

Counting the Number of Minimal Paths in Weighted Coloured–Edge Graphs

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Abstract

A weighted coloured–edge graph is a graph for which each edge is assigned both a positive weight and a discrete colour, and can be used to model transportation and computer networks in which there are multiple transportation modes. In such a graph paths are compared by their total weight in each colour, resulting in a Pareto set of minimal paths from one vertex to another. This paper will give a tight upper bound on the cardinality of a minimal set of paths for any weighted coloured–edge graph. Additionally, a bound is presented on the expected number of minimal paths in weighted bicoloured–edge graphs.

Keywords: graph theory, minimal paths, multimodal network, transportation modes, weighted coloured–edge graph

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1. Introduction

Definition 1.1. A *weighted coloured-edge graph* $\mathbf{G} = \langle V, E, \omega, \lambda \rangle$ consists of a directed multigraph $\langle V, E \rangle$ with vertex set V and edge set E , a *weight* function $\omega: E \rightarrow \mathbb{R}^+$, and a (surjective) *colour* function $\lambda: E \rightarrow M$, where M is a set of possible colours for the edges.

Hence associated with each edge $e \in E$, there is an initial vertex $u \in V$ and a terminal vertex $v \in V$, a positive weight $\omega(e) \in \mathbb{R}^+$, and a colour $\lambda(e) \in M$. The graph \mathbf{G} is said to be *finite* if both V and E are finite sets, in which case M is also finite.

Concepts similar to weighted coloured-edge graphs have received little attention in the literature. Clímaco et al. [1] experimentally studied the number of spanning trees in a weighted graph whose edges are labelled with a colour. In that work, weight and colour are two criteria both to be minimized and the proposed algorithm generates a set of non-dominated spanning trees. The computation of coloured paths in a weighted coloured-edge graph is investigated by Xu et al. [2]. The main feature of their approach is a graph reduction technique based on a priority rule. This rule basically transforms a weighted coloured-edge multidigraph into a coloured-vertex digraph by applying algebraic operations to the adjacency matrix. Additionally, the authors provide an algorithm to identify coloured source-destination paths. Nevertheless, the algorithm is not intended for general instances because its input is a unit-weighted coloured multidigraph and only paths not having consecutive edges equally coloured are considered. In his paper Manoussakis [3] studied the computation of paths with specific colour patterns in unweighted coloured-edge graphs. Particularly, his study focuses on alternating coloured-edge paths in complete coloured-edge graphs. Finally, Bang & Gutin [4] presents a survey of the computation of alternating cycles and paths in coloured-edge multigraphs. Manoussakis as well as Bang & Gutin only focus on unweighted graphs.

As this paper investigates shortest paths and all weights are positive, it is presumed that graphs do not have any self-loops. Similarly, it can be presumed that for any two vertices $x, y \in V$ and colour $c \in M$ there is at most one edge $e_{xy} \in E$ from u to v for which $\lambda(e_{xy}) = c$. Therefore, for a finite graph with $n = |V|$, $m = |E|$, $k = |M|$, there is a bound on the number of edges, given by $m \leq kn(n - 1)$.

A *path* in \mathbf{G} consists of a finite set of edges $\{e_{x_0x_1}, e_{x_1x_2}, \dots, e_{x_{l-1}x_l}\}$ for which the terminal vertex x_i of each $e_{x_{i-1}x_i}$ is the initial vertex of $e_{x_ix_{i+1}}$,

and the path is considered to be from the source x_0 to the destination x_l . The path is called *simple* if no two edges in the path have the same initial vertex nor the same terminal vertex. It is straightforward to verify that a finite graph has at most the following number of simple paths from a chosen source vertex to a destination vertex:

$$k + k^2(n - 2) + k^3(n - 2)(n - 3) + \dots + k^{n-1}(n - 2)(n - 3) \cdots 1.$$

For any path $p_{uv} = \{e_{x_0x_1}, e_{x_1x_2}, \dots, e_{x_{l-1}x_l}\}$ from a vertex $u = x_0$ to a vertex $v = x_l$ and any colour $c \in M$ the path weight in colour c is defined by:

$$\omega_c(p_{uv}) = \sum_{\lambda(e_{x_i x_{i+1}}) = c} \omega(e_{x_i x_{i+1}}),$$

