability 1.

Furthermore, we also have to consider what happens if the demand shock x drops when the leader has already invested in the project. The situation is relatively easier when the follower has not yet invested in the project. We can follow the similar idea to find a break-even point $\underline{x}^P$ such that

$$\begin{aligned} V(x) &< F(x) - E \quad \text{for} \quad x < \underline{x}^P, \\ V(x) &= F(x) - E \quad \text{for} \quad x = \underline{x}^P, \\ V(x) &> F(x) - E \quad \text{for} \quad \underline{x}^P < x < \overline{x}_0^b. \end{aligned}$$

However, as we discussed in Section 3.1.2 the leader may not exit the market at all if the suspension cost E is too large. A more complicated situation is what happens if the demand shock x drops when both firms have invested in the project. In this situation, both firms compete to stay in the market while forcing its rival to exit the market. If both firms choose not to suspend the project, then they should move to the next round of the game which can repeat forever. Eventually, both firms will get $M(x)$ which is the worst payoff. However, if both firms choose to suspend the project, then they enter the investing game, in which the expected payoff of both firms are $(F(x), F(x))$. If one firm suspends the project the other firm is better off since it will earn the monopoly profit. Thus the game can be formalised in Table 5.3.

Table 5.3: Game to suspend the project

		Firm b	
		suspend	wait
Firm a	suspend	$F(x), F(x)$	$V(x), F(x) - E$
	wait	$F(x) - E, V(x)$	$M(x), M(x)$

By solving this game, we find that the mixed strategy equilibrium suggests that each firm should suspend the project with probability $\frac{F(x)-E-M(x)}{V(x)-M(x)}$, and both firms get an expected payoff of $F(x)$. Thus we get the following proposition.

Proposition 5.4. *Suppose that both firms have already invested in the project, i.e., $x \geq \overline{x}_0^b$. If the demand shock x drops blow the exit threshold $\underline{x}_0^b$, then the best strategy for each firm is to suspend the project with probability $\frac{F(x)-E-M(x)}{V(x)-M(x)}$. The expected payoff for each firm is $F(x)$.*

Finally, we may also refer to Subsection 5.1.3 to check that the expected payoffs for each firm in cooperative games are $\frac{V(x)-I-F(x)}{2}$ for the investing game and $\frac{V(x)-F(x)+E}{2}$ for the suspending game respectively.

5.3 Real Option Games with Incomplete Information

In this section, we study mixed strategy equilibria for the asymmetric information model proposed by Graham in [16]. We extend this model by allowing the initial value of the demand shock to occur freely in its region. So we must consider all the non-strategic investment thresholds of the leader. Our model describes a more complicated situation in which a firm can be led to a sub-optimal behaviour. To avoid this, we refer to games with incomplete information, as described in Subsection 2.3.4. Results presented in this section are taken from [8].

First, let's recall the three investment thresholds in the pre-determined leader-follower case which was studied in Chapter 4. Among the three investment thresholds, there exists a region defined by x^{a_2} and x^{a_3} such that for any $x^{a_2} \leq x \leq x^{a_3}$ the leader's value of waiting, denoted by $K(x)$, is greater than the value of investing $V^a(x) - I$. Thus, although we do not specify who are the leader and the follower, both firms are better off if they wait and invest simultaneously when x reaches x^{a_3}. According to the initial level of the demand shock, investment games can be classified into different sub-games, which are defined as follows.

Definition 5.4. An investment game with initial demand shock x_0 that is less than x^{a_2} is called Γ^1. Otherwise it is called Γ^2.

Following Proposition 5.2 and results in [14], we can drive the following propositions.

