Monotonicity criteria

Farida Kachapova, Ilias Kachapov

***Abstract*— Monotonicity is an important type of dependence of random variables. The paper reviews the definitions of comonotonicity and counter-monotonicity, their existing criteria and introduces a new criterion for each of the two concepts, along with some proofs. The new criteria are similar to the definition of a monotone function and clarify the meaning of the monotonicity concept. Stronger criteria are proven for the case of continuous marginals.

***Keywords*—** comonotonic, counter-monotonic, marginal, monotonicity.

I. Introduction

THE research of dependence between random variables is reviewed in several publications (see, for example, [1] and [2]). Most important types of dependence are linear and monotone dependence. Other types of dependence include quadrant dependence (see, for example, [3], [4]) and mutual complete dependence [5].

Monotone dependence of continuous random variables X and Y was introduced by Kimeldorf and Sampson [6]: X and Y are monotone dependent if for some monotone function g, $Y = g(X)$ almost surely. A more general concept of monotonicity was introduced in actuarial science and studied in [7]-[12] but it was not studied in depth in mathematical literature.

This paper reviews the definitions of comonotonicity, counter-monotonicity, and their existing criteria, and introduces a new criterion for each of the two concepts, along with some proofs. The new criteria can be used as definitions; they better reflect the meaning of the concepts. Stronger criteria are proven for the particular case of continuous marginals.

II. Comonotonicity

A. Definition of comonotonicity

We fix an arbitrary probability space $<\Omega, \Sigma, P>$, where Σ is the collection of all events in this space. We consider only random variables defined on this space.

The concept of comonotonicity was introduced in [13] and [14]; other versions of its definition were given in [7]-[12]. Random variables X and Y are comonotonic means there is an increasing dependence between the values of X and Y, or they change in the same direction.

Dhaene et al. [10] give a mathematically rigorous definition of comonotonicity. The following is a particular case of the definition for two random variables.

Definition 1. 1) A set of pairs of real numbers is called **comonotonic** if for any of its elements $<x_1, y_1>$ and $<x_2, y_2>$:

either $(x_1 \le x_2 \;\&\; y_1 \le y_2)$ or $(x_1 \ge x_2 \;\&\; y_1 \ge y_2)$ holds.

2) Random variables X and Y are called **comonotonic** if there exists a comonotonic set A with $P(<X, Y> \in A) = 1$. □

The following is another definition of comonotonicity of two random variables.

Definition 2. The random variables X and Y are called **comonotonic** if there exists an event B of probability 1, such that

$$(\forall \omega_1, \omega_2 \in B)\,[X(\omega_1) < X(\omega_2) \Rightarrow Y(\omega_1) \le Y(\omega_2)]. \qquad (1)$$

□

Note: Clearly Definition 2 is symmetrical with respect to X and Y.

In the following theorem we show that the two definitions are equivalent, so the second one can be considered as a criterion of comonotonicity. Definition 2 is similar to the definition of an increasing function and better reflects the meaning of the comonotonicity concept.

Theorem 1.

Definitions 1 and 2 of comonotonicity are equivalent.

Proof of Theorem 1.

$\Rightarrow$ Suppose X and Y are comonotonic as in Definition 1. Then there is a comonotonic set A with $P(<X, Y> \in A) = 1$.

Denote $B = \{\omega \in \Omega: <X(\omega), Y(\omega)> \in A\}$.

Then $P(B) = P(<X, Y> \in A) = 1$.

Consider $\omega_1, \omega_2 \in B$ with $X(\omega_1) < X(\omega_2)$. This implies $Y(\omega_1) \le Y(\omega_2)$, since both pairs $<X(\omega_1), Y(\omega_1)>$ and $<X(\omega_2), Y(\omega_2)>$ are elements of the comonotonic set A.

$\Leftarrow$ Suppose X and Y are comonotonic as in Definition 2. Then for some event B, $P(B) = 1$ and the condition (1) holds.

Denote $A = \{<X(\omega), Y(\omega)>: \omega \in B\}$. Then $P(<X, Y> \in A) = P(B) = 1$.

