The Isomorphism Problem for Automatic Trees and Linear Orders

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(Joint work with D.Kuske and M.Lohrey)
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From Computable Structures to Automatic Structures

- A structure $\mathcal{A}$ consists of a set D(universes) and relations and functions on D. We assume all structures are countably infinite and relational.
- Computable structures:
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- Automatic Structures [Khoussainov&Nerode 1995]: Replace Turing machines by finite automata.

Automata recognizing relations

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Definition.

For words $w_1, \ldots, w_n \in \Sigma^*$, the convolution is the word $\otimes(w_1, \ldots, w_n)$ in alphabet $(\Sigma \cup \{\diamond\})^n$

$$(w'_1[1], \ldots, w'_n[1])(w'_1[2], \ldots, w'_n[2])(w'_1[3], \ldots, w'_n[3]) \cdots (w'_1[\ell], \ldots, w'_n[\ell]).$$

where $\ell = \max\{|w_i| \mid 1 \leq i \leq n\}$ and $w'_i[j] = w_i[j]$ if $j < |w_i|$ and $\diamond$ otherwise.

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An n-ary relation $R \subseteq (\Sigma^*)^n$ is automatic if the language $\otimes R = \{\otimes(\overline{w}) \mid \overline{w} \in R\}$ is accepted by some automaton M.

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- Any tuple $\mathcal{P}$ of automata that accept the domain and the relations of $\mathcal{S}$ is called an automatic presentation of $\mathcal{S}$.

Examples of automatic structures

- $(\mathbb{N}; <) \cong (0^*; \{\otimes(0^i, 0^j) \mid i < j\})$.
- $(\mathbb{N}; +)$
- $(\mathbb{Q}; \leq) \cong ((0+1)^*1; \leq_{\text{lex}})$.
- The full tree $(\mathbb{N}^*; \leq_{\text{pref}}) \cong (\{1\} \cup 1\{0,1\}^*1; \leq_{\text{pref}})$.
- Configuration graph of a Turing machine.

FO-Decidability

Theorem. [KN]

There is an algorithm that, given an automatic structure $\mathcal{S}$ and a FO-formula $\varphi(\overline{n})$, produces an automaton recognizing precisely those tuples $\overline{a} \in \mathcal{S}$ that make φ true. In particular the FO-theory of $\mathcal{S}$ is decidable.

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[Blumensath, Rubin, Kuske, Lohrey, etc.] FO-decidability also holds if we extend FO by $\exists^{\infty}$, $\exists^{(m,n)}$, and some restricted form of SO existential quantifier.

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Fix a class $\mathcal{K}$ of structures. Decide if two automatic presentations recognize the same structure up to isomorphism, i.e.,

$$\{< P_1, P_2 >| \mathcal{S}(P_1), \mathcal{S}(P_2) \in \mathcal{K} \wedge \exists f : \mathcal{S}(P_1) \cong \mathcal{S}(P_2)\}$$

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- Automatic Structures Σ_1^1-complete [KNRS]
- Automatic well-orders/Boolean algebras Decidable [KNRS]
- Automatic (a) successor trees (b) undirected graphs (c) commutative monoids (d) partial orders (e) lattices of height 4 (f) unary functions. Σ_1^1-complete [Nies]
- Automatic locally finite graphs Π_3^0-complete[Rubin]

Structures with a transitive relation

Questions[KN08]

- What about for other classes of structures? e.g. equivalence structures, order trees, linear orders
- For any level of the arithmetic hierarchy, give a class of automatic structures for which isomorphism problem is complete for that level.

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Theorem.[Kuske,Liu,Lohrey10]

The isomorphism problem is

- Π^0_1-complete for automatic equivalence structures.
- Π^0_{2n-3}-complete for automatic trees of height n ($n \geq 2$).
- computably equivalent to true arithmetic for automatic trees of finite height.
- not arithmetical for automatic linear orders.

- Hilbert's 10th problem:
  $\{p \in \mathbb{Z}[x_1, \ldots, x_k] \mid \exists x_1, \ldots, x_k \in \mathbb{N}^+ : p(x_1, \ldots, x_k) = 0\}$.

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- [Matiyasevich] The following problem is Π^0_1-complete:
$Prob = \{< p_1, p_2 >\mid p_i \in \mathbb{N}[x_1, \ldots, x_k], \forall x_1, \ldots, x_k \in \mathbb{N}^+ :$
$p_1(x_1, \ldots, x_k) \neq p_2(x_1, \ldots, x_k)\}$.