Proposition 5.5. *Given that the current demand shock level $x < x^{a_2}$, the equilibrium strategy for each firm i ($i \in \{a, b\}$) in Γ^1 is given by*

$$G_i(x) = \begin{cases} 0, & \text{if} \quad x < x^P, \\ 1, & \text{if} \quad x^P \leq x < x^{a_2}, \end{cases} \qquad \alpha_i(x) = \begin{cases} 0, & \text{if} \quad x < x^P, \\ \dfrac{V(x) - I - F(x)}{V(x) - I - M(x)}, & \text{if} \quad x^P \leq x < x^{a_2}. \end{cases}$$

Then, the expected payoff of each firm is $F(x)$.

Proposition 5.6. *Given that the current demand shock level $x \geq x^{a_2}$, the equilibrium strategy for each firm i ($i \in \{a, b\}$) in Γ^2, is given by*

$$G_i(x) = \begin{cases} 0, & \text{if} \quad x^{a_2} \leq x < x^{a_3}, \\ \\ 1, & \text{if} \quad \quad x \geq x^{a_3}, \end{cases} \qquad \alpha_i(x) = \begin{cases} 0, & \text{if} \quad x^{a_2} \leq x < x^{a_3}, \\ \\ 1, & \text{if} \quad \quad x \geq x^{a_3}. \end{cases}$$

Then the expected payoff of each firm is $K(x)$ when $x^{a_2} \leq x < x^{a_3}$, and $M(x)$ when $x \geq x^{a_3}$.

In addition to (**A1**) – (**A6**), for our asymmetric information model we must also make the following assumptions:

(A7) There are two types of market dimension, namely N_H and N_L with $N_H > N_L$ and a probability distribution $\mathbb{P}$ such that $\mathbb{P}(N_H) = p$ and $\mathbb{P}(N_L) = 1 - p$, $(0 \leq p \leq 1)$.

(A8) Firm a knows the ture state of the market dimension N_j, $j \in \{H, L\}$, and Firm b only knows the probability distribution $\mathbb{P}$.

For each fixed N_j we can apply the method from Chapter 4 and Subsection 5.1.1 to find the corresponding investment thresholds $x_j^P, x_j^{a_1}, x_j^{a_2}, x_j^{a_3}$. Since $N_H > N_L$, it is easy to check that the thresholds corresponding to N_L are greater than those corresponding to N_H except $x_H^{a_3} > x_L^{a_3}$. Furthermore, for each N_j, $j \in \{H, L\}$, we can construct the corresponding Γ_j^1 and Γ_j^2. Let a state game Γ_j, $j \in \{H, L\}$ be

$$\Gamma_j(x) = \begin{cases} \Gamma_j^1(x), & \text{if} \quad x < x_j^{a_2}, \\ \\ \Gamma_j^2(x), & \text{if} \quad x \geq x_j^{a_2}. \end{cases}$$

From [16], we know that the existence of a pure strategy equilibrium (also called a pure trigger strategy perfect Bayesian equilibrium in [16]) has already been determined when $x_H^{a_1} < x_L^P$. Thus without loss of generality, we can also assume the following:

(A9) $x_L^P < x_H^{a_1}$ and $x_L^{a_1} < x_H^{a_2}$.

Note that the results for $x_L^{a_1} > x_H^{a_2}$ are similar to those for $x_L^{a_1} < x_H^{a_2}$. With the above assumptions, following Example 3 in [46], we can formulate our model as the following game with incomplete information, denoted by Γ:

(1) There are two firms, namely a and b.

(2) There are two possible state games, namely Γ_H and Γ_L.

(3) The common prior probability distribution is given by $\mathbb{P}$, according to which one of the state games will be chosen. The outcome of this chance move is known only to Firm a, who is the informed firm, and Firm b is the uninformed firm.

(4) A set of states of the world is denoted by $Y = \{\omega_H, \omega_L\}$.

(5) For each demand shock level x, a state of the world ω_j, $j \in \{H, L\}$ is denoted by $\omega_j = (\Gamma_j(x); [1\omega_j], [p\omega_H, (1-p)\omega_L])$.

