Applying change of variable to calculus problems

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The paper describes the technique of introducing a new variable in some calculus problems that helps students to master the skills of integration and evaluating limits. This technique is algorithmic and easy to apply; it is especially useful in teaching weak students.

1. Introduction

In our practice of teaching tertiary mathematics we want students to gain both conceptual and procedural knowledge and develop their problem solving abilities. On the tertiary level, the dominance of procedural mathematics is a characteristic of mathematics itself. Tossavainen [1] wrote: "Mathematical theories are based on axioms and derivation rules, thus this knowledge is highly procedural by nature: it must be derived from the fundamental definitions and axioms by a finite sequence of logical steps." Also understanding a mathematical concept often does not provide the

relevant procedural knowledge, for example the definitions of a limit, a derivative or an integral are not directly linked to the methods of their evaluation.

In constructivism, in particular in Piaget's theory of cognitive development [2], conceptual knowledge and procedural knowledge are both integral parts of the learning process. Rittle-Johnson, Siegler, & Alibali [3] developed these views further in the "iterative model". Their research shows the causal relations between conceptual and procedural knowledge: concepts and procedures develop iteratively reinforcing each other. Increased conceptual knowledge leads (through training) to gains in procedural and problem solving abilities. Use of correct procedures leads to improved conceptual understanding.

The authors of this paper share the iterative views. Therefore in our teaching practice we look for effective methods to improve students' procedural knowledge that will in turn enhance their conceptual knowledge. Here we describe a non-traditional technique that we used in calculus courses at the Auckland University of Technology and the Moscow Technological University for several years. The technique is based on a change of variable that simplifies the calculus task. This teaching strategy helps students to master calculus skills faster. The first type of problems that we describe are limits of the form $\infty - \infty$, $\dfrac{\infty}{\infty}$ and $\dfrac{0}{0}$.

2. Evaluating limits

To evaluate a limit of the form $\lim\limits_{x \to a} f(x)$ we recommend to introduce a new variable t with $t \to 0$; the new limit is usually easier to evaluate. When $a = \pm\infty$, the appropriate change of variable is $t = \dfrac{1}{x}$. Here are a few examples.

Example 1. $\displaystyle \lim_{x \to +\infty} \frac{4x^4 + 3x^2\left(1+x^2\right)+4}{2x^4+3} = \left(\frac{\infty}{\infty}\right) = \left|\begin{array}{c} t = \dfrac{1}{x} \\ x = \dfrac{1}{t} \end{array}\right| =$

$$= \lim_{t \to 0} \frac{\dfrac{4}{t^4} + \dfrac{3}{t^2}\left(1+\dfrac{1}{t^2}\right)+4}{\dfrac{2}{t^4}+3} = \lim_{t \to 0} \frac{4 + 3\left(t^2+1\right)+4t^4}{2+3t^4} = \frac{4+3(0+1)+0}{2+0} = \frac{7}{2}. \quad \blacksquare$$

As $t \to 0$, we have $(1+t)^\alpha = 1 + \alpha t + o(t)$. This fact will be used in the following three examples.

Example 2. $\displaystyle \lim_{x \to +\infty}\left(\sqrt{x^2+3x}-x\right) = (\infty - \infty) = \left|\begin{array}{c} t = \dfrac{1}{x} \\ x = \dfrac{1}{t} \end{array}\right| = \lim_{t \to 0+}\left(\sqrt{\dfrac{1}{t^2}+\dfrac{3}{t}}-\dfrac{1}{t}\right) =$

$$= \lim_{t \to 0+}\left(\frac{\sqrt{1+3t}-1}{t}\right) = \lim_{t \to 0+} \frac{1+\dfrac{1}{2}\cdot 3t + o(t)-1}{t} = \lim_{t \to 0+} \frac{\dfrac{3}{2}t + o(t)}{t} = \frac{3}{2}. \quad \blacksquare$$