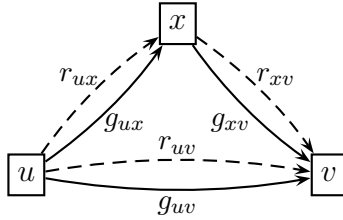
namely the sum of the weights for those edges that have colour c . The *weight* of a path is represented as a k -tuple $(\omega_{c_1}(p_{uv}), \dots, \omega_{c_i}(p_{uv}), \dots, \omega_{c_k}(p_{uv}))$, giving the total weight of the path in each colour. A preorder \leq can be defined on the paths from u to v by $p_{uv} \leq q_{uv}$ if for every colour c one has $\omega_c(p_{uv}) \leq \omega_c(q_{uv})$, essentially using the partial order defined on the weights by the product partial ordering on \mathbb{R}^k .

From the above definitions it is apparent that the concept of a weighted coloured-edge graph with k colours can equivalently be formulated as a multiweighted multigraph where each edge is assigned a k -tuple of non-negative weights $(w_{c_1}, \dots, w_{c_i}, \dots, w_{c_k})$ and exactly one $w_{c_i} > 0$. However, multiweighted graphs are mostly used in multicriteria optimization applications where the weight components correspond to quantities to be optimized, such as cost and time, so edges typically contribute toward more than just one quantity. For this reason multiweighted graphs whose edge weights are zero in all but one component have not received attention in the literature. Furthermore, most multicriteria optimization techniques do not consider multigraphs, although some can be generalized to include such possibilities. By contrast, weighted coloured-edge graphs are suitable for multimodal network applications, which can be modeled using weighted coloured-edge graphs where the weights represent some form of distance in a network and the colours represent the mode of transportation. In the past most multimodal network optimization applications have needed constraints to be placed on the network to make the determination of a shortest path fulfilling application-specific criteria tractable. Often such criteria can be specified based on the total path weights in each mode, so if the set of minimal

paths has manageable cardinality then the application-specific criteria might only need be applied to the set of minimal paths rather than to potentially $O(k^{n-1}(n-2)!)$ paths.

This paper is interested in establishing bounds on the number of paths that can be minimal from one vertex u to another vertex v . Certainly, if the graph is disconnected then there might be no paths from u to v , and if it is connected with edges in each allowed colour then there are at least k minimal paths, since paths that have edges solely in single but distinct colours are incomparable. However, an upper bound might appear to be difficult to establish.

As a simple example, consider the following graph with three vertices, six edges, and two possible colours $M = \{\text{red, green}\}$ where red edges are indicated by dashed arcs.

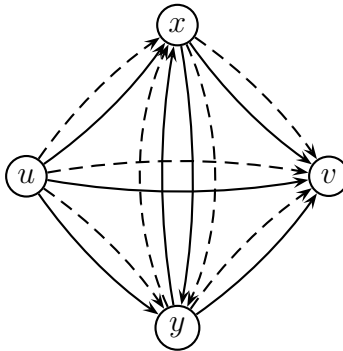


This graph has six paths from u to v with the following weights:

$$(r_{ux} + r_{xv}, 0), (r_{ux}, g_{xv}), (r_{xv}, g_{ux}), (0, g_{ux} + g_{xv}), (r_{uv}, 0), (0, g_{uv}).$$

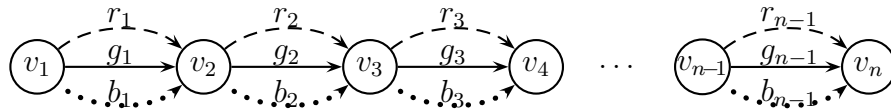
Certainly the path weights $(r_{ux} + r_{xv}, 0)$ and $(r_{uv}, 0)$ must be comparable under the product partial order on \mathbb{R}^2 , as are $(0, g_{ux} + g_{xv})$ and $(0, g_{uv})$. Depending on the six edge weights there can be either two, three, or four minimal paths from u to v with distinct weights.

Similarly, for the following graph with four vertices, fourteen edges and two possible colours:

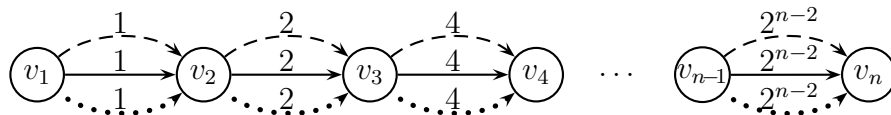


there are 26 paths to consider from u to v . After a surprising amount of effort it can be seen that there are between two and a maximum of eight minimal path weights possible.

