

NOTES ON THE WIJSMAN TOPOLOGY[‡]

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ABSTRACT. In this paper, we show that for an almost locally separable metrizable space X , if its Wijsman hyperspace induced by one compatible metric is Baire, then X itself must be Baire. We also provide an example to demonstrate the necessity of separability.

1. INTRODUCTION

In 1966, Wijsman [16] considered the weak topology on the collection of nonempty closed subsets of a metric space (X, d) generated by the distance functionals viewed as functions of set argument, which is known as the Wijsman topology nowadays. Since then, there has been a considerable effort in exploring various completeness properties of this class of hyperspaces. It was Effros [9] who first showed that a Polish space (i.e., a completely metrizable separable space) admits a metric for which the Wijsman topology is Polish; later, Beer [2], [3] showed that given a separable complete metric space the corresponding Wijsman hyperspace is Polish. Finally, Constantini [7] demonstrated that for a Polish space, the Wijsman hyperspace of any compatible metric is Polish. On the other hand, Constantini [8] further showed that the Wijsman hyperspace of a complete metric space (i.e., the separability is dropped) may fail to be Čech-complete (in this case, the Wijsman hyperspace is Tychonoff, but not metrizable). Since complete metric spaces are Baire by the classic Baire category theorem, it is worth to investigate Baireness of Wijsman hyperspaces. Recall that a topological space X is *Baire* if the intersection of every sequence of dense open subsets in X is dense. Further, if every nonempty closed subspace of X is Baire then X is called *hereditarily Baire*. Zsilinszky [18] showed that the Wijsman hyperspace of a complete metric space is Baire (in fact, is “strongly Choquet”, a property stronger than Baireness), and asked if the conclusion can be strengthened to be hereditarily Baire. Well, Chaber and Pol [5] showed that this is not always the case.

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Now, let us introduce some notation. Throughout the paper, for a given space X , $\tau(X)$ will denote the topology on X , and 2^X will denote the family of all nonempty closed subsets of X . For a metric space (X, d) and $x \in X$, the open ball at x with radius r is denoted by $S_d(x, r)$, and $B_d(x, r)$ denotes the closed ball at x with the radius r , i.e.,

$$S_d(x, r) = \{y \in X : d(x, y) < r\}, \quad B_d(x, r) = \{y \in X : d(x, y) \leq r\}.$$

Let

$$\Delta_d = \{\emptyset\} \cup \{\text{all finite unions of closed balls in } X\}.$$

For $E \subseteq X$, let

$$E^- = \{A \in 2^X : A \cap E \neq \emptyset\}, \quad E^+ := \{A \in 2^X : A \subseteq E\}.$$

Further, we define

$$E^{++} = \{A \in 2^X : S_d(A, \varepsilon) \subseteq E \text{ for some } \varepsilon > 0\},$$

where $S_d(A, \varepsilon) := \bigcup_{x \in A} S_d(x, \varepsilon)$. The complement of E (in X) will be denoted by E^c or $X \setminus E$. Recall that the *Wijsman topology* $\tau_w(d)$ on 2^X has

$$\{U^- : U \in \tau(X)\} \cup \{\{A \in 2^X : d(x, A) > \varepsilon\} : x \in X, \varepsilon > 0\}$$

as a subbase. A topology on 2^X which is closely related to the Wijsman topology is the so-called *ball topology* $\tau_b(d)$, having

$$\{U^- : U \in \tau(X)\} \cup \{(B_d(x, r)^c)^+ : x \in X, r > 0\}$$

as a subbase [3], [20]. It is well-known that for any metric space (X, d) ,

$$\tau_w(d) \subseteq \tau_b(d) \subseteq \tau_v$$

hold on 2^X , refer to [3, page 53].