Suppose $<x_1, y_1> \in A$ and $<x_2, y_2> \in A$. Then for some $\omega_1, \omega_2 \in B$,

F. Kachapova is with the Auckland University of Technology, Private Bag 92006, Auckland 1142, New Zealand (email farida.kachapova@aut.ac.nz)

I. Kachapov is with the University of Auckland, Examination Academic Services, Auckland, New Zealand (e-mail: bill.kachapov@auckland.ac.nz).

$x_1 = X(\omega_1)$, $y_1 = Y(\omega_1)$, $x_2 = X(\omega_2)$, and $y_2 = Y(\omega_2)$.

By (1), $x_1 < x_2$ implies $y_1 \le y_2$, and $x_1 > x_2$ implies $y_1 \ge y_2$. Hence A is a comonotonic set. □

B. Criteria for Comonotonicity

For a random variable X its distribution function is denoted F_X. The following is a well-known definition of **quantile** function F_X^{-1}.

Definition 3. For a random variable X,

1) F_X^{-1} is defined by the following:

$F_X^{-1}(u) = min\ \{x: F_X(x) \ge u\}$ for any $u \in (0, 1)$,

$F_X^{-1}(0) = sup\ \{x: F_X(x) = 0\}$, it is $-\infty$ if the set is empty,

$F_X^{-1}(1) = inf\ \{x: F_X(x) = 1\}$, it is $+\infty$ if the set is empty;

2) denote $D_X = \{ F_X^{-1}(u): u \in (0, 1)\}$. □

Usually $F_X^{-1}(u)$ is defined for $u \in (0, 1)$; the above is a natural extension of the definition to $u = 0$ and $u = 1$. These common conventions were applied:

$$sup\ \varnothing = -\infty \text{ and } inf\ \varnothing = +\infty.$$

The following lemma states basic properties of the quantile function and some related simple facts.

Lemma 1. **1)** $F_X(X) \sim$ Uniform (0, 1), i.e. the random variable $F_X(X)$ has a uniform distribution on (0, 1).

2) For any $u \in (0, 1]$, $x \in \mathbf{R}$:

$$F_X^{-1}(u) \le x \quad \Leftrightarrow \quad u \le F_X(x).$$

3) F_X^{-1} is non-decreasing on [0, 1] and it is left-continuous on (0, 1).

4) For any $x \in D_X$, $F_X^{-1}(F_X(x)) = x$.

5) F_X is increasing on D_X.

6) D_X is a Borel set in $\boldsymbol{R}$ and $P(X \in D_X) = 1$, i.e. D_X is a support of the random variable X.

7) If F_X is continuous, then F_X^{-1} is increasing on [0, 1] and for any $u \in (0, 1)$:

$$F_X(F_X^{-1}(u)) = u. \quad □$$

Dhaene et al. [10] proved three criteria for comonotonicity of n random variables, which are stated in the following theorem for the case of two variables.

Theorem 2. The random variables X and Y are comonotonic if and only if one of the following equivalent conditions holds.

$$(\forall x, y \in \mathbf{R})\ [F_{X,Y}(x, y) = min\ \{F_X(x), F_Y(y)\}], \tag{2}$$

where $F_{X,Y}$ denotes the joint distribution function of X and Y.

For $U \sim$ Uniform (0, 1),

$$<X, Y> =^{d} <F_X^{-1}(U), F_Y^{-1}(U)>, \tag{3}$$

where $=^{d}$ denotes equality in distribution.

There exist a random variable Z and non-decreasing functions g, h, such that

$$<X, Y> =^{d} <g(Z), h(Z)>. \tag{4}$$

□

Next we provide a new proof that the condition (2) implies comonotonicity as in Definition 2. The new proof directly constructs the event B of probability 1 where the condition (1) holds, so the new proof is more constructive than the old one.

Theorem 3. If the random variables X and Y satisfy the condition (2), then they are comonotonic.

Proof of Theorem 3.

Suppose $(\forall x, y \in \boldsymbol{R})\ [F_{X,Y}(x, y) = min\ \{F_X(x), F_Y(y)\}]$. Denote:

$\alpha_0 = F_X^{-1}(0)$, $\alpha_1 = F_X^{-1}(1)$ and $\beta_0 = F_Y^{-1}(0)$, $\beta_1 = F_Y^{-1}(1)$.