As an important special case, a weighted coloured-edge graph is called a *chain* if in some enumeration of its vertices $v_1, v_2, v_3, \dots, v_{n-1}, v_n$ the graph only has edges from a vertex v_x to the next vertex v_{x+1} in the enumeration.



Clearly, a chain can have up to a maximum of k^{n-1} paths from v_1 to v_n . A simple induction argument can be used to show that in the following chain all k^{n-1} paths have minimal weight:



Adding more *forward* edges to the chain from x to y for $y > x+1$ increases the number of paths in the graph but it cannot increase the number in a minimal set of incomparable paths, and might possibly decrease the number. But adding *backward* edges from x to y for $y < x$ appears to greatly complicate the situation. However, it is shown in this paper that essentially chains illustrate the worst possible situation, whereas more general weighted coloured-edge graphs may have a factorial number of paths a minimal set of incomparable paths can only have cardinality up to k^{n-1} (hence in a multimodal network this is a tight bound on the Pareto set of minimal path weights that might need to be considered when applying application-specific constraints). It is interesting to note that an equivalent result for general multiweighted graphs (even restricting attention to only consider unigraphs) does not hold.

This paper also addresses the determination of a bound on the expected number of minimal paths when the edge weights are random variables in a weighted bicoloured-edge graph ($k = 2$). It provides an $O(n^3)$ probabilistic bound on the expected number of minimal paths for bicoloured-edge graphs whose weights are drawn from a bounded probability density function.

2. Upper Bound on Minimal Paths

To prove that k^{n-1} is a bound on the cardinality a special class of weighted coloured-edge graph is first introduced.

Definition 2.1. A weighted coloured-edge graph $\mathbf{G} = \langle V, E, \omega, \lambda \rangle$ is called *canonical* if:

- \mathbf{G} is *complete* in each colour, namely for all vertices $x \neq y$ and colour c there is exactly one edge e_{xy} from x to y with $\lambda(e_{xy}) = c$,
- \mathbf{G} satisfies the *triangle inequality* in each colour, for all distinct vertices x, y, z and colour c , the triangle formed by the three edges e_{xy}, e_{yz}, e_{xz} with $\lambda(e_{xy}) = \lambda(e_{yz}) = \lambda(e_{xz}) = c$ obeys $\omega(e_{xz}) \leq \omega(e_{xy}) + \omega(e_{yz})$.

It is not difficult to verify in a canonical graph that every edge e_{xy} gives a minimal path from x to y . The following lemma shows that it will be sufficient to establish the bound on the class of canonical graphs.

Lemma 2.2. *Given any finite weighted coloured-edge graph $\mathbf{G} = \langle V, E, \omega, \lambda \rangle$ there is a canonical graph $\mathbf{G}^* = \langle V, E^*, \omega^*, \lambda^* \rangle$ with the same vertices and colours as \mathbf{G} , where $E \subseteq E^*$, $\lambda^*|_E = \lambda$, and for which every minimal path in \mathbf{G} is also minimal in \mathbf{G}^* .*

Proof. Firstly note that \mathbf{G} can be made complete by adding edges e_{xy} with weight $n \cdot w$ where $n = |V|$ and w is the maximum weight of any edge in \mathbf{G} . The added edges won't affect any existing minimal paths as those paths contain at most $n - 1$ edges so their path weight in each colour is less than $n \cdot w$. It might however introduce additional minimal paths in the graph if there were no existing path from x to y in some colour. Take E^* to be the resulting set of edges.

Next, the graph can have its weights altered by defining $\omega^*(e_{xy})$ to be the weight of the shortest path from x to y that only uses edges with colour $\lambda^*(e_{xy})$. Thanks to completeness ω^* is well-defined and clearly the resulting graph satisfies the triangle inequality. \square

Theorem 2.3. *Suppose \mathbf{G} is a weighted coloured-edge graph with $n \geq 2$ vertices and k colours. Then a set of incomparable minimal paths in \mathbf{G} from one vertex to another can have at most k^{n-1} paths.*

Proof. The proof uses a counting argument and induction to bound the cardinality $f_c(n)$ of a set of incomparable minimal paths whose first edge has colour c in any weighted coloured-edge graph with n vertices. Trivially, $f_c(2) = 1$ for any graph \mathbf{G} with only two vertices.