(6) Firm a has two types. If it is informed of $\Gamma_H(x)$, then $t^a = [1\omega_H]$. If it is informed of $\Gamma_L(x)$, then $t^a = [1\omega_L]$,. Firm b has only one type, which is $t^b = [p\omega_H, (1-p)\omega_L]$.

For the game Γ, since the initial level of the demand shock can occur anywhere in its domain, Proposition 5.5 and Proposition 5.6 imply that, if we simply apply Bayes' rule in the same way as in [16], a sub-optimal behaviour will occur. For instance, suppose that $x > x_H^{a_2}$, we know that Firm a will choose to wait and earn the value of waiting. By Bayes' rule, the non-strategic investment threshold x_1 becomes a decreasing function of market dimension. If Firm b updates its belief over the state variable, it rules out N_H and concludes that the true state is N_L. Then Firm b will invest as soon as x reaches x_L^P and lead both firms to a worse payoff. To avoid this sub-optimal behaviour, we must find a mixed strategy Nash equilibrium for the game.

Theorem 5.1. *Under* **(A1)** – **(A9)***, there exists a mixed strategy Nash equilibrium for* Γ*, at which Firm a chooses a pair of optimal strategy* (G_{a_j}, α_{a_j}) *for each* N_j, $j \in \{H, L\}$*, and Firm b is always the follower.*

Proof. Based on the concepts in dynamic programming, we should start working from the very last case, i.e., when $x \geq x_L^{a_2}$. In this particular range, Firm b is able to conclude that it is Γ^2 for either market dimension N_j, $j \in \{H, L\}$. Thus according to Proposition 5.6, the optimal strategy for Firm a is

$$G_{a_H}(x) = \begin{cases} 0, & \text{if} \quad x < x_H^{a_3}, \\ \\ 1, & \text{if} \quad x \geq x_H^{a_3}, \end{cases} \qquad \alpha_{a_H}(x) = \begin{cases} 0, & \text{if} \quad x < x_H^{a_3}, \\ \\ 1, & \text{if} \quad x \geq x_H^{a_3}, \end{cases}$$

if $j = H$, and

$$G_{a_L}(x) = \begin{cases} 0, & \text{if} \quad x < x_L^{a_3}, \\ \\ 1, & \text{if} \quad x \geq x_L^{a_3}, \end{cases} \qquad \alpha_{a_L}(x) = \begin{cases} 0, & \text{if} \quad x < x_L^{a_3}, \\ \\ 1, & \text{if} \quad x \geq x_L^{a_3}, \end{cases}$$

if $j = L$. Since Firm b knows it is Γ^2 anyway, its corresponding optimal behaviour strategy will be to wait until Firm a invests in order to capture the value of waiting.

Suppose that $x_H^{a_2} \leq x < x_L^{a_2}$. If $j = H$, Firm a knows it is Γ^2. Thus by Proposition 5.6, its optimal strategy is

$$G_{a_H}(x) = 0, \qquad \alpha_{a_H}(x) = 0.$$

If $j = L$, Firm a knows it is Γ^1. Thus by Proposition 5.5, its optimal strategy is

$$G_{a_L}(x) = 1, \qquad \alpha_{a_L}(x) = \frac{V_L(x) - I - F(x)}{V_L(x) - I - M(x)}.$$

Given that Firm a will choose the above strategy $(G^a(x), \alpha^a(x))$, by using the common prior $\mathbb{P}$, the expected payoff of Firm b is given by

$$\begin{aligned} p(V_H(x) - I) + (1-p)\frac{\alpha_b(x)[\alpha_{a_L}(x)M(x) + (1-\alpha_{a_L}(x))(V_L(x) - I)] + (1-\alpha_b(x))\alpha_{a_L}(x)F(x)}{\alpha_b(x) + \alpha_{a_L}(x) + \alpha_b(x)\alpha_{a_L}(x)} \\ = P(V_H(x) - I) + (1-p)F(x), \end{aligned}$$