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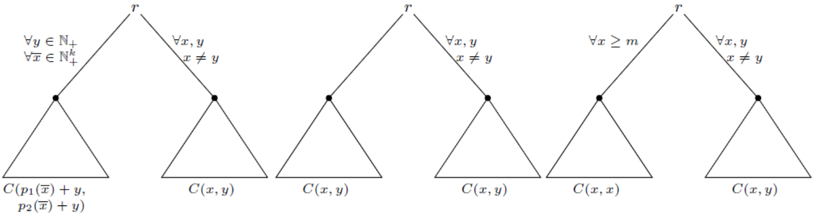
- We can construct automatic height-2 trees $T^2_{p_1,p_2}, T^2_{Good}, T^2_{Bad,m}$ $(m \in \mathbb{N})$ such that
  - $< p_1, p_2 > \in Prob$ if and only if $T^2_{p_1,p_2} \cong T^2_{Good}$
  - $< p_1, p_2 > \notin Prob$ if and only if $T^2_{p_1,p_2} \cong T^2_{Bad,m}$ for some m.

Fix an injective polynomial $C : \mathbb{N}^2 \to \mathbb{N}$.



The tree $T^2_{p_1,p_2}$ The tree T^2_{Good} The tree $T^2_{Bad,m}$

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- For this we only need Σ^0_2-hardness for automatic trees of height 3.

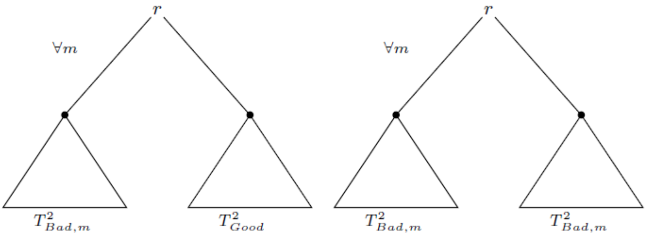
Trees of Height 3

Lemma.

There exists two height-3 trees T^3_{Good} and T^3_{Bad} ($T^3_{Good} \ncong T^3_{Bad}$) such that the following holds: For a given Σ^0_2-set $A \subseteq \{0,1\}^*1$ one can effectively construct an automatic forest F_A of height 3 such that

- The set of roots of F_A is $\{0,1\}^*1$.
- For every $w \in \{0,1\}^*1$, $F_A(w) \cong T^3_{Good}$ if $w \in A$ and $F_A(w) \cong T^3_{Bad}$ if $w \notin A$.

T^3_{Good} and T^3_{Bad}



The tree T^3_{Good} The tree T^3_{Bad}

Σ_1^1-Hardness for Automatic Order Trees

Theorem.[Kuske,Liu,Lohrey, in preparation]

The isomorphism problem for automatic order trees is Σ_1^1-complete.

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- For a computable tree T, construct an automatic order tree *$aut(T)$* as follows:
  - Append to each node $x \in \mathbb{N}^*$ of the full tree a copy of the tree T^3_{Good} if $x \in T$; and append a copy of T^3_{Bad} if $x \notin T$.

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  - Append to each node $x \in \mathbb{N}^*$ of the full tree a copy of the tree T^3_{Good} if $x \in T$; and append a copy of T^3_{Bad} if $x \notin T$.
- $T \cong T'$ if and only if $aut(T) \cong aut(T')$

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- Recall $(\{0,1\}^*1; \leq_{lex})$ is a copy of $(\mathbb{Q}; \leq)$.
- From a computable linear order L, one can compute an index of a computable set $P(L) \subseteq \{0,1\}^*1$ whose complement is dense in $(\{0,1\}^*1; \leq_{lex})$ such that $L \cong L'$ iff $(\{0,1\}^*1; \leq_{lex}, P(L)) \cong (\{0,1\}^*1; \leq_{lex}, P(L'))$.

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- There exist two automatic linear orders M_0 M_1 such that
  - We can construct an automatic linear order $aut(L)$ by replacing each $w \in \{0,1\}^*1$ by a copy of M_0 if $w \in P(L)$, and replacing w by a a copy of M_1 if $w \notin P(L)$.

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  - $(\{0,1\}^*; \leq_{lex}, P(L)) \cong (\{0,1\}^*; \leq_{lex}, P(L'))$ iff $aut(L) \cong aut(L')$.

Non-existence of Hyperarithmetic Isomorphisms

Corollary

There exists two isomorphic automatic order trees (linear orders) without a hyperarithmetic isomorphism.