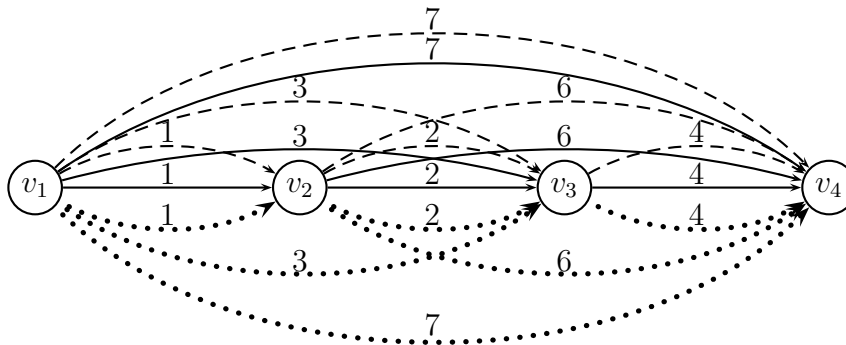
For the inductive step presume that $f_c(m) \leq k^{m-2}$ in any graph with $m \leq n$ vertices and suppose $\mathbf{G} = \langle V, E, \omega, \lambda \rangle$ has $n + 1$ vertices. By Lemma 2.2 \mathbf{G} can be presumed to be canonical. Let u and v be any two distinct vertices of \mathbf{G} and S be a set of incomparable minimal paths from u to v . By the triangle inequality it can be presumed that no minimal path in S has two consecutive edges of the same colour. Furthermore, a useful observation for a minimal path that starts with an edge e_{ux} of colour $\lambda(e_{ux}) = c$ and which passes through some vertex y before reaching v is that for the edge e_{uy} with $\lambda(e_{uy}) = c$ minimality of the path ensures that $\omega(e_{ux}) < \omega(e_{uy})$. To show that $f_c(n + 1) \leq k^{n-1}$ for each colour c order the remaining vertices of V , v_1, v_2, \dots, v_{n-1} so that if $i < j$ then $\omega(e_{uv_i}) \leq \omega(e_{uv_j})$ where e_{uv_i} and e_{uv_j} are the edges of colour c from u to v_i and v_j respectively. By the earlier observation, no minimal path that starts with the edge e_{uv_i} of colour c can pass through any of the vertices v_j for $j < i$. Hence, any minimal path that starts with the edge $e_{uv_{n-1}}$ has only a choice of $k - 1$ edges to reach v (since its consecutive edges are not of the same colour), so there are at most $k - 1$ such paths. Similarly, any minimal path that starts with the edge e_{uv_i} can only utilize $v_i, v_{i+1}, \dots, v_{n-1}$ and v , so by the inductive hypothesis for $m = n - i + 1$ there are at most $\sum_{c' \neq c} f_{c'}(n - i + 1) \leq (k - 1)k^{n-i-1}$. Summing across all the edges $e_{uv_1}, e_{uv_2}, \dots, e_{uv_{n-1}}$ and e_{uv} gives $f_c(n + 1) \leq (k - 1)k^{n-2} + (k - 1)k^{n-3} + \dots + (k - 1) + 1 = k^{n-1}$. Since there are k possible colours in which to start a path this completes the proof. \square

Note that as the proof relies on being able to linearly order the vertices v_1, v_2, \dots, v_{n-1} based on the edge weights $\omega(e_{uv_i})$ in a specific colour c , the proof can not be readily adapted to arbitrary multiweighted graphs. This result gives a tight upper bound on the cardinality of a set of incomparable minimal paths in a weighted coloured-edge graph. The bound is suitable for applications that are primarily interested in determining an optimal path given criteria that depend on the total weight in each mode of transportation. However, the proof can be slightly modified to provide a bound on the total number of minimal paths in the graph from u to v , counting all minimal paths that are comparable with each other (having the same path weight).