if it chooses to invest with any $\alpha_b(x) \in [0, 1]$. If the Firm b chooses to play an isolated jump with $G_b(y) = \lambda, 0 < \lambda < 1$ it gets

$$\begin{aligned} p\,[\lambda(V_H(x) - I) + (1-\lambda)K(x)] + \\ (1-p)[\lambda\,(\alpha_{a_L}(x)M(x) + (1-\alpha_{a_L}(x))(V_L(x) - I)) + (1-\lambda)F(x)] \\ = p\,[\lambda(V_H(x) - I) + (1-\lambda)K(x)] + (1-p)F(x). \end{aligned}$$

If it chooses to wait, it gets $pK_H(x) + (1-p)F(x)$, where $K_H(x)$ represents the value of waiting when $j = H$. Note that when $x \geq x_H^{a_2}$, $V_H(x) - I < K_H(x)$. Thus waiting gives the greatest payoff over all the possible choices for Firm b, and we can conclude that the optimal behaviour strategy of Firm b when $x_H^{a_2} \leq x < x_L^{a_2}$ is $G_b(x) = 0, \alpha_b(x) = 0$.

Suppose that $x < x_H^{a_2}$. This is the case where it is Γ^1 for both N_H and N_L. Then Firm a will choose the optimal strategy according to Proposition 5.5, i.e.,

$$G_{a_H}(x) = \begin{cases} 0, & \text{if} \quad x < x_H^P, \\ 1, & \text{if} \quad x \geq x_H^P, \end{cases} \qquad \alpha_{a_H}(x) = \begin{cases} 0, & \text{if} \quad x < x_H^P, \\ \dfrac{V_H(x) - I - F(x)}{V_H(x) - I - M(x)}, & \text{if} \quad x \geq x_H^P, \end{cases}$$

if $j = H$, and

$$G_{a_L}(x) = \begin{cases} 0, & \text{if} \quad x < x_L^P, \\ \\ 1, & \text{if} \quad x \geq x_L^P, \end{cases} \qquad \alpha_{a_L}(x) = \begin{cases} 0, & \text{if} \quad x < x_L^P, \\ \\ \dfrac{V_L(x) - I - F(x)}{V_L(x) - I - M(x)}, & \text{if} \quad x \geq x_L^P, \end{cases}$$

if $j = L$. Next, we need to check Firm b's optimal strategy given that Firm a will choose the above strategies. We do this by checking Firm b's best expected payoff for each sub interval in $[0, x_H^{a_2})$. First, suppose that $x_L^E \leq x < x_H^{a_2}$. Firm a will choose $(G_{a_H}(x), \alpha_{a_H}(x))$ for $j = H$, and $(G_{a_L}(x), \alpha_{a_L}(x))$ for $j = L$. If Firm b chooses to invest, i.e. $G_b(x) = 1$, and for any $\alpha_b(x) \in [0, 1]$ it gets

$$\begin{aligned} &\frac{p\left[\alpha_b(x)(\alpha_{a_H}(x)M(x) + (1 - \alpha_{a_H}(x))(V_H(x) - I)) + \alpha_{a_H}(x)(1 - \alpha_b(x))F(x)\right]}{\alpha_b(x) + \alpha_{a_H}(x) + \alpha_b(x)\alpha_{a_H}(x)} \\ +&\frac{(1 - p)\left[\alpha_b(x)(\alpha_{a_L}(x)M(x) + (1 - \alpha_{a_L}(x))(V_L(x) - I)) + \alpha_{a_L}(x)(1 - \alpha_b(x))F(x)\right]}{\alpha_b(x) + \alpha_{a_L}(x) + \alpha_b(x)\alpha_{a_L}(x)} = F(x). \end{aligned}$$