2. WIJSMAN AND VIETORIS TOPOLOGIES

For a metrizable space X , and let \mathcal{D} be the family of all compatible metrics on X . It is known that on 2^X , $\tau_v = \sup\{\tau_w(d) : d \in \mathcal{D}\}$. Now, we consider to extend this result to topological spaces. To this end, for a Tychonoff space X , let \mathcal{D}_c be the family of all continuous pseudometric on X . For each fixed $d \in \mathcal{D}_c$ and each point $x \in X$, as usual, $d(x, \cdot) : 2^X \rightarrow \mathbb{R}$ denotes the distance functional defined by

$$d(x, A) = \inf\{d(x, a) : a \in A\}.$$

For each $d \in \mathcal{D}_c$, let $\tau_w(d)$ be the topology on 2^X generated by

$$\{d(x, \cdot)^{-1}((\alpha, \beta)) : x \in X, \alpha, \beta \in \mathbb{R} \text{ and } 0 < \alpha < \beta\}.$$

Theorem 2.1. *For a normal space X , $\tau_v = \sup\{\tau_w(d) : d \in \mathcal{D}_c\}$ on 2^X .*

Proof. Let $\sigma = \sup\{\tau_{w(d)} : d \in \mathcal{D}_c\}$. First, we verify that $\sigma \subseteq \tau_v$. Suppose that $A \in 2^X$, $x \in X$ and $d \in \mathcal{D}_c$ such that $d(x, A) < \varepsilon$, where $\varepsilon > 0$. Then, there is a point $a \in A$ such that $d(a, x) < \varepsilon$. Thus, $A \in B_d(x, \varepsilon)^-$. Further,

$$B_d(x, \varepsilon)^- \subseteq \{B \in 2^X : d(x, B) < \varepsilon\}.$$

Since d is continuous, then $B_d(x, \varepsilon)$ is open in X and $B_d(x, \varepsilon)^- \in \tau_v$. It follows that for each $x \in X$ and each $d \in \mathcal{D}_c$, $\{B \in 2^X : d(x, B) < \varepsilon\}$ is open in 2^X . In a similar way, we can show that for each $x \in X$ and each $d \in \mathcal{D}_c$, $\{B \in 2^X : d(x, B) > \varepsilon\}$ is open in 2^X as well. This means that $\tau_{w(d)} \in \tau_v$ for each $d \in \mathcal{D}_c$, and thus $\sigma \subseteq \tau_v$.

Let V be a nonempty set in X . We shall show that $V^+ \in \sigma$. We may assume that $V \neq X$, otherwise $V^+ = 2^X \in \sigma$. Pick a point $y_0 \in V^c$. For each $A \in V^+$, since $A \cap V^c = \emptyset$ and X is normal, there is a continuous function $f : X \rightarrow [0, 1]$ such that $f(A) = \{1\}$ and $f(V^c) = \{0\}$. Define $d_f : X \times X \rightarrow \mathbb{R}$ by

$$d_f(x, y) = |f(x) - f(y)|.$$

Evidently, $d_f \in \mathcal{D}_c$. If we put

$$\mathcal{N}(A, 1/4) = \{B \in 2^X : d_f(y_0, A) - d_f(y_0, B) < 1/4\},$$

then we have

$$A \in \mathcal{N}(A, 1/4) \in \tau_{w(d_f)}.$$

Further, it can be checked that $\mathcal{N}(A, 1/4) \subseteq V^+$. It follows that $V^+ \in \sigma$. Similarly, we can show that $V^- \in \sigma$ for each nonempty open set V in X . Thus, $\tau_v \subseteq \sigma$. \square

Corollary 2.2. *For a normal space X , $\tau_v = \sup\{\tau_{b(d)} : d \in \mathcal{D}_c\}$ on 2^X .*

3. COUNTABLE SUBCOMPACTNESS OF THE BALL TOPOLOGY

A well-known result of McCoy in [13] states that for a T_1 -space X , if $(2^X, \tau_v)$ is Baire, then so is X . In this section, we show that the same conclusion does not hold for the Wijsman (ball) topology.