Theorem 2.4. *Suppose \mathbf{G} is a weighted coloured-edge graph with $n \geq 2$ vertices and k colours for which there is only at most one edge of each colour between vertices. Then \mathbf{G} has at most $k(k+1)^{n-2}$ minimal paths from one vertex to another.*

Proof. Similar to Theorem 2.3 except that the counting argument bounds $g_c(m) \leq (k+1)^{m-2}$ and paths are allowed to have the same colour on two consecutive edges. \square

The bound established in Theorem 2.4 can be seen to be tight by constructing examples based on the chain example in the previous section but with additional edges, such as the following example for $n = 4$ in which each of the 3×4^2 paths from v_1 to v_4 is minimal:



3. Expected Number of Minimal Paths

This section demonstrates that the expected number of minimal paths for a bicoloured-edge graph is polynomially bounded. The approach is based on some ideas from Röglin & Vöcking [5] and Beier et al. [6] where a bound on the expected number of optimal solutions is estimated for bicriteria problems. Their work focuses on the establishment of a probabilistic bound on the number of Pareto optimal points for bicriteria integer problems, exploiting structural properties of the Pareto frontier that are termed as *winner*s and *loser*s. It is adapted here to estimate a probabilistic bound on the number of minimal paths in weighted bicoloured-edge graphs. However, several arguments have been modified to be applied in the context of coloured-edge graphs.

Suppose \mathbf{G} is a finite weighted bicoloured-edge graph with colours $M = \{\text{red, green}\}$ for convenience and let u, v be vertices of \mathbf{G} for which there is a pure coloured-edge path in colour green from u to v .

Assume e is a red edge of \mathbf{G} whose weight is a random variable with bounded probability density function $f_e : (0, \infty) \rightarrow [0, \phi_e]$ for some $\phi_e > 0$. Define the function $\Delta_e : [0, \infty) \rightarrow (0, \infty]$ for $r \geq 0$ by the following. Consider the paths p_e from u to v that do not include the edge e and for which $\omega_{\text{red}}(p_e) \leq r$. Since there is a pure path in colour green from u to v , there are such paths p_e , and since \mathbf{G} is finite there are only finitely many such paths. Take p_e^{max} to be such a path that has least green weight and let $g_r = \omega_{\text{green}}(p_e^{\text{max}})$. Note that g_r is uniquely defined for r and does not depend in any way on the value of $\omega(e)$. Next, consider the paths q_e from u to v that do include the edge e and for which $\omega_{\text{green}}(q_e) < g_r$. If there is no such path then take $\Delta_e(r) = \infty$ for convenience, otherwise let q_e^{min} denote such a path that has least red weight and take $\Delta_e(r) = \omega_{\text{red}}(q_e^{\text{min}})$. Note that although $\omega_{\text{red}}(q_e^{\text{min}})$ depends on the value of $\omega(e)$, the weight of this edge does not affect the relative red ordering between the various q_e (since they each include e). Hence the choice of q_e^{min} (or another q_e with same red weight) does not depend in any way on the value of $\omega(e)$, and $s_r = \omega_{\text{red}}(q_e^{\text{min}}) - \omega(e)$ (where s_r is the sum of red weights except for e) is uniquely determined by r and does not depend on the choice of $\omega(e)$.

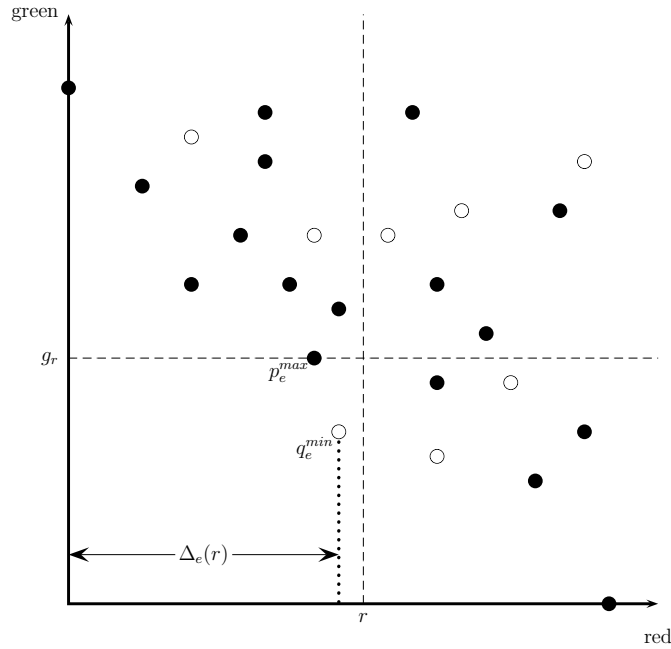


Figure 1: Representation of $\Delta_e(r)$ and associated variables.