Example 3. $\displaystyle \lim_{x \to -\infty}\left(\sqrt{x(x+2)}-\sqrt{x^2-2x+3}\right) = (\infty - \infty) = \left|\begin{array}{c} t = \dfrac{1}{x} \\ x = \dfrac{1}{t} \end{array}\right| =$

$$= \lim_{t \to 0-}\left(\sqrt{\dfrac{1}{t}\left(\dfrac{1}{t}+2\right)}-\sqrt{\dfrac{1}{t^2}-\dfrac{2}{t}+3}\right) = \lim_{t \to 0-} \frac{1}{|t|}\left[\left(1+2t\right)^{\frac{1}{2}}-\left(1-2t+3t^2\right)^{\frac{1}{2}}\right] =$$

$$= \lim_{t \to 0-} \frac{1}{-t}\left[1+t-\left(1-t+\dfrac{3}{2}t^2\right)+o(t)\right] = \lim_{t \to 0-} \frac{2t+o(t)}{-t} = -2. \quad \blacksquare$$

In Examples 1, 2 and 3 other techniques can be used but they are more various and require more inventiveness and memorizing that the change of variable. Also some students have difficulties understanding and applying the concept of infinity; the change of $x \to \infty$ to $t \to 0$ helps them to overcome these difficulties.

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To evaluate a limit of the form $\lim\limits_{x \to a} f(x)$ with a finite a we recommend to introduce the new variable $t = x - a$; then $t \to 0$. The following three examples illustrate the finite case.

Example 4. $\lim\limits_{x \to 1} \dfrac{\sqrt{2x} - \sqrt{1+x}}{\sqrt[3]{x} - 1} = \left(\dfrac{0}{0}\right) = \begin{vmatrix} t = x - 1 \\ x = t + 1 \end{vmatrix} = \lim\limits_{t \to 0} \dfrac{\sqrt{2} \cdot \sqrt{1+t} - \sqrt{2+t}}{\sqrt[3]{1+t} - 1} =$

$$= \lim\limits_{t \to 0} \frac{\left[\sqrt{2} \cdot (1+t)^{\frac{1}{2}} - \sqrt{2} \cdot \left(1 + \dfrac{1}{2}t\right)^{\frac{1}{2}} \right]}{(1+t)^{\frac{1}{3}} - 1} = \lim\limits_{t \to 0} \sqrt{2} \frac{\left[1 + \dfrac{1}{2}t - \left(1 + \dfrac{1}{4}t\right) + o(t) \right]}{\dfrac{1}{3}t + o(t)} =$$

$$= \sqrt{2} \lim\limits_{t \to 0} \frac{\dfrac{t}{4} + o(t)}{\dfrac{t}{3} + o(t)} = \sqrt{2} \cdot \frac{\dfrac{1}{4}}{\dfrac{1}{3}} = \frac{3\sqrt{2}}{4}. \quad \blacksquare$$

Example 5. $\lim\limits_{x \to \frac{\pi}{2}} \dfrac{\cos 5x - \cos 3x}{\cos x} = \left(\dfrac{0}{0}\right) = \begin{vmatrix} t = x - \dfrac{\pi}{2} \\ x = t + \dfrac{\pi}{2} \end{vmatrix} =$

$$= \lim\limits_{t \to 0} \frac{\cos\left(5t + \dfrac{5\pi}{2}\right) - \cos\left(3t + \dfrac{3\pi}{2}\right)}{\cos\left(t + \dfrac{\pi}{2}\right)} = \lim\limits_{t \to 0} \frac{\sin 5t + \sin 3t}{\sin t} = \lim\limits_{t \to 0} \frac{8t + o(t)}{t + o(t)} = 8.$$

(Using the fact that $\sin t = t + o(t)$ as $t \to 0$). $\quad \blacksquare$

Example 6. $\lim\limits_{x \to 3} \dfrac{\ln(2x - 5)}{e^{\sin(\pi x)} - 1} = \left(\dfrac{0}{0}\right) = \begin{vmatrix} t = x - 3 \\ x = t + 3 \end{vmatrix} = \lim\limits_{t \to 0} \dfrac{\ln(2(t+3) - 5)}{e^{\sin(\pi t + 3\pi)} - 1} =$

$$= \lim\limits_{t \to 0} \frac{\ln(1 + 2t)}{e^{-\sin(\pi t)} - 1} = \lim\limits_{t \to 0} \frac{2t + o(t)}{-\sin(\pi t) + o(t)} = \lim\limits_{t \to 0} \frac{2t + o(t)}{-\pi t + o(t)} = -\frac{2}{\pi}.$$