Clearly

$$P(\alpha_0 \le X \le \alpha_1) = 1 \text{ and } P(\beta_0 \le Y \le \beta_1) = 1. \tag{5}$$

For any $x \in \boldsymbol{R}$, denote $g(x) = inf\ \{y: F_Y(y) \ge F_X(x)\}$. We will prove:

$$(\forall x \in (\alpha_0, \alpha_1))\ [g(x) \text{ is finite and } P[X \le x \Rightarrow Y \le g(x)] = 1]. \tag{6}$$

Suppose $\alpha_0 < x < \alpha_1$. Then $0 < F_X(x) < 1$ and

$\{y: F_Y(y) \ge F_X(x)\} \ne \varnothing$. By Lemma 1.2), $F_Y(y) \ge F_X(x)$

implies $y \ge F_Y^{-1}(F_X(x))$; this means that $F_Y^{-1}(F_X(x))$ is a lower bound of the set $\{y: F_Y(y) \ge F_X(x)\}$. Hence $g(x)$ is finite.

For any $y > g(x)$, $F_Y(y) \ge F_X(x)$, so $F_Y(g(x)) \ge F_X(x)$,

since F_Y is continuous on the right. Then $F_{X,Y}(x, g(x)) =$

$= F_X(x)$. So $P[X \le x\ \&\ Y > g(x)] =$

$= P(X \le x) - P[X \le x\ \&\ Y \le g(x)] = F_X(x) - F_{X,Y}(x, g(x)) = 0,$

and $P[X \le x \Rightarrow Y \le g(x)] = 1$.

For any $y \in \mathbf{R}$, denote $h(y) = inf\ \{x: F_X(x) \ge F_Y(y)\}$. Similarly to (6) we get:

$$(\forall y \in (\beta_0, \beta_1))\ [h(y) \text{ is finite and } P[Y \le y \Rightarrow X \le h(y)] = 1]. \quad (7)$$

Denote Q the set of all rational numbers and denote

$$B = \{\alpha_0 \le X \le \alpha_1\} \cap \{\beta_0 \le Y \le \beta_1\} \cap$$
$$\cap \bigcap_{\substack{p \in Q \\ \alpha_0 < p < \alpha_1}} \{X \le p \Rightarrow Y \le g(p)\} \cap$$
$$\cap \bigcap_{\substack{q \in Q \\ \beta_0 < q < \beta_1}} \{Y \le q \Rightarrow X \le h(q)\}.$$

Since this intersection is countable, then $B \in \Sigma$ and $P(B) = 1$ by (5) – (7). It remains to prove for any $\omega_1, \omega_2 \in B$:

$$[\,X(\omega_1) < X(\omega_2) \ \Rightarrow\ Y(\omega_1) \le Y(\omega_2)].$$

Assume the opposite: $\omega_1, \omega_2 \in B$, $X(\omega_1) < X(\omega_2)$ and $Y(\omega_1) > Y(\omega_2)$. Then there exist $p, q \in Q$ such that

$$\alpha_0 \le X(\omega_1) < p < X(\omega_2) \le \alpha_1 \text{ and } \beta_0 \le Y(\omega_2) < q < Y(\omega_1) \le \beta_1.$$

Therefore $\alpha_0 < p < \alpha_1$ and $\beta_0 < q < \beta_1$, so

$\omega_1, \omega_2 \in \{X \le p \Rightarrow Y \le g(p)\}$ and

$\omega_1, \omega_2 \in \{Y \le q \Rightarrow X \le h(q)\}$.

Since $X(\omega_1) < p$, then $Y(\omega_1) \le g(p)$ and $q < g(p)$. By the definition of g, $F_Y(q) < F_X(p)$.

Similarly, since $Y(\omega_2) < q$, then $X(\omega_2) \le h(q)$ and $p < h(q)$. By the definition of h, $F_X(p) < F_Y(q)$. Contradiction. This proves the theorem. □

C. The Case of Continuous Marginals

In this case the criteria for comonotonicity can be made stronger as the following theorem shows.