Figure 1 illustrates Δ_e and its associated variables, where black dots are used to represent paths not using edge e and white dots represent the paths containing e . Note that all q_e paths just shift horizontally depending on the value of $\omega(e)$. However, they do not change their positions relative to each other.

Lemma 3.1. *For any $r \geq 0$ and $\varepsilon \geq 0$, if $\Delta_e(r) < \infty$ then*

$$\mathbb{P}(r < \Delta_e(r) \leq r + \varepsilon) \leq \phi_e \cdot \varepsilon.$$

Proof. Let $r \geq 0$, $\varepsilon > 0$ and q_e^{min} be a path that includes e with $\omega_{\text{green}}(q_e^{min}) < g_r$ and $\omega_{\text{red}}(q_e^{min})$ minimal amongst such paths. As s_r does not depend on the value of $\omega(e)$ one has:

$$\begin{aligned} \mathbb{P}(r < \Delta_e(r) \leq r + \varepsilon) &= \mathbb{P}(r < s_r + \omega(e) \leq r + \varepsilon) \\ &= \mathbb{P}(r - s_r < \omega(e) \leq r - s_r + \varepsilon) \\ &= \int_{r-s_r}^{r-s_r+\varepsilon} f_e(x) dx \\ &\leq \int_{r-s_r}^{r-s_r+\varepsilon} \phi_e dx = \phi_e \cdot \varepsilon. \end{aligned}$$

□

Now, suppose that all the red edges e of the graph \mathbf{G} have weights that are random variables with bounded probability density functions, and suppose that besides a pure green path from u to v there is also a pure coloured-edge path in red from u to v , with a minimal pure coloured-edge path in red having red weight r_{tot} .

Define the function $\Delta : [0, r_{tot}) \rightarrow (0, \infty)$ for $0 \leq r < r_{tot}$ by the following. Consider the minimal paths q from u to v for which $\omega_{\text{red}}(q) > r$. Since there is a pure red path with weight r_{tot} , there are such paths q , and since \mathbf{G} is finite there are only finitely many such paths. Take q^{min} to be a minimal path with $\omega_{\text{red}}(q^{min}) > r$ that has least red weight and take $\Delta(r) = \omega_{\text{red}}(q^{min}) > r$. Figure 2 illustrates $\Delta(r)$ and q^{min} .

Lemma 3.2. *For $0 \leq r < r_{tot}$, there exists a red edge e for which $\Delta(r) = \Delta_e(r)$.*

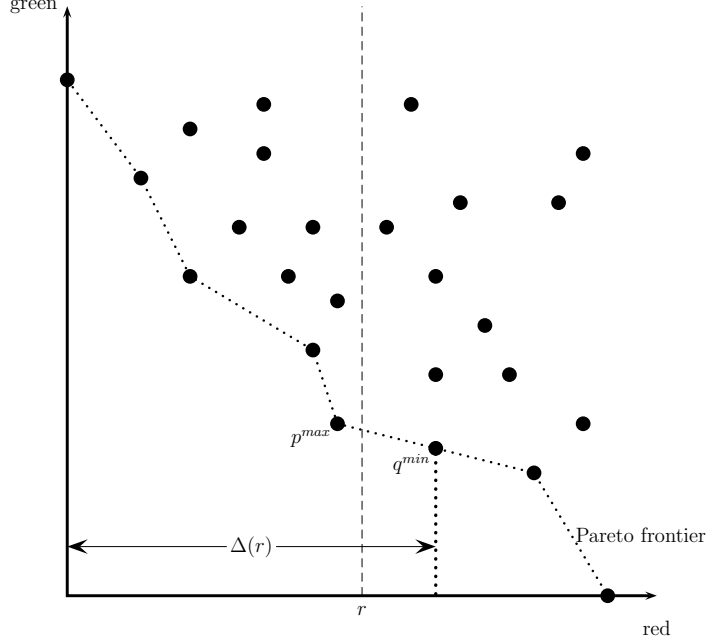


Figure 2: Representation of $\Delta(r)$ and associated variables.