(Using the fact that $\ln(1+t) = t + o(t)$ and $e^t = 1 + t + o(t)$ as $t \to 0$). $\quad \blacksquare$

The change of variable $t = x - a$ is also helpful in evaluating the limits of rational functions of the indeterminate form $\dfrac{0}{0}$. This method allows to avoid factorising, which involves trial and error process and can be sometimes quite difficult. Here are a few examples.

Example 7. $\displaystyle\lim_{x \to 1} \frac{3x^4 - x^3 - x^2 - 1}{2x^2 + x - 3} = \left(\frac{0}{0}\right) = \left|\begin{array}{l} t = x - 1 \\ x = t + 1 \end{array}\right| =$

$$= \lim_{t \to 0} \frac{3(t+1)^4 - (t+1)^3 - (t+1)^2 - 1}{2(t+1)^2 + (t+1) - 3} = \lim_{t \to 0} \frac{3t^4 + 11t^3 + 14t^2 + 7t}{2t^2 + 5t} =$$

$$= \lim_{t \to 0} \frac{3t^3 + 11t^2 + 14t + 7}{2t + 5} = \frac{7}{5}. \quad \blacksquare$$

Example 8. $\displaystyle\lim_{x \to 2} \frac{x^3 - 4x^2 + 7x - 6}{3x^3 - 14x^2 + 21x - 10} = \left(\frac{0}{0}\right) = \left|\begin{array}{l} t = x - 2 \\ x = t + 2 \end{array}\right| =$

$$= \lim_{t \to 0} \frac{(t+2)^3 - 4(t+2)^2 + 7(t+2) - 6}{3(t+2)^3 - 14(t+2)^2 + 21(t+2) - 10} = \lim_{t \to 0} \frac{t^3 + 2t^2 + 3t}{3t^3 + 4t^2 + t} =$$

$$= \lim_{t \to 0} \frac{t^2 + 2t + 3}{3t^2 + 4t + 1} = 3. \quad \blacksquare$$

Example 9. $\displaystyle\lim_{x \to -1} \frac{3x^2 + x - 2}{2x^4 + 5x^2 + 3x - 4} = \left(\frac{0}{0}\right) = \left|\begin{array}{l} t = x + 1 \\ x = t - 1 \end{array}\right| =$

$$= \lim_{t \to 0} \frac{3(t-1)^2 + (t-1) - 2}{2(t-1)^4 + 5(t-1)^2 + 3(t-1) - 4} = \lim_{t \to 0} \frac{3t^2 - 5t}{2t^4 - 8t^3 + 17t^2 - 15t} =$$

$$= \lim_{t \to 0} \frac{3t - 5}{2t^3 - 8t^2 + 17t - 15} = \frac{1}{3}. \quad \blacksquare$$

Clearly most of these limits can be calculated by l'Hôpital's rule. But introducing a new variable gives students the chance to calculate these limits easily before they have learned differentiation.

3. Change of variable in quadratics

The rest of the paper describes several types of problems involving quadratics of the form $y = ax^2 + bx + c$. We suggest to introduce a new variable $t = x + \dfrac{b}{2a}$ in order to eliminate the linear term in the quadratic:

$$y = ax^2 + bx + c = \begin{vmatrix} t = x + \dfrac{b}{2a} \\ x = t - \dfrac{b}{2a} \end{vmatrix} = a\left(t - \dfrac{b}{2a}\right)^2 + b\left(t - \dfrac{b}{2a}\right) + c = at^2 - \dfrac{b^2}{4a} + c.$$

Eliminating the linear term simplifies solutions of different problems with quadratics. We illustrate the method with the following examples; some of them are simple algebraic problems and others are harder integration problems.