Theorem 4. Suppose the marginal distribution functions of X and Y are continuous. The random variables X and Y are comonotonic if and only if one of the following equivalent conditions holds.

$$Y = F_Y^{-1}(F_X(X)) \text{ with probability 1.} \quad (8)$$

There is an event C of probability 1, such that

$$(\forall \omega_1, \omega_2 \in C)[X(\omega_1) < X(\omega_2) \ \Rightarrow\ Y(\omega_1) < Y(\omega_2)]. \quad (9)$$

There is a non-decreasing function g such that $Y = g(X)$ with probability 1. (10)

Proof of Theorem 4.

It is sufficient to prove:

comonotonicity ⇒ (8) ⇒ (9) ⇒ comonotonicity

and

comonotonicity ⇒ (8) ⇒ (10) ⇒ comonotonicity.

We will use Definition 2 of comonotonicity.

Comonotonicity ⇒ (8):

Suppose $<X, Y>$ is comonotonic. Then there is a set B_0 of probability 1, where

$$X(\omega_1) < X(\omega_2) \ \Rightarrow Y(\omega_1) \le Y(\omega_2).$$

Denote $B = \{\omega \in B_0: Y(\omega) \in D_Y\}$. Then $P(B) = 1$. Consider $\omega_0 \in B$ and denote $x_0 = X(\omega_0)$ and $y_0 = Y(\omega_0)$.

Since F_X is continuous, $F_X(X(\omega_0)) = P(X < x_0) =$

$= P(\omega \in B: X(\omega) < X(\omega_0)) \ \le\ P(\omega \in B: Y(\omega) \le Y(\omega_0)) =$

$= P(Y \le y_0) = F_Y(Y(\omega_0))$.

Due to symmetry, $F_Y(Y(\omega_0)) \le F_X(X(\omega_0))$; hence

$F_Y(Y(\omega_0)) = F_X(X(\omega_0))$ and $Y(\omega_0) = F_Y^{-1}(F_X(X(\omega_0)))$ by Lemma 1.4), since $Y(\omega_0) \in D_Y$.

(8) ⇒ (9):

Suppose $Y = F_Y^{-1}(F_X(X))$ on a set B with $P(B) = 1$. Denote $C = \{\omega \in B: X(\omega) \in D_X\}$. Then $P(C) = 1$.

For any $\omega_1, \omega_2 \in C$: if $X(\omega_1) < X(\omega_2)$, then by Lemma 1.5), $F_X(X(\omega_1)) < F_X(X(\omega_2))$. By Lemma 1.7),

$F_Y^{-1}(F_X(X(\omega_1))) < F_Y^{-1}(F_X(X(\omega_2)))$ and $Y(\omega_1) < Y(\omega_2)$.

(9) ⇒ comonotonicity: obvious.

(8) ⇒ (10): we can take $g = F_Y^{-1} \circ F_X$.

(10) ⇒ comonotonicity:

Suppose condition (10) holds. Then there is a set B of probability 1 where $Y = g(X)$. For any $\omega_1, \omega_2 \in B$:

if $X(\omega_1) < X(\omega_2)$, then $g(X(\omega_1)) \le g(X(\omega_2))$, since g is non-decreasing, and $Y(\omega_1) \le Y(\omega_2)$. □

Clearly the condition (9) is a stronger version of Definition 2. The condition (10) means that if X and Y have continuous marginals, then comonotonicity of X and Y is equivalent to their monotone increasing dependence introduced in [6] and briefly described in our introduction.

The next section contains similar material on counter-monotonicity.

III. Counter-monotonicity

A. Definition of counter-monotonicity

Definitions of counter-monotonicity can be found in [11] and [8]. Random variables X and Y are counter-monotonic means there is a decreasing dependence between the values of X and Y, or they change in opposite directions.

Definition 4. 1) A set of pairs of real numbers is called **counter-monotonic** if for any of its elements $<x_1, y_1>$ and $<x_2, y_2>$:

either $(x_1 \le x_2 \ \& \ y_1 \ge y_2)$ or $(x_1 \ge x_2 \ \& \ y_1 \le y_2)$ holds.