If Firm b chooses to play an isolated jump, i.e., $G_b(x) = \lambda, 0 < \lambda < 1$, it gets

$$\begin{aligned} &p\left[\lambda\left(\alpha_{a_H}(x)M(x) + (1 - \alpha_{a_H}(x))(V_H(x) - I)\right) + (1 - \lambda)F(x)\right] \\ +&(1 - p)\left[\lambda(\alpha_{a_L}(x)M(x) + (1 - \alpha_{a_L}(x))(V_L(x) - I)) + (1 - \lambda)F(x)\right] = F(x). \end{aligned}$$

If Firm b chooses not to invest, as long as $G_a(x) = 1$, Firm a guarantees itself as the leader and Firm b receives $F(x)$. Therefore, Firm b is indifferent between all possible choices, and $G_b(x) = 0, \alpha_b(x) = 0$ can be chosen as an optimal behaviour strategy.

Now suppose that $x_H^E \leq x < x_L^E$. If Firm b chooses to invest, for any $\alpha_b(x) \in [0, 1]$, it receives

$$\frac{\left[\alpha_b(x)\left(\alpha_{a_H}(x)M(x) + (1 - \alpha_{a_H}(x))(V_H(x) - I)\right) + \alpha_{a_H}(x)(1 - \alpha_b(x))F(x)\right]}{\alpha_b(x) + \alpha_{a_H}(x) + \alpha_b(x)\alpha_{a_H}(x)} = F(x)$$

if $j = H$, and $V_L(x) - I$ if $j = L$. Thus, the expected payoff of Firm b is

$$pF(x) + (1 - p)(V_L(x) - I) < F(x),$$

where we used the fact that $x < x_L^E$ implies $V_L(x) - I < F(x)$. Again if Firm b plays an isolated jump, i.e., $G_b(x) = \lambda, 0 < \lambda < 1$, then its expected payoff is given by

$$\begin{aligned} p\left[\lambda\left(\alpha_{a_H}(x)M(x) + (1-\alpha_{a_H}(x))(V_H(x)-I)\right) + (1-\lambda)F(x)\right] \\ +(1-p)\left[\lambda(V_L(x)-I) + (1-\lambda)F(x)\right] \\ = pF(x) + (1-p)\left[\lambda(V_L(x)-I) + (1-\lambda)F(x)\right] < F(x). \end{aligned}$$

Thus the optimal behaviour strategy of Firm b is again to choose $G_b(y) = 0$ and $\alpha_b(y) = 0$, so that it gets $F(x)$ for sure as a follower. For the last case where $x < x_H^E$, it is obvious that both firms should not invest. This completes the proof. □

With the the above result established, we are ready to discuss games with finite many types of market dimensions. In this case, assumptions (**A7**), (**A8**), (**A9**) can be reformed as

(A7') There are M many types of market dimension, N_j, $j = \{1, 2, 3, \cdots, M\}$ with $N_j > N_l$ for any $j < l \in M$ and a probability distribution $\mathbb{P}$.

(A8') Firm a knows the ture state of the market dimension N_j, $j = \{1, 2, 3, \cdots, M\}$, and Firm b only knows the probability distribution $\mathbb{P}$.

(A9') $x_l^P < x_j^{a_1}$ and $x_l^{a_1} < x_j^{a_2}$ for any $j < l$.

By the above assumptions, we can extend the analysis and get the following result.

Theorem 5.2. *Under* (**A1**) – (**A6**) *and* (**A7'**) – (**A9'**)*, there exists a mixed-strategy Nash equilibrium for the above game, at which Firm a chooses a pair of optimal strategy* (G_{a_j}, α_{a_j}) *for each* N_j, $j = \{1, 2, 3, \cdots, M\}$*, and Firm b is always the follower.*

The proof is similar to that of Theorem 5.1, thus we will omit the proof.

Chapter 6

Conclusions and Future Work

The purpose of this thesis is to study real options in various scenarios. To do this, several concepts and techniques in Mathematics, Finance, and Economics are employed. In what follows, we summarise main outcomes from our analysis and point out some possible future work. In Section 6.1, we conclude our work into two major aspects. In Section 6.2, we propose some future work based on our existing models.