Example 10. Completing the square. $3x^2 - 5x - 2 = \begin{vmatrix} t = x - \dfrac{5}{6} \\ x = t + \dfrac{5}{6} \end{vmatrix} =$

$$= 3\left(t + \dfrac{5}{6}\right)^2 - 5\left(t + \dfrac{5}{6}\right) - 2 = 3t^2 - \dfrac{49}{12} = 3\left(x - \dfrac{5}{6}\right)^2 - \dfrac{49}{12}. \quad \blacksquare$$

Example 11. Quadratic equation. Solve the equation $12x^2 - 8x - 15 = 0$.

Solution: $\begin{vmatrix} t = x - \dfrac{1}{3} \\ x = t + \dfrac{1}{3} \end{vmatrix}$; $\quad 12\left(t + \dfrac{1}{3}\right)^2 - 8\left(t + \dfrac{1}{3}\right) - 15 = 0, \quad 12t^2 - \dfrac{49}{3} = 0, \quad t^2 = \dfrac{49}{36},$

$$t = \pm\dfrac{7}{6}. \quad x = t + \dfrac{1}{3}. \qquad x_1 = -\dfrac{7}{6} + \dfrac{1}{3} = -\dfrac{5}{6}. \qquad x_2 = \dfrac{7}{6} + \dfrac{1}{3} = \dfrac{3}{2}. \quad \blacksquare$$

Example 12. Quadratic equation. Solve the equation $x^2 + 4x + 1 = 0$.

Solution: $\begin{vmatrix} t = x + 2 \\ x = t - 2 \end{vmatrix}$; $\quad (t - 2)^2 + 4(t - 2) + 1 = 0, \quad t^2 - 3 = 0, \quad t = \pm\sqrt{3}. \quad x = t - 2.$

$$x_1 = -\sqrt{3} - 2. \qquad x_2 = \sqrt{3} - 2. \quad \blacksquare$$

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The change of variable helps to solve quadratic equations in an easier way than traditional factorising or quadratic formula. The factorising method is not very useful in Example 11 because the roots are fractional, and it does not work in Example 12 because the roots are irrational. The quadratic formula works in all cases but it is harder to memorise than the formula $t = x + \dfrac{b}{2a}$.

4. Applications to integration

The technique of change of variable is widely used for integration. Here we will consider the integrals with quadratics, where this technique is not usually applied but can be quite useful. The technique is based on the same change of variable $t = x + \dfrac{b}{2a}$. It is illustrated on the following examples, which use the table of indefinite integrals along with the change of variable.

Example 13. $\displaystyle\int \frac{4x-3}{3x^2+6x+4}\,dx = \begin{vmatrix} t = x+1 \\ x = t-1 \end{vmatrix} = \int \frac{4t-7}{3t^2+1}\,dt =$

$\displaystyle = \frac{4}{6}\int \frac{6t}{3t^2+1}\,dt - \int \frac{7}{\left(\sqrt{3}\,t\right)^2+1}\,dt = \frac{2}{3}\ln\left(3t^2+1\right) - \frac{7}{\sqrt{3}}\tan^{-1}\left(\sqrt{3}\,t\right) + C =$

$\displaystyle = \frac{2}{3}\ln\left(3x^2+6x+4\right) - \frac{7}{\sqrt{3}}\tan^{-1}\left(\sqrt{3}\,(x+1)\right) + C.$ ∎

Example 14. $\displaystyle\int \frac{dx}{2x-5x^2} = \begin{vmatrix} t = x - \dfrac{1}{5} \\ x = t + \dfrac{1}{5} \end{vmatrix} = \int \frac{dt}{-5t^2+\dfrac{1}{5}} = -\frac{1}{5}\int \frac{dt}{t^2-\dfrac{1}{25}} =$

$\displaystyle = -\frac{1}{5}\cdot\frac{5}{2}\ln\left|\frac{t-\dfrac{1}{5}}{t+\dfrac{1}{5}}\right| + C = \frac{1}{2}\ln\left|\frac{5t+1}{5t-1}\right| + C = \frac{1}{2}\ln\left|\frac{5x}{5x-2}\right| + C.$ ∎

Example 15. $\int \dfrac{dx}{\sqrt{x^2-2x+10}} = \left| \begin{matrix} t=x-1 \\ x=t+1 \end{matrix} \right| = \int \dfrac{dt}{\sqrt{(t+1)^2-2(t+1)+10}} =$