2) Random variables X and Y are called **counter-monotonic** if there exists a counter-monotonic set A with $P(<X, Y> \in A) = 1$. □

The following is another definition of counter-monotonicity of two random variables.

Definition 5. The random variables X and Y are called **counter-monotonic** if there exists an event B of probability 1, such that

$$(\forall \omega_1, \omega_2 \in B)\, [X(\omega_1) < X(\omega_2) \ \Rightarrow \ Y(\omega_1) \ge Y(\omega_2)]. \quad (11)$$

□

Note: Clearly Definition 5 is symmetrical with respect to X and Y.

In the following theorem we will show that the two definitions of counter-monotonicity are equivalent, so Definition 5 can be considered as a criterion of counter-monotonicity. Definition 5 is similar to the definition of a decreasing function and better reflects the meaning of the counter-monotonicity concept.

Theorem 5. Definitions 4 and 5 of counter-monotonicity are equivalent.

Proof of Theorem 5.

The variables X and Y are counter-monotonic by Definition 5 if and only if X and $-Y$ are comonotonic by Definition 2. A set A of pairs of real numbers is counter-monotonic if and only if the set $\{<x, -y>: \ <x, y> \in A\}$ is comonotonic.

Hence Theorem 5 follows from Theorem 1. □

B. Criteria for Counter-monotonicity

The following theorem is similar to Theorem 2.

Theorem 6. The random variables X and Y are counter-monotonic if and only if one of the following equivalent conditions holds.

$$(\forall x, y \in \boldsymbol{R})\, [F_{X,Y}(x, y) = max\, \{F_X(x) + F_Y(y) - 1, 0\}]. \quad (12)$$

For $U \sim$ Uniform (0, 1),

$$<X, Y> \ =^{d} \ <F_X^{-1}(U), F_Y^{-1}(1-U)>. \quad (13)$$

There exist a random variable Z, a non-decreasing function g, and a non-increasing function h, such that

$$<X, Y> \ =^{d} \ <g(Z), h(Z)>. \quad (14)$$

□

Note: The bivariate distributions $min\, \{F_X(x), F_Y(y)\}$ and $max\, \{F_X(x) + F_Y(y) - 1, 0\}$ from the conditions (1) and (12), respectively, are called Fréchet bounds. Fréchet [15] showed that they are the upper and lower bounds, respectively, for all bivariate distributions with fixed marginals.

For brevity we will only prove the part of Theorem 6, that the condition (12) implies counter-monotonicity, as follows.

Theorem 7. If the random variables X and Y satisfy the condition (12), then X and Y are counter-monotonic.

Proof of Theorem 7.

Suppose

$(\forall x, y \in \boldsymbol{R})\, [F_{X,Y}(x, y) = max\, \{F_X(x) + F_Y(y) - 1, 0\}]$.

Denote $Z = -Y$. We will prove:

$$(\forall x, z \in \boldsymbol{R})\, [\, F_{X,Z}(x, z) = min\, \{F_X(x), F_Z(z)\}]. \quad (15)$$

Then by Theorem 3, $<X, Z> \ = \ <X, -Y>$ is comonotonic and $<X, Y>$ is counter-monotonic.

Proof of (15)

Fix points $x, z \in \boldsymbol{R}$.

Case 1. The function $F_{X,Y}$ is continuous at point $(x, -z)$ with respect to the second argument.

$F_Z(z) = P(-Y \le z) = P(Y \ge -z) = 1 - P(Y < -z) = 1 - F_Y(-z)$, since F_Y is continuous. So

$$F_Y(-z) - 1 = -F_Z(z). \quad (16)$$

$F_{X,Z}(x, z) = P(X \le x, -Y \le z) = P(X \le x, Y \ge -z) =$

$= P(X \le x) - P(X \le x,\, Y < -z) = F_X(x) - \lim_{t \to -z-} F_{X,Y}(x,t) =$

$= F_X(x) - F_{X,Y}(x,-z) = F_X(x) - max\,\{F_X(x) + F_Y(-z) - 1,\, 0\} =$

$=$ [by (16)] $= F_X(x) - max\,\{F_X(x) - F_Z(z),\, 0\} =$

$= min\,\{F_X(x),\, F_Z(z)\}$.