Recall that a collection \mathcal{F} of nonempty subsets of a space X is called a *regular filterbase* if whenever $F_1, F_2 \in \mathcal{F}$, there is some $F_3 \in \mathcal{F}$ such that $\overline{F_3} \subseteq F_1 \cap F_2$. A regular space X is said to be *countably subcompact* (with respect to \mathfrak{B}) [1] if there is a base \mathfrak{B} of open sets for X such that if $\{B_n : n < \omega\} \subseteq \mathfrak{B}$ is a countable filterbase, then $\bigcap_{n < \omega} \overline{B_n} \neq \emptyset$. It is easy to see that every base-compact space is Baire.

Theorem 3.1 ([20]). *For a metric space (X, d) , $(2^X, \tau_{w(d)})$ is Baire if and only if $(2^X, \tau_{b(d)})$ is Baire.*

Lemma 3.2. *A regular space X is countably subcompact if and only if there is a base \mathfrak{B} of open sets such that for any sequence $\{B_n : n < \omega\} \subseteq \mathfrak{B}$ satisfying $\overline{B_{n+1}} \subseteq B_n$ for all $n < \omega$, $\bigcap_{n < \omega} B_n \neq \emptyset$.*

Theorem 3.3. *Let (X, d) be an ultrametric space. If $S(x, r) \setminus S(x, r')$ is non-separable for all $x \in X$ and $0 < r' < r$, then $(2^X, \tau_{b(d)})$ is countably subcompact.*

Proof. Given a finite collection $\{S_{r_0}(x_0), \dots, S_{r_k}(x_k); S_{t_0}(y_0), \dots, S_{t_m}(y_m)\}$ of pairwise disjoint open balls in (X, d) , we define

$$\mathcal{F}((x_i, r_i)_{i \leq k}, (y_j, t_j)_{j \leq m}) = \bigcap_{i \leq k} S_{r_i}(x_i)^- \cap \left(X \setminus \bigcup_{j \leq m} S_{t_j}(y_j) \right)^+.$$

Since (X, d) is an ultrametric space, each $\mathcal{F}((x_i, r_i)_{i \leq k}, (y_j, t_j)_{j \leq m})$ is a closed and open set in $(2^X, \tau_{b(d)})$. Further, it can be checked that

$$\mathfrak{B} = \{\mathcal{F}((x_i, r_i)_{i \leq k}, (y_j, t_j)_{j \leq m}) : x_i, y_j \in X, r, t_j > 0, k, m < \omega\}$$

is a base for $(2^X, \tau_{b(d)})$. We shall show that $(2^X, \tau_{b(d)})$ is countably subcompact with respect to \mathfrak{B} .

Suppose that $\{\mathcal{F}_n : n < \omega\}$ is a sequence in \mathfrak{B} such that $\mathcal{F}_{n+1} \subseteq \mathcal{F}_n$, and $\mathcal{F}_n = \mathcal{F}((x_i^n, r_i^n)_{i \leq k_n}, (y_j^n, t_j^n)_{j \leq m_n})$ for all $n < \omega$. For convenience, for each $n < \omega$, we write

$$A_n = \bigcup_{i \leq k_n} S_{r_i^n}(x_i^n), \text{ and } B_n = \bigcup_{j \leq m_n} S_{t_j^n}(y_j^n).$$

Note that $\mathcal{F}_{n+1} \subseteq \mathcal{F}_n$ implies that

- (i) $B_n \subseteq B_{n+1}$ for all $n < \omega$;
- (ii) for each $i < k_n$, there must be some $j \leq k_{n+1}$ such that

$$S_{r_{n+1}}(x_j^{n+1}) \subseteq S_{r_n}(x_i^n).$$

Next, we define

$$F_n = A_n \setminus \bigcup \{S_{r_{n+1}}(x_j^{n+1}) : S_{r_{n+1}}(x_j^{n+1}) \not\subseteq S_{r_n}(x_i^n) \text{ for some } i \leq k_n\}.$$

Since $S_{r_n}(x_i^n) \setminus S_{r_{n+1}}(x_j^{n+1})$ is non-separable when $0 < r^{n+1} < r^n$, then $F_n \neq \emptyset$ for all $n < \omega$. By definition, we have

- (iii) $F_n \cap B_n = \emptyset$ for all $n < \omega$.