6.1 Conclusions

In this thesis, we study real options under different scenarios. In Chapter 3, we investigate real options in monopoly. In the classical case, the real option value and the corresponding optimal investment threshold can be easily determined by using the standard dynamic programming method. From theoretical results and numerical examples, we see that an increase in the drift parameter α makes a project more valuable than its real option value, hence decreases the investment threshold, whereas an increase in the volatility parameter σ increases both the real option value and the optimal investment threshold of a firm. For the case where the Heston model is considered, we find the asymptotic solutions using the method of perturbation. Our asymptotic solutions for the real option value and the optimal investment threshold are made up of the first two terms of their asymptotic expansions. We find that the first terms of these solutions are the same as those of the classical real options, in which the constant volatility is replaced by the mean reverting level m of the variance process. Moreover, we also investigate the effects of the extra parameters. Firstly, we find that a positive (resp. negative) correlation ρ between the demand shock and its variance process decreases (resp. increases) the expected payoff and optimal investment threshold. Secondly, we find that volatility parameter σ of the variance process contributes as an amplification factor to the correction terms of the asymptotic solutions. Finally, the effect of the mean reverting rate k^* on the solutions are also discussed. Our results show that the asymptotic solutions approach to the classical solutions as σ and k^* increase. Furthermore, we also study real options with the stochastic volatility under the case where both entry and exit decisions are considered. In Chapter 4, we study real options in a duopoly market where the leader and the follower are pre-designed. In this duopoly market, the follower's real option value are similar to those in monopoly. Once the follower's investment threshold is found, we determine three investment thresholds for the leader and its corresponding real option value. However, most of these results need to be determined numerically.

We also investigate the real options under competitive situations, which are also referred as the strategic real option games. In Chapter 5, we start with an ordinary 2-player non-corporative game. Both the pure strategy equilibrium approach and the mixed strategy equilibrium approach are considered to solve the game. We find that at a pure strategy equilibrium, there exists an unique investment threshold at which the leader and the follower have the same payoff and they must somehow cooperate with each other in order to avoid the simultaneous investment, whereas at a mixed strategy equilibrium, we define a firm's strategy as a pair of probability distributions $(G(x), \alpha(x))$ for each demand shock level x rather than a single investment threshold. Thus the simultaneous investment is allowed. Hence, $G(x)$ represents the probability that a firm invests at this shock level given that its rival has not yet invested in the project. When a firm chooses $G(x) = 1$, it means that the firm locks the current shock level and plays an infinitely repeated investment game. Thus the firm must also choose a positive $\alpha(x)$, which represents the probability of investing in the project in a round. At the mixed strategy equilibrium, both firms are expected to get the same payoff as being the follower. Furthermore, we also investigate strategic real options in 2-player corporative games. Particularly, we consider both maximin solutions and threat solutions under the Nash bargaining scheme. Under bargaining solutions, firms can maximise their joint payoffs and transfer the payoff from one to the other so that their payoffs are better than working by their own. In addition, we also analyse a strategic real option game with entry and exit decisions. In this game, we must solve not only a sub-game for investing in the project but also a sub-game for suspending the project. In the last part of Chapter 5, we study strategic real option games with asymmetric information. Our model is different from the existing asymmetric information models because we allow the demand shock to occur anywhere in its region. Thus all the non-strategic investment thresholds need to be considered and the pure strategy approach can not be applied. Instead, we use the mixed strategy approach. Our result shows that the informed firm has the advantage to become the leader in a game at all times. This is because the informed firm knows the true state of the revenue, it can limit the uninformed firm's expected payoff at $F(x)$ by choosing the optimal mixed strategy. On the other hand, the uninformed firm can not do the same to the informed firm. Due to the uncertainty in the revenue, the uninformed firm's expected payoff will be less if it chooses to preempt the informed firm. Thus the best strategy for the uninformed firm is to wait until the informed firm invests.