Case 2. The function $F_{X,Y}$ is not continuous at point $(x, -z)$ with respect to the second argument.

There is a countable number of points where $F_{X,Y}$ is discontinuous with respect to the second argument, so there exists a sequence $y_n \to z+$ such that $F_{X,Y}$ is continuous at each point (x, y_n) with respect to the second argument. Since $F_{X,Z}$ is right-continuous,

$F_{X,Z}(x, z) = \lim_{y_n \to z+} F_{X,Z}(x, y_n) =$ [by case 1] $=$

$= \lim_{y_n \to z+} min\{F_X(x), F_Z(y_n)\} = min\left\{F_X(x), \lim_{y_n \to z+} F_Z(y_n)\right\} =$

$= min\,\{F_X(x),\, F_Z(z)\}$. □

C. The Case of Continuous Marginals

In this case the criteria for counter-monotonicity can be made stronger as the following theorem shows.

Theorem 8. Suppose the marginal distribution functions of X and Y are continuous. The random variables X and Y are counter-monotonic if and only if one of the following equivalent conditions holds.

$$Y = F_Y^{-1}[1 - F_X(X)] \text{ with probability 1.} \tag{17}$$

There is an event C of probability 1, such that

$$(\forall \omega_1, \omega_2 \in C)[X(\omega_1) < X(\omega_2) \Rightarrow Y(\omega_1) > Y(\omega_2)]. \tag{18}$$

There is a non-increasing function h such that $Y = h(X)$ with probability 1. (19)

Proof of Theorem 8.

It follows from Theorem 4 and the following fact:

$<X, Y>$ is counter-monotonic $\Leftrightarrow$ $<-X, Y>$ is comonotonic.

□

Clearly the condition (18) is a stronger version of Definition 5. The condition (19) means that if X and Y have continuous marginals, then counter-monotonicity of X and Y is equivalent to the monotone decreasing dependence introduced in [6] and briefly described in our introduction.

IV. Discussion

When there is no linear dependence between random variables it can be useful to investigate the case of monotone dependence. The concepts of monotonicity describe monotone dependence between two random variables: comonotonicity means increasing dependence and counter-monotonicity means decreasing dependence. There is also research on monotonicity of more than two variables (see, for example, [10]). Monotonicity has useful applications in actuarial science.

When a relationship of two random variables is studied it can be useful to measure the degree of their monotonicity. One of such possible measures, the monotonicity coefficient was introduced in [16]:

$$\rho m(X,Y) = \begin{cases} \dfrac{Cov(X,Y)}{Cov(X^*,Y^*)} & if\ Cov(X,Y) > 0, \\ 0 & if\ Cov(X,Y) = 0, \\ -\dfrac{Cov(X,Y)}{Cov(X^*,Y')} & if\ Cov(X,Y) < 0. \end{cases}$$

Here $Cov(X, Y)$ is the covariance of X and Y, $X^* = F_X^{-1}(U)$ and $Y' = F_Y^{-1}(1-U)$ for a fixed random variable U with the uniform distribution on (0, 1).

It satisfies the following natural conditions for such a measure. For the random variables X and Y,

1) $|\,\rho\,(X, Y)\,| \le |\,\rho m\,(X, Y)\,| \le 1$, where $\rho\,(X, Y)$ is the Pearson correlation of X and Y;

2) if X and Y are independent, then $\rho m\,(X, Y) = 0$;

3) $\rho m(X, Y)=1$ if and only if the pair $<X, Y>$ is comonotonic;

4) $\rho m\,(X, Y) = -1$ if and only if the pair $<X, Y>$ is counter-monotonic.

This is the sample version of the monotonicity coefficient for a two-dimensional sample (x, y):

$$rm(x,y) = \begin{cases} \dfrac{s(x,y)}{s(x^*,y^*)} & if\ s(x,y) > 0, \\ 0 & if\ s(x,y) = 0, \\ -\dfrac{s(x,y)}{s(x^*,y')} & if\ s(x,y) < 0, \end{cases}$$

where $s(x, y)$ is the sample covariance, x^* is the sample x with its values in ascending order and y' is the sample y with its values in descending order. The properties of rm are studied in detail in [17].

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