$= \int \dfrac{dt}{\sqrt{t^2+9}} = \ln\left|t+\sqrt{t^2+9}\right|+C = \ln\left(x-1+\sqrt{x^2-2x+10}\right)+C.$ ∎

Example 16. $\int \sqrt{4x-x^2}\ dx = \left| \begin{matrix} t=x-2 \\ x=t+2 \end{matrix} \right| = \int \sqrt{4-t^2}\ dt = \left| \begin{matrix} t=2\sin\theta \\ dt=2\cos\theta\, d\theta \end{matrix} \right| =$

$= \int 4\cos^2\theta\, d\theta = \int 2(1+\cos 2\theta)d\theta = 2\theta+\sin 2\theta+C = 2\theta+2\sin\theta\cos\theta+C =$

$= 2\sin^{-1}\left(\dfrac{t}{2}\right)+2\cdot\dfrac{t}{2}\cdot\dfrac{\sqrt{4-t^2}}{2}+C = 2\sin^{-1}\left(\dfrac{x-2}{2}\right)+\dfrac{x-2}{2}\sqrt{4x-x^2}+C.$ ∎

Example 17. $\int \dfrac{x-1}{\sqrt{1-4x-x^2}}\,dx = \left| \begin{matrix} t=x+2 \\ x=t-2 \end{matrix} \right| = \int \dfrac{t-3}{\sqrt{5-t^2}}\,dt =$

$= -\dfrac{1}{2}\int \dfrac{-2t}{\sqrt{5-t^2}}\,dt -3\int \dfrac{dt}{\sqrt{5-t^2}} = -\sqrt{5-t^2}-3\sin^{-1}\left(\dfrac{t}{\sqrt{5}}\right)+C =$

$= -\sqrt{1-4x-x^2}-3\sin^{-1}\left(\dfrac{x+2}{\sqrt{5}}\right)+C.$ ∎

Example 18. $\int \dfrac{dx}{\sqrt{(x^2+8x+7)^3}} = \left| \begin{matrix} t=x+4 \\ x=t-4 \end{matrix} \right| = \int \dfrac{dt}{\sqrt{(t^2-9)^3}} = \left| \begin{matrix} t=3\cosh\theta \\ dt=3\sinh\theta\, d\theta \end{matrix} \right| =$

$= \int \dfrac{3\sinh\theta\, d\theta}{(3\sinh\theta)^3} = \dfrac{1}{9}\int \dfrac{d\theta}{\sinh^2\theta} = -\dfrac{1}{9}\coth\theta+C = -\dfrac{\cosh\theta}{9\sinh\theta}+C =$

$= -\dfrac{t}{9\sqrt{t^2-9}}+C = -\dfrac{x+4}{9\sqrt{x^2+8x+7}}+C.$ ∎

5. Discussion

The authors used the described teaching strategy for several years in calculus courses at the Auckland University of Technology and the Moscow Technological University. While a formal statistical analysis of the results is yet to be done, verbal responses

from students and their assessment results showed effectiveness of this strategy. The students who applied the suggested procedures are more successful than the ones using traditional procedures, in technical manipulations as well as in learning the relevant concepts. While applying this method the students get answers faster and make fewer mistakes on the way.

The method of introducing a new variable was used to teach the calculus problems, where other methods were traditionally applied. The method simplifies mastering calculus techniques, especially for weaker students. Also the method is algorithmic and eliminates most of guessing and memorizing.

References

[1]   T.Tossavainen, *Conceptualising procedural knowledge of mathematics – or the other way around.* (2006). Available at

http://www.distans.hkr.se/rikskonf/Grupp%206/Tossavainen.pdf

[2]   J.Piaget, *The equilibrium of cognitive structures*, Harvard University Press, Cambridge, 1985.

[3]   B.Rittle-Johnson, R.S.Siegler, and M.W.Alibali, *Developing conceptual understanding and procedural skill in mathematics: an iterative process*, J. of Educational Psychology 93, No. 2 (2001), pp. 346-362.