Proof. Consider the minimal paths p from u to v for which $\omega_{\text{red}}(p) \leq r$. Since there is a pure green path from u to v there are such paths, and since \mathbf{G} is finite there are only finitely many such paths. Take p^{max} to be a minimal path with $\omega_{\text{red}}(p) \leq r$ that has least green weight. Then p^{max} and q^{min} are adjacent minimal paths on the Pareto frontier, so there can be no path between p^{max} and q^{min} in the sense that no path can have both its red weight less than $\omega_{\text{red}}(q^{\text{min}})$ and its green weight less than $\omega_{\text{green}}(p^{\text{max}})$.

Since $\omega_{\text{red}}(p^{\text{max}}) \leq r < \omega_{\text{red}}(q^{\text{min}})$ there must be some red edge e that is in q^{min} but not in p^{max} . As p^{max} has the least green weight amongst paths p_e that do not include e and for which $\omega_{\text{red}}(p_e) \leq r$, $\omega_{\text{green}}(p_e^{\text{max}}) \leq \omega_{\text{green}}(p^{\text{max}})$. As $\omega_{\text{red}}(p_e^{\text{max}}) \leq r < \omega_{\text{red}}(q^{\text{min}})$, and there are no paths between p^{max} and q^{min} it follows that $\omega_{\text{green}}(p_e^{\text{max}}) = \omega_{\text{green}}(p^{\text{max}})$. Hence $g_r = \omega_{\text{green}}(p^{\text{max}})$.

As p^{max} and q^{min} are incomparable $\omega_{\text{green}}(q^{\text{min}}) < \omega_{\text{green}}(p^{\text{max}}) = g_r$. Next, as q^{min} has the least red weight amongst paths q_e that do include e and for which $\omega_{\text{green}}(q_e) < g_r$, $\omega_{\text{red}}(q_e^{\text{min}}) \leq \omega_{\text{red}}(q^{\text{min}})$. But since $\omega_{\text{green}}(q_e^{\text{min}}) < g_r = \omega_{\text{green}}(p^{\text{max}})$ and there are no paths between p^{max} and q^{min} , it follows that $\omega_{\text{red}}(q_e^{\text{min}}) = \omega_{\text{red}}(q^{\text{min}})$. Hence one has $\Delta(r) = \omega_{\text{red}}(q^{\text{min}}) = \omega_{\text{red}}(q_e^{\text{min}}) = \Delta_e(r)$. \square

Corollary 3.3. *For any $0 \leq r < r_{tot}$ and $\varepsilon > 0$*

$$\mathbb{P}(\Delta(r) \leq r + \varepsilon) \leq \left(\sum_{\text{red edge } e} \phi_e \right) \cdot \varepsilon.$$

Proof. If $r < \Delta(r) \leq r + \varepsilon$ then by Lemma 3.2, there exists a red edge e for which $r < \Delta_e(r) \leq r + \varepsilon$. Hence using a union bound and Lemma 3.1,

$$\begin{aligned} \mathbb{P}(r < \Delta(r) \leq r + \varepsilon) &\leq \sum_{\text{red edge } e} \mathbb{P}(r < \Delta_e(r) \leq r + \varepsilon) \\ &\leq \sum_{\text{red edge } e} \phi_e \cdot \varepsilon. \end{aligned}$$

□

Corollary 3.3 provides the main argument to establish a bound on the expected number of minimal paths for a bicoloured-edge graph.

Theorem 3.4. *Let \mathbf{G} be a finite weighted bicoloured-edge graph and let u, v be vertices of \mathbf{G} for which there is a pure colour path from u to v in each of the two colours. Suppose that the weights of all edges e in one of the colours are random variables with probability density functions bounded above by ϕ_e , and let r_{tot} denote the weight of the minimal pure colour path in that colour. Then the expected number of Pareto minimal elements with distinct weights is bounded above by $\sum_e \phi_e \cdot r_{tot} + 1$.*

Proof. As previously denote the colours by {red, green} for convenience. As \mathbf{G} is finite there are only finitely many minimal paths q from u to v , and they all have red weight between 0 and r_{tot} inclusive.