6.2 Open Problems with the Existing Models

In what follows, we shall also discuss some possible extensions based on our existing results. This thesis mainly focuses on real options under the Heston model. However, there are many other stochastic processes can be considered in real option analysis. For instance, in [43], Ting et al. also considered a single firm real option model which assumes the project value $\{x_t : t \geq 0\}$ satisfies

$$\begin{aligned} dx_t &= \eta(\overline{x} - x_t)x_t dt + \sqrt{y_t}x_t dW_t^0, \\ dy_t &= \alpha(m - y_t)dt + \beta\sqrt{y_t}dW_t^1, \end{aligned}$$

where W_t^0 and W_t^1 are correlated Brownian motions with correlation ρ, η and $\overline{x}$ are the mean reverting rate and the mean reverting level of the project's value respectively. The rest of the parameters are similar to those in (2.8) and (2.9). This is called the Heston-GMR process. Thus, a possible extension would be to consider the Heston-GMR process in real options under duopoly. In [13], another very interesting stochastic volatility model was mentioned. This model is called the multiscale stochastic volatility model, which is defined as follows:

$$\begin{aligned} dx_t &= \alpha x_t dt + f(y_t, z_t)x_t dW_t^0, \\ dy_t &= \frac{1}{\varepsilon}(m - y_t)dt + \frac{\nu\sqrt{2}}{\sqrt{\epsilon}}dW_t^1, \\ dz_t &= \sigma c(z_t)dt + \sqrt{\sigma}g(z_t)dW_t^2. \end{aligned}$$

In this model, $\{x_t : t \geq 0\}$ is in the form of a geometric Brownian motion with a volatility process that is driven by two diffusion processes $\{y_t : t \geq 0\}$ and $\{z_t : t \geq 0\}$. Furthermore, the function f is also assumed to be smooth, positive, and bounded away from zero. The diffusion process $\{y_t : t \geq 0\}$ is a fast mean reverting process with mean reversion rate $\frac{1}{\epsilon}$ with $\epsilon > 0$ and mean reverting level m. The volatility of $\{y_t : t \geq 0\}$ is denoted by $\frac{\nu\sqrt{2}}{\sqrt{\epsilon}}$. W_t^1 is a standard Brownian motion that correlated with W_t^0, i.e., $\left[dW_t^0, dW_t^1\right] = \rho_1 dt$. The diffusion process $\{z_t : t \geq 0\}$ is a slowly varying process, where $\sigma > 0$ is small, $c(z)$ and $g(z)$ are smooth and at most linearly growing at infinity. Again W_t^2 is a standard Brownian motion that correlated with W_t^0 and W_t^1, i.e., $\left[dW_t^0, dW_t^2\right] = \rho_2 dt$ and $\left[dW_t^1, dW_t^2\right] = \rho_{12} dt$. We may consider real options with the multiscale stochastic volatility.

In Chapter 4 we consider the leader-follower duopoly model for real options that only involves investing in the project. Since we only discuss real options with entry and exit decision under monopoly, we may also consider a real option duopoly model with entry and exit decisions.

In Chapter 5, we refer to [14] for the mixed strategy approach to solve games for investment decisions. However, such an approach assumes that once the demand shock level is given, firms are able to play a game infinitely many times until the game reaches its absorbing state. This assumption is somewhat unrealistic. In fact, it is more appropriate to suppose that if both firms do not invest in the project, then the demand shock should move to its next level and a similar game will start corresponding to the new demand shock level. To model such a situation, we must refer to differential games. The idea of differential games is similar to that of dynamic programming but with more control variables. However, most differential games are very difficult to solve. Furthermore, we only consider a model with asymmetric information on the revenue. In fact, there are many other possible scenarios that asymmetric or incomplete information can occur. We may also extend our analysis to those scenarios.

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