Furthermore, (ii) implies that

- (iv) $\{F_n : n < \omega\}$ is a mutually disjoint family.

Now, fix an $n_0 < \omega$. By (i) and definition, $F_{n_0} \cap B_n = \emptyset$ for all $n \leq n_0$. Suppose $n > n_0$. Since (X, d) is an ultrametric space, by (ii), for each $j \leq m_n$, either $A_{n_0} \cap S_{t_j^n}(y_j^n) = \emptyset$ or $S_{t_j^n}(y_j^n) \subseteq A_{n_0}$ with $t_j^n < r^{n_0}$. On

the other hand, $A_{n_0} \setminus S_{t_j^n}(y_j^n)$ is non-separable whenever $S_{t_j^n}(y_j^n) \subseteq A_{n_0}$ and $0 < t_j^n < r^{n_0}$. It follows that $F_{n_0} \setminus \bigcup_{n < \omega} B_n$ is nonempty. If we define

$$F = \overline{\bigcup_{n < \omega} F_n} \setminus \bigcup_{n < \omega} B_n,$$

then $F \in 2^X$. Further, since $F \in (B_n^c)^+$ and

$$\left(F_n \setminus \bigcup_{j < \omega} B_j \right) \cap S_{r^n}(x_i^n) \neq \emptyset$$

for all $i \leq k_n$ and $n < \omega$, we conclude that $F \in \bigcap_{n < \omega} \mathcal{F}_n$. Finally, by Lemma 3.2, $(2^X, \tau_{b(d)})$ is countably subcompact. \square

Corollary 3.4. *Let (X, d) be an ultrametric space. If $S(x, r) \setminus S(x, r')$ is non-separable for all $x \in X$ and $0 < r' < r$, then $(2^X, \tau_{b(d)})$ is Baire.*

Given a cardinal κ , let $Y(\kappa) = \kappa^\omega$ be equipped with the Baire metric ϱ_κ , that is,

$$\varrho_\kappa(x, y) = \begin{cases} 0, & \text{if } x(n) = y(n) \text{ for all } n < \omega; \\ 2^{-n}, & \text{if } x \neq y \text{ and } n \text{ is the least with } x(n) \neq y(n). \end{cases}$$

Let $Z(\kappa) = \kappa^{<\omega}$. If each element in $Z(\kappa)$ is identified with a sequence which is eventually 0, then $Z(\kappa) \subseteq Y(\kappa)$. In [20], Zsilinszky considered the space $X = Z(\omega) \times Y(\omega)$ equipped with the product metric d . Evidently, (X, d) is of first category. It is shown in [20] that $(2^X, \tau_{w(d)})$ is Baire. However, we could not follow this example, it may contain a gap. Nevertheless, we give another example of the same nature in the following.

Example 3.5. *There is a metric space (X, d) which is of first category such that $(2^X, \tau_{w(d)})$ is Baire.* Let $X = Z(\omega_1)$ and d be the restriction of ϱ_{ω_1} on $Z(\omega_1)$. For each $n < \omega$, let

$$F_n = \{z \in Z(\omega_1) : z(i) = 0 \text{ when } i \geq n\}.$$

Then for each $n < \omega$, F_n is closed nowhere dense in (X, d) . Since $X = \bigcup_{n < \omega} F_n$, then (X, d) is of first category. On the other hand, it is easy to see that (X, d) satisfies the condition stated in Theorem 3.3. Thus, by Corollary 3.4, $(2^X, \tau_{b(d)})$ is Baire. Finally, by Theorem 3.1, $(2^X, \tau_{w(d)})$ is also Baire.

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