Partition the interval $(0, r_{tot}]$ into κ equal subintervals and note that since only minimal paths with distinct (red) weights are considered, there is a threshold κ_{min} above which each interval can contain at most one minimal path. Hence for all $\kappa \geq \kappa_{min}$, the expected number of distinct minimal paths is

$$1 + \sum_{i=0}^{\kappa-1} \mathbb{P} \left(\exists \text{ minimal path } q \text{ with } \frac{r_{tot}}{\kappa} i < \omega_{\text{red}}(q) \leq \frac{r_{tot}}{\kappa} (i + 1) \right)$$

including the minimal pure green path that has red weight 0. Figure 3 depicts the partition of $(0, r_{tot}]$.

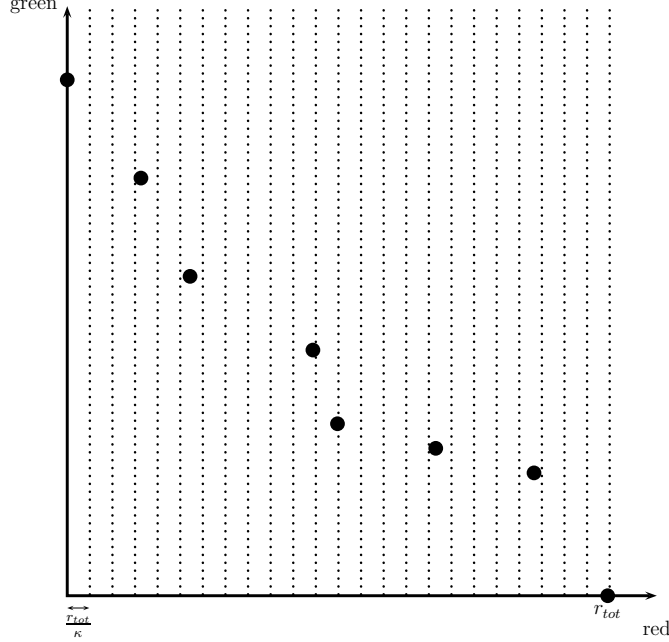


Figure 3: Pareto minimal elements and partition of the interval $(0, r_{tot}]$.

Now for each i there is a minimal path q with $\frac{r_{tot}}{\kappa}i < \omega_{red}(q) \leq \frac{r_{tot}}{\kappa}(i+1)$ if and only if $\Delta(\frac{r_{tot}}{\kappa}i) \leq \frac{r_{tot}}{\kappa}(i+1)$ which has probability bounded above by $(\sum \phi_e) \frac{r_{tot}}{\kappa}$ by Corollary 3.3. Hence the expected number of minimal paths with distinct weights is bounded by

$$1 + \sum_{i=0}^{\kappa-1} \left(\sum \phi_e \right) \frac{r_{tot}}{\kappa} = 1 + \left(\sum \phi_e \right) \cdot r_{tot}.$$

□

Note that if each red edge has weight bounded by r_{max} then r_{tot} is $O(n)$, so the expected number of minimal paths is $O(mn) + 1$ where m denotes the number of red edges. Furthermore, there are only at most $n^2 - 3n + 3$ red edges to consider in a graph with n vertices so the order is bounded by $O(n^3)$.

4. Conclusions

For weighted coloured-edge graphs to be successfully utilized in multi-modal network applications the number of minimal paths needs to be man-

ageable, as each minimal path may need to be further investigated to determine a sought optimal path or paths in a particular application. The chain example from Section 1 with exponentially growing weights illustrates that there can be an exponential number of incomparable minimal paths in a weighted coloured-edge graph. However, Theorem 2.3 shows that this is a tight bound, and the fact that there can only be an exponential number k^{n-1} of minimal paths rather than potentially a factorial number $O(k^{n-1}(n-2)!)$ is surprising. As a consequence, even in the worst case moderately sized graphs (with around 20 – 30 vertices) can still be feasibly tackled.

Experimental studies undertaken by the authors indicate that the number of minimal paths in real networks is typically a low-order polynomial function of n , so very large networks can be studied in practice. Theorem 3.4 justifies this observation for bicoloured-edge graphs whose edge weights are randomly drawn from a bounded probability density function, showing that only $O(n^3)$ minimal paths are expected. It is presumed that this polynomial bound can be substantially reduced and that a similar result is also true for coloured-edge graphs with $k > 2$ colours.

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